Failure of Standard Thermodynamics in Planck Scale Black Hole System

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Abstract

The final stage of the black hole evaporation is a matter of debates in the existing literature. In this paper, we consider this problem within two alternative approaches: noncommutative geometry (NCG) and the generalized uncertainty principle (GUP). We compare the results of two scenarios to find a relation between parameters of these approaches. Our results show some extraordinary thermodynamical behavior for Planck size black hole evaporation. These extraordinary behavior may reflect the need for a fractal nonextensive thermodynamics for Planck size black hole evaporation process.

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Key Words: Black Hole Thermodynamics, Noncommutative Geometry, Generalized Uncertainty Principle
1 Introduction

Usual uncertainty principle of Heisenberg should be modified to incorporate quantum gravitational effects. Existence of a minimal physical length ($\sim 10^{-33}$ cm) which is a common feature of alternative approaches to quantum gravity problem, restricts the resolution of spacetime points[1,2]. This finite resolution of spacetime can be addressed by the generalized uncertainty principle(GUP). Consequences of GUP for various aspects of quantum gravity problem have been studied extensively [3-7]. Among these studies, black hole thermodynamics has found considerable attentions[8-11]. Adler et al have argued that contrary to standard viewpoint, GUP may prevent small black holes from total evaporation in exactly the same manner that the uncertainty principle prevents the Hydrogen atom from total collapse[12]. Embedding of black hole in a space-time with higher dimensions has been studied in both compact and infinitely extended extra dimensions[13]. The results of these studies are summarized as follows

*Black hole evaporates by emission of Hawking radiation. This evaporation process terminates when black hole reaches a Planck size remnant. This remnant has zero entropy, zero heat capacity and a finite nonzero temperature.*

Of course, there are other alternative proposals to solve problem of final stage of black hole evaporation such as approach based on trace anomaly[14].

The idea of making spacetime noncommutative goes back to the early days of quantum field theory, at least as early as 1947 [15]. The idea was that considering a noncommutative structure of spacetime at very small length scale, one could introduce an effective ultraviolet cutoff. One of the main motivations is the hope that a nontrivial structure of spacetime at small distances may led one to quantum field theories with better ultraviolet behaviors. So, it seems that both GUP and noncommutative geometry(NCG) provide suitable framework for short distance behavior of physical systems. Quantum correction of black hole thermodynamics within GUP and NCG shows a similarity between the results of two approaches[16]. This feature has its own importance since it can provide a better understanding of the ultimate quantum gravity scenario. The purpose of this paper is to consider the effect of space noncommutativity and the generalized uncertainty principle on the short distance thermodynamics of an evaporating Schwarzschild black hole. Our analysis shows that extension of ordinary Boltzmann-Gibbs thermodynamics to very short distance systems such as Planck size black holes encounters severe difficulties. These difficulties may reflect the need for a nonextensive thermodynamics such as
Tsallis thermodynamics[17]. As a possible connection between the results of two scenarios, we compare our results obtained in NCG with the results of GUP to find a relation between parameters of corresponding theories.

2 Preliminaries

The study of the structure of spacetime at Planck scale, where quantum gravity effects are non-negligible, is one of the main open challenges in fundamental physics. Since the dynamical variable in Einstein general relativity is spacetime itself (with its metric structure), and since in quantum mechanics and in quantum field theory the classical dynamical variables should be noncommutative in principle, one is strongly led to conclude that noncommutativity of spacetime is a feature of Planck scale physics. This expectation is further supported by Gedanken experiments that aim at probing spacetime structure at very small distances. They show that due to gravitational back reaction, one cannot test spacetime at Planck scale. Its description as a (smooth) manifold becomes therefore a mathematical assumption no more justified by physics. It is then natural to relax this assumption and conceive a more general noncommutative spacetime, where uncertainty relations and discretization naturally arise. Noncommutativity is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables, such as position and momentum. One can just as easily imagine that position measurements might fail to commute and describe this using noncommutativity of the coordinates. The noncommutativity of spacetime can be encoded in the commutator[18-20]

\[
[\hat{x}^i, \hat{x}^j] = i\theta^{ij}
\]

where \(\theta^{ij}\) is a real, antisymmetric and constant tensor, which determines the fundamental cell discretization of spacetime much in the same way as the Planck constant \(\hbar\) discretizes the phase space. In \(d = 4\), by a choice of coordinates, this noncommutativity can be brought to the form

\[
\theta^{ij} = \begin{pmatrix}
0 & \theta & 0 & 0 \\
-\theta & 0 & \theta & 0 \\
0 & -\theta & 0 & \theta \\
0 & 0 & -\theta & 0
\end{pmatrix}
\]

This was motivated by the need to control the divergences showing up in theories such as quantum electrodynamics. Although there has been a long held belief that in theories of
quantum gravity, space-time must change its nature at distances comparable to the Planck scale, but instead of trying to modify spacetime, focus was directed to the fields defined on it. The outcome of these efforts was what is known as string theory. The strings serve to smear out the interaction in space-time and, in a sense, make the notion of a point meaningless. There is a smallest distance that one can probe. For this reason, in the context of string theories, this observable distance is referred to generalized uncertainty principle- usual uncertainty principle of quantum mechanics, the so-called Heisenberg uncertainty principle, should be reformulated due to noncommutative nature of spacetime at Planck scale. A GUP can be written as follows[3]

\[
\Delta x \Delta p \geq \frac{1}{2} \left(1 + \alpha^2 l_p^2 (\Delta p)^2\right).
\]

Where \(\alpha\) is a dimensionless and positive parameter of order unity (for simplicity, we set \(G = \hbar = c = 1\)). The main consequence of GUP is that measurement of the position is possible only up to Planck length. So one can not setup a measurement to find more accurate particle position than the Planck length, and this means that the notion of locality breaks down. In other words, we cannot look inside the region of minimal length. This minimal length provides a natural cut off for underlying quantum field theory[21]. Based on this idea, it seems that the laws of physics should be reformulated in very short distance systems. As we will show, the need for a reformulation of the Planck size black hole thermodynamics is inevitable.

After a brief overview of the conceptual preliminaries, we discuss the issue of black hole thermodynamics in two alternative approaches: GUP and NCG and finally we compare the results of these two approaches. This comparison results an interesting relation between parameters of these two scenario.

## 3 Black Hole Thermodynamics with GUP

In this section we derive black hole thermodynamics in GUP framework based on our previous works[10,11,16]. Here we focus on mass dependence of black hole thermodynamical properties. Using GUP as our primary input, we obtain temperature, entropy and heat capacity of a microscopically large Schwarzschild black hole. The results of this calculations are interesting since they reflect some unusual features of systems in very short distances.

In the current standard viewpoint, small black holes emit black body radiation at the
Hawking temperature. This temperature may be obtained in a heuristic way with the use of the standard uncertainty principle and general properties of black holes [22]. In this way, we estimate the characteristic energy $E$ of the emitted photons from the standard uncertainty principle. In the vicinity of the black hole surface, there is an intrinsic uncertainty in the position of any particle of about the Schwarzschild radius $r_s$, due to the behavior of its field lines [23], as well as on dimensional grounds. This leads to momentum uncertainty

$$\Delta p \approx \frac{1}{\Delta x} = \frac{1}{r_s} = \frac{1}{2M},$$

and to an energy uncertainty of $\Delta E \approx \frac{1}{2M}$. We identify this as the characteristic energy of the emitted photon, and thus as a characteristic temperature; it agrees with the Hawking temperature up to a factor of $4\pi$, which we will henceforth include as a ”calibration factor” and write, with $k_B = 1$,

$$T_H \approx \frac{1}{8\pi M},$$

The related entropy is obtained by integration of $dS = \frac{dM}{T}$ which is the standard Bekenstein entropy,

$$S_B = 4\pi M^2.$$  

(5)

However, if one consider the GUP as given by equation (2), the last two equations become respectively,

$$T_{GUP} = \frac{M}{4\pi} \left[ 1 \mp \sqrt{1 - \frac{1}{M^2}} \right],$$

(6)

and

$$S_{GUP} = 2\pi \left[ M^2 + \sqrt{1 - \frac{1}{M^2}} - \ln \left( M + \sqrt{M^2 - 1} \right) - 1 \right].$$

(7)

In equation (6), to recover the well-known results in the large mass limit, one should consider the minus sign. As these equation show, when the size of black hole approaches the Planck scale size, it will cease radiation and its temperature reaches a maximum. This can be seen from the behavior of the heat capacity. From equation (4), we obtain standard heat capacity as follows

$$C_H = \frac{dM}{dT_H} = -8\pi M^2.$$  

(8)
If we consider $T_{\text{GUP}}$ as given by equation (6), we obtain the following generalized heat capacity

$$C_{\text{GUP}} = \frac{dM}{dT_{\text{GUP}}} = -\frac{4\pi \sqrt{1 - \frac{1}{M^2}}}{1 - \sqrt{1 - \frac{1}{M^2}}}.$$ (9)

These equations strongly suggest the existence of black holes remnants. As it is evident from figures 1, 2 and 3, in the framework of GUP black hole can evaporate until when it reaches a remnant with Planck mass. This remnant has zero entropy, zero heat capacity and a non-zero maximal temperature. In the existing literatures there is no obvious reason for this maximum temperature. As we will show, within noncommutative geometry considerations, this maximum temperature of remnant decreases and finally reaches to zero. Vanishing of entropy for Planck size remnant seems to be strange and needs more careful considerations.

One can show that within this viewpoint black hole remnants are stable[12].

4 Black Hole Thermodynamics in Noncommutative Spaces

In this section we describe the effects of space noncommutativity on the black hole thermodynamics.

There are two relatively different viewpoint to incorporate space noncommutativity in the issue of black hole thermodynamics. In one of these viewpoints, one considers the effect of space noncommutativity on the radius of event horizon by a simple analysis on coordinates noncommutativity[24,25]. In this framework one can show that up to second order of noncommutativity parameter, the noncommutative radius of event horizon has the following form[16]

$$\dot{r}_s = r_s - \frac{\zeta}{2r_s} + \frac{27}{8} \frac{\zeta^2}{r_s^3}, \quad r_s = 2M$$ (10)
Figure 1: Temperature of a black hole versus its mass. Mass is in the units of Planck mass and temperature is in the units of the Planck energy. The lower curve (dashed line) is the well-known Hawking result, while the upper curve (line) is the result of GUP.

Figure 2: Entropy of a black hole versus its Mass. Entropy is dimensionless and mass is in the units of the Planck mass. The upper curve (dashed line) is the Hawking result, while the lower curve (line) is the result of GUP.
Figure 3: Heat capacity of a black hole versus the mass. The lower curve is the Hawking result (dashed line), while the upper curve (line) is the result of GUP.

where

$$\zeta = \frac{1}{16(1 + \beta p^2)^2} \left( p_x^2 + p_y^2 \right) \theta^2$$

and $p_i$ are components of black hole linear momentum and $\beta$ is string theory parameter related to minimal length[3]. Within this framework, perturbative thermodynamics of noncommutative Schwarzschild black hole based on analysis presented in [16] can be summarized as follows:

Black hole temperature:

$$\hat{T}_H \approx \frac{1}{8\pi M} \left[ 1 + \frac{\zeta}{4M^2} - \frac{3\zeta^2}{8M^4} \right].$$

(11)

Black hole entropy:

$$S \simeq \frac{A}{4} - \frac{\pi \zeta}{2} \ln \frac{A}{4} + \sum_{n=1}^{\infty} c_n \left( \frac{A}{4} \right)^n + C$$

(12)

where $C$ is a constant of integration and $c_n$ are constant. This framework considers the effect of noncommutativity on geometric properties of black hole (such as black hole event horizon radius) as starting point and one can obtain a modified Schwarzschild line element using modified event horizon radius.

There is another viewpoint based on Nicolini et al. works[26]. It has been shown [27] that noncommutativity eliminates point-like structures in favor of smeared objects in
flat spacetime. As Nicolini et al have shown, the effect of smearing is mathematically implemented as a substitution rule: position Dirac-delta function is replaced everywhere with a Gaussian distribution of minimal width $\sqrt{\theta}$. In this framework, they have chosen the mass density of a static, spherically symmetric, smeared, particle-like gravitational source as follows

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4\theta}\right) \tag{13}$$

As they have indicated, the particle mass $M$, instead of being perfectly localized at a point, is diffused throughout a region of linear size $\sqrt{\theta}$. This is due to the intrinsic uncertainty as has been shown in the coordinate commutator (1). This matter source results the following static, spherically symmetric, asymptotically Schwarzschild solution of the Einstein equations[26]

$$ds^2 = \left(1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right)dt^2 - \left(1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right)^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{14}$$

where $\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)$ is the lower incomplete Gamma function:

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \equiv \int_0^{\frac{r^2}{4\theta}} t^\frac{1}{2} e^{-t} dt \tag{15}$$

The event horizon of this metric can be found where $g_{00}(r_s) = 0$,

$$r_s = \frac{4M}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r_s^2}{4\theta}\right) \tag{16}$$

As it is obvious from this equation, the effect of noncommutativity in the large radius regime can be neglected, while at short distance one expects significant changes due to the spacetime fuzziness. Now the black hole temperature can be calculated as follows

$$T_{NCG} \equiv \left(\frac{1}{4\pi\sqrt{-g_{00}g_{11}}} \right)_{r=r_s} \frac{dg_{00}}{dr} = \frac{1}{4\pi r_s} \left[1 - \frac{r_s^3}{4\theta^2} \exp\left(-\frac{r_s^2}{4\theta}\right) \gamma\left(\frac{3}{2}, \frac{r_s^2}{4\theta}\right)\right], \tag{17}$$

where, $M$ has been expressed in terms of $r_s$ from the horizon equation (16). For large black holes, where $\frac{r_s^2}{4\theta} >> 1$, one recovers the standard result for the Hawking temperature

$$T_H = \frac{1}{4\pi r_s} \tag{18}$$

At the initial stages of radiation, the black hole temperature increases while the horizon radius is decreasing. It is important to investigate what happens as $r_s \to \sqrt{\theta}$. In the
commutative case $T_H$, which is given by (18), diverges and this puts limit on the validity of the conventional description of Hawking radiation. Against this scenario, temperature (17) includes noncommutative effects which are relevant at distances comparable to $\sqrt{\theta}$[26].

Here, for numerical calculation purposes, for a moment we set $\theta = 1$, and rewrite equation (17) as follows

$$T_{NCG} = \frac{1}{8\pi M \left[ 1 - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, M^2 \right) \right]} \times$$

$$\left[ 1 - \frac{8M^3 \left[ 1 - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, M^2 \right) \right]^3 \exp \left( -M^2 \left[ 1 - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, M^2 \right) \right]^2 \right)}{4\Gamma\left(\frac{3}{2}, M^2 \right) \left[ 1 - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}, M^2 \right) \right]} \right],$$

(19)

where $\Gamma$ is upper Gamma function[26]. Behavior of the noncommutative space temperature $T_{NCG}$ as a function of black hole mass is plotted in figure 4 and 5 in two different limits.

![Figure 4](image-url)

**Figure 4:** Temperature of a black hole versus its mass. Mass is in the units of Planck mass and temperature is in the units of Planck energy. Curve (b) is the Hawking result, while curve (a) is the result of NCG.
Figure 5: Temperature of a black hole versus its mass. Mass is in the units of Planck mass and temperature is in the units of Planck energy. The lower curve (b) is the Hawking result, while the upper curve (a) is the result of NCG. In this figure we have considered the possibility of total evaporation.

As figure 4 shows, within noncommutative geometry, temperature of black hole grows during its evaporation until it reaches to a maximum extremal value and then falls down to zero. In figure 5 we have considered the possibility of complete evaporation of black hole. If black hole evaporate completely, noncommutative geometry consideration leads to a negative temperature for black hole. We know from thermodynamics that negative temperature can be reached by crossing high temperatures. Therefore an extraordinary result is obtained for very short distance system of Planck scale black hole. Note that Nicolini et al have not considered the possibility of total evaporation. They have plotted their figures for evaporation process which continues only to a Planck size remnant[26]. They have not considered the possibility of total evaporation. One point should be stressed here: from GUP view point total evaporation is forbidden while space noncommutativity consideration do not restrict evaporation process to a Planck size remnant. In fact, equation of black hole thermodynamics in noncommutative space allow the possibility of total evaporation. For this reason we have considered the possibility of total evaporation. The entropy of the black hole can be obtained using the following relation

$$S_{NCG} = \int_0^M \frac{dM}{T_{NCG}}.$$  

(20)
The numerical result of this integration is shown in figure 6.

Figure 6: Entropy of a black hole versus its mass. Entropy is dimensionless and mass is in the units of Planck mass. The upper curve (a) is the Hawking result, and the lower curve (b) is the result of NCG. This figure shows the failure of standard Boltzmann-Gibbs thermodynamics for Planck scale black holes. Negative entropy is a signature of this failure.

As figure 6 shows, entropy of black hole in noncommutative space has some unusual behavior, specially it attains negative values for some intervals of mass variation. This is physically meaningless. This unusual thermodynamical behavior may be related to fractal structure of spacetime in very short distances. In other words, application of ordinary thermodynamics to situations such as Planck scale black hole seems to be impossible. Due to non-extensive and non-additive nature of such systems, one should apply non-extensive formalism such as Tsallis thermodynamics[17]. Fractal nature of spacetime at very short distances encourages the use of non-extensive thermodynamics for Planck size black hole. The purpose of this paper is to show the need for these non-extensive thermodynamics for Planck size black holes. The formulation of such a thermodynamics remains for future. We believe that theories such as E-infinity[28] and scale relativity[29] which are based on fractal structure of spacetime at very short distances, provide a possible framework for thermodynamics of these short distance systems. This will be the subject of our forthcoming paper.
Heat capacity of black hole is given by

\[ C_{NCG} = \left( \frac{dT_{NCG}}{dM} \right)^{-1}, \]  

and its variation as a function of black hole mass is plotted in figure 7.

Figure 7: Heat capacity of a black hole versus the mass. The lower curve (a), is the Hawking result(a) while the upper curve, (b) is the result of NCG. Failure of standard thermodynamics is evident from this figure.

5 Extra Dimensional Considerations

One surprising prediction of string theory is that several extra dimension should be exist. This possible existence has opened up new and exciting directions of research in quantum gravity. One of the most significant sub-fields is the study of black hole production at particle colliders, such as the Large Hadronic Collider (LHC)[30] and the muon collider [31], as well as in ultrahigh energy cosmic ray (UHECR) airshowers [32]. Furthermore, detection and vastly production of such black holes at LHC could then be examined experimentally in some details[33]. In such a scenario it would be natural that GUP and NCG or both combination would now also be of order TeV allowing it to be accessible to
colliders, because the properties of such $TeV$-scale black hole may be influenced by NCG and GUP or both combination effects, which originate at a similar scale, and if these effects are large enough to be observable in collider data output.

In a scenario with large extra dimensions (such as Arkani-Hamed, Dimopoulos and Dvali (ADD) model [34]), GUP can be written as follows

$$\Delta x_i \Delta p_i \geq \frac{1}{2} \left( 1 + \alpha^2 L_{Pl}^2 (\Delta p_i)^2 + \frac{\beta^2}{L_{Pl}^2} (\Delta x_i)^2 + \gamma \right). \quad (22)$$

Here $\alpha$, $\beta$ and $\gamma$ are dimensionless, positive and independent of $\Delta x$ and $\Delta p$ but may in general depend on the expectation values of $x$ and $p$. Planck length now is defined as $L_{Pl} = G_d^{1/2}$ where $G_d$ is gravitational constant in $d$-dimensional spacetime which in ADD model is given by $G_d = G_4 L^{d-4}$ ($L$ is the extension of the compactified dimensions).

In which follows, we use this general form of GUP as our primary input and construct a perturbational calculations to find thermodynamical properties of black hole and its quantum gravitational corrections. It should be stressed that since GUP is a model independent concept, the results which we obtain are consistent with any fundamental theory of quantum gravity.

A $d$-dimensional spherically symmetric BH of mass $M$ (to which the collider BHs will settle into before radiating) is described by the metric [9],

$$ds^2 = -\left( 1 - \frac{16\pi G_d M}{(d-2) \Omega_{d-2} r^{d-3}} \right) dt^2 + \left( 1 - \frac{16\pi G_d M}{(d-2) \Omega_{d-2} r^{d-3}} \right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \quad (23)$$

where $\Omega_{d-2}$ is the metric of the unit $S^{d-2}$ as $\Omega_{d-2} = \frac{2\pi^{d-1}}{\Gamma(\frac{d-1}{2})}$. Since the Hawking radiation is a quantum process, the emitted quanta should satisfy the generalized uncertainty principle (which has quantum gravitational nature) in its general form. Therefore, we consider equation (22), where $x_i$ and $p_i$ with $i = 1...d-1$, are the spatial coordinates and momenta respectively. By modeling a BH as a $(d-1)$-dimensional cube of size equal to its Schwarzschild radius $r_s$, the uncertainty in the position of a Hawking particle at the emission is,

$$\Delta x_i \approx r_s = \omega_d L_{Pl} m^\frac{1}{d-1}, \quad (24)$$

where

$$\omega_d = \left( \frac{16\pi}{(d-2) \Omega_{d-2}} \right)^{\frac{1}{d-1}}.$$
\[ m = \frac{M}{M_{Pl}} \text{ and } M_{Pl} = G_d^{-\frac{1}{2}} = L^{-1}_{Pl}. \] Here \( \omega_d \) is dimensionless area factor. A simple calculation based on equation (22) gives,

\[ \Delta x_i \approx \frac{L^2_{Pl}\Delta p_i}{\beta^2} \left[ 1 \pm \sqrt{1 - \beta^2 \left( \frac{\alpha^2 + \frac{\gamma + 1}{L^2_{Pl}(\Delta p_i)^2}}{1 - \alpha^2 \beta^2} \right)^2} \right]. \tag{25} \]

where to achieve standard values (for example \( \Delta x_i \Delta p_i \geq 1 \)) in the limit of \( \alpha = \beta = \gamma = 0 \), we should consider the minus sign. One can minimize \( \Delta x \) to find

\[ (\Delta x_i)_{min} \approx r_s(min) \approx \pm \alpha L_{Pl} \sqrt{1 + \frac{\gamma}{1 - \alpha^2 \beta^2}}. \tag{26} \]

This is the minimal observable length of the order of Planck length. Here we should consider the plus sign whereas the negative sign has no evident physical meaning. Equation (22) gives also

\[ \Delta p_i \approx \frac{\Delta x_i}{\alpha^2 L^2_{Pl}} \left[ 1 \pm \sqrt{1 - \alpha^2 \left( \beta^2 + \frac{L^2_{Pl}(\gamma + 1)}{(\Delta x_i)^2} \right)^2} \right]. \tag{27} \]

To achieve correct limiting results we should consider the minus sign in bracket. From a heuristic argument based on Heisenberg uncertainty relation, one deduces the following equation for Hawking temperature of black holes

\[ T_H \approx \frac{(d - 3) \Delta p_i}{4\pi} \] \tag{28}

where we have set the constant of proportionality equal to \( \frac{(d - 3)}{4\pi} \) in extra dimensional scenario. Based on this viewpoint, but now using generalized uncertainty principle in its general form (22), modified black hole temperature in GUP is,

\[ T_H^{GUP} \approx \frac{(d - 3) \Delta x_i}{4\pi\alpha^2 L^2_{Pl}} \left[ 1 \pm \sqrt{1 - \alpha^2 \left( \beta^2 + \frac{L^2_{Pl}(\gamma + 1)}{(\Delta x_i)^2} \right)^2} \right]. \tag{29} \]

Since \( \Delta x_i \) is given by (24), this relation can be expressed in terms of black hole mass in any stage of its evaporation. Figure 8 shows the relation between temperature and mass of the black hole in different spacetime dimensions. Following results can be obtained from this analysis : In scenarios with extra dimensions, black hole has higher temperature. This feature leads to faster decay and less classical behaviors for black holes. It is evident that in extra dimensional scenarios final stage of evaporation( black hole remnant) has mass more than its four dimensional counterpart. Therefore, in the framework of GUP, it
seems that quantum black holes are hotter, shorter-lived and tend to evaporate less than classical black holes. Note that these results are applicable to both ADD and RS brane world scenarios.

Figure 8: Temperature of black hole Versus its mass in different spacetime dimensions.

Now consider a quantum particle that starts out in the vicinity of an event horizon and then ultimately absorbed by black hole. For a black hole absorbing such a particle with energy $E$ and size $l$, the minimal increase in the horizon area can be expressed as

$$ (\Delta A)_{\text{min}} \geq \frac{8\pi L_{\text{Pl}}^{d-2} El}{(d-3)}, $$

(30)

then one can write

$$ (\Delta A)_{\text{min}} \geq \frac{8\pi L_{\text{Pl}}^{d-2} \Delta p_i l}{(d-3)}, $$

(31)

where $E \sim \Delta p_i$ and $l \sim \Delta x_i$.

$$ (\Delta A)_{\text{min}} \simeq \frac{8\pi L_{\text{Pl}}^{d-4} (\Delta x_i)^2}{(d-3)\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2 \left( \frac{\beta^2 + L_{\text{Pl}}^2 (\gamma + 1)}{(\Delta x_i)^2} \right)^{\frac{1}{2}}} \right], $$

(32)

Now we should determine $\Delta x_i$. Since our goal is to compute microcanonical entropy of a large black hole, near-horizon geometry considerations suggests the use of inverse surface
gravity or simply the Schwarzschild radius for $\Delta x_i$. Therefore, $\Delta x_i \approx r_s$ and defining $\Omega_{d-2} r_s^{d-2} = A$ or $r_s^2 = \Omega_{d-2} A^{\frac{2}{d-2}}$ and $(\Delta S)_{\text{min}} = b$, then it is easy to show that,

$$
(\Delta A)_{\text{min}} \simeq \frac{8\pi L_{\text{Pl}}^{d-4} \Omega_{d-2}^{-\frac{2}{d-2}} A^{\frac{2}{d-2}}}{(d-3)\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2 \left( \beta^2 + \frac{L_{\text{Pl}}^2 (\gamma + 1)}{\Omega_{d-2}^{-\frac{2}{d-2}} A^{\frac{2}{d-2}}} \right)} \right], \quad (33)
$$

and,

$$
\frac{dS}{dA} \simeq \frac{(\Delta S)_{\text{min}}}{(\Delta A)_{\text{min}}} \simeq \frac{8\pi L_{\text{Pl}}^{d-4} A^{\frac{2}{d-2}}}{1 - \sqrt{1 - \alpha^2 \left( \beta^2 + \frac{\Omega_{d-2}^{-\frac{2}{d-2}} L_{\text{Pl}}^2 (\gamma + 1)}{A^{\frac{2}{d-2}}} \right)}}. \quad (34)
$$

Note that $b$ can be considered as one bit of information since within standard thermodynamics entropy is an extensive quantity. Note also that in our approach we consider microcanonical ensemble since we are dealing with Schwarzschild black hole of fixed mass.

Now we should perform integration. There are two possible choices for lower limit of integration, $A = 0$ and $A = A_p$. Existence of a minimal observable length leads to existence of a minimum event horizon area, $A_p = \Omega_{d-2} (\Delta x_i)_{\text{min}}^{d-2}$. So it is physically reasonable to set $A_p$ as lower limit of integration. This is in accordance with existing picture[12]. Based on these arguments, we can write

$$
S \simeq \varepsilon \int_{A_p}^{A} \frac{A^{-\frac{2}{d-2}}}{1 - \sqrt{\eta + \kappa A^{-\frac{2}{d-2}}}} dA \quad (35)
$$

or

$$
S \simeq \varepsilon \int_{r_s(\text{min})}^{r_s} \frac{(d-2) \Omega_{d-2}^{\frac{d-6}{d-2}} r_s^{-d-5}}{1 - \sqrt{\eta + \kappa \left( \Omega_{d-2}^{\frac{d-6}{d-2}} r_s^{-2} \right)^{-2}}} dr_s \quad (36)
$$

where,

$$
\varepsilon = \frac{\Omega_{d-2}^{\frac{2}{d-2}} b \alpha^2 (d-3)}{8\pi L_{\text{Pl}}^{d-4}}, \quad \kappa = -\Omega_{d-2}^{\frac{2}{d-2}} \alpha^2 L_{\text{Pl}}^2 (\gamma + 1), \quad \eta = 1 - \alpha^2 \beta^2,
$$

$$
A_p = \Omega_{d-2} (\alpha L_{\text{Pl}})^{d-2} \left( \frac{1 + \gamma}{1 - \alpha^2 \beta^2} \right)^{(d-2)} \quad (37)
$$

This integral can be solved numerically. The results are shown in figure 9. These figures show that: In scenarios with extra dimensions, black hole entropy decreases. The classical picture breaks down since the degrees of freedom of the black hole, i.e. its entropy,
is small. In this situation one can use the semiclassical entropy to measure the validity of the semiclassical approximation. It is evident that in extra dimensional scenarios final stage of evaporation (black hole remnant) has event horizon area greater than its four dimensional counterpart. Therefore, higher dimensional black hole remnants have less classical features relative to their four dimensional counterparts. In addition, as figure 9 shows, for large $d$ (for example $d \geq 8$), one find a linear area-entropy relation but this linear entropy-area relation differs with standard Bekenstein-Hawking result since it has greater slope.

![Figure 9: Entropy of black hole versus the area of its event horizon in different spacetime dimensions.](image)

We assumed that the metric of our $d$-dimensional space is given by

$$ds^2 = e^\nu dt^2 - e^\mu dr^2 - r^2 d\Omega^2_{d-2}. \quad (39)$$

$$\rho_{\theta,d}(r) = \frac{M}{(4\pi\theta)^{\frac{d-1}{2}}} e^{-\frac{r^2}{4\theta}}. \quad (38)$$

Now, by considering noncommutative geometry in large extra dimensional scenario, a static, spherically symmetric, Gaussian-smeared matter source is given by[35]
where
\[
g_{00} = e^\nu = 1 - \frac{1}{M_{\text{Pl}}^{d-2}} \frac{M}{(d-2)\pi} \frac{1}{r^{d-3}} \int_0^{r^2} e^{-t^{(d-3)}} dt. \tag{40}
\]
The horizon radius, \( r_s \), occurs at values of \( r \) where \( g_{00} = 0 \). Then, \( r_s \) can be obtained by solving the equation
\[
r_s = L_{\text{Pl}} \left[ \frac{m}{c_d} \frac{g_d\left( \frac{r_s}{\sqrt{\theta}} \right)}{\theta} \right]^\frac{1}{d-3}, \tag{41}
\]
where \( c_d = \frac{(d-2)\pi^{(d-1)}}{\Gamma\left(\frac{d-1}{2}\right)} \), \( m = \frac{M}{M_{\text{Pl}}} \) and the functions \( g_d\left( \frac{r_s}{\sqrt{\theta}} \right) \), are given by the integrals
\[
g_d\left( \frac{r_s}{\sqrt{\theta}} \right) = \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{r_s^2} e^{-t^{(d-3)}} dt. \tag{42}
\]
Using arguments presented in preceding discussion about horizon area and black hole entropy, relative integral can be written as
\[
S \simeq \varepsilon \int_{r_s(\text{min})}^{r_s} \frac{(d-2)\Omega_{d-2}^{\frac{d-3}{2}} \theta^{d-5} g_d\left( \frac{r_s}{\sqrt{\theta}} \right)}{1 - \sqrt{\eta + \kappa \theta^{d-2} \frac{\frac{1}{\theta} \theta^{\frac{d-3}{2}} g_d\left( \frac{r_s}{\sqrt{\theta}} \right)}{\frac{1}{\theta^2} \theta^{\frac{d-3}{2}} g_d\left( \frac{r_s}{\sqrt{\theta}} \right)}} dr_s, \tag{43}
\]
where,
\[
\theta = \left( \frac{ML_{\text{Pl}}^{d-2}}{c_d} \right)^{\frac{1}{d-3}} \tag{44}
\]
Numerical calculation of this integral for different spacetime dimensions are shown in figure 10.

6 The Relation Between Parameters of GUP and NCG

Now we are going to compare the results of two approaches: GUP and NCG. Using relation (2), we find
\[
\Delta p \simeq \frac{\Delta x}{\alpha^2 l_p^2} \left( 1 - \frac{1}{\left( \Delta x \right)^2} \right). \tag{45}
\]
Within the original Bekenstein-Hawking framework and using the previous results, one finds that there is a characteristic temperature where agrees with the Hawking temperature up to a factor of $4\pi$,

$$T_H \approx \frac{\Delta p}{4\pi},$$

or

$$T_{H}^{GUP} \approx \frac{\Delta x}{4\pi \alpha^2 l_p^2} \left( 1 - \sqrt{1 - \frac{\alpha^2 l_p^2}{(\Delta x)^2}} \right).$$

for temperature of radiated photons in GUP framework. As relation (2) shows, the position uncertainty has a minimum value of $(\Delta x)_{min} = \alpha l_p$, so the string theory parameter of GUP times the Planck distance play the role of a minimum or fundamental distance which we take to be the Schwarzchild radius $r_s$. In the region where $r_s = \alpha l_p$, $T_{H}^{GUP}$ deviates from the standard hyperbola (18). Instead of exploding with shrinking $r_s$, $T_{H}^{GUP}$ reaches a maximum,

$$T_{H}^{GUP}(max) = \frac{1}{4\pi \alpha l_p}.$$
Comparing our two results (48) (with $M|_{T_{\text{max}}} = 3.63\sqrt{\theta}$) and (17), we find the following relation between GUP and NCG parameters,

$$T_{H}^{\text{GUP}}(\text{max}) = T_{H}^{\text{NCG}}(\text{max}) \implies \frac{1}{4\pi\alpha l_{p}} = \frac{0.0144}{\sqrt{\theta}}.$$  \hspace{1cm} (49)

So we find the following interesting relation between parameters of GUP and NCG,

$$\theta \simeq 0.033\alpha^{2}l_{p}^{2}.$$  \hspace{1cm} (50)

Since $l_{p} \sim 10^{-33}\text{cm}$ and $\alpha \sim 1$, we find

$$\theta \sim 10^{-68}\text{cm}^{2}.$$  

In a model universe with spacelike extra dimensions, we find the following generalization for $\Delta p_{i}$

$$\Delta p_{i} \simeq \frac{\Delta x_{i}}{\alpha^{2}L_{P}^{2}} \left[ 1 - \sqrt{1 - \frac{\alpha^{2}L_{P}^{2}}{(\Delta x_{i})^{2}}} \right].$$  \hspace{1cm} (51)

Now GUP can be written as

$$\Delta x_{i}\Delta p_{i} \geq \frac{1}{2} \left[ 1 + \alpha^{2}L_{P}^{2}(\Delta p_{i})^{2} \right],$$  \hspace{1cm} (52)

and we find the minimum of $\Delta x$ as follows

$$(\Delta x_{i})_{\text{min}} \simeq \alpha L_{P}.$$  \hspace{1cm} (53)

Based on Heisenberg uncertainty relation, one deduces the following equation for Hawking temperature of black holes in a universe model with large extra dimensions

$$T_{H} \approx \frac{(d - 3)\Delta p_{i}}{4\pi}$$  \hspace{1cm} (54)

where we have set the constant of proportionality equal to $\frac{(d-3)}{4\pi}$ in extra dimensional scenario. Therefore, substitution of momentum uncertainty with the minimum value of $\Delta x$, we obtain the maximum value for Hawking temperature,

$$T_{H}^{\text{GUP}}(\text{max}) \approx \frac{(d - 3)}{4\pi\alpha L_{P}^{2}}.$$  \hspace{1cm} (55)

In this extra dimensional scenario, the relation between parameters of GUP and NCG is as follows

$$\theta \simeq \frac{0.033\alpha^{2}L_{P}^{2}}{(d - 3)^{2}}.$$  \hspace{1cm} (56)
7 Summary

There are several alternative approaches for treating black hole evaporation process. This process is a quantum gravitational effects and its thorough understanding provides suitable framework toward a complete formulation of quantum gravity proposal. In this paper, based on our previous works and some recent literature, we have discussed the final stage of black hole evaporation within two alternative approaches: generalized uncertainty principle and the space noncommutativity. Our calculations and the corresponding enlightening figures show some unusual thermodynamical features when the mass of the black hole becomes of the order of Planck mass or less than it. Negative entropy, negative temperature, anomalous heat capacity are some of these unusual features of this short distance system. The origin of these unusual features may be on the failure of standard thermodynamics at quantum gravity level. It seems that standard formulation of thermodynamics breaks down at very short distance systems. Due to fractal nature of spacetime at very short distances, a new formulation of short distance thermodynamics is inevitable. Theories such as $E$-infinity and scale relativity which are based on fractal structure of spacetime at very short distances may provide a suitable framework for formulation of this short distance thermodynamics. The signature of this non-standard thermodynamics has been seen in other problems such as very early universe\cite{36}. Currently there is no explicit formulation of such a short distance formalism but in any case it should be based on fractal nature of spacetime at quantum gravity level. As another important result, we have compared temperature of black hole calculated from two different viewpoint and we have found an interesting relation between parameters of GUP and NCG. This relation reveals the conceptual correspondence of GUP and NCG. We are going to formulate fractal thermodynamics of Planck size black hole in our forthcoming paper.

References


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