Modified teleparallel gravity: inflation without inflaton

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Born-Infeld strategy to smooth theories having divergent solutions is applied to teleparallel equivalent of General Relativity. Differing from other theories of modified gravity, modified teleparallelism leads to second order equations, since teleparallel Lagrangian only contains first derivatives of the vierbein. We show that Born-Infeld-modified teleparallelism solves the particle horizon problem in a spatially flat FRW universe by providing an initial exponential expansion without resorting to an inflaton field.

In 1934 Born and Infeld \textsuperscript{1} proposed the following scheme for modifying a field theory governed by a Lagrangian density \( \mathcal{L} = \sqrt{-g} L \):

\[
\mathcal{L} \longrightarrow \mathcal{L}_{BI} = \sqrt{-g} \lambda \left[ \frac{\sqrt{1 + 2 L}}{\lambda} - 1 \right].
\] (1)

The basic idea was to introduce a new scale \( \lambda \) with the aim of smoothing singularities. The scheme \( \mathcal{L}_{BI} \) is essentially the way for going from the classical free particle Lagrangian to the relativistic one; in such case, the scale is \( \lambda = -mc^2 \), which smooths the particle velocity by preventing its unlimited growing. Besides, Born and Infeld subtracted the “rest energy” to get that \( \mathcal{L}_{BI} \) vanishes when \( \lambda = 0 \). We can then expect that Born-Infeld dynamics will differ from the original dynamics for those configurations where \( \lambda \) is large. In fact, Born and Infeld looked for a reformulation of Maxwell’s electrodynamics in order to smooth the divergence of the point-like charge electric field, and they have succeeded in obtaining a finite self-energy for this configuration. On the other hand, the original Lagrangian is recovered if \( \lambda \ll L \); hence the solutions of both theories do not appreciably differ in these regions. Nowadays Born-Infeld Lagrangians have reappeared in developments of string theories at low energies \textsuperscript{2,3,4,5,6,7}; they have also been used in quintessence theories for modeling matter fluids able to drive both inflation and the present accelerated expansion \textsuperscript{8}. However, although the subject have been received some attention \textsuperscript{9}, no gravitational BI analogue was yet proposed in four dimensions.

A wide variety of modified gravity theories have been considered in the last decades. For instance, Lovelock Lagrangian is a polynomial in Riemann curvature which leads to second order equations for the metric tensor \textsuperscript{10}. Nevertheless, Lovelock Lagrangian only differs from the Einstein-Hilbert Lagrangian, \( \mathcal{L}_{EH}(g_{\mu\nu}(x)) = -(16\pi G)^{-1}\sqrt{-g} \mathcal{R} \) (\( \mathcal{R} \) being the scalar curvature), for dimension larger than 4. On the other hand “\( f(\mathcal{R}) \)” theories are being currently studied, mostly connected with the attempts to explain the cosmic acceleration without resorting to quintessence models \textsuperscript{11,12}. For instance, a \( f(\mathcal{R}) \) theory could be obtained by using the Born-Infeld scheme:

\[
\mathcal{L} = -\frac{1}{16\pi G} \sqrt{-g} \lambda \left( \sqrt{1 + \frac{2 \mathcal{R}}{\lambda}} - 1 \right).
\] (2)

However we find this strategy unsatisfactory because: 1) fourth order dynamical equations will result, since \( \mathcal{R} \) contains second derivatives of the metric (a feature that is common to \( f(\mathcal{R}) \) theories); 2) this strategy is unable to smooth black holes, since they have \( \mathcal{R} = 0 \) (then the scale \( \lambda \) could not play any role).

Concerning the first objection, it is well known that the second derivatives of the metric in Einstein-Hilbert Lagrangian do not lead to fourth order equations because they only give rise to surface terms in the action. This characteristic only remains valid in Lovelock Lagrangians but is lost in \( f(\mathcal{R}) \) theories.

In order to build a modified gravity leading to second order equations in four dimensions, we will start not from Hilbert-Einstein Lagrangian but from the teleparallel equivalent of General Relativity (TEGR). While General Relativity uses Levi-Civita connection (curvature but no torsion), Teleparallelism uses Weitzenböck connection \textsuperscript{13} (torsion but no curvature). In this sense Teleparallelism \textsuperscript{14,15} is a sector of Einstein-Cartan theories \textsuperscript{16,17}, which describe gravity by means of a connection having both torsion and curvature. In Teleparallelism the dynamical object is the vierbein field \( \{h_i(x)\} \), \( i = 0, 1, 2, 3 \). Each vector \( h_i \) is described by its components \( h^\mu_i, \mu = 0, 1, 2, 3 \), in some coordinate basis. The matrix \( (h^\mu_i) \) is invertible; i.e. there exist a matrix \( (h^i_\mu) \) fulfilling

\[
h^\mu_i h^i_\nu = \delta^\mu_\nu, \quad h^i_\mu h^\mu_i = \delta^i_\nu.
\] (3)

Weitzenböck connection,

\[
\tilde{\Gamma}^\lambda_{\mu\nu} = -h^\lambda_\nu \partial_\mu h^i_i = h^\lambda_\lambda \partial_\mu h^i_\mu,
\] (4)

is such that the Weitzenböck covariant derivative of a
vector \( \mathbf{V} = V^i \mathbf{h}_i = V^i h_i^\mu \partial_\mu \) becomes
\[
\nabla_\nu V^\mu = \partial_\nu V^\mu + \tilde{\Gamma}_\lambda^{\mu \nu} V^\lambda
= \partial_\nu (V^i h_i^\mu) + \tilde{\Gamma}_\lambda^{\mu \nu} V^\lambda h_i^\mu = h_i^\mu \partial_\nu V^i. \tag{5}
\]
Hence a vector \( \mathbf{V} \) will be autoparallel if its components \( V^i = h_i^\mu V^\mu \) are constant.

Weitzenböck connection has zero Riemann curvature and non-null torsion:
\[
T^\lambda_{\mu \nu} = \Gamma^\lambda_{\mu \nu} - \Gamma^\lambda_{\nu \mu} = h_i^\lambda (\partial_\mu h_i^\nu - \partial_\nu h_i^\mu), \tag{6}
\]
i.e., \( h_i^\lambda T^\lambda_{\mu \nu} \) are the components of the 2-form \( dh^i \), where \( \{ h^i \} \) is the dual basis (whose elements have components \( h_i^\mu \)). The TEGR Lagrangian is
\[
\mathcal{L}_T[h_i^\mu(x)] = \frac{1}{16 \pi G} h S_{\rho}^{\mu \nu} T^\rho_{\mu \nu}, \tag{7}
\]
where \( h \equiv \text{det}(h_i^\mu) \) and \( S_{\rho}^{\mu \nu} \) is given by
\[
S_{\rho}^{\mu \nu} = \frac{1}{2} [K_{\rho}^{\mu \nu} + \delta_\rho^{\mu} T^{\theta \nu} - \delta_\rho^{\nu} T^{\theta \mu}]. \tag{8}
\]
In this last equation, the contorsion tensor is
\[
K_{\rho}^{\mu \nu} = -\frac{1}{2} (T^\rho_{\mu \nu} - T^\nu_{\mu \rho} - T^\rho_{\nu \mu}). \tag{9}
\]
In Eq. (8) and (9), indexes have been risen and lowered with the metric
\[
g_{\mu \nu}(x) = \eta_{ij} h_i^\mu(x) h_j^\nu(x), \quad g^{\mu \nu}(x) = \eta^{ij} h_i^\nu(x) h_j^\mu(x) \tag{10}(\eta_{ij} = \text{diag}(1, -1, -1, -1)), \text{ so it is } h = (\text{det } g_{\mu \nu})^{1/2}.
\]
Note that the tetrad is orthonormal in this metric:
\[
g_{\mu \nu}(x) h_i^{\nu}(x) h_j^{\nu}(x) = \eta_{ij}. \tag{11}
\]
Moreover, Weitzenböck connection results to be metric. It is easy to prove that the contorsion equals the difference between Levi-Civita connection associated with the metric (10) and Weitzenböck connection:
\[
\tilde{\Gamma}^\lambda_{\mu \nu} = \Gamma^\lambda_{\mu \nu} + K^\lambda_{\mu \nu}. \tag{12}
\]
Taking into account the Weitzenböck connection definition (11), this means that
\[
\nabla_\nu h_i^\mu = h_i^\lambda K^\lambda_{\mu \nu}. \tag{13}
\]
Eq. (12) also means that Weitzenböck four-acceleration of a freely falling particle is not zero but it is
\[
\frac{d^2 x^\lambda}{d \tau^2} + \Gamma^\lambda_{\mu \nu} \frac{dx^\mu}{d \tau} \frac{dx^\nu}{d \tau} = K^\lambda_{\mu \nu} \frac{dx^\mu}{d \tau} \frac{dx^\nu}{d \tau}. \tag{14}
\]
Thus, the contorsion can be regarded as a gravitational force which moves particles away from Weitzenböck autoparallel lines.

Actually, Lagrangian \( \mathcal{L}_T \) (7) is not the most general Lagrangian quadratic in the torsion. In Ref. [13] it has been proved that a general quadratic theory combines three quadratic pieces, each of them associated with each of the three irreducible parts of the torsion: vectorial, axial and traceless-symmetric. The coefficients of this combination are strongly constrained by physics in solar system. Lagrangian \( \mathcal{L}_T \) is just the Einstein-Hilbert Lagrangian for metric (10) plus a divergence (see, for instance, [14]). Hence Euler-Lagrange equations result to be equivalent to Einstein equations. However, the fact that the values of \( \mathcal{L}_T \) and \( \mathcal{L}_{\text{EH}} \) are different is essential for smoothing vacuum solutions by means of a Born-Infeld scheme: vacuum solutions have \( \mathcal{L}_{\text{EH}} = 0 \), but the scale \( \lambda \) in (11) does not play any role if the Lagrangian is zero.

The vierbeins solving the teleparallel Euler-Lagrange equations for given sources establish an orthogonal grid of autoparallel lines on the manifold, which defines an absolute parallelism of vectors (“a vector \( \mathbf{V} \) is autoparallel if its components \( V^i = h_i^\mu V^\mu \) are constant”). This absolute parallelism is invariant under global Lorentz transformations of the vierbein: \( h_i^\mu(x) \rightarrow \Lambda_i^i h_i^\mu(x) \). So a dynamical theory for the vierbein must have the same invariance. Since torsion and the determinant \( h \) are invariant under global Lorentz transformations, we can be sure that teleparallel action has this kind of symmetry. (It has been proved in Ref. [18] that the equations resulting from Lagrangian (7) exhibit a partial local Lorentz invariance that keeps null the axial part of torsion. This ambiguity of TEGR does not affect the dynamics of matter coupling to the metric, since metric (10) is invariant under local Lorentz transformations of the vierbeins).

Teleparallel dynamical equations are
\[
\frac{1}{h} \partial_\sigma (h S_i^{\sigma \rho} - 4 \pi G j_i^{\rho}) + 4 \pi G h_i^{\rho} T^\rho_{\sigma} = 0, \tag{15}
\]
where \( S_i^{\sigma \rho} = h_i^\lambda S^{\lambda \sigma \rho} \), \( T^\rho_{\sigma} = h^{-1} h_{ij}^\rho (\delta \mathcal{L}_{\text{matter}} / \delta h_{ij}^\rho) \) is the symmetric energy-momentum tensor of the (non-spinning) sources and
\[
\dot{j}_i^{\rho} - \frac{1}{h} \frac{\partial \mathcal{L}_T}{\partial h_{ij}^\rho} = \frac{h_i^\lambda}{4 \pi G} \left( S_i^{\mu \rho} T^\mu_{\lambda \sigma} - \frac{1}{4} \delta_i^\rho S_{\eta}^{\mu \nu} T^\eta_{\mu \nu} \right). \tag{16}
\]
Due to the antisymmetry of \( S_i^{\sigma \rho} \), a conserved current appears:
\[
\partial_\rho (h j_i^{\rho} + h h_i^{\rho} T^\rho_{\sigma}) = 0, \tag{17}
\]
so \( j_i^{\rho} \) is associated with the vierbein energy-momentum.

The structure of teleparallel Lagrangian (7) resembles the one of a gauge field: it is quadratic in the torsion \( h_i^\mu V^\mu \). In particular, it has only first derivatives of the vierbein field. This is the feature we will exploit for building a modified teleparallel gravity leading to second order dynamical equations. Concretely we are going to use not the Lagrangian \( \mathcal{L}_T \) but a teleparallel Lagrangian alla
Born-Infeld:
\[ \mathcal{L}_{BI} = \frac{\lambda}{16\pi G} h \left( \sqrt{1 + 2S_{\mu\nu} T^{\mu\nu}} - 1 \right). \] (18)

We are aimed to test Born-Infeld modified teleparallelism in a cosmological framework. For this, we will replace a solution of the form
\[ h^i_\mu = \text{diag}(N(t), a(t), a(t), a(t)) \] (19)
in the Euler-Lagrange equations emerging from Lagrangian (18). The proposed solution implies a metric
\[ g_{\mu\nu} = \text{diag}(N^2(t), -a(t)^2, -a(t)^2, -a(t)^2), \] (20)
i.e., a spatially flat FRW cosmological model. Then we will use as source a homogeneous and isotropic fluid; so \( T^\mu_\sigma = \text{diag}(\rho, -p, -p, -p) \) in the comoving frame. Of course, the dynamical equations get more involved than the RG-equivalent ones (15). The high symmetry of the proposed solution makes that some of the sixteen equations become trivial. Finally only two independent equations are left: a first order equation
\[ (1 - \frac{12H^2}{N^2\lambda}) - \frac{1}{2} = \frac{16\pi G}{\lambda} N^2 \rho, \] (21)
which results from varying with respect to \( h^0_\sigma \) (\( H(t) = \dot{a}(t)/a(t) \) is the Hubble parameter), and a second order one,
\[ (16H^2 + 4H^2q - 1)(1 - \frac{12H^2}{N^2\lambda}) - \frac{\dot{\chi}^2}{2} = \frac{16\pi G}{\lambda} \rho, \] (22)
which results from varying with respect to \( h^i_\sigma \) with \( i = \sigma = 1, 2 \) or \( 3 \) (\( q = -\ddot{a}/a^2 \) is the deceleration parameter).
Actually Eqs. (21) and (22) could also be obtained by replacing the proposed solution (19) in Lagrangian (18) and then varying with respect to \( N(t) \) and \( a(t) \); this is a typical feature of high symmetry solutions. Note that \( S_{\mu\nu} T^{\mu\nu} = -6H(t)^2/N(t)^2 \), so \( \lambda \) in (18) will prevent the Hubble parameter from becoming infinite. As it was expected, Eq. (21) is not a dynamical equation for \( N(t) \) but a constraint for \( a(t) \) (“initial value equation”), as a consequence of the fact that \( N(t) \) is not a genuine degree of freedom: \( N(t) \) can be absorbed by redefining the \( t \) coordinate, so we will choose \( N(t) = 1 \).

By differentiating Eq. (21) with respect to \( t \) and combining it with Eq. (22), it is obtained the conservation of the fluid energy-momentum:
\[ \frac{d}{dt}(\rho a^3) = -\rho \frac{d}{dt}a^3. \] (23)
If the fluid is described by the state equation \( p = \omega \rho \) then it will result
\[ a^{3(1+\omega)} \rho = \text{constant} = a_o^{3(1+\omega)} \rho_o, \] (24)
where \( a_o \) and \( \rho_o \) indicate the present-day values.
Combining Eqs. (21) and (22) it results
\[ 1 + q = \frac{3}{2} \left( 1 + \frac{\rho}{\rho_o} (1 + \omega) \right), \] (25)
In General Relativity (\( \lambda \to \infty \)) an accelerated expansion (\( q < 0 \)) is only possible if \( \omega < -1/3 \) (negative pressure). Instead, in Born-Infeld modified teleparallelism an accelerated expansion can be handled without resorting to negative pressure; a large density \( \rho \) is sufficient:
\[ \frac{32\pi G}{\lambda} \rho > -3 + \sqrt{13 + 12\omega}. \] (26)
Actually, for \( \rho \to \infty \) in (26), it is \( q \to -1 \) and the expansion becomes exponential.

Eq. (21) defines the critical density \( \rho_c \) making the universe spatially flat. As usual, the contributions to the density coming from different constituents will be measured as fractions \( \Omega_i = \frac{\rho_i}{\rho_c} \). In this way, by combining Eqs. (21) and Eq. (22) we obtain
\[ \left( 1 - \frac{12\lambda^2}{\rho} \right)^{-\frac{1}{2}} - 1 = \frac{16\pi G}{\lambda} \sum_i \rho_{oi} \left( \frac{a}{a_o} \right)^{-3(1+\omega_i)}, \] (27)
which can be rewritten in the form
\[ \dot{x}^2 + V(x) = 0, \quad x = \frac{a}{a_o}, \] (28)
being \( V(x) \) an effective potential given by
\[ V(x) = \frac{\lambda}{12} x^2 \left( 1 + \beta_o \sum_i \Omega_{oi} x^{-3(1+\omega_i)} \right)^{-2} - 1 \] (29)
where \( \beta_o = (1 - 12H_0^2/\lambda)^{-1/2} - 1 \), is a constant. The potential is always negative and vanish with null derivative when \( a \to 0 \), for any value of \( \omega \). Moreover, if \( \omega > -1/3 \) the potential will asymptotically approach to zero when \( x \) goes to infinite. Instead if \( \omega < -1/3 \) then \( V \) will be a decreasing function. More relevant is the fact that \( |V| \) is proportional to \( x^2 \) when \( x \) goes to zero, so giving an exponential expansion for the early universe, as was anticipated. If \( \omega > -1/3 \) then the initial behavior is \( a(t) \propto \exp[(\lambda/12)^{1/2}t] \). Therefore, Hubble constant is equal to the maximum value \( H_{\text{max}} = (\lambda/12)^{1/2} \) at the early stage. Eq. (28) also says that
\[ H(z)^2 = H_{\text{max}}^2 \left[ 1 - (1+\beta_o \sum_i \Omega_{oi} (1+z)^{3(1+\omega_i)})^{-2} \right], \] (30)
where \( z = a_o/a(t) - 1 \) is the redshift. Eq. (28) for only one constituent (\( \Omega = 1 \)) can be easily integrated to obtain the evolution in an implicit way:
\[ \ln \left[ 2(1 + v) + 2\sqrt{v(2 + v)} - \sqrt{v^{-1}(2 + v)} \right] = T, \] (31)
being \( v = \beta_o (a/a_o)^{-3(1+\omega)} \) and \( T = -3(1 + \omega) H_{\text{max}} t \).
standard evolution of the universe from the epoch of nucleosynthesis. This means that \( H(z) \) at \( z_{\text{nuc}} \sim 10^9 - 10^{10} \) should not appreciably differ from its standard value. Then we will consider a radiation dominated universe (\( \omega = 1/3 \)) and we will expand Eq. (30) in \( \alpha^2 \equiv H_0^2/H_{\text{max}}^2 \):

\[
H(z)^2 = H_{\text{std}}^2 \left[ 1 + \frac{3}{4} (1 - (1+z)^4) \alpha^2 \right] + \mathcal{O}(\alpha^4), \quad (32)
\]

where \( H_{\text{std}}^2 = H^2(1+z)^4 \) is the standard GR expression. In order that the correcting term results to be negligible at \( z_{\text{nuc}} \sim 10^9 - 10^{10} \), it must be \( 10^{18} \alpha \ll 1 \), i.e., \( \beta_0 < 10^{-36} \).

Figure 1 shows the adimensional cosmological time \( H_0 t \) as a function of \( a(t)/a_0 \) for several values of \( \alpha \), as result from Eq. (31) with \( \omega = 1/3 \). The standard \( (H_0 t = (a/a_0)^2/2) \) behavior is plotted as a reference (dashed) curve. Remarkably, modified teleparallelism smoothes the singularity because the scale factor goes to zero asymptotically.

The main feature of the scale factor behavior is its asymptotic exponential character for any value of \( \omega \). This means that \( H(z) \) becomes a constant when \( z \) goes to infinite (see Figure 2). This feature implies that the particle horizon radius \( \sigma = a_0 \int_0^z \frac{dz'}{H(z')} \) diverges. Hence the whole space-time results to be causally connected, in agreement with the isotropy of the cosmic microwave background radiation. This fact appears as an essential property of modified teleparallelism which does not require any special assumption about the sources of the gravitational field.

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