Using long-term transit timing to detect terrestrial planets

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ABSTRACT
We propose that the presence of additional planets in extrasolar planetary systems can be detected by long-term transit timing studies. If a transiting planet is on an eccentric orbit then the presence of another planet causes a secular advance of the transiting planet’s pericenter over and above the effect of general relativity. Although this secular effect is impractical to detect over a small number of orbits, it causes long-term differences in when future transits occur, much like the long-term decay observed in pulsars. Measuring this transit-timing delay would thus allow the detection of either one or more additional planets in the system or the first measurements of non-zero oblateness ($J_2$) of the central stars.

Key words: planetary systems – celestial mechanics – gravitation – extrasolar planets – stellar oblateness

1 INTRODUCTION
The study of long-term orbital precession was one of the triumphs of celestial mechanics, when the planetary theories of Laplace and Lagrange showed that essentially all the known long-term precessions of the planetary orbits could be explained by their mutual gravitational interaction. The perturbation caused by the small planetary masses ‘breaks’ the perfect central force character of the Sun’s gravitational field, causing the planetary orbital nodes to regress and their perihelia to slowly advance, with typical periods of $10^4$–$10^5$ years. It was this advanced understanding of celestial mechanics that permitted LeVerrier and Adams to detect a new planet in our Solar System (Neptune) by inverting its observable effect on the known planets to predict Neptune’s mass and position.

The only known exception at the start of the 20th century to these theories was the mystery that Mercury’s perihelion longitude advanced $0.43''/yr$ faster than the predicted rate of $5.31''/yr$ produced by the perturbative effects of all the other planets. The obvious possibility was that the Sun’s mass distribution was not spherically symmetric, but explaining the mercurian advance would require a solar oblateness (measured by the parameter $J_2$) of several percent, which was uncomfortably large (e.g. Hall 1900). General relativity provided the solution, explaining essentially the entire discrepancy. In fact the Sun’s oblateness is left with only an empirical upper bound of $J_2 < 1 \times 10^{-6}$, with a current theoretical estimate for its true value of of $J_2 \approx 1 \times 10^{-7}$ (Pireaux & Rozelot 2003).

As of the beginning of 2006, nearly two hundred extrasolar planets have been discovered and several of them exhibit transits (Butler et al. 2006). The longitude of periastron of eccentric hot Jupiters (3-day periods) will show a secular advance of the instant of transit. If the orbital period can be well established (via transit timing or radial velocity) then the long-term drift of the transit centers will allow one to measure the slow advance of periastron. Observations on 10-year baselines should certainly show the relativistic advance, which is much more important for close-in hot Jupiters since they are closer to their parent stars than Mercury is to our Sun. In this paper we discuss the possibility of using the periastron advance rate to measure either a host-star oblateness or the presence (and thus discovery) of additional planets in the system.

2 SECULAR ADVANCE
In the Newtonian two-body problem the Laplace Runge-Lenz vector (also known as Hamilton’s vector, or the eccentricity vector), which points from the star to the planetary orbit’s pericenter, is stationary, so the location of periastron is constant. Several effects can cause the periastron to advance. In our Solar system in increasing order of importance we have:

(i) Stellar oblateness,
(ii) General relativity, and
(iii) Other planets.

The amplitude of these effects depends on several parameters, and thus in extrasolar planetary systems the order of importance may differ. In the following sections we provide expressions for the rate of periastron advance for these effects.

2.1 Stellar contributions

The central star can cause periastron advance by either being non-spherical or due to general relativistic effects caused by its mass.

According to Misner et al. (1973) we have an advance of periastron of

$$\delta \omega = \frac{6\pi GM_*}{c^2 a(1-e^2)} + J_2 \frac{3\pi R_*^2}{a^2(1-e^2)^2}$$

per radial period due to the star itself. Here, $c$ is the speed of light, $G$ Newton’s gravitational constant, $a$ the semimajor axis of the planet’s orbit, $e$ its eccentricity, and $M_*$, $R_*$, and $J_2$ are the mass, radius, and oblateness parameter of the star, respectively.

Since $a = (P^2GM_*/4\pi^2)^{1/3}$, we get a time rate of change for the relativistic component of

$$\dot{\omega}_{GR} = \frac{12^{1/3} \text{ yr}^{-1} (M_*/M_5)^{2/3} (P/3 \text{ day})^{-5/3}}{1 - e^2}$$

and for the stellar oblateness

$$\dot{\omega}_{J_2} = \frac{3.1^{1/7} \text{ yr}^{-1} J_2 (R_*/R_0)^2 (M_*/M_5)^{-2/3} (P/3 \text{ day})^{-7/3}}{1 - e^2}$$

where we have scaled these effects to typical values appropriate for a hot Jupiter. For Mercury the relativistic effect is 0.43″/yr. This drops as $a^{-5/2}$ for more distant orbits, being 0.086″/yr for Venus, 0.038″/yr for Earth, and 0.013″/yr for Mars. The scaling value of $J_2 = 10^{-6}$ used in Eq. (3) is essentially a firm upper limit for the oblateness of our sun that can be obtain from theoretical and observational considerations (Pireaux & Rozelot 2003).

2.2 Planetary contributions

Just as Mercury’s pericenter advance is affected the other planets, an exoplanet’s orbit will precess due to the perturbations of an unseen planet. We will generally assume that a second planet in the system is an external one, although the theory is almost identical if the perturber is interior.

A simple way to estimate the contribution to the periastron advance from another planet in the system is to assume that the mass of the second planet is smeared out over circular ring coplanar (e.g. Price & Rush 1979) with the first planet’s orbit and calculate the contribution of this ring to the potential.

An additional planet in the system causes the periastron of an observed planet to advance by

$$\delta \omega = \frac{\pi}{2} \frac{m_2}{M_*} \frac{a_1^3(a_2^2 - a^2)}{R(a_2^2 - a^2)^2} \approx \frac{3}{2} \frac{m_2}{M_*} a_2^3$$

during each radial orbit. The approximation holds if $a \ll a_2$ and $a_2$ is the radius of the second planet’s orbit. The mass of second planet is $m_2$. This result comes from Price & Rush (1973).

We can do a bit better if we relax the assumption of $a \ll a_2$. In this case we find that the potential due to the second planet is

$$V_2 = -\frac{G M_2}{a_2} \frac{2}{\pi (\alpha + 1)} K\left(\frac{2\alpha^{1/2}}{\alpha + 1}\right)$$

where $K$ is the complete elliptic integral of the first kind and $\alpha = a/a_2$. From Price & Rush (1973), the advance of periastron per radial orbit is

$$\delta \omega = 2\psi - 2\pi$$

where the increase in azimuthal angle for half a radial orbit is

$$\psi = \pi \left\{ 3 + a \left[ V''(a)/V'(a) \right] \right\}^{-1/2}$$

and $V(a)$ is the potential due to the star and the perturbing planet. For $M_2 \ll M_*$ we have

$$\delta \omega = \frac{M_2}{M_*} \frac{\alpha}{(\alpha + 1)(\alpha - 1)^2} \left[ (\alpha + 1) E\left(\frac{2\alpha^{1/2}}{\alpha + 1}\right) - (\alpha - 1)^2 K\left(\frac{2\alpha^{1/2}}{\alpha + 1}\right) \right]$$

$$= \pi \frac{M_2}{M_*} \frac{\alpha^2 b_{3/2}^{(1)}(\alpha)}{2} \text{ for } a < 1$$

where $E$ is the complete elliptic integral of the second kind. The function $b_{3/2}^{(1)}(\alpha)$ is a Laplace coefficient from classical perturbation theory; it is equal to 30 for $a \ll 1$ and is order unity for $a=0.1–0.5$, and then increases rapidly to $>100$ for $a > 0.9$. In the limit of a large ratio between the two semimajor axes we obtain

$$\delta \omega \approx \frac{M_2}{M_*} \left\{ \begin{array}{ll}
\frac{\pi a^3}{2} & a \ll 1 \\
\frac{4}{\alpha^2} a_2^{3/2}(\alpha) & a \gg 1
\end{array} \right\}$$

This is perhaps more transparently expressed by noticing that $N_p = 2\pi/\delta \omega$ is the number of inner planet revolutions required for a full precession of its orbit:

$$N_p = 2\pi \frac{2\pi}{\delta \omega} = \frac{4}{\alpha^2} \frac{M_*}{b_{3/2}^{(1)}(\alpha) M_2}$$

for $a_1 < a_2$. Fig. I gives the exact value of $N_p \times (M_2/M_*)$ and its asymptotic form. As examples, an unseen Jupiter-mass planet at ten times the semimajor axis of an interior hot-Jupiter (3-day period) will cause the hot Jupiter’s orbit to precess completely in about $10^6$ orbits $= 3 \times 10^6$ days, or about 8,000 years; a 3-Earth mass planet 1.5 times more distant than the hot Jupiter would cause precession in only about $10^5$ orbits (800 years).

The analysis above assumes that no resonant or near-resonant terms are important to the dynamics. It is unclear how valid this assumption is, as some extrasolar planetary systems already exhibit near-resonant behaviour (Rasio et al. 1992; Ford et al. 2003; Aghel et al. 2003 and Holman & Murray 2005) include the effects of orbital resonances and the eccentricity of the perturber’s orbit; however,
Figure 1. The number of orbital periods of an inner planet required for that planet’s orbit to precess by $2\pi$ under the secular advance induced by an external perturber, times the mass ratio of the perturber to the central star ($M_2/M_\star$). The $x$-axis is the semimajor axis ratio of the outer to inner planet. For an unseen Jovian perturber the number of orbits for precession is the vertical axis time $10^3$ and for a 3 Earth-Mass perturber one multiplies by $10^5$. The dashed line is the approximation using the small-$\alpha$ limit of the Laplace coefficient.

Holman & Murray (2005) focus on the stochastic variation of the inter-transit interval and Agol et al. (2005) do not consider secular terms in their calculations, which is the focus of the analysis here.

An explicit expression for the precession rate of the longitude of periastron, if the semimajor axis of the outer planet is much larger than that of the inner observed planet ($a \ll a_2$), is

$$\dot{\omega} = 355'' \text{ yr}^{-1} \frac{M_2 M_\odot}{M_\star M_\odot} \left(\frac{a}{a_2}\right)^3 \frac{3 \text{ day}}{P}$$

(12)

where $P$ is the period of the inner planet. This asymptotic estimate of the rate of periastron advance as shown in Fig. 1 underestimates significantly the rate for $\alpha > 1/4$. In fact for $\alpha = 0.5$ the estimate falls short by a factor of two (c.f. Miralda-Escudé 2002).

Figure 2. The periastron precession rate of a hot Jupiter in a low-eccentricity, three-day orbit around a solar-mass star. The vertical dotted line shows the minimum separation of an exterior Jupiter-mass planet (see text).

3 WHAT IS THE SENSITIVITY TO OTHER PLANETS IN THE SYSTEM?

Fig. 2 shows the precession contribution from the three effects on an observed transiting planet. The most interesting component is the contribution from other planets in the system, so we would like to have firm upper limits on the contributions from the other two effects so that we can estimate the excess that might be due to an unseen planet.

The relativistic contribution is the most certain. The mass of the star can generally be estimated to a few percent and the period of the orbit of the planet can be measured to a part in 10,000 or better; therefore, the value in the foregoing equation can be estimated to a high degree of precision.

Since the GR-induced precession rate is simply proportional to the mass of the star, the fractional precision of this mass estimate will set the sensitivity limit of the periastron advance method. If we take 3% as a nominal $M_\star$ precision then, if one observes a transiting hot-Jupiter with $P=3$ days, a precession rate different from the GR-rate by less than 3% will not provide a reliable detection of a planet or an oblate star; the parameters in Fig. 2 are such that by chance the induced rate from the host star with $J_2 = 10^{-6}$ falls close to the detection limit.
The value of the oblateness ($J_2$) for the host stars of extrasolar planets is unknown—we don’t even know it well for our Sun. However, we can estimate an upper limit to its value from the solar estimates and a scaling of the rotation rate of our sun to that of the host star if it is known. From observations of solar oscillations and theoretical considerations, the value of $J_2$ for the sun is probably around a few times $10^{-7}$ (Pfaffenberger et al. 2003; Winn et al. 2005) recently determined the spin rate of the star in the transiting planetary system HD 209458. The value that they obtain $v \sin \theta = (4.70 \pm 0.16) \text{ km/s}$ is not much larger that the typical values for the sun of 1.4 – 2.0 km/s, so it is unlikely that the value of $J_2$ for at least this system is much larger than that of the sun. The value $10^{-6}$ provides a firm upper limit; this yields a oblateness contribution of about 3º per year given, a order of magnitude less that the contribution from a nearby Earth-scale planet. It is unlikely that stars have values of $J_2 > 10^{-6}$; detected precession rates more than ~3º faster than the GR-induced rate are strong evidence for the presence of an unseen planet.

Consequently, the detection of an advance of periastron of an extrasolar planet with a three-day period that exceeded of the relativistic amount by 31” per year would either indicate the presence of an unseen planet or a stellar $J_2$ of about $10^{-5}$, an order of magnitude larger than the upper limits for the Sun. Either result is interesting. Assuming that the oblate star hypothesis is ruled out, such upper limits for the Sun. Either result is interesting. As- suming that the oblate star hypothesis is ruled out, such a detection either implies (Fig. 2) a terrestrial-scale planet a few times further than the hot Jupiter, or a jovian-mass planet even more distant (for a given precession rate there is a one-parameter family of mass-distance for the external perturber).

The jovian-mass case would likely be limited by several constraints. Very large precession rates (many times the GR-induced precession) will not occur; if both planets were jovian mass, then the analysis of Gladman (1993) shows that an outer planet would not be stable if $(a_2 - a_1)/a_2 \lesssim 0.24$ (i.e., the outer planet’s semimajor axis must be than >125% of the inner planet’s or the system would be unstable; the vertical dotted line of Fig. 2). Notice that a jovian perturber an order of magnitude further away than the hot Jupiter induces a precession rate comparable to the GR-induced rate. However, such a perturber should be trivially detectable already in the radial velocity data from the system; thus unless the host star is so noisy that radial-velocity techniques cannot be applied, the periastron technique is less sensitive to massive planets.

However, while terrestrial-mass planets cannot be seen by the radial-velocity technique, the periastron advance they would produce on a hot Jupiter is in principle detectable if they are out to ~5 times more distant.

4 HOW CAN ONE OBSERVE THE ADVANCE OF PERIASTRON?

We can have the following information (loosely in order of difficulty)

(i) Timing of the primary transit,

(ii) Timing of the secondary transit and

(iii) Timing of the radial velocity data

4.1 Primary transit

As the orbit precesses the location of the transit relative to the orbit changes; specifically, the value of the true anomaly ($\nu$) at the center of the transit decreases at a rate of $\dot{\nu} = \delta \omega / P$. To get the time of the transit, we have to relate the true anomaly to the mean anomaly ($M$ — the phase of the orbit increasing linearly in time from zero at one periastron to $2\pi$ at the next) through Kepler’s equation,

$$M = E - e \sin E,$$

$$\tan \frac{E}{2} = \sqrt{1 - e} \tan \frac{\nu}{2}$$

where $E$ is the eccentric anomaly.

For simplicity we work in the frame rotating with the precessing orbit, so we take $P$ to be the time between successive periastrons and the true anomaly that corresponds to the transit changes as the orbit precesses. The interval between two transits is simply the difference in the mean anomalies at each transit divided by the mean motion $2\pi/P$. If the orbit is not precessing, this time is simply the period of the orbit. However, if the periastron advances, the timing between the transits is

$$\Delta t = \frac{2}{P} \left(1 - \frac{dM \delta \omega}{d\nu} \right) + \mathcal{O}((\delta \omega)^2)$$

where we have assumed that the periastron advance per orbit is small. As the orbit precesses, the time between transits will change, according to

$$\frac{d\Delta t}{dt} = \left(1 + \frac{dM \delta \omega}{d\nu} \right)^{-2} \frac{\delta \omega}{2\pi} \frac{d^2M}{d\nu^2}$$

At face value this gradual change of the timing of the transits appears hopeless to detect because the change is proportional to the very small square of the advance of periastron per orbit. However, the difference in the timing accumulates from orbit to orbit as the orbit precesses, so in practice one would predict the timing of the transits from a few observations and look for a difference from that prediction after many hundreds of orbits had passed. Essentially we are interested in the integral of $\Delta t$ over many periods.

If one observes several initial transits and determines a value of $\Delta t$, one can predict the timing of future transits. These predictions will be incorrect by the following amount

$$t_{\text{pred}} - t_{\text{actual}} = M(\nu) - M(\nu_0) - \left. \frac{dM}{d\nu} \right|_{\nu_0} (\nu - \nu_0) \frac{P}{2\pi}$$

where $\nu_0$ is the true anomaly during the transit at the initial epoch and $\nu$ is the true anomaly during the transit at the later epoch.
One can calculate
\[ \frac{d^2 M}{d\nu^2} = \frac{\sqrt{1 - e^2} \sin \nu (1 - e \cos \nu)}{(1 + e \cos \nu)^2} \]
(19)
or expanding for small eccentricities
\[ \frac{d^2 M}{d\nu^2} = 2e \sin \nu - 3e^2 \sin 2\nu + 3e^3 \sin 3\nu + O(e^4) \]
(20)
From Eq. 17, one can see that for an error in the predictions to accumulate the second derivative the mean anomaly with respect to the true anomaly must not vanish. Thus, how quickly the error accumulates depends on the initial epoch of the observations. If the transit is initially occurring at periastron (\(\nu = 0\)) or apastron (\(\nu = \pi\)), the second derivative vanishes so it will take significantly longer for the time delay to become observable.

To lowest order in the change in the periastron advance (or the true anomaly at transit), we have
\[ t_{\text{pred}} - t_{\text{actual}} \approx \frac{P}{4\pi} \left( \nu - \nu_0 \right)^2 \left. \frac{d^2 M}{d\nu^2} \right|_0 \]
(21)
so the delay accumulates quadratically in time. Using reasonable values for the various numbers we have
\[ t_{\text{pred}} - t_{\text{actual}} = 1 \text{ ms} \left( \frac{e \sin \nu_0}{0.1} \right) \left( \frac{P}{3 \text{ days}} \right) \left( \frac{t - t_0}{1 \text{ year} 100^\circ \text{ yr}^{-1}} \right)^2 \]
(23)

### 4.1.1 Error Analysis

Eq. 23 makes it seem hopeless to detect the timing delay because one can determine the time of a particular transit to possibly ten seconds; therefore, naively one would expect to have to wait one hundred years before detecting the advance. Fortunately, one can detect the periastron advance in the series of transit times long before one could detect it in the timing of an individual transit.

In practice one characterizes the timing of the transits with a formula of the following form
\[ t_n = A + Bn + Cn^2 = t_0 + \Delta t_0 n + \frac{P}{4\pi} (\delta \pi)^2 \left. \frac{d^2 M}{d\nu^2} \right|_0 n^2 \]
(24)
where \(n\) is the number of the transit, \(t_0\) is the time of an initial reference transit, \(\Delta t_0\) is the initial time between transits and the quadratic term contains the periastron advance.

Using the standard results for \(\chi^2\) fitting, we obtain
\[ \sigma_C = \sigma \left( \frac{180^2 \sigma_0^2}{(r_0 N)(r_0 N)^4 - 5(r_0 N)^2 + 4} \right)^{1/2} \]
(25)
\[ \approx 13.41 \sigma_0^{-1/2} N^{-5/2} (1 + O(N^{-2})) \]
(26)
where \(N\) is the number of the last transit sampled, \(r_0\) is the fraction of transits with times and \(\sigma_0\) is the timing error on each transit.

The upper limit obtained for the value of \(\delta \pi\) is
\[ \sigma_{\delta \pi} = \sigma_C^{1/2} \left( \frac{P}{4\pi} \left. \frac{d^2 M}{d\nu^2} \right|_0 \right)^{-1/2} \]
(27)
where we have ignored the fractional error in the values of \(P\) and \(d^2 M/d\nu^2\). This yields
\[ \sigma_{\delta \pi} \approx 1.6 \times 10^{-5} \left( \frac{N}{1000} \right)^{-5/4} \left( \frac{r_0}{1000} \right)^{-1/4} \left( \frac{P}{3 \text{ day}} \right)^{2/3} \left( 10 \frac{\sigma_0}{\text{ms}} \right)^{1/2} \]
(28)
as an upper limit on the advance per orbit. This is not much larger than the expected relativistic contribution of
\[ (\delta \pi)_GR = 5 \times 10^{-6} \left[ \frac{M_*}{M_{\odot}} \right]^{2/3} \left( \frac{P}{3 \text{ day}} \right)^{-2/3} \]
(29)
so for \(N\) greater than a few thousand the errors in the determination of the relativistic term will dominate over the statistical errors in the timing. These estimates agree with the results of Miralda-Escudé (2003) who considered the effects of periastron advance on the timing of the primary transit and the duration of the primary transit.

One thousand transits of a planet with a three-day orbit takes just a shade under eight and a quarter years. The upper limit on \(\delta \pi\) decreases with time as \(t^{-5/4}\) until a value is detected. After this time, the errors decrease as \(t^{-5/2}\). The time to achieve the desired sensitivity scales as the \(P^{3/5}\) so this technique is also applicable to planets with longer orbital periods. These estimates assume that every transit is timed \((r_0 = 1)\). The error analysis assumes that the observed transits are evenly spaced in time; it may be possible to devise an observing strategy that achieves errors similar to the \(r_0 = 1\) case with many fewer observations — this is beyond the scope of this paper.

#### 4.1.2 Why don’t observations of the primary transit tell us more?

Each time the planet orbits the star the time is takes a bit less than an orbital period for it to reach the point of primary transit, because the orbit is shifting a bit. However, since we don’t know the orbital period itself, this time is essentially unobserved. The time for the planet to cover the missing angular distance is related to the distance from the star to the planet at transit. As the orbit precesses, this distance will change which in turn will change the time between transits. It is this change in the time between transits that we try to observe. The correction in the time between transits is proportional to the periastron advance. The change in the distance between the star and the planet from orbit to orbit is also proportional to the periastron advance. Combining these facts indicates that the change in the time between transits is second-order in the small periastron advance; consequently, it takes a relatively many orbits to detect the periastron shift, if one times only one type of transit.

### 4.2 Secondary transit

Looking at the secondary transit (when the planet goes behind the star) does not just provide a new set of times to fit but also provides new information and possibly a faster way of detecting unseen planets in the system. The secondary transit occurs when the true anomaly is 180 degrees away...
from where the primary transit occurs; therefore, we will denote quantities that describe the secondary transit with the subscript \( \pi \). The primary transit is given by the subscript 0. Let us examine look at the time between two successive primary and two successive secondary transits. Using the earlier formulae we have to lowest order in the advance of periastron
\[
\Delta t_0 = P \left( 1 - \frac{dM}{dv} \right) \frac{\delta \pi}{2\pi}
\]
and
\[
\Delta t_\pi = P \left( 1 - \frac{dM}{dv} \right) \frac{\delta \pi}{2\pi},
\]
where
\[
\frac{dM}{dv} = \sqrt{1 - e^2} \left( 1 - e \cos \nu \right)
= 1 - 2e \cos \nu + \frac{3}{2} e^2 \cos 2\nu - e^3 \cos 3\nu + O(e^4)
\]
If we take the difference between these two values we get
\[
\Delta t_0 - \Delta t_\pi = \left( \frac{dM}{dv} \right)_0 - \left( \frac{dM}{dv} \right)_\pi \frac{\delta \pi}{2\pi} P
\]
so the interval between two successive primary and two successive secondary transits differs by an amount proportional to the advance of periastron per orbit. This should be compared with observations of the primary transit alone in which the advance of periastron only enters to high order.

Furthermore, the difference in the time between the primary and secondary transits also contains some valuable information. We have
\[
t_\pi - t_0 = [M(\nu_0 + \pi) - M(\nu_0)] \frac{P}{2\pi}.
\]
If the orbit is eccentric this will differ from half of the orbital period. Expanding in the eccentricity we have
\[
t_\pi - t_0 = \frac{P}{2} + \frac{P}{2\pi} \left[ 4e \sin \nu_0 + \frac{2}{3} e^3 \sin 3\nu_0 + O(e^5) \right].
\]
The first term in the series is twice the value of the first term in the series for \( d^2M/dv^2 \), so the interval between the primary and secondary transits helps to calculate the periastron advance when one uses the timing of the primary transits.

### 4.2.1 Error Analysis

How well can we determine the time of the transits and the time between successive transits? Fitting the transit times to a timing model
\[
t_{0,n} = t_0 + \Delta t_0 n
\]
and
\[
t_{\pi,n} = t_\pi + \Delta t_\pi n.
\]
Because the periastron advance now enters in the difference between the interval between the successive transits, we only need to fit the times to first order in the number of the transit “n”. From the \( \chi^2 \)-analysis we obtain the following error estimates
\[
\sigma_{t_\pi} = \sigma_\pi \left( \frac{2[2(r_\pi N) + 1]}{(r_\pi N)([r_\pi N] - 1)} \right)^{1/2}
\]
and
\[
\sigma_{\Delta t_\pi} = \sigma_\pi \left( \frac{12e^2}{(r_\pi N)([r_\pi N] - 1)} \right)^{1/2}
\]
where \( r_\pi \) and \( \sigma_\pi \) are the fraction of secondary transits with times and the error in the timing of the secondary transit. The error in the time of the initial transit scales as \( N^{-1/2} \) where \( N \) is the number of orbits that have elapsed between the first and last one observed.

We are interested in the difference
\[
\Delta t_0 - \Delta t_\pi = \left( -4e \cos \nu_0 - 2e^3 \cos 3\nu_0 + O(e^5) \right) \delta \pi \frac{P}{2\pi}
\]
and especially the error in the difference
\[
\sigma_{\Delta t_0 - \Delta t_\pi} \approx \left[ \left( \frac{\sigma_\pi^2}{r_0} + \frac{\sigma_\pi^2}{r_\pi} \right) \frac{12}{N^3} \right]^{1/2}
\]
for large \( N \) and the error in
\[
\sigma_{\pi} \approx 7 \times 10^{-8} \left( \frac{1}{e \cos \nu_0} \right) \left( \frac{N}{1000} \right)^{-3/2} \frac{3 \text{ day}}{P} \times \left( \frac{\sigma_\pi^2}{(10 \text{ s})^2 r_0} + \frac{\sigma_\pi^2}{(10 \text{ s})^2 r_\pi} \right)^{1/2}.
\]
Combining results for the secondary transit with those from the primary transit yields an increase in sensitivity of a factor of 160. If we could time the secondary transit to the same precision of ten seconds it would take \( N \sim 100 \) to detect an Earth-like planet within twice the semimajor axis of the observed planet. The time to achieve the desired sensitivity scales as \( P^{1/3} \) so this technique is also applicable to planets with longer orbital periods; furthermore, one is sensitive to smaller planets in systems with larger eccentricities.

#### 4.2.2 Why does the secondary transit help so much?

After analyzing the primary transit, one saw how difficult it was to disentangle the change in the angle of periastron from the observations of the orbital period of the system. The timing of the secondary transit breaks this degeneracy, and it is straightforward to understand why. Unless the orbit is perfectly circular (or if we are really unlucky in the epoch of observations), the planet is a different distance from the star at the primary and secondary transit, so according to Kepler’s Second Law (conservation of angular momentum) its angular velocity along the orbit is different at these two times. If the periastron shifts as the planet orbits, it takes a different amount of time to cover the missing angle at the primary than at the secondary transit; consequently, the time between secondary transits differs from that between the primary transits — if we can detect this time difference we can detect the advance of periastron.
4.3 Radial Velocity Information

The radial velocity information is arguably the most difficult to obtain. It turns out that it is essentially the least useful (at least in quantity) for the purposes of characterizing the periastron shift. It is difficult to imagine obtaining timing of the radial velocity data with a precision of tens of seconds, so it is not directly useful in getting additional timing points as we did with the secondary transit. In principle, one would find that the time interval between when the star passed through a particular radial velocity and when it repeated itself would depend on the radial velocity in question.

However, the period found by fitting the radial velocity curve is typically precise to about one hundred milliseconds. This time interval would only differ from the interval between transits by a tiny amount on the order of the periastron advance. Determining accurately the relationship between the time between periastrons (what we have called the period) and the period found by fitting the radial velocities requires Monte Carlo simulations of the observed data. One can also gain some insight into what time interval emerges from fitting radial velocities by considering orbits that are nearly circular.

When one fits the radial velocity measurements one is most sensitive to parts of the orbit with large accelerations to or from the observer. The acceleration reaches an extreme when the jerk vanishes. To first order in the eccentricity of the orbit,

$$\frac{d^2 v_{\text{radial}}}{dt^2} \approx \left(\frac{2\pi}{P}\right)^3 a \left[ \sin(\nu - \nu_0) + 2\epsilon (11 \cos \nu \sin(\nu - \nu_0) + 5 \sin \nu_0) \right]$$

so the jerk vanishes where

$$\sin(\nu - \nu_0) = -10\epsilon \sin \nu_0 + \mathcal{O}(e^2)$$

To lowest order in the eccentricity, the radial velocity measurements are equally sensitive to the timing at the primary ($\nu - \nu_0 = 0$) and secondary transits ($\nu - \nu_0 = \pi$), so we assume that the time interval determined by fitting the radial velocity is given by the average of the two intervals discussed earlier

$$\Delta t_{\text{RV}} = \frac{1}{2} (\Delta t_0 + \Delta t_\pi) + \mathcal{O}(e^2) = P \left(1 - \frac{\delta \pi}{2\pi}\right) + \mathcal{O}(e^2).$$

The ”period” obtained by fitting the radial velocity data differs slightly by the period between periastrons or the period between primary transits. We confirmed this by generating radial velocity data with an advancing periastron and fitting these data with purely Keplerian radial velocity curves. These simulations gave Eq. 13 for small eccentricities.

If we take the difference between the two observable quantities we get

$$\Delta t_{\text{RV}} - \Delta t_0 = 2\epsilon \cos \nu_0 \frac{P}{2\pi}.$$  

The error in this quantity is given by

$$\sigma_{\delta \pi} \approx \left(\frac{\sigma_{\Delta t_{\text{RV}}}^2 + \sigma_0^2}{12 \nu_0 r_0 \sqrt{3}}\right)^{1/2} \frac{2\pi}{P} \frac{1}{2|\cos \nu_0|}.$$  

We see that the timing errors in the radial velocity measurements dominate over the transit timing for a quoted precision of 100ms. Analysis of radial velocity measurements over longer baselines would provide a more precise estimate of this period.

Even without a detailed understanding of the relationship between the velocity and transit timing, the radial velocity data is crucial to convert a observed timing solution into a periastron shift by determining the values of the eccentricity and the true anomaly at transit, and in combination with the timing of either the primary or secondary transit could yield hints of the periastron advance due to other planets in the system.

As the orbit precesses, the radial velocities observed over a orbit will also shift. However, over the typical one hundred orbits required to detect a Earth-like planet the orbit will only precess about an arcminute. It is difficult to imagine that radial velocity measurements will become so sensitive as to characterize an orbit to the required precision of $10^{-4}$. If they did, one could probably detect the planet causing the precession in the radial velocity data already.

4.4 Rapid Precession

If the transiting planet’s orbital eccentricity goes to zero, the precession rate of its pericenter longitude will formally go to infinity, and Eq. 23 misleadingly indicates that the precession rate will be trivial to detect. However, Eq. 13 is not correct in the case of rapid precession.

If the rate of the precession is a constant ($\delta \pi$) per radial orbit we have following equation for the true anomalies of the transits

$$\nu \mod 2\pi = \left[\nu_0 - \frac{\delta \pi}{P} t\right] \mod 2\pi.$$  

If we assume that the orbit is a precessing ellipse and that the angular momentum of the observed planet is conserved, we can use Kepler’s equations to determine the time corresponding to each true anomaly. To lowest order in the eccentricity we have

$$t = \frac{P \nu}{2\pi} - \frac{P \epsilon^2}{2\pi} \sin 2\nu.$$  

If we first ignore the eccentricity, we find that the time between two successive transits is given by the angular period

$$\Delta t^{(0)} = P \left(1 + \frac{\delta \pi}{2\pi}\right)^{-1}.$$  

$$\frac{d^2 v_{\text{radial}}}{dt^2} \approx \left(\frac{2\pi}{P}\right)^3 a \left[ \sin(\nu - \nu_0) + 2\epsilon (11 \cos \nu \sin(\nu - \nu_0) + 5 \sin \nu_0) \right]$$

$$\sin(\nu - \nu_0) = -10\epsilon \sin \nu_0 + \mathcal{O}(e^2)$$

$$\Delta t_{\text{RV}} = \frac{1}{2} (\Delta t_0 + \Delta t_\pi) + \mathcal{O}(e^2) = P \left(1 - \frac{\delta \pi}{2\pi}\right) + \mathcal{O}(e^2).$$

$$\sigma_{\delta \pi} \approx \left(\frac{\sigma_{\Delta t_{\text{RV}}}^2 + \sigma_0^2}{12 \nu_0 r_0 \sqrt{3}}\right)^{1/2} \frac{2\pi}{P} \frac{1}{2|\cos \nu_0|}.$$
and
\[
\Delta \nu = \nu_2 - \nu_1 = -\delta \varpi \left( 1 + \frac{\delta \varpi}{2\pi} \right)^{-1}
\] (52)

Looking at Kepler’s equation we find that the correction to this quantity introduced by the eccentricity of the orbit is limited by \( \epsilon^2 P/(4\pi) \). We have
\[
\Delta t^{(2)} = \Delta t^{(0)} - \frac{P}{2\pi} \frac{\epsilon^2}{4\pi} (\sin 2\nu_2 - \sin 2\nu_1)
\] (53)
\[
= \Delta t^{(0)} \left[ 1 - \frac{a}{2\pi} \frac{\epsilon^2}{4\pi} \left( \sin \Delta \nu \cos (2\nu_1 + \Delta \nu) \right) \right]
\] (54)

For a perturber in an elliptical orbit \( \delta \varpi \) is inversely proportional to the eccentricity of the observed planet (Murray & Dermott 2000); however, the observable correction to the transit time in this case is proportional to the eccentricity, so even if the orbit precesses arbitrarily quickly, the observations will not be strongly affected.

5 SPECIAL RELATIVISTIC CORRECTIONS

The foregoing analysis focused on the angles necessary for a transit to occur. It was essentially geometry with any kinematics. Specifically it neglected the time for light to travel across the system. The variation in the distance of the planet and the star from transit to transit as the orbit precesses would affect the times that we observe the transits to occur. To proceed we have to be clear not only when but where each transit occurs.

- The primary transit occurs as the planet blocks the light from the star. It occurs at the location of the planet.
- The secondary transit occurs as the star blocks the light from the planet. It occurs at the location of the star.

Most obviously the light travel time will affect the observed time difference between the primary transit and the secondary transit that immediately follows it.
\[
[t_s - t_0]_{\text{obs}} = \left[ M(v_0 + \pi) - M(v_0) \right] \frac{P}{2\pi} +
\frac{1}{c} \left[ \frac{M_*}{m + M_*} r(v_0) - \frac{m}{m + M_*} r(v_0 + \pi) \right]
\] (55)
where \( m \) is the mass of the planet and
\[
r(\nu) = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \nu}
\] (56)
The light-travel time will cause the primary transit to appear to occur about 20 s (\( P/3 \) days) \({2/3} \left( \frac{M_*}{M} \right)^{1/3} \) earlier. The appearance of the secondary transit will be delayed by an interval a factor of \( m/M_* \sim 10^{-3} \) smaller. The difference in the inter-transit interval due to geometry is on order of the period of the orbit, a factor of 10\(^4\) larger.

Other quantities that we have examined are the the interval from primary to primary transit and from secondary to secondary transit, and the time derivative of these quantities. If the orbit did not precess the light travel time would not affect either of these intervals, so we know that the relativistic correction to these intervals will be proportional to the change in a angle of periastron during an orbit.

We have
\[
[\Delta t_{\text{obs}}] = \left[ 1 - \left( \frac{P}{2\pi} \frac{dM}{d\nu} \frac{d\nu}{d\psi} + \frac{M_*}{c m + M_*} \frac{dr}{d\nu} \frac{d\nu}{d\psi} \right) \right] \varpi
\] (57)
and
\[
[\Delta t_{\text{rel}}] = \left[ 1 - \left( \frac{P}{2\pi} \frac{dM}{d\nu} \frac{d\nu}{d\psi} + \frac{m}{c m + M_*} \frac{dr}{d\nu} \frac{d\nu}{d\psi} \right) \right] \varpi
\] (58)

The ratio of the two corrections to the interval between primary transits is
\[
\frac{\Delta t_{\text{rel}}}{\Delta t_{\text{obs}}} = \frac{M_* \tan \nu_0}{m + M_*} \frac{a P}{\pi c}
\] (59)
\[
\approx 8 \times 10^{-7} \tan \nu_0 \left( \frac{M_* 3 \text{ days}}{M_0 0 \text{ P}} \right)^{1/3}.
\] (60)
The ratio is smaller by a factor of \( m/M_* \sim 10^{-3} \) for the secondary transits; therefore, as found by [Agol et al. 2002], it is safe to ignore the relativistic corrections to the intervals between two similar transits.

6 OUTLOOK

6.1 Systems with a transiting planet

Nearly two hundred extrasolar planets have been detected as of the beginning of 2009 (Butler et al. 2009). The eccentricity and the longitude of periastron have been measured for most of these planets; unfortunately, for the few transiting planets, this information is lacking, so we will use the orbital elements for all the nearby exoplanets to calculate the sensitivity of timing measurements to periastron advance and more importantly how long of a timing series would be required to achieve a precision of \( 10^{-7} \) in the measurement of \( \delta \varpi \); this is sufficient to detect an Earth-mass planet orbiting within a factor of four of the semimajor axis of the transiting planet orbiting around a one-solar-mass host star. Fig. 4 shows that for about ten systems this level of sensitivity could be achieved within ten years (these results have assumed that the sampling rate \( r \) is unity; determining the optimal sampling rate and schedule considering observational constraints is beyond the scope of this paper).

To be certain of the presence of an unseen planet one has to be sure that the periastron precession exceeds that expected from the star. For this discussion we shall assume that the contribution due to stellar oblateness is small. According to Fig. 4 even for a three-day period, the expected contribution from oblateness is a factor of 30–300 below the relativistic value. This ratio increases as \( P^{2/3} \) so oblateness does not make the dominant contribution for any of the systems in Fig. 4. On the other hand, Fig. 4 shows the expected relativistic precession is greater that \( 10^{-7} \) for most of the systems in the catalogue; consequently, the mass of

1 We used the updated catalogue at [http://exoplanets.org](http://exoplanets.org)
the host star for these systems must be determined to better than about a tenth of a solar mass. Otherwise, the error in the determination of the mass of the host star will dominate the statistical error in the timing.

Fig. 5 shows the sensitivity to finding additional planets in the transiting systems with a ten-year time series assuming that the mass of the host star is known with a precision of 0.03M\textsubscript{\odot}. Four objects stand out in Figs. 3–5: GJ 436b, HD 118203b, HD 33263 and HD 74156b require the shortest time series to achieve a sensitivity of $10^{-7}$ in $\delta \pi$. These systems have moderate eccentricities between 0.2 and 0.6. Because of its short 2.6-day orbit, the orbital precession of GJ 436b would be dominated by relativistic effects with $(\delta \pi)_{GR} \approx 2.3 \times 10^{-8}$ assuming a mass of 0.41M\textsubscript{\odot} for GJ 436 (Butler et al. 2006). There are several planets in the catalogue whose orbits will precess at a rate similar to GJ 436b due to GR, but the geometry and eccentricity of the orbits are not as favourable for detecting the precession by transit timing as for GJ 436b. On the other hand, the maximum sensitivity to unseen planets are found in the HD 74156 and HD 168443 systems because their relatively long orbits (52 days and 58 days) reduces the expected contribution due to GR and stellar oblateness. The minimum detectable value of $(\delta \pi)_{planet}$ is about $5 \times 10^{-8}$, sufficient to detect an Earth-mass planet with a period less than about 460 days.

Figure 3. The duration of the time series of primary and secondary transits (or primary transits and radial velocity timing) to yield a detectable $\delta \pi$ of $10^{-7}$ for the planets in the catalogue of [Butler et al. 2006], assuming a ten-second error in the timing. This is the sensitivity required to detect an Earth-mass planet at the three-sigma level within a factor of four in semimajor axis of the transiting planet orbiting a solar mass star. The required length of the time series increases as $\sigma_{\delta \pi}^{-2/3}$.

Figure 4. The advance of the longitude of periastron per orbit for the planets of [Butler et al. 2006] from general relativity.

Figure 5. The sensitivity to the advance of the longitude of periastron caused by additional planets (or stellar oblateness) in the planetary systems of [Butler et al. 2006] assuming that the mass of the host star is known to 0.03M\textsubscript{\odot} with a ten-year baseline of transit timing. To detect an Earth-mass planet with an orbit up to a factor of four larger than the observed planet the sensitivity to the planetary contribution to the precession must be less than $10^{-7}$. 

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6.2 Systems with more than one observed planet

A system that already has more than one planet detected provides a chance to characterize an additional planets and verify the stellar oblateness does not dominate over the planetary signal. Of course, each observed planet will induce a periastron precession in the other. This must be subtracted from the observed shifts along with the relativistic shifts. Because both planets orbit the same star the contribution due to the stellar oblateness is proportional $P^{-7/3}$, consequently, unless the residual periastron precession rate in each planet that remains is proportional to $P^{-7/3}$, there must be other planets in the system.

If one can argue from other data that the value of $J_2$ is small, one can use the residual precession rates to find the mass of the unseen planet and its semimajor axis. With a single observed planet one can only constrain the combination the determines $\delta \varpi$. With three observed planets, one could unravel $J_2$ and the properties of an unseen planet or alternatively the properties of an unseen planet and the possibility of further planets!

6.3 Systems with fewer than one observed planet

It may seem odd to suggest measuring the periastron advance in systems without any planets yet discovered. However, the techniques outlined here could be used in eclipsing binary systems. In single-lined systems it could be used to constrain the total mass of the system through the relativistic term. In double-lines systems or if the total mass of the system is known, measurements of the periastron advance through timing of the radial velocity data and one or both transits could be used to constrain the values of $J_2$ for the stars or to look for planets in the systems. One could search for planets with the long-term timing measurements of eclipsing binaries that have already been taken.

7 CONCLUSIONS

Accurate timing data (of pulsars) allowed the discovery of the first Earth-mass planets (Wolszczan & Frail 1992). Careful and accurate timing of planet transits and radial velocity data is sensitive to additional planets down to the mass of Earth and below. The combination of two sets of timing data provides much stronger constraints on the presence of additional bodies in the system than looking at the primary transits alone (c.f. Miralda-Escudé 2002, Schneider 2003, Agol et al 2003, Holman & Murray 2003). The timing signature of an Earth-mass planet is an induced shift in the periastron of the orbits of the known planets. There are generally two dominant contributions to this shift: general relativity and other planets. The stellar oblateness can also contribute but only competes with the other effects if the oblateness is nearly two orders of magnitude larger than that of the Sun. Consequently, if the observed periastron shift exceeds the relativistic expectation, either the system has additional planets or the parent star has an unusually large oblateness. Even the less likely possibility of a large oblateness would give tantalizing hints to the origins of these close-in extrasolar planets. More likely would be the presence of a Earth-mass or even a Mars-mass planet in a nearby orbit to the known planet.

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