Equivalence Classes for Gauge Theories

MÁRCIO A. M. GOMES(*)1 AND R. R. LANDIM(**)1

1 Departamento de Física, Universidade Federal do Ceará, Caixa Postal 6030, 60455-900, Fortaleza, Ceará, Brazil

PACS. 11.15.-q, 11.10.Kk –

Abstract. – In this paper we go deep into the connection between duality and fields redefinition for general bilinear models involving the 1-form gauge field $A$. A duality operator is fixed based on “gauge embedding” procedure. Dual models are shown to fit in equivalence classes of models with the same fields redefinitions.

Beside the importance of duality in understanding various non-perturbative aspects of field theory and its fundamental role played in strings theories, several interesting generalizations of self-dual Chern-Simons-Proca model [1] and its equivalent model, [2] the three dimensional topologically massive electrodynamics [3–5], has been studied in literature. Soon after the work of Deser and Jackiw, it was realized that self-duality can also occur in Maxwell-Chern-Simons-Proca model [6]. This model has also been used in the study of bosonization in higher dimensions [7, 8]. Recently it was shown that there exists a unified theory [9] from which self-dual model [1], topologically massive electrodynamics [3–5] and Maxwell-Chern-Simons-Proca systems [6] can be recovered as special cases.

More recently, Lemes et al [10, 11], showed that both abelian and non-abelian Maxwell-Chern-Simons models can be reset into a pure Chern-Simons term through a suitable local field redefinition. This was used to evaluate a fermionic determinant [12] and to study the large-mass behavior of the linking-number in Maxwell-Chern-Simons theory (MCS) [13]. In a recent paper [14], we put the question if two dual theories do share the same fields redefinition. We showed that MCS and Maxwell-Proca and their dual models do. A more profound investigation about this apparently obvious property reveals that in fact one can construct equivalence classes embodying all theories with same fields redefinitions, dual models included.

We begin with the more general bilinear action with a global symmetry. By construction, it is divided into three parts: a masslike term, and two terms involving derivatives, one of which is gauge-invariant and another term that is not. These two sectors are also shown to be orthogonal to each other. Naturally, the non-gauge-invariant part drops out in the process of establishing duality through a ”gauge embedding” algorithm [15]. We then present a more direct way to dualize this kind of model by introducing a duality operator.

(*) Electronic address: gomes@fisica.ufc.br
(**) E-mail: renan@fisica.ufc.br
© EDP Sciences
Next, we extract group properties from redefining fields by introducing an abelian group whose action on the connection gauge field produces another connection gauge field. We call the space generated by the group action on $A$ ”the space of redefined fields”. Gauge theories having the same field redefinition are collected into classes of equivalence and a criterion is established to built these classes. Through all paper we lay hold of functional calculus with differential forms rules introduced in [14].

In order to have a better understanding about duality and field redefinition, we will derive general properties of local bilinear actions constructed with a gauge field $A$.

Lemma 1 The most general local bilinear action constructed with a real one-form field $A$ is

$$S = \frac{b_0}{2}(A, A) + \frac{1}{2}(A, \hat{B} A) + \frac{1}{2}(A, \hat{C} A),$$

(1)

where $b_0$ is a constant with canonical dimension $C(b_0) = 3 - 2C(A)$,

$$\hat{B} = \sum_{i=1}^{M} b_i (d * d^*)^i,$$  

(2a)

$$\hat{C} = \sum_{i=1}^{N} c_i (\ast d)^i,$$  

(2b)

with $C(b_i) = 3 - 2C(A) - 2i$ and $C(c_i) = 3 - 2C(A) - i$.

Proof:

A bilinear action on field $A$ has the general form $(A, O A)$, where $O$ is an operator that maps a one-form into another, i.e. $O : \omega_1 \rightarrow \Omega_1$. The operator $O$ must be constructed with the exterior derivative $d$ and the Hodge operator $\ast$. Since $d^2 = 0$ and $\ast \ast = \pm 1$, there are only two operators in three dimension that maps a one-form into another:

$$\ast d : \omega_1 \rightarrow \Omega_1,$$  

(3a)

$$d * d^* : \omega_1 \rightarrow \Omega_1,$$  

(3b)

hence $O$ has the most general form

$$O = b_0 + \sum_{i=1}^{M} b_i (d * d^*)^i + \sum_{i=1}^{N} c_i (\ast d)^i.$$

(4)

Note that the Laplacian operator $\Delta = dd^\dagger + d^\dagger d$ is a particular case of (4). The operators $\hat{B}$ and $\hat{C}$ given by (2) satisfy the following properties

$$\hat{B} \hat{C} = \hat{C} \hat{B} = 0,$$  

(5a)

$$(\hat{B} \omega_1, \Omega_1) = (\omega_1, \hat{B} \Omega_1),$$  

(5b)

$$(\hat{C} \omega_1, \Omega_1) = (\omega_1, \hat{C} \Omega_1),$$  

(5c)

for any one-forms $\omega_1$ and $\Omega_1$. The first property of (5) is due to the fact that $(\ast d)(d * d^*) = 0$ and $(d * d^*) (\ast d) = 0$. The others follow from $(\ast d \omega_1, \Omega_1) = (\omega_1, \ast d \Omega_1)$ and $(d * d * \omega_1, \Omega_1) = (\omega_1, d * d * \Omega_1)$. $\hat{B}$ and $\hat{C}$ form two orthogonal spaces in sense that $(\hat{B} \omega_1, \hat{C} \Omega_1) = (\omega_1, \hat{B} \hat{C} \Omega_1) =$.
Let us observe that if $\omega_1 = \hat{C} A$, then $\delta \omega_1 = \hat{C} d \omega_0 = 0$. Consequently the space generated by $\hat{C}$ is gauge invariant. Conversely, the space generated by $\hat{B}$ is not gauge invariant. The $\hat{B}$ term is gauge fixing.

For the bilinear action given by $\text{Eq.} (1)$ with $b_0 \neq 0$, we can write the operator $O$ in a more suitable way

$$O = b_0 \left( 1 + \hat{B} \left( \frac{1}{b_0} \right) + \hat{C} \left( \frac{1}{b_0} \right) \right) = b_0 O_1^2 O_2^2,$$

(6)

where

$$O_1 = \left( 1 + \frac{\hat{C}}{b_0} \right)^{1/2} = 1 + \sum_{j=1}^{\infty} \alpha_j \left( -\frac{\hat{C}}{b_0} \right)^j,$$

(7a)

$$O_2 = \left( 1 + \frac{\hat{B}}{b_0} \right)^{1/2} = 1 + \sum_{j=1}^{\infty} \alpha_j \left( -\frac{\hat{B}}{b_0} \right)^j,$$

(7b)

with $\alpha_j = \frac{(2j)!}{2^{2j}(2j-1)(j!)^2}$ being the expansion coefficients of the power series. The operators $O_1$ and $O_2$ satisfy the following properties

$$[O_1, O_2] = 0,$$

(8a)

$$[\ast d, O_1] = 0,$$

(8b)

$$[d \ast, O_2] = 0,$$

(8c)

$$\ast d O_2 = O_2 \ast d = \ast d,$$

(8d)

$$O_1 d \ast = d \ast O_1 = d \ast.$$  

(8e)

Using (6), the action in $\text{Eq.} (1)$ now reads

$$S = \frac{b_0}{2} (O_1 O_2 A, O_1 O_2 A).$$

(9)

We may wonder what type of gauge invariant action is dual to this action. The gauge embedding procedure is used. We define the first iterative action as

$$S^{(1)} = \frac{b_0}{2} (O_1 O_2 A, O_1 O_2 A) - \left( \frac{\delta S}{\delta A} \right) B,$$

(10)

where $B$ is an auxiliary one-form field with $\delta B = \delta A = d \omega_0$. Then

$$\delta S^{(1)} = -b_0 (O_2^2 \hat{C} b_0 \hat{C} A) = \frac{1}{2} \left( \frac{\hat{C} A}{b_0} \right) - \frac{1}{2b_0} (\hat{C} A, \hat{C} A),$$

(11)

where we used (9). The next iterative action is gauge invariant:

$$S^{(2)} = \frac{b_0}{2} (O_1 O_2 A, O_1 O_2 A) - b_0 (O_1 O_2 A, O_1 O_2 B) + b_0 (O_2 B, O_2 B).$$

(12)

Since $O_1$ and $O_2$ are invertible operators, we can eliminate the auxiliary field $B$ to rewrite (12) as a gauge invariant action depending only on field $A$,

$$S_{\text{dual}} = -\frac{b_0}{2} (O_1 A, \frac{\hat{C}}{b_0} O_1 A) = -\frac{1}{2} (A, \hat{C} A) - \frac{1}{2b_0} (\hat{C} A, \hat{C} A).$$

(13)
This is the dual action of (9). Note that the dual action does not depend on the operator $\hat{B}$. An important conclusion was found: The dual action depends only on the gauge invariant sector and the mass term. In other words a gauge fixing term is invisible under duality.

We can express the duality map in a more suitable way. From (9),
\[
\frac{\delta S}{\delta A} = b_0 \hat{C} \hat{O}_2^2 A,
\]
and (13) can be written as
\[
S_{\text{dual}} = -\frac{1}{2b_0} (\hat{C} A, \frac{\delta S}{\delta A}) = -\frac{1}{2b_0} (\hat{C} A, \frac{\delta}{\delta A}) S,
\]
where $\hat{C} \hat{O}_2 = \hat{C}$ was used. Then the duality map is now an operator
\[
\hat{D} = -\frac{1}{2b_0} (\hat{C} A, \frac{\delta}{\delta A}),
\]
that acts on $S$ giving its dual. For the self-dual model given by
\[
S_{SD} = \int \left( \frac{m^2}{2} A^* A + \frac{1}{2} m A d A \right) = \frac{m^2}{2} (A, A) - \frac{m}{2} (A, *d A),
\]
the duality operator is
\[
\hat{D} = \frac{1}{2m} (*d A, \frac{\delta}{\delta A}).
\]
When $b_0 = 0$, the best way to find the dual model is writing the operator $\mathcal{O}$ in terms of the Laplacian and a gauge invariant operator. Since $\Delta = (d^*)^2 - (*d)^2$, we have $(d^*)^2 = \Delta + (*d)^2$, then
\[
\mathcal{O} = \sum_{i=1}^{N} b_i \Delta^i + \hat{C}' = \hat{B}' + \hat{C}',
\]
where $\hat{C}'$ is the gauge invariant part. Then applying the same procedure we find the dual action,
\[
S_{\text{dual}} = -\frac{1}{2} (A, \hat{C}' A) - \frac{1}{2} (\hat{C}' A, \hat{C}' \hat{B}'^{-1} A) = \hat{D} S,
\]
where
\[
\hat{D} = -\frac{1}{2} (\hat{C}' A, \hat{B}'^{-1} \frac{\delta}{\delta A}),
\]
is a non-local duality operator, since $\hat{B}'^{-1}$ is non-local. In this work we are only taking into account the local operators.

Let us go back to local operator $\mathcal{O}_1$. Its action on a gauge field $A$ produces another gauge field with the same gauge transformation, i.e, if $\delta A = d\omega_0$, then $\delta \mathcal{O}_1 A = d\omega_0$. Let us denote this set of operators by $G_1$. An element $g_1$ of $G_1$ has the form
\[
g_1 = (1 + \hat{P}),
\]
with $\hat{P} d = 0$ and $d * \hat{P} = 0$. We shall prove that $G_1$ is an abelian group. In fact, for $g_1$ and $g'_1 \in G_1$,
\[
g_1 g'_1 = (1 + \hat{P})(1 + \hat{P}') = (1 + \hat{P} + \hat{P}' + \hat{P} \hat{P}') = (1 + \hat{P}''),
\]
where $g''_1 \in G_1$. 

since

$$(\hat{P} + \hat{P}' + \hat{P}\hat{P}')d = \hat{P}d + \hat{P}'d + \hat{P}\hat{P}'d = 0$$

and

$$d * (\hat{P} + \hat{P}' + \hat{P}\hat{P}') = d * \hat{P} + d * \hat{P}' + d * \hat{P}\hat{P}' = 0.$$ 

The inverse and identity elements are well defined:

$$(1 + \hat{P})^{-1} = (1 + \sum_{n=0}^{\infty} (-1)^n \hat{P}^n) = (1 + \hat{P}_{-1}), \quad (1 + \hat{P})(1 + \hat{P}_{-1}) = 1.$$ 

The action of $G_1$ on $A$ defines a space of redefined fields. Any redefined field has the same gauge transformation of $A$. An element of the space of redefined fields is $\hat{A} = (1 + \hat{P})A$. A redefined field is completely specified by the action of an element of $G_1$. If $\hat{A} = (1 + \hat{P})A$ and $\hat{A}' = (1 + \hat{P}')A$, then $\hat{A} = \hat{A}' \iff \hat{P} = \hat{P}'$.

For a gauge invariant theory the bilinear action is

$$S = \frac{1}{2}(\hat{C}A, A),$$

with $\hat{C}$ being a polynomial in $*d$ with at least one zero root. We can always express $\hat{C}$ in the form

$$\hat{C} = c_0(*d)^j g_1^2,$$

for some $j \in N$ and $g_1 \in G_1$ with $c_0$ a constant. Then any gauge invariant bilinear action can be written as

$$S = \frac{c_0}{2}(\hat{A}, (*d)^j \hat{A}),$$

where $\hat{A} = g_1 A = (1 + \hat{P})A$ is a redefined field. Let us observe that $(*d)^j g_1$ can not belongs to $G_1$. Therefore another gauge invariant action can present the same redefined field. For example

$$S' = \frac{c_0'}{2}(\hat{A}, (*d)^k \hat{A}),$$

for $k \neq j$, has the same redefined field of (24), but it is a different gauge theory. $S$ and $S'$ are related since they have the same redefined field. We are now ready to construct a relation between these two gauge theories.

**Definition 1** Two gauge actions $S$ and $S'$ are said to belong to the same equivalence class if there exist $p$ and $p' \in N$ and a constant $a_0 \neq 0$ such that

$$a_0(*d)^p \frac{\delta S}{\delta A} = (*d)^{p'} \frac{\delta S'}{\delta A}.$$ 

We shall denote this relation by $\sim$.

It is easy to see that this relation $\sim$ defined above is indeed an equivalence relation: i.e. (i) $S \sim S$ (reflexive); (ii) if $S \sim S'$, then $S' \sim S$ (symmetric); and (iii) if $S \sim S'$ and $S' \sim S''$, then $S \sim S''$ (transitive). Let us verify the last one. If $S \sim S'$ and $S' \sim S''$, then there are numbers $p, p'$ and $q, q''$ and constants $a_0$ and $b_0$ such that

$$a_0(*d)^p \frac{\delta S}{\delta A} = (*d)^{p'} \frac{\delta S'}{\delta A},$$

$$b_0(*d)^q \frac{\delta S'}{\delta A} = (*d)^{q''} \frac{\delta S''}{\delta A}.$$
and
\[
\delta S' = (\ast d) \frac{\delta S'}{\delta A} = (\ast d) \frac{\delta S''}{\delta A}.
\]
Then
\[
\delta S = (\ast d) \frac{\delta S}{\delta A} = (\ast d) \frac{\delta S'}{\delta A} = (\ast d) \frac{\delta S''}{\delta A},
\]
implies that \( S \sim S'' \). The actions that fulfill condition (25) form an equivalence class of gauge theories. We now prove that \( S \) and \( S' \) belonging to the same equivalence class must have the same field redefinition. To see this, let \( S \) and \( S' \) given by
\[
S = \frac{c_0}{2} (\hat{A}, (\ast d)^j \hat{A}),
\]
and
\[
S' = \frac{c'_0}{2} (\hat{A}', (\ast d)^k \hat{A}'),
\]
be two actions that have in principle different field redefinitions. Since by hypothesis they belong to the same equivalent class, \( \exists \, p, p' \) and a constant \( a_0 \) such that
\[
a_0 c_0 (\ast d)^{j+p} g_1^2 A = c'_0 (\ast d)^{k+p'} g_1^2 A,
\]
holds. Now using the fact that \( g_1^2 = 1 + \hat{P} \) and \( g_1^2 = 1 + \hat{P}' \) we have
\[
a_0 c_0 (\ast d)^{j+p} A + a_0 c_0 (\ast d)^{j+p} \hat{P} A = c'_0 (\ast d)^{k+p'} A + c'_0 (\ast d)^{k+p'} \hat{P}' A,
\]
which implies that \( a_0 c_0 = c'_0, j + p = k + p' \) and \( \hat{P} = \hat{P}' \). Consequently, \( \hat{A} = \hat{A}' \). Any member of a class has the same redefined field and the same solution of the equation of motion. For instance, the self-dual and Maxwell-Chern-Simons actions are members of the same class, since they have the same redefined field. This class may be called the class of self-dual model. These two models considered are also dual to each other. This is not coincidence. Looking at equation (25), we can see that \( S_{\text{dual}} \) and \( S \) belong to the same class if \( \hat{C} = c_0 (\ast d)^j \). This is just the case of the self-dual model with \( \hat{C} = -m \ast d \).

Another interesting equivalence class is obtained from the Maxwell-Proca action
\[
S_{MP} = \frac{m^2}{2} (A, A) - \frac{1}{2} (\ast d A, \ast d A) = \frac{m^2}{2} (A, (1 - (\ast d/m)^2) A)
\]
(26)
In this case \( \hat{C} = - (\ast d)^2 \), and the group element is
\[
g_1 = (1 - (\ast d/m)^2)^{1/2} = 1 + \sum_{j=1}^{\infty} \alpha_j (\ast d/m)^{2j}.
\]
The Maxwell-Proca action and its dual action belong to the same class
\[
S_{\text{dual-MP}} = \frac{1}{2m^2} (\ast d A, \frac{\delta S_{MP}}{\delta A}) = \frac{1}{2} (\ast d A, \ast d A) - \frac{1}{m^2} (\ast d \ast d A, \ast d \ast d A),
\]
(27)
or in terms of redefined fields
\[
S_{MP} = \frac{m^2}{2} (\hat{A}, \hat{A}),
\]
(28)
and
\[
S_{\text{dual-MP}} = \frac{1}{2} (\ast d \hat{A}, \ast d \hat{A}).
\]
(29)
Let us underline that if two models belongs to the same class, its bilinear terms have in common an invertible operator given by \( \mathbf{7} \). This means that the propagators are related in some way. We know that if two models are dual they represent same physics in different scales of energy. We can argue that if two models belong to the same equivalence class, they represent the same physics in certain scales of energy. This point will be analyzed in a future work.

An important point is how to construct the non-abelian version of our results. For this, we would begin from the requirement that the field redefinition must be local and the redefined field remains a connection under BRST transformation, namely

\[
s \hat{A} = dc + g[\hat{A}, c] = \hat{D}c,
\]

where \( \hat{A} = A + \sum_{i=1}^{\infty} A_i/m_i \), \( m \) is a mass parameter and \( g \) is the coupling constant. It follows that

\[
s A_i = g[A_i, c].
\]

Since that the cohomology of the non-abelian version is isomorphic to its abelian counterpart, the knowledge of the abelian redefinition allows us to discover the non-abelian expansion. A detailed analysis will be given in a future work.

In conclusion, in this paper we study the connection between duality and field redefinition. We show that a general bilinear three-dimensional action for the 1-form gauge field \( A \) can be rewritten as a unique term with a suitable field redefinition. We construct equivalence classes for models sharing same field redefinition. Dual models belong to the same class. Another important issue is the derivation of a duality operator.

Acknowledgments

Conselho Nacional de Desenvolvimento Científico e tecnológico-CNPq is gratefully acknowledged for financial support.

REFERENCES