Neutrino oscillations: theory and phenomenology

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Abstract. A brief overview of selected topics in the theory and phenomenology of neutrino oscillations is given. These include: oscillations in vacuum and in matter; phenomenology of 3-flavour neutrino oscillations and effective 2-flavour approximations; CP and T violation in neutrino oscillations in vacuum and in matter; matter effects on $\nu_\mu \leftrightarrow \nu_\tau$ oscillations; parametric resonance in neutrino oscillations inside the earth; oscillations below and above the MSW resonance; unsettled issues in the theory of neutrino oscillations.

1. A bit of history...
The idea of neutrino oscillations was first put forward by Pontecorvo in 1957 [1]. Pontecorvo suggested the possibility of $\nu \leftrightarrow \bar{\nu}$ oscillations, by analogy with $K^0 \bar{K}^0$ oscillations (only one neutrino species — $\nu_e$ — was known at that time). Soon after the discovery of muon neutrino, Maki, Nakagawa and Sakata [2] suggested the possibility of neutrino flavour transitions (which they called “virtual transmutations”).

Figure 1. Bruno Pontecorvo (1913 - 1993), Shoichi Sakata (1911 - 1970), Ziro Maki (1929 – 2005) and Masami Nakagawa (1932 - 2001).

2. Theory
2.1. Neutrino oscillations in vacuum
Neutrino oscillations are a manifestation of leptonic mixing. For massive neutrinos, weak (flavour) eigenstates do not in general coincide with mass eigenstates but are their linear

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2 On leave from the National Research Center “Kurchatov Institute”, Moscow, Russia
combinations; the diagonalization of the leptonic mass matrices leads to the emergence of the leptonic mixing matrix in the expression for the charged current interactions, the relevant part of the leptonic Lagrangian being

\[- L_{w+m} = \frac{g}{\sqrt{2}} \bar{e}_{L\gamma_{\mu}} V_{L_{\mu}}^{\dagger} U_{L} \nu_{L} W^{\mu} + \text{diag. mass terms}. \quad (1)\]

Here \(V_{L}\) and \(U_{L}\) are the left-handed unitary transformations that diagonalize the mass matrices of charged leptons and neutrinos. The leptonic mixing matrix \(U_{L} = V_{L}^{\dagger} U_{L}\) relates the left-handed components of the neutrino mass eigenstates and flavour eigenstates according to

\[|\nu_{a}^{\text{fl}}\rangle = \sum_{i} U_{ai}^{*} |\nu_{i}^{\text{mass}}\rangle \quad (a = e, \mu, \tau; \ i = 1, 2, 3). \quad (2)\]

For relativistic neutrinos, the oscillation probability in vacuum is

\[P(\nu_{a} \rightarrow \nu_{b}; L) = \left| \sum_{i} U_{bi} e^{-i m_{i}^{2} L} U_{ai}^{*} \right|^{2}. \quad (3)\]

For 2-flavour (2f) oscillations, which are a good first approximation in many cases, one has

\[|\nu_{e}\rangle = \cos \theta |\nu_{1}\rangle + \sin \theta |\nu_{2}\rangle, \quad (4)\]

\[|\nu_{\mu}\rangle = -\sin \theta |\nu_{1}\rangle + \cos \theta |\nu_{2}\rangle, \quad (5)\]

and eq. (1) yields the 2f transition probability

\[P_{12} = \sin^{2} 2\theta \sin^{2} \left( \frac{\Delta m^{2}}{4E L} \right). \quad (6)\]

The modes of neutrinos oscillations depend on the character of neutrino mass terms:

- Dirac mass terms \((\bar{\nu}_{L_{a}} m_{D} N_{R} + \text{h.c.})\): active - active oscillations \(\nu_{aL} \leftrightarrow \nu_{bL} (a, b = e, \mu, \tau)\)
  Neutrinos are Dirac particles.

- Majorana mass terms \((\bar{\nu}_{L_{a}} m_{L} \nu_{L})^{c} + \text{h.c.})\): active - active oscillations \(\nu_{aL} \leftrightarrow \nu_{bL}\).
  Neutrinos are Majorana particles.

- Dirac + Majorana mass terms \((\bar{\nu}_{L_{a}} m_{D} N_{R} + \bar{\nu}_{L_{a}} m_{L} \nu_{L})^{c} + \bar{N}_{R} M(N_{R})^{c} + \text{h.c.})\): active - active oscillations \(\nu_{aL} \leftrightarrow \nu_{bL}\); active - sterile oscillations \(\nu_{aL} \leftrightarrow (N_{bR})^{c}\).
  Neutrinos are Majorana particles.

Would an observation of active - sterile neutrino oscillations mean that neutrinos are Majorana particles? Not necessarily! In principle, one can have active - sterile oscillations with only Dirac - type mass terms at the expense of introducing additional species of sterile neutrinos with opposite lepton number \(L\).

2.2. Neutrino oscillations in matter – The MSW effect

Matter can change the pattern of neutrino oscillations drastically. In particular, a resonance enhancement of oscillations and resonance flavour conversion become possible (Wolfenstein, 1978; Mikheyev & Smirnov, 1985).

Matter effect on neutrino oscillations is due to the coherent forward scattering of neutrinos on the constituents of matter (fig. 3). The neutral current interactions, mediated by the \(Z^{0}\) boson exchange, are the same for active neutrinos of all three flavours (modulo tiny radiative corrections) and therefore do not affect neutrino oscillations. In contrast to this, charged current
interactions, mediated by the $W^\pm$ exchanges, are only possible for electron neutrinos because there are no muons or tauons in normal matter. This yields an effective potential of the electron neutrinos

$$V_{CC}^{ee} \equiv V = \sqrt{2} G_F N_e,$$

which leads to a modification of the nature of neutrino oscillations in matter. The 2f neutrino evolution equation in matter is

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2}{4E} \cos 2\theta + V & \frac{\Delta m^2}{4E} \sin 2\theta \\ \frac{\Delta m^2}{4E} \sin 2\theta & -\frac{\Delta m^2}{4E} \cos 2\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \quad (7)$$

The mixing angle in matter $\theta_m$, which diagonalizes the Hamiltonian on the r.h.s. of eq. (7), is different from the vacuum mixing angle $\theta$:

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta \cdot (\frac{\Delta m^2}{2E})^2}{[\frac{\Delta m^2}{2E} \cos 2\theta - \sqrt{2} G_F N_e]^2 + (\frac{\Delta m^2}{2E})^2 \sin^2 2\theta}. \quad (8)$$

The flavour eigenstates can now be written as

$$|\nu_e\rangle = \cos \theta_m |\nu_{1m}\rangle + \sin \theta_m |\nu_{2m}\rangle, \quad (9)$$

$$|\nu_\mu\rangle = -\sin \theta_m |\nu_{1m}\rangle + \cos \theta_m |\nu_{2m}\rangle, \quad (10)$$

where $|\nu_{1m}\rangle$ and $|\nu_{2m}\rangle$ are the eigenstates of the neutrino Hamiltonian in matter (matter eigenstates). The Mikheyev - Smirnov - Wolfenstein (MSW) resonance condition is

$$\sqrt{2} G_F N_e = \frac{\Delta m^2}{2E} \cos 2\theta. \quad (11)$$

At the resonance $\theta_m = 45^\circ$ ($\sin^2 2\theta_m = 1$), i.e. the mixing in matter becomes maximal.

If the matter density changes slowly enough (adiabatically) along the neutrino trajectory, neutrinos can undergo a flavour conversion (see fig. 3). In the adiabatic regime the transitions between the matter eigenstates $|\nu_{1m}\rangle$ and $|\nu_{2m}\rangle$ are strongly suppressed, i.e. these states evolve independently. However, their flavour composition, which is determined by the mixing angle $\theta_m$, varies with density: $\theta_m(N_e \gg (N_e)_{res}) \approx 90^\circ$, $\theta_m(N_e = (N_e)_{res}) = 45^\circ$, $\theta_m(N_e \ll (N_e)_{res}) \approx \theta$. Therefore the state produced at high densities as, e.g., $\nu_e \approx \nu_{2m}$ will end up at low densities as
Figure 4. Adiabatic neutrino flavour conversion. Solid curves show the energy levels of neutrino matter eigenstates, dashed curves illustrate level crossing in the absence of mixing. Black and white filling corresponds to the weights of neutrino flavour eigenstates in given matter eigenstates.

A superposition of $\nu_e$ and $\nu_\mu$ with the weights $\sin^2 \theta$ and $\cos^2 \theta$, respectively. The adiabaticity (slow density change) condition can be written as

$$\frac{\sin^2 2\theta \Delta m^2}{\cos 2\theta \cdot 2E} L_\rho \gg 1,$$

where $L_\rho$ – electron density scale height: $L_\rho = \left|(1/N_e)dN_e/dx\right|^{-1}$.

A simple and useful formula for 2f conversion probability, averaged over production/detection positions (or small energy intervals), was derived in [4]:

$$P_{tr} = \frac{1}{2} - \frac{1}{2} \cos 2\theta_i \cos 2\theta_f \cos 2P' (1 - 2P').$$

Here $\theta_i$ and $\theta_f$ are the mixing angles in matter in the initial and final points of the neutrino path, and $P'$ is the hopping probability, which takes into account possible deviations from adiabaticity: In the adiabatic regime $P' \ll 1$, whereas in the extreme non-adiabatic regime $P' = \sin^2(\theta_i - \theta_f)$.

The evolution equation for the neutrino system can be also written as

$$\frac{d\vec{S}}{dt} = 2(\vec{B} \times \vec{S}), \quad \text{where} \quad \vec{S} = \{\text{Re}(\nu_e^* \nu_\mu), \text{Im}(\nu_e^* \nu_\mu), \nu_e^* \nu_e - 1/2\},$$

$$\vec{B} = \{(\Delta m^2/4E) \sin 2\theta_m, \ 0, \ V/2 - (\Delta m^2/4E) \cos 2\theta_m\}.$$

The first equation here coincides with the equation for spin precession in a magnetic field. This analogy can be used for a graphical illustration of neutrino oscillations in matter (see fig. 5).

Figure 5. Analogy between neutrino oscillations in matter and spin precession in a magnetic field. Left panel: oscillations in constant-density matter, middle and right panels – adiabatic and non-adiabatic conversions in matter of varying density (adopted from [5]).
Another analogy of neutrino flavour conversion in matter is provided by a system of two coupled pendula [6] (see fig. 6). When the right pendulum gets a kick, it starts oscillating, but the left pendulum is almost at rest because the eigenfrequencies of the two pendula are very different. With the length \( l_2 \) of the right pendulum slowly decreasing, its eigenfrequency approaches that of the left one, and when \( l_2 = l_1 \) the two frequencies coincide (the resonance occurs): both pendula oscillate with the same amplitude. When the length of the right pendulum decreases further, the amplitude of its oscillations decreases too, while the left pendulum starts oscillating with a large amplitude. This adiabatic transfer of the oscillation energy from one pendulum to another is analogous to the adiabatic neutrino flavour conversion.

Figure 6. Mechanical analogue of neutrino flavour conversion in matter – two coupled pendula of variable lengths.

Analysis of the solar neutrino data and the results of the KamLAND and CHOOZ reactor neutrino experiments has convincingly demonstrated that the (large mixing angle) MSW effect is responsible for the flavour conversion of solar neutrinos, thus resolving the long-standing problem of the deficiency of the observed flux of solar neutrinos. This is illustrated by the analysis of the Bari group, in which the strength of the matter-induced potential of electron neutrinos was considered a free parameter (fig. 7). For more on MSW effect, see the talk of A. Friedland [8].

Figure 7. Results of the analysis of the solar, CHOOZ and KamLAND data with the standard matter-induced potential rescaled by a factor \( a_{MSW} \), treated as a free parameter. The value \( a_{MSW} \approx 1 \) is strongly favoured [7].

3. Phenomenology
All the available neutrino data except those of the LSND experiment can be explained in terms of oscillations between the 3 known neutrino species – \( \nu_e \), \( \nu_\mu \) and \( \nu_\tau \). If the LSND results are correct, they would most likely require the existence \( \geq 1 \) light sterile neutrinos \( \nu_s \) (though some exotic scenarios also exist: CPT violation, violation of Lorentz invariance, mass-varying neutrinos, shortcuts in extra dimensions, decaying neutrinos, etc.). The MiniBooNE experiment was designed to confirm or refute the LSND claim, and the results are expected very soon. One should remember, however, that even if the LSND result is not confirmed, this would not rule out the existence of light sterile neutrinos and \( \nu_s \leftrightarrow \nu_e \) oscillations, which is an intriguing possibility with important implications for particle physics, astrophysics and cosmology. From
now on I will concentrate on 3-flavour (3f) oscillations of active neutrinos. For more on sterile neutrinos, see the talk of A. Kusenko [9].

3.1. 3-flavour neutrino mixing and oscillations

For 3 neutrino species the mixing matrix depends in general on 3 mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, one Dirac-type CP-violating phase $\delta_{\text{CP}}$, and two Majorana-type CP violating phases $\sigma_{1,2}$. The Majorana-type phases can be factored out in the mixing matrix according to $\nu_3$. 3-flavour neutrino mixing and oscillations

neutrinos, see the talk of A. Kusenko [9].

For more on sterile reasons for this are (i) the hierarchy of $\Delta m^2_{31}$, separated from $\nu_1$ and $\nu_2$ by the largest mass gap, is the heaviest one, and inverted hierarchy, when $\nu_3$ is the lightest state (see fig. 8).

Normal hierarchy:

![Diagram of normal hierarchy]

$$U = \begin{pmatrix} c_{12} c_{13} & -s_{12} c_{23} c_{13} e^{i \delta_{\text{CP}}} & s_{12} s_{23} + c_{12} c_{23} e^{i \delta_{\text{CP}}} \\ s_{12} c_{23} c_{13} & c_{12} c_{23} - s_{12} s_{23} c_{13} e^{i \delta_{\text{CP}}} & s_{13} e^{-i \delta_{\text{CP}}} \\ s_{12} s_{23} - c_{12} c_{23} e^{i \delta_{\text{CP}}} & -s_{12} c_{23} - s_{12} s_{23} c_{13} e^{i \delta_{\text{CP}}} & c_{13} c_{23} \end{pmatrix}, \quad (16)$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$.

Neutrino oscillations probe the neutrino mass squared differences, which satisfy $\Delta m^2_{\text{sol}} \equiv \Delta m^2_{21} \ll \Delta m^2_{32} \equiv \Delta m^2_{\text{atm}}$. Accordingly, there are two possible orderings of the neutrino masses: normal hierarchy, when the mass eigenstate $\nu_3$, separated from $\nu_1$ and $\nu_2$ by the largest mass gap, is the heaviest one, and inverted hierarchy, when $\nu_3$ is the lightest state (see fig. 8).

Inverted hierarchy:

![Diagram of inverted hierarchy]

Figure 8. Normal and inverted neutrino mass orderings. The different fillings show the relative weights of different flavour eigenstates in given mass eigenstates.

3.2. 2f and effective 2f approximations

In many cases 2f description of neutrino oscillations gives a good first approximation. The reasons for this are (i) the hierarchy of $\Delta m^2$: $\Delta m^2_{\text{sol}} \ll \Delta m^2_{\text{atm}}$, and (ii) the smallness of $|U_{ee}|$. There are exceptions, however: when oscillations due to the solar frequency ($\propto \Delta m^2_{\text{sol}}$) are not frozen, the probabilities $P(\nu_\mu \leftrightarrow \nu_\tau)$, $P(\nu_\mu \rightarrow \nu_\mu)$ and $P(\nu_\tau \rightarrow \nu_\tau)$ do not have a 2f form [10]. However, even for the probabilities of oscillations involving $\nu_e$, the corrections due to 3-flavourness can be as large as $\sim 10\%$, i.e. are at the same level as the accuracy of the present-day data, and so cannot be ignored. In addition, there is a number of very interesting pure 3f effects in neutrino oscillations. Therefore, 3f analyses are now a must.

For oscillations driven by $\Delta m^2_{\text{sol}}$, the third neutrino mass eigenstate $\nu_3$ essentially decouples. However, there is still a “memory” of this state through unitarity, which results in the emergence of various powers of $c_{13}$ in the expressions for transition probabilities. An example is the survival probability of solar $\nu_e$ [11] (the same expression also applies for the survival probability of reactor $\bar{\nu}_e$ observed in KamLAND):

$$P_D(\nu_e \rightarrow \nu_e) \simeq c_{13}^4 P_{2ee}(\Delta m^2_{21}, \theta_{12}, c_{13}^2 V) + s_{13}^4. \quad (17)$$
The term \( s_{13}^4 \) is tiny and can be safely neglected. Another example is the day-night effect for solar \( \nu_e \); while the day-time survival probability \( P_D(\nu_e) \propto c_{13}^4 \), the difference of the night-time and day-time probabilities \( P_N(\nu_e) - P_D(\nu_e) \propto c_{13}^6 \) \cite{12,13}. Deviations from 2f expressions (which correspond to the limit \( \theta_{13} \to 0 \)) may be substantial: for the maximal currently experimentally allowed values of \( \sin^2 2\theta_{13} \) one has \( (1 - c_{13}^4) \simeq 0.1 \), \( (1 - c_{13}^6) \simeq 0.13 \).

For oscillations of reactor \( \bar{\nu}_e \), the survival probability can to a very good accuracy be written as

\[
P_{\bar{\nu}_e \bar{\nu}_e} \simeq 1 - \sin^2 2\theta_{13} \cdot \sin^2 \left( \frac{\Delta m^2_{31} L}{4E} \right) - c_{13}^4 \sin^2 2\theta_{12} \cdot \sin^2 \left( \frac{\Delta m^2_{21} L}{4E} \right).
\]  

(18)

Since the average energies of reactor antineutrinos \( E \sim 4 \text{ MeV} \), for experiments with relatively short baseline \( (L \lesssim 1 \text{ km}) \), such as CHOOZ, Palo Verde and Double CHOOZ, one has \( (\Delta m^2_{21}/4E)L \ll 1 \). Eq. (18) then reduces to

\[
P(\bar{\nu}_e \to \bar{\nu}_e; L) = 1 - \sin^2 2\theta_{13} \cdot \sin^2 \left( \frac{\Delta m^2_{31} L}{4E} \right),
\]

(19)
i.e. takes the 2f form. Note that the “solar” term \( \sim \sin^2 2\theta_{12} \) in eq. (18) cannot be neglected if \( \theta_{13} \lesssim 0.03 \), which is about the reach of currently discussed future reactor experiments.

For the unique long-baseline reactor neutrino experiment KamLAND \( (L \simeq 170 \text{ km}) \) one has \( (\Delta m^2_{21}/4E)L \gtrsim 1 \), \( (\Delta m^2_{31}/4E)L \gtrsim 1 \), and the \( \bar{\nu}_e \) survival probability takes the effective 2f form as in eq. (19). Note that matter effects in KamLAND should be of order a few per cent, i.e. can be comparable with 3f corrections due to \( \theta_{13} \neq 0 \).

3.3. Genuine 3f effects in neutrino oscillations

These are, first of all, CP and T violation. CP violation results in \( P(\nu_a \to \nu_b) \neq P(\nu_a \to \nu_b) \), whereas T violation leads to \( P(\nu_a \to \nu_b) \neq P(\nu_b \to \nu_a) \). Under the standard assumptions of locality and normal relation between spin and statistics, quantum field theory conserves CPT. CPT invariance of neutrino oscillations in vacuum gives \( P(\nu_a \to \nu_b) = P(\bar{\nu}_b \to \bar{\nu}_a) \). Therefore CP violation implies T violation and vice versa.

One can consider the following probability differences as measures of CP and T violation:

\[
\Delta P_{ab}^{CP} \equiv P(\nu_a \to \nu_b) - P(\bar{\nu}_a \to \bar{\nu}_b), \quad \Delta P_{ab}^{T} \equiv P(\nu_a \to \nu_b) - P(\nu_b \to \nu_a).
\]

(20)

From CPT invariance, for oscillations in vacuum one has

\[
\Delta P_{ab}^{CP} = \Delta P_{ab}^{T}, \quad \Delta P_{aa}^{CP} = 0.
\]

(21)

In the 3f case there is only one Dirac-type CP-violating phase \( \delta_{CP} \) and therefore only one CP and T violating probability difference:

\[
\Delta P_{e\mu}^{CP} = \Delta P_{\mu\tau}^{CP} = \Delta P_{e\tau}^{CP} \equiv \Delta P, \quad \text{where}
\]

\[
\Delta P = -4s_{12}s_{13}c_{13}^2s_{23}c_{23} \sin \delta_{CP}
\]

\[
\times \left[ \sin \left( \frac{\Delta m^2_{12} L}{2E} \right) + \sin \left( \frac{\Delta m^2_{23} L}{2E} \right) + \sin \left( \frac{\Delta m^2_{31} L}{2E} \right) \right].
\]

(22)

(23)

This probability difference vanishes when one or more of the following conditions are satisfied: at least one \( \Delta m^2_{ij} = 0 \); at least one \( \theta_{ij} = 0 \) or \( 90^\circ \); \( \delta_{CP} = 0 \) or \( 180^\circ \); in the regime of complete averaging; in the limit \( L \to 0 \) \( (\Delta P \to 0 \text{ as } L^3) \). Obviously, the effects of CP and T violating are very difficult to observe! For more on that, see the talk of O. Mena \cite{14}.
**CP violation and T violation in \( \nu \) oscillations in matter.** Normal matter (with number of particles ≠ number of antiparticles) violates C, CP and CPT, which leads to a fake (extrinsic) CP violation in neutrino oscillations. It exists even in the 2f limit and may complicate the study of the fundamental (intrinsic) CP violation.

The situation with T-violation in matter is different: matter with density profile symmetric w.r.t. the midpoint of neutrino trajectory does not induce any fake T violation. Asymmetric profiles do, but only for \( N > 2 \) flavors [15, 16]. Matter-induced T violation is an interesting pure 3f effect; it may fake fundamental T violation and complicate its study (extraction of \( \delta_{CP} \) from the experiment). However, it is absent when either \( U_{e3} = 0 \) or \( \Delta m^2_{sol} = 0 \) (2f limits) and thus is doubly suppressed by both these small parameters. Therefore its effects in terrestrial experiments are expected to be very small [16].

**Matter effects on \( \nu_\mu \leftrightarrow \nu_\tau \) oscillations.** In the 2f limit, matter does not affect \( \nu_\mu \leftrightarrow \nu_\tau \) oscillations (because the matter-induced potentials \( V(\nu_\mu) \) and \( V(\nu_\tau) \) coincide up to tiny radiative corrections). However, this is not true in the full 3f framework [17]. In particular, for oscillations inside the earth there are ranges of baselines and neutrino energies for which the matter effect can be very large (fig. 9, left panel, \( E \sim 5 – 10 \) GeV). If one ignores them, one may end up with a negative expected flux of oscillated \( \nu_\mu \) in atmospheric neutrino experiments (fig. 9, right panel).

![Figure 9](image-url)

**Figure 9.** Left panel: \( P_{\mu\tau} \). Right panel: oscillated flux of atmospheric \( \nu_\mu \). \( \Delta m^2_{31} = 2.5 \times 10^{-3} \) eV\(^2\), \( \sin^2 \theta_{13} = 0.026, \theta_{23} = \pi/4, \Delta m^2_{21} = 0, L = 9400 \) km. Red (dark) curves – with matter effects, green (light) curves – without matter effects on \( P_{\mu\tau} \).

### 3.4. Parametric resonance in neutrino oscillations in matter

The MSW effect is not the only possible way matter can influence neutrino oscillations. Another interesting possibility is a parametric enhancement of neutrino oscillations in matter [18, 19]. Parametric resonance in oscillating systems with varying parameters occurs when the rate of the parameter change is correlated in a certain way with the values of the parameters themselves. A well-known mechanical example is a pendulum with vertically oscillating suspension point (fig. 10): when the frequency \( \Omega \) and amplitude \( A \) of these oscillations are in a special correlation with the eigenfrequency \( \omega \) and amplitude \( a \) of the pendulum, the pendulum can turn upside down and start oscillating around the vertical, normally unstable, equilibrium point. Neutrino oscillations

![Figure 10](image-url)

**Figure 10.** Parametric resonance in oscillations of a pendulum with vertically oscillating point of support. For small-amplitude oscillations the resonance condition is \( \Omega_{res} = 2\omega/n \) (\( n = 1, 2, 3... \)).
in matter can undergo parametric enhancement if the length and size of the density modulation is correlated in a certain way with neutrino parameters. This enhancement is completely different from the MSW effect; in particular no level crossing is required. An example admitting an exact analytic solution is the "castle wall" density profile \[19, 20\] (see fig. 11). The resonance condition in this case can be written as \[20\]

\[ X_3 \equiv -(\sin \phi_1 \cos \phi_2 \cos 2\theta_{1m} + \cos \phi_1 \sin \phi_2 \cos 2\theta_{2m}) = 0, \]  

(24)

where \(\phi_{1,2}\) are the oscillation phases acquired in layers 1 and 2 and \(\theta_{m1,2}\) are the corresponding mixing angles in matter.

**Figure 11.** Parametric resonance in the case of a "castle wall" density profile. Coordinate dependence of the potential \(V\) (left panel) and of the transition probability \(P\) (right panel).

The earth’s density profile seen by neutrinos with core-crossing trajectories can be well approximated by a piece of this castle wall profile (fig. 12). Interestingly, the parametric resonance condition \[21\] can be satisfied for oscillations of core-crossing neutrinos in the earth for a rather wide range of zenith angles both at intermediate energies \[22, 23, 20\] and high energies \[24\] (see figs. 13 14). The parametric resonance of neutrino oscillations in the earth can be observed in future atmospheric or accelerator experiments if \(\theta_{13}\) is not too much below its current upper limit.

3.5. Some recent developments

When \(V \ll \Delta m^2/2E\) (oscillations of low-\(E\) neutrinos in matter or, equivalently, oscillations in low-density matter), matter effects on neutrino oscillations are small and can be considered in perturbation theory. This gives simple and transparent formulas describing, in particular,
oscillations of solar and supernova neutrinos in the earth. The earth matter effects can be expressed through the regeneration factor

$$f_{\text{reg}} = P^\oplus_{2e} - P^\text{vac}_{2e},$$

where

$$P^\oplus_{2e} = \frac{1}{2} c^2_{13} \sin^2 \theta_{12} \int_0^L dx V(x) \sin \left[ 2 \int_0^x \omega(x') \, dx' \right],$$

and

$$\omega(x) = \sqrt{\cos 2 \theta_{12} \delta - c^2_{13} V(x) / 2} + \delta^2 \sin^2 \theta_{12}, \quad \delta = \frac{\Delta m^2_{21}}{2E}. \tag{26}$$

The 2f ($\theta_{13} = 0$) version of these equations was derived in \cite{25} (see also \cite{26}).

The regeneration factor as the function of the cosine of the nadir angle $\Theta_z$ is shown in fig. 15. As can be seen in the left panel, in the case of perfect energy resolution one could expect a significant increase of the regeneration factor for core-crossing trajectories. However, experimentally no such an increase was observed in the $\cos \Theta_z$ dependence of the day-night signal difference for solar neutrinos, which is rather flat. As was shown in \cite{25}, this comes about because of the finite energy resolution $\Delta E$ of the detectors, which leads to a suppression of the effects of the earth density variations that are far from the detector (see the right panel of fig. 15), the attenuation length being $d \simeq l_{\text{osc}}(E/\Delta E)$.

Oscillations of high energy neutrinos in matter or, equivalently, oscillations in dense matter ($V > \delta \equiv \Delta m^2/4E$), can also be very accurately described analytically. The transition
Figure 15. Left panel: earth regeneration factor for $E = 10$ MeV neutrinos, perfect energy resolution. Black (solid) curve – numerical calculation, red (dashed) curve – analytic result. Right panel: regeneration factors averaged over three different intervals of energy [26].

The probability for oscillations in a matter of an arbitrary density profile is given by [24]

$$P = \delta^2 \sin^2 2\theta \left| \int_0^L dx e^{-2i\phi(x)} \right|^2, \quad \phi(x) = \int_0^x dx' \omega(x').$$  (27)

The accuracy of this approximation is quickly increasing with neutrino energy (see the right panel of fig. 13, where the exact results are shown by solid curves and the analytic results, by dashed curves). Eq. (27) also allows a simple analytic interpretation of the two prominent parametric peaks in the core region, seen in this figure [24].

4. Unsettled issues?

The theory of neutrino oscillations is quite mature and well developed now. However, it is far from being complete or finished, and a number of basic questions are still being debated. Below I list some of these questions (given in italics), along with my short answers to them: 3

- **Equal energies or equal momenta?**
  - Neither equal $E$ nor equal $p$ assumptions normally used in the derivations of the oscillation probability are exact. But for relativistic neutrinos, both give the correct answer.

- **Evolution in space or in time?**
  - This is related to the previous question. For relativistic neutrinos both are correct and equivalent. Fortunately, in practice we only deal with relativistic neutrinos. In the non-relativistic case the very notion of the oscillation probability is ill-defined (the probability depends on both the production and detection processes).

- **Claim: evolution in time is never observed**
  - Incorrect. Examples: K2K, MINOS (and now also CNGS) experiments, which use the neutrino time of flight in order to suppress the background.

- **Is wave packet description necessary?**
  - Yes, if one wants to rigorously justify the standard oscillation probability formula. Once this is done, the wave packets can be forgotten unless the issues of coherence become important.

- **Do charged leptons oscillate?**
  - No, they don’t.

3 Detailed discussion could not be given for the lack of time.
Is the standard oscillation formula correct?
- Yes, it is. In particular, there is no extra factor of two in the oscillation phase, which is sometimes claimed to be there. However, it would be theoretically interesting and important to study the limits of applicability of the standard approach.

A number of subtle issues of the neutrino oscillation theory still remain unsettled (e.g., rigorous wave packet treatment, oscillations of non-relativistic neutrinos, etc). At present, this is (rightfully) of little concern for practitioners.

What are interesting future tasks for the theory and phenomenology of neutrino oscillations? These include the search for the best strategies for measuring neutrino parameters, study of subleading effects and effects of non-standard neutrino interactions and of the domains of applicability and limitations of the current theoretical framework. Future experimental results may also bring some new surprises and pose more challenging problems for the theory!

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References
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