Bethe-Salpeter equation with cross-ladder kernel in Minkowski and Euclidean spaces

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Some results obtained by a new method for solving the Bethe-Salpeter equation are presented. The method is valid for any kernel given by irreducible Feynman graphs. The Bethe-Salpeter amplitude, both in Minkowski and in Euclidean spaces, and the binding energy for ladder + cross-ladder kernel are found. We calculate also the corresponding electromagnetic form factor.

Bethe-Salpeter (BS) equation \[1\] is an important tool for studying the relativistic bound state problem in a field theory framework. The BS amplitude in Minkowski space is singular, what makes the numerical resolution of BS equation difficult. Therefore it is usually solved in Euclidean space to find the binding energy. However, to describe some observables, we need the original BS amplitude in Minkowski space as well. The methods proposed so far to calculate this amplitude are valid either for the ladder kernel \[2\] or for a separable one \[3\].

We give here a brief review of results obtained by solving BS equation using a new method, developed in \[4\]. It allows to find the BS amplitude both in Minkowski and Euclidean spaces as well as the light-front (LF) wave function. The BS amplitude is written in terms of the Nakanishi integral representation \[5\]:

\[
\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{-ig(\gamma',z')}{[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p \cdot k \ z' - i\epsilon]^3}
\]

and substituted in the BS equation. The resulting equation is then projected on the LF plane, i.e., both parts of it are integrated over the variable \(k_- = k_0 - k_z\). This integration eliminates singularities of the original BS amplitude. An integral equation for the weight function \(g(\gamma, z)\) is derived, which for spinless particles has the form:

\[
\int_{0}^{\infty} \frac{g(\gamma', z) d\gamma'}{[\gamma' + \gamma + m^2 - \frac{1}{4}(1 - z^2)M^2]^2} = \int_{0}^{\infty} d\gamma' \int_{-1}^{1} dz' V(\gamma, z; \gamma', z')g(\gamma', z') ,
\]

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The projected kernel $V$ is obtained in [11] in terms of the BS interaction kernel. This derivation does not contain any approximation.

For the massless ladder exchange the function $g(\gamma,z)$ turns into $g(\gamma,z) = \delta(\gamma)\tilde{g}(z)$ and equation (2) is reduced to the well-known Wick-Cutkosky equation [6] for $\tilde{g}(z)$. For massive ladder exchange, the numerical calculations by equation (2) and by the BS equation in Euclidean space give the same binding energy [4].

![Figure 1. Ground state mass $M$ obtained by eq. (2) with L and L+CL kernels, compared with the Feynman-Schwinger representation results for an exchanged mass $\mu = 0.15$.](image)

Figure 1 shows the ground state mass $M$ obtained for an exchanged mass $\mu = 0.15$ (in units of the constituent mass), by eq. (2) with ladder (L) and ladder+cross-ladder (L+CL) kernels together with Feynman-Schwinger representation results [7]. The latter incorporates all the higher order cross box contributions in the kernel, but not the self energy. The effect of CL diagrams is large and attractive, but an even larger contribution remains to be included in the kernel, due to the higher order terms. The L+CL calculation performed in the framework of the light-front dynamics (LFD) provides binding energy very close to the BS ones.

BS equation with L+CL kernel was also solved in Euclidean space in [8,9] and corresponding LFD equation – in [10]. Our results are in close agreement with these references. In [11], an equation obtained by projecting the BS equation on the light-front plane was also derived.

Once $g(\gamma,z)$ is known, we can find by means of eq. (1) the BS amplitude both in Minkowski and Euclidean spaces. The latter is obtained by substituting in this equation $k_0 = ik_4$. An example is given in Figure 2. It is worth noticing the difference between the smooth behavior of the Euclidean amplitude and the singularities displayed in the Minkowski one. The amplitude in Euclidean space found by eq. (1) for imaginary $k_0 = ik_4$ coincides with one obtained by the direct solution in Euclidean space. We have also carried out precise calculations (accuracy better than 0.1%) of the binding energy for L+CL kernel independently by eq. (2) and in Euclidean space. The results are presented in Table 1. Their coincidence demonstrates both the validity of our approach and the possibility of Wick rotation for the CL kernel. The latter point had not been proved as it was the case for the ladder kernel (see a discussion in [12]).
Figure 2. Left: The Euclidean BS amplitude obtained by solving eq. (1) with imaginary $k_0 = ik_4$ (for ladder kernel), for different values of $k$. The amplitude calculated by direct resolution of BS equation in Euclidean space is indistinguishable. Right: The same solution in Minkowski space, obtained by eq. (1) for real $k_0$.

Table 1
Comparison for BS L+CL kernel and $\mu = 0.5$, of the coupling constant $\alpha$ obtained by eq. (2) and by the Euclidean space calculations, for given values of the binding energy $B$. The accuracy on the coupling constants is better than 0.1 % for both calculations.

<table>
<thead>
<tr>
<th>$B$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$, eq. (2)</td>
<td>1.206</td>
<td>1.607</td>
<td>1.930</td>
<td>2.416</td>
<td>3.446</td>
<td>4.549</td>
</tr>
<tr>
<td>$\alpha$, Euclid</td>
<td>1.205</td>
<td>1.608</td>
<td>1.930</td>
<td>2.417</td>
<td>3.448</td>
<td>4.551</td>
</tr>
</tbody>
</table>

Though the binding energy can be found from the Euclidean space calculation, to describe many observables, we need the Minkowski space BS amplitude. We need it even to calculate EM form factors where, at first glance, we could use the Euclidean BS amplitude, which is determined in the rest frame $\vec{p} = 0$. Indeed, the form factors are expressed through the Euclidean BS amplitude $\Phi_E(k_4, \vec{k}; p)$ for non-zero total momentum $\vec{p}$. To express the BS amplitude for non-zero $\vec{p}$ via the rest frame solution $\Phi_E(k_4, \vec{k}; p_0 = M, \vec{p} = 0)$, we need to make a boost of four-momentum $k$: $\Phi_E(k_4, \vec{k}; p) = \Phi_E(k'_4, \vec{k'}; p_0 = M, \vec{p} = 0)$. In Minkowski space this boost reads: $k'_0 = \frac{1}{M}(p_0k_4 - \vec{k} \cdot \vec{p})$ and similarly for $\vec{k'}$. After replacing $k_0 = ik_4$ (with still real $p_0$), it results in a complex value of boosted variables $k'_4 = \frac{1}{M}(p_0k_4 + i\vec{p} \cdot \vec{k})$. This requires the knowledge of the BS amplitude in the full complex plane, and not only on the imaginary axis. The problem can be solved in the static approximation [13], where the form factor is calculated approximately, through the BS amplitude obtained from the rest frame by a non-relativistic boost. An estimation of accuracy, done in [13] perturbatively, using a decomposition in momentum transfer, shows that the correction is not negligible and increases with momentum transfer.
We calculate the form factor exactly, through the Minkowski space BS amplitude for \( L+CL \), and compare our result with the static approximation. This comparison is shown in Figure 3. The curves considerably differ from each other and the difference increases with momentum transfer. The zero of form factor in this model is an artifact of static approximation. The LF calculation gives result which is almost indistinguishable from the Minkowski space one.

We conclude that the results presented above confirm the validity of the method [4] and demonstrate the necessity for using the BS amplitude in Minkowski space.

Figure 3. Dashed curve: Form factor in static approximation. Solid curve: Exact form factor (BS in Minkowski). Dot-dashed: LFD calculation.

REFERENCES