Inflation without Inflaton(s)

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We propose a model for early universe cosmology without the need for fundamental scalar fields. Cosmic acceleration and phenomenologically viable reheating of the universe results from a series of energy transitions, where during each transition vacuum energy is converted to thermal radiation. We show that this 'cascading universe' can lead to successful generation of adiabatic density fluctuations and an observable gravity wave spectrum in some cases, where in the simplest case it reproduces a spectrum similar to slow-roll models of inflation. We also find the model provides a reasonable reheating temperature after inflation ends. This type of model can also be used to explain the smallness of the vacuum energy today.

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I. INTRODUCTION

Despite the simplicity and promising phenomenology of scalar driven, slow-roll inflation, much remains to make the idea theoretically viable. In particular, vexing issues such as the required flatness of the inflationary potential and the very existence of a fundamental scalar (which must be both extremely light and weakly interacting) remain elusive (see however [1, 2, 3] and more recently [4, 5]). In recent years, a substantial effort has been invested in understanding how to embed such models in a quantum theory of gravity [6, 7], and there has also been the suggestion of removing the need for slow-roll completely [8] (see [9] for earlier work). However, in this paper we will take a different and yet complimentary approach to inflation model building based on fundamental scalars.

Recent progress in string theory [10] seems to suggest the possibility of a large number $N$ of (nearly) degenerate vacua [11], many of which could correspond to de Sitter (dS) space [6]. In this paper, we ask whether using such a near degeneracy of vacua can lead to a successful model of early universe cosmology. The universe is assumed to begin in a robust quantum superposition of the dS vacua. Assuming the degeneracy to be large $N \gg 1$ and given the finite extent of dS (more precisely Euclidean dS) the universe develops a band-like structure. This is in analogy to the Bloch energy spectra in a solid, and we have a universe that can go through a series of cascades until reaching the true ground state, whose value is determined by taking into consideration all the string vacua – not just a single, local, perturbative vacuum. While the universe is in a given level the scale factor will undergo nearly exponential expansion, i.e. there will be inflation. As transitions commence the vacuum energy is reduced and radiation in the form of massless string states (e.g. gravitons) is created. Radiation production is not completely homogeneous, and the inhomogeneities generated during the transition process lead to both density perturbations and inhomogeneous production of gravitational waves. Transitions will continue until the vacuum energy reaches the true ground state, however cosmic acceleration

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terminates once the radiation density exceeds the vacuum density and the universe emerges into the radiation dominated epoch. In this way the cascade leads to a natural exit from inflation and a hot radiation filled universe emerges.

In the late universe, following radiation and matter domination, the vacuum energy will come to dominate again, and the degeneracy of the vacuum can be used to explain the smallness and perhaps even the value of the vacuum energy density that we observe today. This approach to part of the cosmological constant problem(s) was advocated by two of us (with A. Zytkow) in [12], where it was shown that the vacuum degeneracy seems to suggest that the naive vacuum density \( \rho_\Lambda \approx H^2 M_p^2 \) given by expanding around a single perturbative vacuum should be replace qualitatively with \( \rho_\Lambda \rightarrow H^2 M_p^2 / N \) as in the case of a Bloch solid. This can be understood as a mixed state resulting from the non-trivial mixing of nearly degenerate vacua with the result a small, but non-zero vacuum density today. Although this possibility for addressing the cosmological constant problem was part of our initial motivation, we stress that it is not essential for the cascading model we shall consider in what follows, which is more general.

Despite the simplicity of this approach, there are immediately many questions that must be addressed if such a model should even be taken seriously. In this paper, we begin to address many of these issues. In Section II we attempt to motivate the initial conditions for the model and the subsequent cascading from level to level. This section represents the most speculative part of the paper, given our lack of knowledge of a complete theory of quantum gravity. Recent works suggest [14, 15, 16] that it may be possible to get some intuition for how cascades will proceed within the confines of a semi-classical quantum theory of gravity, since the transitions of interest require only a knowledge of the IR (long wavelength) behavior. However, this section is not essential for the phenomenology of the model and only serves to address the issue of initial conditions.

In Section III we turn to the cosmology of the model, and demonstrate that acceleration and successful reheating are possible with reasonable parameters. In Section IV we turn to the issue of cosmological perturbations and demonstrate that a nearly scale invariant spectrum of both density and tensor perturbations results in the simplest case of constant decay rate. We also find that in the more realistic case of a varying decay rate it may be possible to distinguish this model from the usual slow-roll models in that the evolving adiabatic sound speed can result in an observable tensor to scalar ratio. We conclude in Section V where we summarize our results and discuss future considerations.

II. SPONTANEOUS SYMMETRY BREAKING OF DE SITTER SPACE

De Sitter space is the maximally symmetry solution to Einstein’s equations in vacuum with positive cosmological constant \( \Lambda \). This space-time is classically stable (no Jean’s instability), however semi-classically it was argued long ago to possess an instability towards decay (see e.g. [13]). Naively, of course, the existence of quantum fluctuations in any interacting theory of particles, electrons, gravitons or otherwise, implies one does not have a perfect dS space. Qualitatively the instability can be understood by the fact that dS possesses a finite horizon and an associated temperature, the Gibbons-Hawking temperature. This temperature is associated with the existence of tensor metric fluctuations, i.e., gravitons, that do not vanish at the boundary [13]. The question remains what physical effect these gravitons can have and whether they can alter the classical solution in any significant way.

To address this question we begin by considering the Euclidean path integral\(^1\),

\[
Z = \int D[g_{\mu\nu}] e^{-S_E},
\]

where the sum is over all compact four geometries. The Euclidean action is given by

\[
S_E = -\frac{M_p^2}{16\pi} \int d^4x (R - 2\Lambda),
\]

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\(^1\) Here we will use the Hartle-Hawking \((t \rightarrow -i\tau)\) prescription [13], however it has been argued that other approaches may be more appropriate [14, 15].
where the metric of Euclidean dS (EdS) is
\[ ds^2 = d\tau^2 + a^2(\tau)d\Omega_3^2, \] (3)
with \( d\Omega_3^2 \) the volume of a unit three sphere and the space-time posses an \( SO(5) \) symmetry. In terms of this metric the action becomes
\[ S_E = -\frac{3\pi M_p^2}{4} \int d\tau \left( \dot{a}^2 + K_0 a - \frac{\Lambda_0}{3} a^3 \right) \] (4)
where \( K_0 \) and \( \Lambda_0 \) are the bare values of the curvature and cosmological constant and we are free to use scale invariance of \( a(\tau) \) to set \( K_0 = 1 \).

We then extremize the action and solve the resulting equation of motion for \( a(\tau) \) finding
\[ a(\tau) = \sqrt{\frac{3}{\Lambda}} \cos \left( \sqrt{\frac{\Lambda}{3}} \tau \right). \] (5)
Evaluating the action for this instanton solution, we have a semi-classical (saddle point) approximation for the ground state wave function whose square gives the decay rate of dS,
\[ \Gamma = A_0 e^{-S_E}, \] (6)
\[ = A_0 \exp \left( \frac{3\pi M_p^2}{\Lambda} \right). \] (7)

The purpose of this section is to discuss how to understand and estimate the dS decay rate \( \Gamma \). However, the above expression for the decay rate is well known to be immediately bothersome on two accounts. It is not normalizable and we see that as \( \Lambda \to 0 \) we have \( \Gamma \to \infty \). Thus, even if we take such an expression seriously, we see that a \( \Lambda = 0 \) universe is overwhelmingly favored.

However, recently it has been argued in a series of papers, \[17, 18, 19\] that this divergence must be regulated by considering the effect of quantum fluctuations on the classical dS vacuum\(^2\). This requires going beyond the mini-superspace approximation, taking into consideration the inhomogeneities associated with the tensor fluctuations of the metric which give rise to the Gibbons-Hawking temperature. Of course, while it seems obvious that such fluctuations should break the \( SO(5) \) dS symmetry it is not certain, but the observation that we do not live in an eternal dS space (as verified by the CMBR) seems to be a good motivation for considering such a possibility.

This is spontaneous symmetry breaking of the \( SO(5) \) de Sitter symmetry by radiative corrections coming from the quantum effects of gravitons and other particles (e.g. massless closed string modes). The authors of \[17, 18, 19\] argue that when considering the path integral \(1\), one must go beyond the mini-superspace approximation and include the inhomogeneities resulting from metric fluctuations. These metric fluctuations were studied previously by Halliwell and Hawking in \[24\]. The authors then include the effects of these modes on the effective theory by integrating them out, taking a UV cutoff at the string scale \( M_s \) and an IR cutoff at the horizon scale \( H^{-1} = \sqrt{\frac{3}{\Lambda}} \). Once the modes are integrated out this results in a correction term in the effective action
\[ S_{eff} = -\frac{3\pi M_p^2}{4} \int d\tau \left( \dot{a}^2 + K_0 a - \frac{\Lambda_0}{3} a^3 + \frac{N}{4a} \right), \] (8)
where
\[ N = \frac{H^{-1}}{M_s} f(a), \] (9)
where we see the explicit appearance of the cutoffs and \( f(a) \) is a calculable fourth degree polynomial in \( a(\tau) \) and depends on the assumed symmetries of the fluctuations. In particular, if one assumes that the metric fluctuations respect the \( SO(5) \) symmetry, then one can take \( f(a) = a \) and the effect of the new term is simply to renormalize the bare values \( K_0 \) and \( \Lambda_0 \) as one expects. However, if we allow for the metric fluctuations to spontaneously break the \( SO(5) \) symmetry the generic form of \( f(a) \) is then \[17, 18, 19\]
\[ f^4(a) = Aa^4 + Ba^2 + C, \] (10)
where the constants \( A, B \) and \( C \) must be determined from a renormalization scheme and the absence of odd terms is due to the fact that the metric fluctuations are Gaussian random variables (i.e. \( \langle \delta a \rangle = \langle \delta a^3 \rangle = 0 \)). Considering this form for \( f(a) \) in \(8\) we see that in addition to the renormalization of the bare terms, we get an additional radiation term coming from the \( C \) term in \( f(a) \). The

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\(^2\) For other recent attempts at addressing this issue see \[20, 21, 22, 23\]
curvature can again be rescaled so that $K_{ren} = 1$, however the presence of radiation now implies that the vacuum term must decrease to conserve energy. This result agrees with the early approach in [13], where it was argued that the value of $\Lambda$ could be lowered by tunneling to a dS universe of lower $\Lambda$ but with a gas of blackholes. In the presence of the radiation term, one finds that the corrected decay rate is now given by

$$\Gamma \sim \exp \left( \frac{3\pi M_p^2}{\Lambda} - \frac{6M_4^4}{\Lambda^2} \right),$$  
(11)

which is finite. The authors of [17, 18, 19] then argue that because the decay rate favors a small $\Lambda$ (large $H$) the semi-classical approach is justified and quantum effects can be safely ignored.

Although we find the basic approach above intuitively compelling, the details remain a bit controversial. In particular, the explicit cutoff dependence in the correction entering the effective action and the manner in which the graviton modes are traced out have been rightfully criticized [25, 26]. We will not attempt to resolve these issues here, we simply present this argument as a motivation for our initial starting point. In the next section we will take $\Gamma$ to be phenomenologically determined by the constraints of adequate inflation and a realistic spectrum of density perturbations. We can then ask whether such a choice makes sense, given the scales that naturally arise, remembering that if the approach discussed in this section is developed further it may be used to precisely determine the value of $\Gamma$.

### III. A CASCADING UNIVERSE

Let us consider the case of a universe dominated by vacuum energy and an additional radiation component that is subdominant. We begin by first considering the case of a single transition, followed by the case of interest, namely multiple sequential transitions leading to an effectively time dependent vacuum density. For a homogeneous and isotropic universe the Einstein equations are

$$3H^2 = \frac{8\pi}{M_p^2} \rho,$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3M_p^2} (\rho + 3p),$$

$$\dot{H} = -\frac{4\pi}{3M_p^2} (\rho + p),$$  
(12)

where we work with the Planck mass, which is related to the Newton constant by $G_N = M_p^{-2}$. The continuity equation is given by

$$\nabla \mu T^{\mu\nu} = 0,$$

$$\Rightarrow \dot{\rho} = -3H (\rho + p),$$  
(13)

We consider a two component fluid composed of radiation and vacuum energy. The energy density and pressure are given by

$$\rho = \rho + \rho_r, \quad p = \rho + p_r,$$

$$\rho = \frac{\Lambda M_p^4}{8\pi}, \quad \rho_r = \rho_0 a^4,$$

$$p = -\rho, \quad p_r = \frac{1}{3} \rho_r,$$  
(15-17)

with $\Lambda > 0$. For these sources the equations of motion (12) and (13) become

$$3H^2 = \frac{8\pi}{M_p^2} (\rho + \rho_r),$$

$$\frac{\ddot{a}}{a} = \frac{8\pi}{3M_p^2} (\rho - \rho_r),$$

$$\dot{H} = -\frac{16\pi}{3M_p^2} \rho_r,$$

$$\dot{\rho}_r = -4H \rho_r,$$  
(18)

where we have used $\dot{\rho}_\Lambda = 0$. We see that in order for acceleration to occur we need $\rho_r < \rho$. In fact, the amount of radiation present is a measure of how far the universe is from an exactly de Sitter phase. Quantitatively this can be seen by considering

$$\frac{d}{dt} (H^{-1}) = -\frac{\dot{H}}{H^2} = \dot{\epsilon},$$  
(19)

where $\epsilon$ is a parameter measuring the deviation
from a pure dS space-time\textsuperscript{3}. For inflation to occur we thus expect \( \dot{\epsilon} \ll 1 \), which for this background gives

\[
\dot{\epsilon} = \frac{2 \rho_r}{\rho_\Lambda + \rho_r} \ll 1. \tag{20}
\]

One can solve the background equations \textsuperscript{18} in the absence of a coupling and we find

\[
a^2(t) = c_0^2 \sinh \left( \sqrt{\frac{4 \Lambda}{3}} t + c_1 \right), \tag{21}
\]

\[
H(t) = \sqrt{\frac{\Lambda}{3}} \coth \left( \sqrt{\frac{4 \Lambda}{3}} t + c_1 \right), \tag{22}
\]

where

\[
c_0^2 = \left( \frac{8 \pi \rho_0}{M_p^2 \Lambda} \right)^{1/2} = \frac{1}{\sinh(c_1)} \tag{23}
\]

are constants chosen so that when \( t = 0 \) we have \( a = 1 \).

Since this solution is derived for the case of \( \Lambda = \text{constant} \), inflation of course does not end, and this solution does not provide for a successful inflationary epoch. When we include time varying values of the vacuum energy, which results from multiple transitions or cascades, we will see the result becomes satisfactory.

The case of a single transition was argued for in Section II. The transition resulted in a lower cosmological vacuum density and an additional radiation term \( \rho_r \). We now consider the case of interest, namely multiple transitions producing radiation as the cascades proceed.

The energy transfer from the vacuum energy density to radiation (massless string states) is given by

\[
Q_\Lambda = -\Gamma \rho_\Lambda, \quad Q_r = \Gamma \rho_\Lambda. \tag{24}
\]

and the modified continuity equation becomes

\[
\nabla_\mu T^\mu_\Lambda = \dot{\rho}_\Lambda = Q_\Lambda, \tag{25}
\]

\[
\nabla_\mu T^\mu_r = \dot{\rho}_r + 4H \rho_r = Q_r, \tag{26}
\]

\[
\Rightarrow \nabla_\mu (T^\mu_\Lambda + T^\mu_r) = 0. \tag{27}
\]

The equations of motion for the background are then

\[
3H^2 = \frac{8 \pi}{3M_p^2} (\rho_\Lambda + \rho_r), \quad \tag{29}
\]

\[
\dot{\rho}_\Lambda = -\Gamma \rho_\Lambda, \quad \tag{30}
\]

\[
\dot{\rho}_r = -4H \rho_r + \Gamma \rho_\Lambda. \tag{31}
\]

From \textsuperscript{30} we immediately find

\[
\rho_\Lambda = \rho_{\Lambda 0} e^{-\Gamma t}. \tag{32}
\]

Using this result in the above equations we find

\[
H + 2H^2 = \frac{16 \pi}{3M_p^2} \rho_{\Lambda 0} e^{-\Gamma t}. \tag{33}
\]

The solutions are related to modified Bessel functions. They can be simply expressed by introducing the dimensionless quantity

\[
\tau = \sqrt{\frac{128 \pi \rho_{\Lambda 0}}{3M_p^2 \Gamma^2}} e^{-\Gamma t/2} \equiv \tau_0 e^{-\Gamma t/2}. \tag{34}
\]

The scale factor and Hubble parameter are then given by

\[
a^2 = \frac{4}{\Gamma} \left( \alpha_1 I_0(\tau) + \alpha_2 K_0(\tau) \right), \tag{35}
\]

\[
H = \frac{\Gamma \tau}{4} \left( \alpha_2 K_1(\tau) - \alpha_1 I_1(\tau) \over \alpha_2 K_0(\tau) + \alpha_1 I_0(\tau) \right), \tag{36}
\]

where the functions \( I_\nu \) and \( K_\nu \) are modified Bessel functions of order \( \nu \) (see e.g. \textsuperscript{29}). The constants are given by

\[
\alpha_1 = \frac{\Gamma \rho_{\Lambda 0}}{4} K_1(\tau_e) - H_e K_0(\tau_e), \tag{37}
\]

\[
\alpha_2 = \frac{\Gamma \rho_{\Lambda 0}}{4} I_1(\tau_e) + H_e I_0(\tau_e), \tag{38}
\]

with \( \tau_e = \tau(t_e) \), \( H_e \) the Hubble parameter at the end of inflation \( (t = t_e) \) and we normalize so that

\textsuperscript{3} This is analogous to the slow-roll parameter \( \epsilon \) in models of scalar field inflation, but because our model does not contain any scalar fields we will avoid this terminology. Moreover, in contrast to the slow-roll case, the definition of \( \dot{\epsilon} \) is exact and does not depend on any approximation.
FIG. 1: The deformation parameter for various values of the decay rate $\Gamma$. Time is measured in units of the Planck time and we take $M_p^4 \gg \rho_\Lambda \gg \rho_r$. The various curves are given by the values $\Gamma = 0.01, 0.02, 0.03, 0.05$ from left to right. As discussed in the text, $\dot{\epsilon}$ is initially small and proportional to the density of radiation $\rho_r$. As inflation proceeds, more and more energy is dumped into radiation via the coupling $\Gamma$. At the very end of inflation we are left with a radiation dominated universe corresponding to $\dot{\epsilon} = 2$.

the number of e-foldings is measured from the end of inflation, $N = \ln(a_e/a) = -\ln(a)$ where $a_e = 1$.

We can again introduce a deformation parameter as in (20), however now it is time dependent,
$$\dot{\epsilon}(t) = \frac{2\rho_r}{\rho_\Lambda + \rho_r} = 2\Omega_r. \quad (39)$$

Using this in (29) we find
$$\dot{\epsilon}(t) = 2 - \frac{16\pi\rho_\Lambda}{3H^2M_p^2} e^{-\Gamma t}, \quad (40)$$

with $H$ being given by (30). At the beginning of inflation we have $H^2M_p^2 \sim \rho_\Lambda$ and so $\dot{\epsilon} \ll 1$. As the energy is transferred from the vacuum energy density to radiation via particle production, the deformation parameter increases as can be seen in Fig. 1. Inflation then ends when $\rho_r \approx \rho_\Lambda$ and $\dot{\epsilon} \approx 1$, which can be seen in Figures 1 and 3. The final result is $\dot{\epsilon} = 2$ and we are left with a universe filled by the radiation dominated universe corresponding to $\dot{\epsilon} = 2$.

FIG. 2: The number of e-folds of inflation $N = \int H dt$ for various values of the decay rate $\Gamma$. Time is measured in units of the Planck time and we take $M_p^4 \gg \rho_\Lambda \gg \rho_r$. The various curves are given by the values $\Gamma = 0.01, 0.02, 0.03, 0.05$ from top to bottom. We see that the requirement of sufficient inflation places a constraint $\Gamma \leq 0.02$.

amount of inflation or number of e-folds. Using the above expression for the deformation parameter the Hubble equation can be rewritten as
$$3H^2 = \frac{16\pi\rho_\Lambda}{M_p^2(2 - \dot{\epsilon})} e^{-\Gamma t/2}. \quad (41)$$

This can then be integrated to find the number of e-foldings,
$$N = \left( \frac{16\pi\rho_\Lambda}{3M_p^2} \right)^{1/2} \int_{t_e}^{t_0} \frac{e^{-\Gamma t/2}}{\sqrt{2 - \dot{\epsilon}}} \, dt_e \approx \frac{2}{\Gamma} \left( \frac{16\pi\rho_\Lambda}{3M_p^2} \right)^{1/2} \sim \frac{8\rho_\Lambda}{\Gamma M_p} \sim \frac{8H_b}{\Gamma}, \quad (42)$$

where $t_0 = 0$ is the beginning of inflation, $t_e$ is the end and $H_b$ is the initial Hubble scale. In the second line we use the fact that the denominator varies smoothly from $\sqrt{2}$ to 1 and $\exp(-\Gamma t_e/2) \approx 0$. As we may have anticipated the amount of inflation depends on the initial vacuum density and the decay rate.

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4 Much later, of course, the radiation is diluted as the volume increases and the small remaining constant energy density again dominates.
FIG. 3: The graphs above show the evolution of the vacuum energy density $\Omega_\Lambda = \rho_\Lambda / \rho$ and the radiation energy density $\Omega_r = \rho_r / \rho$ relative to the total density. We present the evolution for two values of the coupling $\Gamma = 0.05$ (top) and $\Gamma = 0.01$ (bottom), where it can be seen that stronger coupling means inflation ends faster, through faster dissipation.

As an example, consider inflation with a Hubble scale near the GUT scale $H_b \sim M_{\text{GUT}} \approx 10^{15} \text{GeV}$. We see to get adequate inflation the decay rate need only be slightly below the initial Hubble scale $\Gamma \sim 10^{14} \text{GeV}$. This condition is required in order that cascading lasts long enough so that the cosmic acceleration can resolve the horizon and flatness problems. In Figures 1-3, we examine the evolution numerically and find adequate inflation is possible given modest values of the parameters.

Another important consideration is the reheat temperature of the model. The cosmic acceleration ends at the moment $t_r$ when $\rho_r = \rho_\Lambda$ and radiation comes to dominate. At this moment we have $3H^2 = 16\pi G \rho_\Lambda$ where $\rho_\Lambda = \rho_r e^{-\Gamma t_r}$. Using the exact solution (35) and (36) and assuming the minimal amount of efoldings ($N = 60$) we find $\Gamma t_r \approx 10$ so that the reheating temperature can be approximated as

$$T_r \approx \rho_r^{1/4} e^{-\Gamma t_r/4},$$

$$\approx 10^{15} \text{GeV},$$

where we have used $\rho_\Lambda = \Lambda_0 M_p^2 / 8\pi$ and we have taken the initial Hubble scale $H_b \approx 10^{14} \text{GeV}$. We will see in the next section that this is consistent with producing the observed temperature anisotropies in the cosmic microwave background and avoiding over production of gravity waves.

IV. COSMOLOGICAL PERTURBATIONS

Let us consider density and tensor fluctuations about the background solution (18). We will be primarily interested in modes that leave the Hubble radius 50–60 e-folds before the end of inflation, since these are the modes that are responsible for the CMBR anisotropies observed today. Therefore, instead of working with the exact solution (18) for the study of perturbations, it will often be simpler to work in the conformal time $\eta = -\infty \ldots 0$ where $d\eta = a^{-1}dt$ and with approximate solution

$$a(\eta) = (-\eta)^{-(1+\hat{\epsilon})},$$

$$\mathcal{H}(\eta) = \frac{1 + \hat{\epsilon}}{-\eta},$$

where $\mathcal{H}$ is the conformal Hubble parameter $\mathcal{H} = aH$ and is related to the deformation parameter by $\hat{\epsilon} = 1 - \frac{3\mathcal{H}^2}{2H^2}$. The approximate solution treats the deformation parameter as a constant, since its rate of change is small ($\dot{\hat{\epsilon}} \approx \hat{\epsilon}$), but of course this solution must break down towards the end of inflation when $\hat{\epsilon} \sim 1$.

We now consider linearized perturbations about the background (18) and in what follows we will adopt the conventions of [27]. We work in longitudinal gauge with the perturbed line element

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 + 2\Psi + h_{ij})dx^i dx^j,$$
where the tensor perturbation is traceless and transverse (i.e., \( h^i_1 = \nabla_i h^0_j = 0 \)) and can be broken into its two polarizations \( h^\pm \). Our background contains no anisotropic stress so one finds from the Einstein equations \( \Phi = \Psi \). Thus, we have only one scalar metric degree of freedom associated with the density perturbation and two tensor metric degrees of freedom for the gravity waves. Because we are considering linearized perturbations the scalar and tensor metric fluctuations decouple and we will treat each one in-turn.

### A. Density Fluctuations

Working in conformal time the equations for density perturbations are,

\[
\begin{align*}
\nabla^2 \Phi - 3\mathcal{H}(\mathcal{H} \Phi + \Phi') &= 4\pi G a^2 \delta \rho \\
\partial_i (\mathcal{H} \Phi + \Phi') &= 4\pi G a (\rho + p) \delta u_i, \\
\Phi'' + 3\mathcal{H} \Phi' + (2\mathcal{H}' + \mathcal{H}^2) \Phi &= 4\pi G a^2 \delta \rho,
\end{align*}
\]

where \( \delta \rho \), \( \delta p \), and \( \delta u_i \) are the perturbations of the total energy density, pressure, and velocity, respectively and \( \mathcal{H} = \frac{\dot{a}}{a} \) is the conformal Hubble parameter and \( \nabla \) is the comoving gradient. The pressure perturbation is related to the energy density and entropy density perturbations by

\[
\delta p = \left. \frac{\partial p}{\partial \rho} \right|_s \delta \rho + \left. \frac{\partial p}{\partial s} \right|_\rho \delta s, \tag{50}
\]

\[
= c_s^2 \delta \rho + \tau \delta S, \tag{51}
\]

with \( c_s \) the adiabatic sound speed at which the perturbations evolve and should not be confused with the equation of state parameter \( w = p/\rho \), which depends on the background quantities.

Indeed, for the model we consider here these quantities are quite different. The adiabatic sound speed for long-wavelength perturbations is given by

\[
c_s^2 = \frac{\dot{\rho}}{\rho} = 1 - \frac{4\rho_r}{3(\rho_\Lambda + \rho_r)}, \tag{52}
\]

where we see in the limit \( \Gamma \to 0 \) our perturbations will evolve like pure radiation. This is consistent with the fact that a true cosmological constant does not propagate, i.e. is constant. This allows us to see the importance of the graviton and other particle production (\( \Gamma \) term), since in an quasi-exponentially expanding background with no transfer the perturbations will be immediately damped away.

The equation of state parameter is given by

\[
w = \frac{p}{\rho} = -1 + \frac{4\rho_r}{3(\rho_\Lambda + \rho_r)}, \tag{53}
\]

\[
= -1 + \frac{2}{3} \epsilon, \tag{54}
\]

which reduces to the pure de Sitter solution if \( \rho_r = \epsilon = 0 \) as we have noted. We see that \( w(t) \) is explicitly time dependent and in general this implies that there can be a significant contribution from non-adiabatic pressure in \( \mathcal{L} \) by the entropy term \( \tau \delta S \). This could result in significant generation of entropy perturbations, a possibility that we will analyze in Section [IV.A.2].

We now return to solving the system \( \text{(47)}-\text{(49)} \). Combining equation \( \text{(47)} \) with \( \text{(49)} \) and working in momentum space \( \nabla^2 \Phi \to -k^2 \Phi \), we find a second order differential equation,

\[
\Phi'' + 3\mathcal{H}(1 + c_s^2) \Phi' + \left[ c_s^2 k^2 + 2\mathcal{H}' + (1 + 3c_s^2) \mathcal{H}^2 \right] \Phi = 4\pi G a^2 \tau \delta S. \tag{55}
\]

subject to the constraint \( \text{(51)} \). We can simplify

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5 Strictly speaking it is incorrect to think of \( c_s^2 = \frac{\dot{\rho}}{\rho} \) as the sound speed during inflation, since this is only true on large scales where \( c_s^2 = \frac{\dot{\rho}}{\rho} \big|_{s} \approx \frac{\dot{\rho}}{\rho} \) and these modes evolve on scales beyond the sound horizon. On small scales during inflation the metric perturbation \( \Phi \) oscillates and the effective adiabatic sound speed is found to be \( c_s^2 = 1 \), in agreement with causality.
this equation by introducing the field redefinition,
\[
\Phi_k = \frac{4\pi G \rho^{1/2} \sqrt{1 + w}}{\sqrt{1 + w}}, \quad (57)
\]
\[
\theta^2 = \frac{8\pi M_p^{-2}}{3a^2(1 + w)}, \quad (58)
\]
where \( w = p/\rho \). Then (55) becomes
\[
u_k'' + \left( k^2 c_s^2 - \frac{\theta''}{\theta} \right) u_k = N \quad (59)
\]
with \( N \) giving the contribution from entropy modes as
\[
N = a^2 \rho^{1/2} \sqrt{1 + w} \tau \delta S. \quad (60)
\]

1. Adiabatic Fluctuations

We will first consider solutions to (59) in the absence of entropy modes, i.e. \( N = 0 \). For modes that are far inside the sound horizon \( kc_s \gg \mathcal{H} \) we find that \( u \) oscillates with a constant amplitude that is to be found by the initial conditions after quantization. Far outside the horizon \( kc_s \ll \mathcal{H} \) and by inspection we have the solution \( u \sim \theta \). However, this solution corresponds to a decaying mode for the metric perturbation \( \Phi \). Instead, it is the growing mode that is of interest, which can be found by noting \( \theta'' \approx 0 \) during inflation, so that \( u \) constant is also a solution. Assuming that we are deep in the inflation epoch where \( \dot{\epsilon} \) is nearly constant an exact solution can be found by integration. In terms of the original metric perturbation \( \Phi \) one finds
\[
\Phi = \frac{\mathcal{H}}{a^2} \left( A_1 + A_2 \int (1 + w) a^2 d\eta \right), \quad (61)
\]
\[
= \frac{\mathcal{H}}{a^2} \left( A_1 + \frac{2}{3} A_2 \int \dot{\epsilon} a^2 d\eta \right), \quad (62)
\]
\[
= A_1 \frac{\mathcal{H}}{a^2} + \Phi_0 \dot{\epsilon}, \quad (63)
\]
where we have used \( w = -1 + \frac{2}{3} \dot{\epsilon} \) is nearly constant during inflation. The first term in (63) corresponds to the decaying mode found above \( u \sim \theta \), whereas the second mode is nearly constant with \( \Phi_0 = \frac{2}{3} A_2 \) to be determined by matching to the oscillating mode inside the sound horizon. We note the importance of the graviton production in this model resulting in a non-zero radiation density, since in the pure de-Sitter case where \( \rho_r = 0 \) so that \( \dot{\epsilon} = 0 \) we see no density metric perturbation would remain since \( \Phi \to 0 \) as \( \dot{\epsilon} \to 0 \) and all that is left is the decaying mode. This is an illustration of the no hair theorem for pure de Sitter space.

In summary, we have found that during inflation the metric perturbation is nearly constant on super-horizon scales, whereas on sub-horizon scales we find \( u \sim \Phi \) undergoes constant amplitude oscillations. What remains is to quantize the perturbations in order to determine the unknown constant \( \Phi_0 \). However, we must first justify neglecting the entropy mode term (i.e. \( N \)) in (59).

2. Entropy Fluctuations

In this section we consider the role of entropy fluctuations in the model. We will follow [28] where a systematic procedure for the study of perturbations in multi-fluid systems was described. It will be useful to introduce \( \zeta \), which is curvature perturbation on constant energy density hypersurfaces. We will drop the momentum index in what follows, writing \( \zeta \equiv \zeta_k \).

In the presence of multiple fluids, the total curvature perturbation can be expressed as a sum of the curvature perturbation due to each fluid component as
\[
\zeta = \sum_{\alpha} \frac{\rho_{\alpha}'}{\rho_{\alpha}} \zeta_{\alpha}, \quad (64)
\]
where
\[
\zeta_{\alpha} = \Phi + \frac{\mathcal{H}}{\rho_{\alpha}} \delta \rho_{\alpha}, \quad (65)
\]
and we have used the lack of anisotropic stress to again write \( \Phi = \Psi \) as before. For entropy fluctuations we are interested in the non-adiabatic contribution to the pressure perturbation in (51), which is given by
\[
\delta p_{\text{nad}} \equiv \tau \delta S = \delta p - c_s^2 \delta \rho. \quad (66)
\]
As discussed in [28], there are two sources of non-adiabatic pressure
\[
\delta p_{\text{nad}} = \delta p_{\text{nad}}^c + \delta p_{\text{nad}}^i, \quad (67)
\]
which are the relative and intrinsic non-adiabatic pressures, respectively. In the model we are considering here the two fluid components have fixed equation of state, i.e. \( \delta p_\Lambda = -\delta \rho_\Lambda \) and \( \delta p_r = 1/3\delta \rho_r \) so that there is no intrinsic non-adiabatic pressure, i.e. \( \delta p_{nad}^{int} = 0 \). The contribution to the relative non-adiabatic pressure is 

\[
\delta p_{nad}^{rel} = -\frac{1}{6H\rho} \sum_{\alpha,\beta} \rho_\alpha \dot{\rho}_\beta (c^2_\alpha - c^2_\beta) S_{\alpha\beta},
\]

\[
= -\frac{1}{3H\rho} \rho_\Lambda \dot{\rho}_r (c^2_\Lambda - c^2_r) S_{\Lambda r},
\]

where we have introduced the relative entropy perturbation

\[
S_{\alpha\beta} = 3(\zeta_\alpha - \zeta_\beta),
\]

\[
= -3H \left( \frac{\delta \rho_\alpha}{\rho_\alpha} - \frac{\delta \rho_\beta}{\rho_\beta} \right),
\]

with the factor of three due to the convention of normalizing to baryons. For the model we consider here

\[
S_{\Lambda r} = -S_{r\Lambda} = -3H \left( \frac{\delta \rho_\Lambda}{\rho_\Lambda} - \frac{\delta \rho_r}{\rho_r} \right).
\]

Returning to \( \ref{eq:68} \) we see that the relative non-adiabatic pressure is proportional to \( \dot{\rho}_r \), the rate of change of the radiation density. During inflation one finds from the background solution \( \ref{eq:35} \) and \( \ref{eq:36} \) that \( \dot{\rho}_r \approx 0 \). That is, the transfer of vacuum energy to the radiation density via the coupling \( \Gamma \) is just enough to counter the dilution of the radiation by the exponential expansion. Thus, the non-adiabatic pressure is negligible and we need not worry about the presence of entropy perturbations during inflation.

However, there is a more fundamental reason to expect entropy perturbations to be absent from this model. The crucial point is that a relative entropy perturbation is produced when two fluids generate different curvature perturbations. This difference can then be mediated from one fluid to the other via the gravitational background. A well known example is the perturbation in the baryon-photon ratio

\[
S_{B\gamma} = 3(\zeta_B - \zeta_\gamma) = \frac{\delta \rho_B}{\rho_B} - \frac{3}{4} \frac{\delta \rho_\gamma}{\rho_\gamma},
\]

which does not vanish because the two fluids are perturbed differently.

However, in the case we consider here things are different. In the absence of the coupling \( \Gamma \) there is only one fluid with propagating fluctuations, namely the radiation density with fluctuations \( \delta \rho_r \). In this case the long-wave fluctuations propagate at \( c^2_s = 1/3 \) and the cosmological constant remains a constant, i.e. \( \delta \rho_\Lambda = 0 \). In the presence of the coupling \( \Gamma \) the fluctuations now propagate at a different adiabatic sound speed \( \ref{eq:52} \), but the two fluids are coupled through their equations \( \ref{eq:30} \) and \( \ref{eq:31} \) through the term \( \pm \Gamma \rho_\Lambda \). Thus, there is really only one propagating degree of freedom and the two fluids do not evolve independently, resulting in \( S_{\Lambda r} = 0 \).

In fact, in this regard this is not unlike the case of inflation by a single scalar field where it is known that there are only adiabatic perturbations. Instead of working with the scalar directly, we could consider two fluids, one representing the kinetic energy with a stiff equation of state \( p_1 = \rho_1 = 1/2\dot{\phi}^2 \) and a second fluid composed of the potential \( p_2 = -\rho_2 = V(\phi) \). Insisting on this two-fluid description and demanding that the full equations of motion are satisfied we are led to an energy exchange term \( Q_\pm = \pm \phi V'(\phi) \), similar to the case we have above. However, since we know there is only one degree of freedom, we certainly know that there are no entropy perturbations and no non-adiabatic pressures. This can be seen by examining the perturbation equations in full detail, and in particular one finds that the two fluids do not evolve independently due to the coupling \( Q_\pm \) and the fact that the second fluid does not propagate in the absence of the coupling (i.e. \( \delta \rho_2 = 0 \) for \( Q = 0 \)).

In sum, we see that entropy perturbations in the cascading model are negligible during inflation for the case of a constant decay rate \( \Gamma \). For the case of a time varying \( \Gamma \), this issue must be revisited, which is work in progress.

### B. Spectrum of Fluctuations

Having shown that entropy perturbations are negligible, we proceed to find the spectrum of the density fluctuations. In order to find the power spectrum all that remains is to determine the un-
known constant $\Phi_0$ in \[ 63 \]. We can then find the gauge invariant, comoving curvature perturbation

$$ R_k = \Phi_k + \frac{2}{3} \left( \frac{\Phi_k' + \mathcal{H} \Phi_k}{1 + w} \right), \quad (73) $$

which is related to the curvature perturbation $\zeta_k$ from the last section by

$$ R_k = \zeta_k + \frac{1}{3} \frac{k^2 \Phi_k}{\mathcal{H}' - \mathcal{H}^2}, \quad (74) $$

so that for large scales modes (which are the ones of interest) $k \to 0$ and $R_k \to \zeta_k$. The density power spectrum is then defined as

$$ P_\zeta = \frac{k^3}{2\pi^2} |\zeta_k|^2, \quad (75) $$

which can be compared with observations.

Finding the constant $\Phi_0$ is accomplished by enforcing the correct initial condition on the modes. However, these modes are born in their vacuum state far below the Hubble radius. This requires us to quantize the perturbations, starting their evolution in the standard adiabatic vacuum. Then we have seen that the solution inside ($kc_s \gg \mathcal{H}$) oscillates with constant amplitude until Hubble radius crossing where it can be matched to the solution outside ($kc_s \ll \mathcal{H}$) providing us with the required normalization constant.

The only obstacle to quantization is finding the canonical field which diagonalizes the action. For the case of hydrodynamical fluids, as we consider here, this was done in \[ 27 \]. There it was found that the canonical field $v_k$ (the so-called Mukhanov variable) which is related to $u_k$ by

$$ u_k = -\frac{(vk\theta)'}{c_s k^2 \theta}, \quad (76) $$

and the curvature perturbation \[ 73 \] by $v_k = z\zeta_k$, reduces the action to that of a harmonic oscillator with time dependent frequency. In terms of this variable the equation of motion \[ 59 \] becomes

$$ v_k'' + \left( k^2 c_s^2 - \frac{z''}{z} \right) v_k = 0, \quad (77) $$

where $z = (c_s \theta)^{-1}$. In terms of the new variable $v_k$ the solutions on large scales ($kc_s \gg \mathcal{H}$) are given by $v_k \sim z$. Notice this is the growing mode of interest in contrast to the classical case where $u_k \sim \theta$ decayed and it is this squeezing of the quantum state that will result in classical fluctuations on large scales. On small scales the momentum term dominates and we again have oscillations with constant amplitude.

We could now proceed with the approximate solution, however in the case $\dot{\epsilon} \ll 1$ we can solve \[ 77 \] exactly. We find $z'' = \frac{\nu^2 - 1/4}{\sigma^2}$ where $\nu = 3/2 + \dot{\epsilon}$ and the solutions can be expressed in terms of Hankel functions. We require that the modes begin in the adiabatic vacuum, which amounts to the condition

$$ v_k = \frac{1}{\sqrt{2c_s k}} e^{-ikc_s \eta} \quad \text{as} \quad kc_s \eta \to -\infty. \quad (78) $$

The appropriate solution is then given by

$$ v_k(\eta) = \frac{\alpha}{2} \sqrt{-\eta} H_{\nu}^{(1)}(-kc_s \eta), \quad (79) $$

where $H_{\nu}^{(1)}$ is a Hankel function of the first kind and $|\alpha| = 1$. We can then immediately find the curvature perturbation

$$ \zeta_k = \frac{|v_k|}{z} = \frac{1}{2z} \sqrt{-\eta} H_{\nu}^{(1)}(-kc_s \eta) \quad (80) $$

On large scales using the asymptotic expansion of the Hankel function we have

$$ |\zeta_k| \approx \frac{1}{z\sqrt{2\pi kc_s}} \left( -kc_s \eta \right)^{1/2 - \nu}, \quad (81) $$

and using $z = (c_s \theta)^{-1}$ the power spectrum \[ 75 \] is

$$ P_\zeta = \frac{1}{4\pi^2 c_s^3} \left( \frac{H}{M_P} \right)^2 (-kc_s \eta)^{-2\dot{\epsilon}}. \quad (82) $$

We note that this reduces to the standard slow-roll inflation result for the case $|c_s| = 1$.

The tilt of the power spectrum is given by

$$ n_s = 1 + \frac{d \ln P_\zeta}{d \ln k} = 1 - 2\dot{\epsilon}, \quad (83) $$

$$ = 1 - 4\Omega_r, \quad (84) $$

where $\Omega_r = \rho_r/\rho$. By noting that $\dot{\rho}_r \approx 0$ during the time modes of interest exit the Hubble radius (i.e. $N \sim 50$), we see that the tilt of the spectrum
is set by the initial abundance of radiation since \( \dot{\epsilon} \approx 2\Omega_\epsilon_0 \) is constant during inflation. Comparing (82) to the best fit WMAP3 data [30],

\[
P_\zeta = 19.9^{+1.3}_{-1.8} \times 10^{-10}, \tag{85}
\]

we find that

\[
\frac{H}{c_s \dot{\epsilon}^{1/2}} \lesssim 10^{-5} M_p. \tag{86}
\]

Since \( \dot{\epsilon} \ll 1 \) during inflation this implies an upper bound on the Hubble scale during inflation \( H \lesssim 10^{14} \) GeV. Combining this with the constraint for adequate inflation from (42), i.e. \( \Gamma / H \lesssim 1/N \) we find an upper bound on the decay rate of the vacuum energy \( \Gamma \lesssim 10^{13} \) GeV consistent with our earlier results and our general approach.

1. Gravity Waves

The gravitational wave spectrum is found in much the same way as the spectrum of density perturbations. One first decomposes the graviton into its two polarizations \( h_k^{(+)} \) and \( h_k^{(-)} \). The modes then obey the same equation (77) as the density fluctuations except in this case we have \( v_k = a h_k M_p \) (where we set \( h \equiv h_{\pm} \)) and \( z \) is replaced by the scale factor \( a \). The solution is again in terms of Hankel functions, and one finds for the long-wavelength fluctuations

\[
h_k \approx \frac{1}{a \sqrt{2\pi k}} (-k\eta)^{1/2-\nu}, \tag{87}
\]

and the power spectrum is

\[
P_h = \frac{8}{\pi^2} \left( \frac{H}{M_p} \right)^2 (-k\eta)^{-2\nu}, \tag{88}
\]

with the tilt of the tensor spectrum \( n_T = -2\nu \). Thus, we see that the main difference between the tensor and density spectrum is the deformation factor and the presence of the adiabatic sound speed in the spectrum of density perturbations. Our tensor to ‘scalar’ ratio is then

\[
r = \frac{P_h}{P_\zeta} = 16\dot{\epsilon} c_s, \quad \text{Scalar Free Model} \quad (89)
\]

which contains the adiabatic sound speed \( c_s \) evaluated at the time of Hubble radius crossing.

This is an important result and is similar to models of kinetic inflation [9], where the adiabatic sound speed offers a way to distinguish this model from standard slow-roll inflation which gives instead

\[
r = 16\epsilon, \quad \text{Scalar Slow-roll Inflation,} \quad (90)
\]

where we recall that \( \epsilon \approx \dot{\epsilon} \) is the usual slow-roll parameter which measures the slope of the scalar field potential in units of the Hubble scale.

We have seen the adiabatic sound speed does not differ greatly from the usual slow-roll inflation case for the choice of \( \Gamma = \text{constant} \) that we have considered here. However, for the case of non-constant \( \Gamma \) this could dramatically change, since the adiabatic sound speed could differ greatly from one. This could allow for an observable tensor to scalar ratio, where standard models of scalar driven inflation starting near the string scale seem to generically predict an unobservable spectrum [32]. This is work in progress.

V. FURTHER CONSIDERATIONS AND CONCLUSIONS

In this paper we have considered a cascading model for the early universe that provides a period of cosmological acceleration, which can account for the required number of efoldings and ends in a radiation dominated universe with a temperature \( T_r \approx 10^{15} \) GeV. As the universe cascades, vacuum energy is converted into radiation inhomogeneously, resulting in a nearly scale invariant spectrum of cosmological density perturbations and a small amount of gravitational waves. Naively, the model also does not contain any fine-tunings, beyond the need to begin cascading at or near the string or GUT scale, which is expected in the approach of [12]. Finally, once the universe has evolved beyond the radiation and matter epochs the vacuum density will once again dominate the energy density with its small value being determined by the degeneracy of the dS vacua, which need only be a small number of the many degenerate vacua predicted by string theory.

We have seen that our approach has one basic (in principle calculable) parameter, the level decay
rate $\Gamma$. The number of e-foldings, the reheating, and the density fluctuations all depend on $\Gamma$, and we find there does exist a range of values of $\Gamma$ consistent with the data for all of these, which might not have happened.

Although these preliminary findings are promising, much remains to be addressed. A particularly pressing issue is a concrete derivation of the decay rate $\Gamma$ or equivalently a better understanding of the level spacing and the time spent in a given energy (density) level. In fact, we argued in Section V that if $\Gamma$ is not taken constant, the result is a varying adiabatic sound speed which can result in density perturbations and gravity waves that would further distinguish the cascading model presented here from usual slow-roll inflation.

We argued in Section II that the issue of $\Gamma$ may be addressable in a semi-classical approach to quantum gravity. However, as noted there one must be careful when introducing arbitrary cut-offs in the effective action, since this lies at the heart of the difficulties associated with the non-renormalizability of gravitational theories. An alternative possibility would be to embed this model directly in string theory. For example, it was found that the topological string partition function can give a string realization of the Hartle-Hawking wave function of the universe. One can then imagine the cascading model here resulting from a similar analysis, although the issue of time dependence and dS space present well known challenges for cosmological model building in string theory. Regardless of viewpoint, it is clear that a better understanding of the decay rate $\Gamma$ is needed, which is work in progress.

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