Optimal quantum chain communication by end gates

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The scalability of solid state quantum computation relies on the ability of connecting the qubits to the macroscopic world. Quantum chains can be used as quantum wires to keep regions of external control at a distance. However even in the absence of external noise their transfer fidelity is too low to assure reliable connections. We propose a method of optimizing the fidelity by minimal usage of the available resources, consisting of applying a suitable sequence of two-qubit gates at the end of the chain.

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Introduction:— It is often noted that the advantage of solid state computation is its scalability. This is because a typical chip can contain millions of qubits and because the fabrication of many qubits is in principle no more difficult than the fabrication of a single one. In the last couple of years, remarkable progress was made in experiments with quantum dots \cite{1} and superconducting qubits \cite{2}. It should however be emphasized that for initialization, gating and readout, those qubits have to be connected to the macroscopic world. For example, in a typical flux qubit gate, microwave pulses are applied onto specific qubits of the sample. This requires many (classical) wires on the chip, which is thus a compound of quantum and classical components. The macroscopic size of the classical control is likely to be the bottleneck of the scalability as a whole.

In this situation, quantum chains may turn out to be extremely useful in order to keep some distance between the controlled quantum parts. They consist of lines of coupled single qubits \textit{without external classical control}. In many cases, such permanent couplings are easy to build in solid state devices (in fact a lot of effort usually goes into \textit{suppressing} such couplings). We imagine a setup for a quantum computer similar to the scheme of Fig. \ref{fig:1}. It is built out of blocks of qubits, some of which are dedicated to communication and therefore connected to another block through a quantum chain. Within each block, arbitrary unitary operations can be performed in a fast and reliable way (they may be decomposed into single and two-qubit operations). Such blocks do not currently exist, but they are focus of much work in solid state quantum computer architecture. The distance between the blocks is determined by the length of the quantum chains between them. It should be large enough to allow for classical control wiring of each block, but short enough such that the timescale of the quantum chain communication is well below the timescale of decoherence in the system.

Many interesting aspects of quantum chain communication were investigated in the last years \cite{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}, both from a physics point of view and from a quantum information point of view. Here, we would like to concentrate on those schemes \cite{4, 5, 6, 7} which require no further resources than those outlined in Fig. \ref{fig:1}. The chain couplings may be engineered \cite{8, 9} to improve the theoretical communication fidelity, but coupling fluctuations and energy mismatches will lower the fidelity in practice \cite{8, 9, 10, 11}. Hence even without the contribution of external noise \cite{11, 12} the quality of transfer may well be too low to yield a scalable system.

In this article we will show that the fidelity can be improved easily using the gates available in the regions of quantum control. The main idea is to apply in certain time-intervals two-qubit gates at the receiving end of the chain. The resulting sequence is determined \textit{a priori} by the Hamiltonian of the system. As we shall see, the maximal fidelity that can be reached this way is limited only by external noise, and not by the spatial fluctuations of the couplings (cf. \cite{11}). This is similar in spirit to the dual-rail \cite{11} and memory protocols \cite{12}, but here we give a protocol that is \textit{optimal} in the resources used: a single spin chain and a two-qubit gate at the each end. It is optimal because two-qubit gates at the send-
A crucial ingredient to our protocol is the operator \( t \) time the chain. As mentioned in the introduction, the fidelity of transporting from the first to the last qubit of the realization of the blocks from Fig. 1. Furthermore, we show numerically that our protocol could also be realized by a simple switchable interaction. This means that quantum state transfer experiments with our protocol could be performed well before the realization of the blocks from Fig. 1.

**Arbitrarily Perfect State Transfer:** Let us now concentrate on a single chain from the setup of Fig. 1 and show how the receiving block can improve the fidelity to an arbitrarily high value by applying two-qubit gates between the end of the chain and a “target qubit” of the block. We label the qubits of the chain by 1, 2, \ldots, \( N \) and the target qubit by \( N + 1 \). We define the states

\[
|0\rangle \equiv |00\rangle \quad |n\rangle \equiv \sigma^+_n|0\rangle \quad n = 1, 2, \ldots, N + 1,
\]

where \( \sigma^+_n \) is the Pauli \( \sigma^+ \) operator acting on the \( n \)th qubit.

The coupling of the chain is described by a Hamiltonian \( H \). We assume that the Hamiltonian \( H \) has a \( N \)-dimensional invariant subspace spanned by the vectors

\[
\{ |n\rangle; \quad n = 0, 1, 2, \ldots, N + 1 \}
\]

and that

\[
H|0\rangle = H|N + 1\rangle = 0. \tag{1}
\]

The first assumption corresponds to a Hamiltonian that conserves the number of excitations along the chain, which would be the case for example for a Heisenberg or XY chain. Note that in some cases this may only be an approximation \[14\]. For what follows we restrict all operators to the \( N + 2 \)-dimensional Hilbert space

\[
\mathcal{H} = \text{span} \{ |n\rangle; \quad n = 0, 1, 2, \ldots, N + 1 \}.
\]

Note that by Eq. \[14\], also the space \( \mathcal{H} \) is invariant under \( H \). Our final assumption about the Hamiltonian of the system is that there exists a time \( t \) such that

\[
\langle N | \exp \{-itH\} | 1 \rangle \neq 0.
\]

Physically this means that the Hamiltonian has the capability of transporting from the first to the last qubit of the chain. As mentioned in the introduction, the fidelity of this transport may be very bad in practice.

We denote the unitary evolution operator for a given time \( t_k \) as \( U_k \equiv \exp \{-it_kH\} \) and introduce the projector

\[
P = \mathbb{1}_\mathcal{H} - |0\rangle\langle 0| - |N\rangle\langle N| - |N + 1\rangle\langle N + 1|.
\]

A crucial ingredient to our protocol is the operator

\[
W(c, d) \equiv P + |0\rangle\langle 0| + d|N\rangle\langle N| + d'|N + 1\rangle\langle N + 1| + c'|N + 1\rangle\langle N| - c|N\rangle\langle N + 1|,
\]

where \( c \) and \( d \) are complex normalized amplitudes. It is easy to check that

\[
WW^\dagger = W^\dagger W = \mathbb{1}_\mathcal{H},
\]

so \( W \) is a unitary operator on \( \mathcal{H} \). \( W \) acts as the identity on all but the last two qubits, and can hence be realized by a local two-qubit gate on the qubits \( N \) and \( N + 1 \). Furthermore we have \( WP = P \) and

\[
W(c, d) \left[ \{ |c\rangle N \rangle + d|N + 1\rangle \right] = |N + 1\rangle. \tag{2}
\]

The operator \( W(c, d) \) has the role of moving probability amplitude \( c \) from the \( N \)th qubit to target qubit. It can be applied locally by the receiving block.

Using the time-evolution operator and two-qubit unitary gates on the qubits \( N \) and \( N + 1 \) we will now develop a protocol that transforms the state \( |1\rangle \) into \( |N + 1\rangle \). Let us first look at the action of \( U_1 \) on \( |1\rangle \). Using the projector \( P \) we can decompose this time-evolved state as

\[
U_1|1\rangle = PU_1|1\rangle + \langle N|NU_1|1\rangle
\]

\[
= PU_1|1\rangle + \sqrt{p_1}\{c_1|N\rangle + d_1|N + 1\rangle\},
\]

where \( p_1 = \langle N|U_1|1\rangle^2 \), \( c_1 = \langle N|U_1|1\rangle/\sqrt{p_1} \) and \( d_1 = 0 \). Let us now consider the action of \( W_1 \equiv W(c_1, d_1) \) on the time-evolved state. By Eq. \[14\] it follows that

\[
W_1U_1|1\rangle = PU_1|1\rangle + \sqrt{p_1}|N + 1\rangle. \tag{3}
\]

Hence with a probability of \( p_1 \), the excitation is now in the position \( N + 1 \), where it is “frozen” (since that qubit is not coupled to the chain. We will now show that at the next step, this probability is increased. Applying \( U_2 \) to Eq. \[3\] we get

\[
U_2W_1U_1|1\rangle
\]

\[
= PU_2PU_1|1\rangle + \langle N|U_2PNU_1|1\rangle|N\rangle + \sqrt{p_1}|N + 1\rangle
\]

\[
= PU_2PNU_1|1\rangle + \sqrt{p_2}\{c_2|N\rangle + d_2|N + 1\rangle\}
\]

with \( c_2 = \langle N|U_2PNU_1|1\rangle/\sqrt{p_2} \), \( d_2 = \sqrt{p_1}/\sqrt{p_2} \) and

\[
p_2 = p_1 + \langle N|U_2PNU_1|1\rangle^2 \geq p_1.
\]

Applying \( W_2 \equiv W(c_2, d_2) \) we get

\[
W_2U_2W_1U_1|1\rangle = PU_2PU_1|1\rangle + \sqrt{p_2}|N + 1\rangle.
\]
Repeating this strategy \( \ell \) times we get
\[
\left( \prod_{k=1}^{\ell} W_k U_k \right) |1\rangle = \left( \prod_{k=1}^{\ell} P U_k \right) |1\rangle + \sqrt{p_\ell} |N + 1\rangle, \tag{4}
\]
where the products are arranged in the time-ordered way. Using the normalization of the r.h.s. of Eq. (4) we get
\[
p_\ell = 1 - \left( \left\| \left( \prod_{k=1}^{\ell} P U_k \right) |1\rangle \right\|^2.
\]
From Ref. [11] we know that there exists a \( \tau > 0 \) such that for equal time intervals \( t_1 = t_2 = \ldots = t_k = \tau \) we have \( \lim_{\ell \to \infty} p_\ell = 1 \). Therefore the limit of infinite gate operations for Eq. (4) is given by
\[
\lim_{\ell \to \infty} \left( \prod_{k=1}^{\ell} W_k U_k \right) |1\rangle = |N + 1\rangle. \tag{5}
\]
It is also easy to see that \( \lim_{k \to \infty} d_\ell = 1 \), \( \lim_{k \to \infty} c_\ell = 0 \) and hence the gates converge to the identity operator
\[
\lim_{k \to \infty} W_k = \mathbb{I}_\mathcal{H}.
\]
Furthermore, since \( W_k U_k |0\rangle = |0\rangle \) it also follows that an arbitrary and unknown qubit at the first site,
\[
|\psi_{\text{initial}}\rangle = \alpha |0\rangle + \beta |1\rangle,
\]
is transferred to the last site,
\[
|\psi_{\text{final}}\rangle = \alpha |0\rangle + \beta |N + 1\rangle,
\]
i.e. an arbitrarily perfect state transfer is achieved. As discussed in [12], this convergence is asymptotically exponentially fast in the number of gate applied (a detailed analysis of the relevant scaling can be found in [11]). Equation (5) is a surprising result, which shows that any non-perfect transfer can be made arbitrarily perfect by only applying two-qubit gates on one end of the quantum chain. It avoids restricting the gate times to specific times (as opposed to the scheme in [11]) while requiring no additional memory qubit (as opposed to the scheme in [13]).

The sequence \( W_k \) that needs to be applied to the end of the chain to perform the state transfer is only depending on the Hamiltonian of the quantum chain. The relevant properties can in principle be determined a priori by preceding measurements and tomography on the quantum chain (as discussed in [11]).

**Practical Considerations:** Motivated by the above result we now investigate how the above protocol may be implemented in practice, well before the realization of the quantum computing blocks from Fig. 1. The two-qubit gates \( W_k \) are essentially rotations in the \( \{|01\rangle, |10\rangle\} \) space of the qubits \( N \) and \( N + 1 \). It is therefore to be expected that they can be realized (up to a irrelevant phase) by a switchable Heisenberg or \( XY \)-type coupling between the \( N \)th and the target qubit. However in the above, we have assumed that the gates \( W_k \) can be applied instantaneously, i.e. in a time-scale much smaller than the time-scale of the dynamics of the chain. This corresponds to a switchable coupling that is much stronger than the coupling strength of the chain. Here, we numerically inves-

Figure 3: Numerical example for the convergence of the success probability. Simulated is a quantum chain of length \( N = 20 \) with the Hamiltonian from Eq. (6) (dashed line) and Eq. (7) with \( B/J = 20 \) (solid line). Using the original protocol [1], the same chain would only reach a success probability of 0.63 in the above time interval.

We have investigated two types of switching. For the first type, the coupling itself is switchable, i.e.
\[
H(t) = J \sum_{n=1}^{N-1} \sigma_n^- \sigma_{n+1}^+ + \Delta(t) \sigma_N^- \sigma_{N+1}^+ + \text{h.c.}, \tag{6}
\]
where \( \Delta(t) \) can be 0 or 1. For the second type, the target qubit is permanently coupled to the remainder of the chain, but a strong magnetic field on the last qubit can be switched,
\[
H(t) = J \sum_{n=1}^{N} \sigma_n^- \sigma_{n+1}^+ + \text{h.c.} + B \Delta(t) \sigma_N^z, \tag{7}
\]
where again $\Delta(t)$ can be 0 or 1 and $B \gg 1$. This suppresses the coupling between the $N$th and $N+1$th qubit due to an energy mismatch.

In both cases, we first numerically optimize the times for unitary evolution $t_k$ over a fixed time interval such that the probability amplitude at the $N$th qubit is maximal. The algorithm then finds the optimal time interval during which $\Delta(t) = 1$ such that the probability amplitude at the target qubit is increased. In some cases the phases are not correct, and switching on the interaction would result in probability amplitude floating back into the chain. In this situation, the target qubit is left decoupled and the chain is evolved to the next amplitude maximum at the $N$th qubit. Surprisingly, even when the time-scale of the gates is comparable to the dynamics, near-perfect transfer remains possible (Fig 3). In the case of the switched magnetic field, the achievable fidelity depends on the strength of the applied field. This is because the magnetic field does not fully suppress the coupling between the two last qubits. A small amount of probability amplitude is lost during each time evolution $U_k$, and when the gain by the gate is compensated by this loss, the total success probability no longer increases.

Conclusions and Acknowledgments:— Arbitrarily perfect state transfer can be achieved by applying a sequence of two-qubit gates at the receiving end of a quantum chain. Surprisingly, the gates can be realized by a switchable interaction of the same strength as the chain coupling. We would like to thank Floor Pauw, Andriy Lyakhov and Christoph Bruder for stimulating discussions. DB acknowledges the support of the UK Engineering and Physical Sciences Research Council, Grant Nr. GR/S62796/01. VG is grateful to SB for the hospitality at UCL.