N=1/2 Supersymmetric gauge theory in noncommutative space

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A formulation of (non–anticommutative) N=1/2 supersymmetric $U(N)$ gauge theory in noncommutative space is studied. We show that at one loop UV/IR mixing occurs. Supersymmetric Seiberg–Witten map for noncommutative superspace is employed to obtain an action in terms of commuting fields at the first order in the noncommutativity parameter $\theta$. This leads to a gauge invariant theory for $U(1)$ gauge group whose $\theta$ deformed supersymmetry transformations are presented. Non–abelian case is also discussed.

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1 Introduction

Deformation of superspace where fermionic coordinates are non-anticommuting appeared in some different contexts \cite{1}–\cite{6}. At the start one can simultaneously deform bosonic coordinates allowing them to be noncommuting, in terms of a star product embracing both of the deformations\(\text{(e.g.}\ \cite{7})\). However, as far as gauge theories are concerned usually non-anticommutativity is considered alone. Instead of introducing noncommutativity of bosonic coordinates and non–anticommutativity of fermionic ones simultaneously from the beginning, we may do it in two steps: \(\text{N=1/2 supersymmetric gauge theory action in components, includes ordinary fields and non–anticommutativity parameter}\ \cite{2}. Thus its noncommutative generalization can be obtained as usual. However, the same action would result using the superfield formulation given in \cite{7}. Hence, two approaches are equivalent. We study this non–anticommutative as well as noncommutative theory.

One of the most important features arising in field theories in noncommutative space is the UV/IR mixing\cite{8}. In supersymmetric gauge theory in noncommuting space, linear and quadratic poles in the noncommutativity parameter \(\theta\) are absent at one loop, due to the fact that contributions from fermionic and bosonic degrees of freedom cancel each other. First loop Feynman graph calculations for noncommutative supersymmetric gauge theory with abelian gauge group was studied in \cite{9}–\cite{11} and \(\text{U}(N)\) case was considered in \cite{12, 13}.

Renormalization of \(\text{N=1/2 supersymmetric Yang-Mills theory was discussed in}\ \cite{7, 14}–\cite{22}. For the gauge group \(\text{U}(N)\), renormalizability at one loop requires to alter the original action. In \cite{15} it was commented that in supersymmetric gauge theory where both noncommutativity and non–anticommutativity are present, there would be UV/IR mixing. Although we do not study renormalizability properties of the non–anticommutative and noncommutative theory, we will show that UV/IR mixing is present by an explicit calculation for \(\text{U}(1)\) case.

Seiberg and Witten\cite{23} introduced an equivalence relation between the gauge fields \(\hat{A}\) taking values in noncommutative gauge group and the ordinary gauge fields \(A\) as

\[
\hat{A}(A) + \hat{\delta}_\phi \hat{A}(A) = \hat{A}(A + \delta_\phi A).
\]

Here \(\hat{\phi}\) and \(\phi\) denote gauge parameters of the noncommutative and ordinary cases. Seiberg–Witten (SW) map allows one to deal with noncommutative gauge theory in terms of an action expanded in the noncommutativity parameter \(\theta\) with ordinary gauge fields. Gauge transformations of \(\text{N=1/2 supersymmetric theory in components fields does not depend on the non–anticommutativity parameter } C, \text{ owing to the parametrization of the vector superfield } V \text{ given in } \cite{2}. \text{ In this parametrization } V \text{ has bilinear terms in component fields which also show up in } V^2. \text{ Thus generalization of SW map should be given by replacing gauge fields with } \Sigma = V + V^2/2 \text{ in } \cite{1}. \text{ This is not in conflict with the definition adopted in } \cite{24}, \text{ where gauge field is replaced with only } V \text{ in } \cite{1} \text{ to obtain an equivalence between ordinary and } C \text{ deformed fields.}
They are two different approaches. We show that our generalization of SW map results in supersymmetric SW map of noncommutative supersymmetric gauge theory in terms of component fields where only bosonic coordinates are deformed [25]–[27]. This is not surprising, because SW map refers only to gauge transformations not to actions. Our definition of SW map is between, $C$ independent, noncommutative and commutative gauge transformations of a theory which depends on $C$. We adopt this definition to acquire $\theta$–expanded commuting theory up to the first order in $\theta$. A gauge invariant $U(1)$ theory results. We will also present $\theta$ deformed supersymmetry transformations of this theory. Unfortunately, non–abelian theory does not yield a gauge invariant action. This may be cured by adding some nonlocal terms at the order $C\theta$ to SW map.

In section 2 we present N=1/2 supersymmetric gauge theory action in noncommutative space exhibiting its gauge and supersymmetry invariance. Moreover, we show that UV/IR mixing occurs. In section 3 first we discuss how to generalize SW map to N=1/2 supersymmetric noncommuting theory. Then, we adopt the map given in [25]–[27] to obtain $\theta$–expanded action in commuting fields at the first order in $\theta$.

2 Noncommutative N=1/2 supersymmetric gauge theory

In terms of constant, antisymmetric parameter $C_{\alpha\beta}$, let the Grassmann coordinates $\theta_\alpha, \alpha = 1, 2$, satisfy the non–anticommutativity relation

$$\{\theta_\alpha, \theta_\beta\} = C_{\alpha\beta}. \tag{2}$$

However, the other Grassmann coordinates $\bar{\theta}_\dot{\alpha}, \dot{\alpha} = 1, 2$, are retained anticommuting. This is possible only in euclidean space. Although we deal with euclidean $\mathbb{R}^4$, we use Minkowski space notation and follow the conventions of [28]. In the formulation where $y^\mu = \tilde{x}^\mu + i \theta^\mu \bar{\theta}$ coordinates are taken as commuting[2], bosonic $\tilde{x}^\mu, \mu = 0, \cdots, 3$, coordinates should satisfy

$$[\tilde{x}_\mu, \tilde{x}_\nu] = \theta \theta C_{\mu\nu}. \tag{3}$$

We denoted $C^{\mu\nu} = C^{\alpha\beta} \epsilon_{\beta\gamma} \sigma^{\mu\nu\gamma}_{\alpha}$ which satisfies the self–duality property

$$C^{\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\rho\lambda} C_{\rho\lambda}. \tag{4}$$

A vector superfield was written in a Wess–Zumino gauge leading to a parametrization of nondeformed part of vector superfield which is different from the usual supersymmetric theory. This vector superfield was employed to write the action in commuting coordinates as

$$S = \frac{1}{g^2} \int d^4x \text{tr} \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - i \lambda \partial \bar{\lambda} + \frac{1}{2} D^2 - i C^{\mu\nu} F_{\mu\nu} \bar{\lambda} \lambda + \frac{|C|^2}{8} (\bar{\lambda} \lambda)^2 \right\}. \tag{5}$$
$F_{\mu\nu}$ is the non-abelian field strength related to the gauge field $A_\mu$. $\lambda$, $\bar{\lambda}$ are independent fermionic fields and $D$ is auxiliary bosonic field. Covariant derivative is defined as $D_\mu = \partial_\mu + i[A_\mu, \cdot]$. The action (5) is invariant under the usual gauge transformations and it possesses N=1/2 supersymmetry.

Obviously, (5) is a theory in commuting coordinates though the constant parameter $C$ appears. Hence, considering it in noncommuting space letting the coordinates satisfy

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta_{\mu\nu}$$

is legitimate. $\theta_{\mu\nu}$ is a constant, antisymmetric deformation parameter. As usual, instead of dealing with operators $\hat{x}_\mu$ we introduce the star product

$$f(x) \star g(x) = f(x)e^{\frac{i}{2}g_{\mu\nu}\overrightarrow{\partial_\mu}\overleftarrow{\partial_\nu}}g(x)$$

and work with the commuting coordinates $x_\mu$ satisfying the Moyal bracket

$$[x^\mu, x^\nu]_* = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}. \quad (8)$$

By replacing ordinary products with the star product (7) in (5), one obtains the action

$$I = \frac{1}{g^2} \int d^4x \tr \left\{ -\frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - i\hat{\lambda} \hat{D} \star \hat{\lambda} + \frac{1}{2} \hat{D}^2 - \frac{i}{2} G_{\mu\nu} \hat{F}_{\mu\nu} \hat{\lambda} \star \hat{\lambda} + \frac{|C|^2}{8} (\hat{\lambda} \star \hat{\lambda})^2 \right\}. \quad (9)$$

Here we adopted the definitions

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[A_\mu, \hat{A}_\nu]_*,$$

$$\hat{D} \star \hat{\lambda} = \phi \hat{\lambda} + i[A, \hat{\lambda}]_*.$$

This noncommutative gauge theory would also be resulted from the superfield formulation of N=1/2 supersymmetric theory given in [7] by making use of the parametrization given in [2] for vector superfields.

We assume that surface terms are vanishing, so that the following properties are satisfied

$$\int d^4 x f(x) \star g(x) = \int d^4 x f(x)g(x),$$

$$\int d^4 x f(x) \star g(x) \star h(x) = \int d^4 x (f(x) \star g(x))h(x) = \int d^4 x f(x)(g(x) \star h(x)).$$

Gauge transformations of the fields are

$$\delta \hat{A}_\mu = \partial_\mu \hat{\phi} - i[\hat{\phi}, \hat{A}_\mu]_*,$$

$$\delta \hat{\lambda}_\alpha = -i[\hat{\phi}, \hat{\lambda}_\alpha]_*,$$

$$\delta \hat{\lambda}_\dot{\alpha} = -i[\hat{\phi}, \hat{\lambda}_{\dot{\alpha}}]_*,$$

$$\delta \hat{D} = -i[\hat{\phi}, \hat{D}]_*,$$ 

$$\delta \hat{\phi} = -i[\hat{\phi}, \hat{\phi}]_*.$$ 

(10)
where \( \hat{\phi} \) denotes gauge parameter. Making use of (10) one can observe the following transformations

\[
\delta \hat{F}_{\mu\nu} = -i[\hat{\phi}, \hat{F}_{\mu\nu}]_*, \\
\delta (\hat{\rho} \star \hat{\lambda}) = -i[\hat{\phi}, \hat{\rho} \star \hat{\lambda}]_*, \\
\delta (\hat{\lambda} \star \hat{\lambda}) = -i[\hat{\phi}, \hat{\lambda} \star \hat{\lambda}]_*.
\]

Therefore, we can conclude that the action (9) is gauge invariant under noncommutative \( U(N) \) gauge transformations.

On the other hand, supersymmetry transformations of the component fields can be defined as

\[
\delta S_{\hat{\lambda}} = i\xi \hat{D} + \sigma^{\mu\nu}\xi(\hat{F}_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\hat{\lambda} \star \hat{\lambda}), \\
\delta S_{\hat{A}_\mu} = -i\hat{\lambda}\tilde{\sigma}_\mu\xi, \\
\delta S_{\hat{D}} = -\xi\sigma^\mu D_\mu \star \hat{\lambda}, \\
\delta S_{\hat{\lambda}} = 0.
\]

where \( \xi^a \) is a constant Grassmann parameter. To discuss supersymmetry properties of the action (9) one needs to make use of the relation

\[
\sigma^{\rho\lambda} \sigma^\mu = \frac{1}{2}(\eta^{\mu\lambda} \sigma^\rho + \eta^{\mu\rho} \sigma^\lambda + i\epsilon^{\mu\rho\lambda\kappa} \sigma^\kappa).
\]

The \( C = 0 \) part can be shown to be supersymmetric using the Bianchi identity \( \epsilon^{\mu\nu\lambda\rho}D_\mu \star \hat{F}_{\nu\lambda} = 0 \), which is due to the associativity of star product. On the other hand the \( C_{\mu\nu} \) dependent terms yield

\[
\delta_S I = \int d^4x \left\{ \frac{\xi}{2}(\sigma_\nu C^{\mu\nu} D_\mu \star \hat{\lambda})(\hat{\lambda} \star \hat{\lambda}) + \frac{i\xi}{4}\epsilon^{\mu\nu\rho\lambda}\sigma_\nu C_{\rho\lambda}(\hat{\lambda} \star \hat{\lambda})(D_\mu \star \hat{\lambda}) \\
-\xi(\sigma_\nu C^{\mu\nu} D_\mu \star \hat{\lambda})(\hat{\lambda} \star \hat{\lambda}) \right\} = 0,
\]

where the self-duality condition (4) is utilized. Hence, (9) is a noncommutative N=1/2 supersymmetric \( U(N) \) gauge theory action.

To perform perturbative calculations one should introduce ghost fields to fix the gauge. Moreover, matter fields may also be added. Let us consider noncommutative \( U(1) \) gauge group. In this case Feynman rules can be read from the N=1/2 supersymmetric \( U(N) \) gauge theory\[14\] by the replacement of the structure constants:

\[
f_{a_1a_2a_3} \rightarrow 2 \sin \left( k_2k_3 \right), \quad (15) \\
d_{a_1a_2a_3} \rightarrow 2 \cos \left( k_2k_3 \right), \quad (16)
\]
where we denoted $k^\mu \equiv \theta^{\mu\nu} k_\nu$. Here, $k_2$ and $k_3$ are the momenta of the lines corresponding to the indices $a_2$ and $a_3$, respectively. Instead of giving a full discussion of one loop calculations, we would like to consider only the following non-planar one loop diagram

\[
\begin{aligned}
\lambda & \quad \lambda \\
p_1 & \quad p_2 & \quad p_3 \quad A_{\mu} \\
\bar{\lambda}_\beta & \quad \bar{\lambda}_{\dot{\alpha}}
\end{aligned}
\]

which is typical of the $N=1/2$ supersymmetric gauge theory. The amplitude is proportional to

\[
\propto g^3 C^{\kappa\nu} \sigma^\kappa_{\gamma\dot{\beta}} \sigma^\nu_{\gamma\delta} \epsilon^\alpha_{\dot{\gamma}} \int \frac{d^4k}{(2\pi)^4} \frac{k_\nu (\vec{k} - \vec{p}_1)^{\gamma\dot{\gamma}} (\vec{k} + \vec{p}_2)^{\delta\beta}}{k^2 (k - p_1)^2 (k + p_2)^2} \cos(\vec{k}p_1) \sin(\vec{k}p_2) \sin(\vec{k}p_3). \tag{17}
\]

Using the calculation methods of [9], one can observe that this amplitude produces low momenta poles as

\[
g^3 C^{\kappa\nu} \sigma^\kappa_{\gamma\dot{\beta}} \sigma^\nu_{\gamma\delta} \epsilon^\alpha_{\dot{\gamma}} \frac{\tilde{l}_\nu (\tilde{l})^{\gamma\dot{\gamma}} (\tilde{l})^{\delta\beta}}{\tilde{l}^4}, \tag{18}
\]

where $l_\mu$ are some definite functions of $p$ :

\[
l = l(p_1, p_2, p_3).
\]

To get the correct factors we should take into account contributions coming from all of the diagrams including ghosts and also matter if they are coupled. Nevertheless, calculation of the above diagram shows that UV/IR mixing occurs.

### 3 $\theta_{\mu\nu}$–Expanded action

SW map (1) clearly alludes only to gauge transformations. Hence, to discuss how it may be generalized to noncommutative $N=1/2$ supersymmetric gauge theory, we would like to recall how gauge transformations of $N=1/2$ supersymmetric theory were obtained in [2]: Infinitesimal gauge transformations are obtained from

\[
\delta e^V = -i\bar{\Lambda} e^V + ie^V \Lambda, \tag{19}
\]

where the vector superfield is parametrized as

\[
V = -\theta \sigma^\mu \theta A_\mu + i \theta \bar{\theta} \bar{\lambda} - i \bar{\theta} \theta \alpha \left( \lambda_\alpha + \frac{1}{4} \epsilon_{\alpha\beta\gamma} C^{\beta\gamma} \sigma^\mu_\gamma \{\bar{\lambda}, A_\mu\} \right) + \frac{1}{2} \theta \bar{\theta} \bar{\theta} (D - i\partial_\mu A^\mu). \tag{20}
\]
Hence, one can show that
\begin{align}
V^2 & = -\frac{1}{2} \bar{\theta} \theta \left( \theta \theta A_\mu A^\mu + C^{\mu \nu} A_\mu A_\nu - i \theta_\alpha C^{\alpha \beta} \sigma^\mu_{\beta \dot{\alpha}} [A_\mu, \bar{\Lambda}^{\dot{\alpha}}] + \frac{1}{4} |C|^2 \bar{\Lambda} \Lambda \right), \\
V^3 & = 0.
\end{align}

Gauge parameters are given by
\begin{align}
\Lambda &= \phi, \\
\bar{\Lambda} &= \phi - i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi + \frac{1}{2} \theta \theta \bar{\theta} \partial^2 \phi + \frac{i}{4} \bar{\theta} \theta C^{\mu \nu} \{ \partial_\mu \phi, A_\nu \}.
\end{align}

All the component fields are functions of \( y^\mu \) coordinates and we suppress star product of non–anticommuting coordinates \( \theta_\alpha \). Contrary to the usual case, to obtain gauge transformations we need to deal not only with \( V \) but \( \Sigma = V + \frac{1}{2} V^2 \). Because, now \( V \) and \( V^2 \) possess terms bilinear in fields which should be considered on an equal footing. Indeed, the infinitesimal gauge transformation (19) reads
\begin{align}
\delta \Lambda \Sigma = -i \left( \bar{\Lambda} - \Lambda + \bar{\Lambda} \Sigma - \Sigma \Lambda \right),
\end{align}
where we should keep \( V^2 \) and \( \bar{\Lambda} V^2 \) terms. We define generalization of SW map to noncommutative N=1/2 supersymmetric gauge theory by the equivalence relation
\begin{align}
\hat{\Sigma}(\Sigma) + \hat{\delta} \hat{\Lambda} \hat{\Sigma}(\Sigma) = \hat{\Sigma}(\Sigma + \delta \Lambda \Sigma).
\end{align}

This is obtained by replacing the gauge field \( A \) with the vector superfield \( \Sigma \) and the gauge parameter \( \phi \) with the supergauge parameter \( \Lambda \) in (1). The noncommutative gauge transformations (10) can be obtained by replacing in (20) multiplication of the bilinear components with the star product and defining
\begin{align}
\delta \hat{\Sigma}_\Lambda = -i \left( \hat{\Lambda} - \hat{\Lambda} + \hat{\Lambda} \star \hat{\Sigma} - \hat{\Sigma} \star \hat{\Lambda} \right).
\end{align}

Here the star product is in terms of \( y \) coordinates. This is not in conflict with the definition of an equivalence relation:
\begin{align}
\hat{V}(V) + \hat{\delta} \hat{\Lambda} \hat{V}(V) = \hat{V}(V + \delta \Lambda V),
\end{align}
where the deformation is only in terms of \( C^{\alpha \beta} \) and one seeks a solution in powers of the non–anticommutativity parameter \( C^{\alpha \beta} \) as studied in [24]. They are two different approaches: Our definition (26) is an equivalence relation between \( \theta_{\mu \nu} \) deformed and non deformed gauge transformations of a theory which is \( C^{\alpha \beta} \) dependent but where already a kind of SW map has done to obtain gauge transformations independent of \( C^{\alpha \beta} \).

Let us deal with abelian gauge group to solve the equivalence relation (26). Keeping terms first order in \( \theta_{\mu \nu} \) which are denoted as
\begin{align}
\hat{\Sigma} = \Sigma + \Sigma^{(1)}, \quad \hat{\Lambda} = \Lambda + \Lambda^{(1)}, \quad \hat{\bar{\Lambda}} = \bar{\Lambda} + \bar{\Lambda}^{(1)},
\end{align}
leads to

$$
\Sigma_{(1)} (\Sigma + \partial_\lambda \Sigma) - \Sigma_{(1)} (\Sigma) + i \hat{\Lambda}_{(1)} - i \Lambda_{(1)} = \frac{1}{2} \theta \theta \left[ \theta \theta (\partial_\mu \phi_{(1)} A^\mu + \partial_\mu \phi A^\mu_{(1)}) + \theta \sigma (\partial_\mu \phi_{(1)} \lambda^\mu + \partial_\mu \phi \lambda^{\mu}_{(1)}) \right]
$$

By denoting the component fields $V_i \equiv (A, \lambda, \lambda, D)$, it yields

$$
A_{(1)}(V_i + \partial V_i) - A_{(1)}(V_i) + \partial_\mu \phi_{(1)} = \theta^\rho_\sigma \partial_\mu \phi \partial_\sigma A_\mu,
$$

$$
\lambda_{(1)}(V_i + \partial V_i) - \lambda_{(1)}(V_i) = \theta^\rho_\sigma \partial_\mu \phi \partial_\sigma \lambda^\rho,
$$

$$
\hat{\lambda}_{(1)}(V_i + \partial V_i) - \hat{\lambda}_{(1)}(V_i) = \theta^\rho_\sigma \partial_\mu \phi \partial_\sigma \hat{\lambda}^\rho,
$$

$$
D_{(1)}(V_i + \partial V_i) - D_{(1)}(V_i) = \theta^\rho_\sigma \partial_\mu \phi \partial_\sigma D,
$$

which are independent of $C$ and indeed they are the same with the equations obtained in [25, 27] for noncommutative supersymmetric gauge theory with only bosonic coordinates are noncommuting. This would have been expected, because as a matter of fact (26) alludes only to gauge transformations, it does not refer to any action. Hence, also for $U(N)$ gauge group we adopt the ordinary supersymmetric generalization of SW map given in [26] which, at the first order in $\theta$ are given as

$$
\hat{A}_\mu = A_\mu + \frac{\theta^\rho_\sigma}{4} \{ A_\rho, \partial_\sigma A_\mu + F_\mu \},
$$

$$
\hat{F}_{\mu \nu} = F_{\mu \nu} - \frac{\theta^\rho_\sigma}{4} \{ 2 \{ F_\mu \rho, F_\nu \sigma \} - \{ A_\rho, (D_\sigma + \partial_\sigma) F_{\mu \nu} \} \},
$$

$$
\hat{D} = D + \frac{\theta^\rho_\sigma}{4} \{ A_\rho, (D_\sigma + \partial_\sigma) D \},
$$

$$
\hat{\lambda}_\alpha = \lambda_\alpha + \frac{\theta^\rho_\sigma}{4} \{ A_\rho, (D_\sigma + \partial_\sigma) \lambda_\alpha \},
$$

$$
\hat{\lambda}^\alpha = \hat{\lambda}^\alpha + \frac{\theta^\rho_\sigma}{4} \{ A_\rho, (D_\sigma + \partial_\sigma) \hat{\lambda}^\alpha \}.
$$

We would like to write $\theta$–expanded N=1/2 supersymmetric gauge theory action up to the first order in $\theta$. For $U(1)$ we have the gauge invariant $\theta$–expanded action

$$
I^{(1)} = \int d^4 x \left[ - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D^2 - i \lambda \phi \lambda - \frac{i}{2} C^{\mu \nu} F_{\mu \nu} \lambda^2 - \theta^\rho_\sigma \left( - \frac{1}{2} F_{\mu \nu} F_{\nu \rho} F_{\mu \rho} + \frac{1}{8} F_{\rho \sigma} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} D^2 F_{\rho \sigma} + \frac{i}{2} F_{\rho \sigma} \lambda \phi \lambda \right) + i \lambda \sigma^\mu \partial_\sigma \lambda F_{\mu \rho} + \frac{i}{2} C^{\mu \nu} F_{\mu \rho} F_{\nu \sigma} \lambda^2 - \frac{i}{4} C^{\mu \nu} F_{\rho \sigma} F_{\mu \nu} \lambda^2 \right].
$$

Although (32) possesses the usual $U(1)$ gauge invariance its supersymmetry transformations should be altered. The transformations (11)–(14) can be used to define
the \(\theta\)-expanded supersymmetry transformations as

\[
\delta_S A_\mu = i \xi \sigma_\mu \lambda - i 2 \theta^{\rho\sigma} \xi \sigma_\rho \lambda (\partial_\sigma A_\mu + F_{\sigma \mu}) + i \frac{1}{2} \theta^{\rho\sigma} \xi \sigma_\sigma A_\rho \partial_\mu \lambda;
\]

\[
\delta_S \lambda = i \xi D - \xi \sigma^{\mu\nu} F_{\mu \nu} + \theta^{\rho\sigma} \xi \sigma^{\mu\nu} F_{\mu \rho} F_{\nu \sigma} + i \frac{1}{2} \sigma^{\mu\nu} \xi C_{\mu \nu} \lambda^2 - i \xi \theta^{\rho\sigma} \partial_\rho \lambda \lambda \sigma_\sigma \lambda,
\]

\[
\delta_S \bar{\lambda} = - i \theta^{\rho\sigma} \xi \partial_\rho \lambda \lambda \sigma_\sigma \lambda,
\]

\[
\delta_S D = - \xi \sigma^{\mu\nu} \partial_\mu \lambda + \theta^{\rho\sigma} \xi \sigma^{\mu\nu} \partial_\rho \lambda F_{\mu \sigma} + i \theta^{\rho\sigma} \xi \sigma_\sigma \partial_\rho D \lambda,
\]

which can be shown to yield

\[
\delta_S F_{\mu \nu} = i \xi (\sigma_\mu \partial_\nu \lambda - \sigma_\nu \partial_\mu \lambda) + i \xi \theta^{\rho\sigma} \sigma_\rho (\partial_\nu \lambda F_{\sigma \nu} - \partial_\nu \lambda F_{\sigma \mu}) - i \xi \theta^{\rho\sigma} \sigma_\rho \partial_\mu \lambda F_{\nu \rho}.
\]

In fact, we explicitly checked that the action (32) is invariant under the \(\theta\)-expanded supersymmetry transformations (33).

The \(\theta\)-expanded \(U(1)\) gauge theory action (32) can be utilized to study some different aspects of noncommuting \(N=1/2\) supersymmetric gauge theory. Similar to noncommuting electrodynamics one can calculate one loop renormalization properties of this theory[29] and find solutions of equations of motion[30]. Moreover, using the master action of \(N=1/2\) supersymmetric \(U(1)\) gauge theory given in [31] one can study duality properties of the action (33).

When we employ the map (31) in the action (9), up to some surface terms, we attain

\[
I = \int d^4 x \text{tr} \left[ - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{\theta^{\rho\sigma}}{2} F^{\mu \nu} F_{\mu \rho} F_{\nu \sigma} - \frac{\theta^{\rho\sigma}}{8} F_{\sigma \rho} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} D^2 
\right.
\]

\[
+ \frac{\theta^{\rho\sigma}}{4} D^2 F_{\rho \sigma} - i \lambda \sigma^{\mu\nu} D_{\mu} \lambda - i \frac{\theta^{\rho\sigma}}{4} \left( \{ F_{\rho \sigma}, \lambda \} \sigma^{\mu\nu} D_{\mu} \lambda + 2 \lambda \sigma^{\mu\nu} \{ F_{\rho \sigma}, D_{\mu} \lambda \} \right)
\]

\[
- \frac{1}{2} C^{\mu \nu} \left( F_{\mu \rho} \lambda^2 - \theta^{\rho\sigma} F_{\mu \rho} F_{\nu \sigma} \lambda^2 - \frac{\theta^{\rho\sigma}}{4} \{ F_{\rho \sigma}, F_{\mu \nu} \} \lambda^2 
\right.
\]

\[
+ \frac{\theta^{\rho\sigma}}{4} F_{\mu \nu} \{ [\lambda_\alpha, A_\rho], (\partial_\sigma + D_\sigma) \lambda_\alpha \} + \frac{|C|^2}{8} \left( \lambda^2 \lambda^2 + \frac{1}{4} \{ F_{\rho \sigma}, \lambda^2 \} \right). \tag{34}
\]

Unfortunately, this is not gauge invariant for non–abelian groups (there may be some exceptions). As we have already emphasized SW map does not refer to any action but it is an equivalence relation between gauge transformations. Hence, a priori one cannot guarantee that a noncommutative gauge theory will remain gauge invariant under SW map. Nevertheless, it seems plausible to modify the map (31) with some terms which are at the order of \(C \theta\). In fact if one considers the formal nonlocal map

\[
\dot{\lambda}_\alpha = \lambda_\alpha + \frac{\theta^{\rho\sigma}}{4} \{ A_\rho, (D_\sigma + \partial_\sigma) \lambda_\alpha \} - \frac{\theta^{\rho\sigma}}{8} C^{\mu \nu} F_{\mu \nu} \{ [\lambda_\alpha, A_\rho], (\partial_\sigma + D_\sigma) \} (\sigma^\kappa D_\kappa \lambda)^{-1} \tag{35}
\]

with the other components unaltered, \(\theta^{\rho\sigma}\) expanded action will become gauge invariant. Obviously, adding some \(C \theta\) depended terms to the map (31) deserves a detailed study.
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References


