Quantum Kolmogorov Complexity and its Applications

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Kolmogorov complexity is a measure of the information contained in a binary string. We investigate here the notion of quantum Kolmogorov complexity, a measure of the information required to describe a quantum state. We show that for any definition of quantum Kolmogorov complexity measuring the number of classical bits required to describe a pure quantum state, there exists a pure n-qubit state which requires exponentially many bits of description. This is shown by relating the classical communication complexity to the quantum Kolmogorov complexity. Furthermore we give some examples of how quantum Kolmogorov complexity can be applied to prove results in different fields, such as quantum computation and thermodynamics, and we generalize it to the case of mixed quantum states.

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I. INTRODUCTION

In 1948, Shannon \cite{Shannon} laid the foundation of modern theory of information and communication by defining mathematically the concept of information. In particular, Shannon was interested in the amount of information contained in typical messages emitted from a source (a random variable). The Shannon entropy of a random variable $X$ measures, in fact, how much information we gain on average if we learn the value of $X$. A different, but related issue, regards the idea of quantifying the information contained in a single message (or, more generally, a single object), independently of the statistical properties of the source that emits it. This problem has been first considered by Solomonoff in 1960, but its most relevant developments are due to independent works by Kolmogorov and Chaitin \cite{Solomonoff,Chaitin}. The underlying idea, common to all these approaches, is that the amount of information contained in a finite object (bit string) is the size of the shortest program that, without additional data, computes the string and halts. To give full credit to all those who made pioneering contributions to the subject, one should probably refer to such a measure of information as “Solomonoff-Kolmogorov-Chaitin complexity”. For brevity we adopt the most common choice in literature, and refer to it simply as Kolmogorov complexity.

The Kolmogorov complexity of a $n$-bit string, $\omega$, is defined as the length of the shortest binary program that computes that string, i.e.

$$K_U(\omega) = \min_p \{l(p) | U(p) = \omega \}, \quad (1)$$

where $U$ denotes a Turing machine, which computes $\omega$, given the program $p$ as input. One can prove that the Kolmogorov complexity, also called algorithmic, or descriptive complexity, is largely independent of the particular choice of the Turing machine. That is, changing Turing machine will change the algorithmic complexity at most by a constant, which becomes negligible in the limit of very long strings ($n \to \infty$) \cite{Kolmogorov}. For this reason one usually omits the subscript “$U$” and writes simply $K(\omega)$, or $K_C(\omega)$ to underline the fact that we are considering the classical quantity.

Note that this quantity is not computable \cite{Kolmogorov}. However, it turned out that already upper bounds on the Kolmogorov complexity of asymptotically long bit strings, where they can be computed, are very fruitful. A trivial upper bound on the Kolmogorov complexity of an $n$-bit string $\omega = \omega_1, \ldots, \omega_n$ is $n$, since there always exists a program of the form: "Print $\omega_1, \ldots, \omega_n$". A string $\omega$ is said to be compressible by $k$ bits if there exists a program with less than or equal to $n-k$ bits that gives $\omega$ as an output. A simple counting argument shows that at most $2^k$ strings have a complexity smaller than or equal to $k$. Thus, for any $n$, there exists at least one $n$-bit complex (incompressible) string. Actually, one can show that almost all bit strings are complex \cite{Lempel}. 

The definition of Kolmogorov complexity satisfies the intuition that it is much easier to describe a regular object than a random one. Moreover, it is a good measure of the information content of a bit string. A complex object contains a large amount of information (albeit not necessarily structure \cite{Lempel}) and thus cannot be compressed. On the contrary, regular strings have a much smaller complexity: for example, the complexity of a periodic string is at most logarithmic in its length. This logarithmic dependency is only due to the fact that one has to specify the number $n$, which requires $\log(n)$ bits of description. Notice that this dependency has no deeper meaning than the mere specification of the length of the output string. One can thus remove it by considering the conditional quantity $K(\omega|n)$, which measures the number of bits required to describe $\omega$ given the fact that the computer is given the length $n$ of the string \cite{Lempel}.

The theory developed in the context of Kolmogorov complexity has become very powerful in classical information theory. It has been successfully employed as a general proof method, known as “incompressibil-
ity method" \[4\], in as diverse fields as learning theory, complexity theory, and combinatorics, to name just a few. The notion of Kolmogorov complexity has been used \[3\] to solve in a simple and elegant way many otherwise difficult problems, such as, e.g., Gödel’s theorem \[5\]. Recently it has also been used to establish a connection between classical information and thermodynamics \[8\] \[8\] \[11\] \[11\]. How to generalize the concept of Kolmogorov complexity to the quantum world has become a natural question with the development of the theory of quantum information. Not only is it a good notion of quantum Kolmogorov complexity deep fundamental interest, but one also hopes it will allow to develop a theory similarly powerful as its classical counterpart.

In recent years, a number of different definitions of quantum Kolmogorov complexity have been proposed \[12\] \[13\] \[12\] \[14\] \[15\] \[15\], each generalizing the classical quantity in a different way. Some measure the number of qubits required to describe a quantum state, while others count instead the number of classical bits. The aim of this paper is to shed new light on the different definitions of quantum Kolmogorov complexity. In particular the results presented here will allow us to better understand the physical meaning of some of the definitions. Furthermore we will show how Kolmogorov complexity can, also in the quantum scenario, be used as a proof method.

The paper is organized as follows. In Section II we review some of the existing definitions of quantum Kolmogorov complexity of pure states. Then we review the notion of communication complexity. We will focus on the equality problem and, more precisely, on its solution by means of the fingerprinting protocol studied in \[16\]. In Section III we relate the number of classical bits required to describe a quantum state (as a general definition) to a problem in communication complexity. This allows us to prove that there exist quantum states of \(n\) qubits whose description require exponentially many classical bits. In Section IV we use the relation studied in Section III to derive some implications for any definition of Kolmogorov complexity, including a natural generalization to mixed states. Furthermore we will provide a non-trivial relation between the so-called network Komogorov complexity of a class of pure quantum states \[17\] and the classical Kolmogorov complexity of bit strings. In the last section we apply the notion of quantum Kolmogorov complexity to a number of problems, including computation complexity and thermodynamics.

II. REVIEW OF QUANTUM KOLMOGOROV

COMPLEXITY AND COMMUNICATION

COMPLEXITY

In this section we first review the different definitions of quantum Kolmogorov complexity and their properties. Second, we review the notion of communication complexity. In particular we will describe the fingerprinting protocol studied in \[16\] to solve the equality problem. In Section III we will relate these two concepts to each other. This section will also serve to introduce our notation.

A. Quantum Kolmogorov Complexity

Many classical quantities, e.g. the Shannon entropy, have been successfully generalized to quantum information and have become useful and powerful tools to understand and further develop quantum information theory. In the case of Kolmogorov complexity, though, the way to do so is not straightforward. Indeed, recently a number of different definitions for quantum Kolmogorov complexity have been introduced by Gäcs \[12\] Berthiaume et al. \[13\] Vitanyi \[14\] and Mora and Briegel \[17\]. Contrary to the classical case, where different definitions have been shown to be equivalent \[3\], in the quantum setting they are not. The definition proposed in \[12\] differs from the others in that it is a generalization of an alternative definition of algorithmic complexity, introduced by Levin \[18\]. The general idea of such approach is not to search for the shortest program that reproduces the string \(x\) (as one does in the definition proposed by Kolmogorov), but rather to look at the probability of obtaining \(x\) when a random program (that is, a sequence of 0 and 1 generated, e.g., by coin tosses) is given as input to a Turing machine. It has been shown that the negative logarithm of such a probability coincides with the Kolmogorov complexity of the string \(x\). The probability distribution over the set of binary strings generated by running random programs on a Turing machine is called universal probability, and it can be characterized by precise mathematical properties. In his work, Gäcs proposes a universal density matrix, \(\mu\), that can be viewed as a generalization of the universal probability distribution. The definition of the Komogorov complexity of a quantum state follows, thus, quite naturally from the classical theory. The two possible choices, whose properties are analyzed in detail in \[12\], are

\[
\overline{H}(|\phi\rangle) = -\langle \phi | \log \mu |\phi\rangle \quad \text{and} \quad \underline{H}(|\phi\rangle) = -\log \langle \phi | \mu |\phi\rangle .
\] (2)

It is easy to see that, given an \(n\)-qubit state \(|\phi\rangle\), both quantities are upper bounded by \(n\).

The other definitions mentioned above are, instead, based on a generalization of complexity, as introduced by Kolmogorov and Chaitin. In \[13\] the Kolmogorov complexity of a quantum state \(\rho\) is defined as the size of the smallest quantum program (state) that, given as input to a quantum Turing machine, yields \(\rho\) as output. In this work, the “length of a quantum state” is defined as the logarithm of the dimension of the smallest Hilbert space that contains the state itself. One can then define the complexity of \(\rho\) as

\[
K^\epsilon_{BLV}(\rho) = \min_{\pi} \{ \| \pi \| F(U(\pi), \rho) > 1 - \epsilon \} ,
\] (3)

where \(1 - \epsilon\) measures the fidelity, \(F\), with which the state \(\rho\) must be reproduced. It is clear from the definition, that
ally required to describe the state information [20] is given, then only assume that Shannon-Fano code can be used to obtain a (almost optimal) description [5]. In order to explain the code, let us prepare for a bad approximation of the state. Analogously to the definition of Berthiaume et al., the complexity of a state $|\varphi\rangle$ can grow at most linearly with the number of qubits whose state is described by $|\varphi\rangle$.

Another approach, which is related to Vitaly’s one has been introduced by Zurek, first, and later by Cavies and Schack [10, 11, 13, 21]. Since we will discuss this approach in more detail in the last section of this paper, we also want to briefly review it here. Suppose that on is given a set of possible states, $S = \{|\Psi_i\rangle\}$ and an associated probability distribution $P = \{p_i\}_i$, with $p_i = P(\{|\Psi_i\rangle\})$. In order to specify a particular element of this set, one just has to specify the index, which is a purely classical task. It is well known that the so-called Shannon-Fano code can be used to obtain a (almost optimal) description [3]. In order to explain the code, let us assume that $p_1 \geq p_2 \ldots$ and define $F_i = \sum_{k=1}^{i} p_k$. The codeword $c_i$ for index $i$ is $r(F_i)$, which is $F_i$ rounded off to $l_i = \lfloor \log_2(1/p_i) \rfloor$, i.e. the floor of $\log_2(1/p_i)$, bits. Thus, if the information of the set $S$, also called the background information, is given, then only $l_i$ bits are additionally required to describe the state $|\Psi_i\rangle$ [3]. Note that the logarithmic term in the definition before, can be understood in this way.

A completely different approach, based on the identification of a state with its abstract preparation procedure, was investigated in [13, 17]. In this case the Kolmogorov complexity of a state $|\varphi\rangle$ is defined as the amount of the classical information required to describe a circuit that prepares $|\varphi\rangle$ with the required precision $\varepsilon$ from a fixed initial state, e.g. $|0\rangle^\otimes n$. To be more precise, first a complete and finite gate basis $B$ is fixed. Any state $|\varphi\rangle$ can then be prepared, with arbitrary finite precision, $\varepsilon$, by a circuit, $C^{B,\varepsilon}(|\varphi\rangle)$, built with gates from such a basis [33]. Furthermore, as the basis is finite, it is possible to define a finite code and to encode any circuit in a classical sequence. Thus it is possible to associate to each state $|\varphi\rangle$ a set of strings $\omega^{B,\varepsilon}(|\varphi\rangle)$, each of which encodes a circuit that prepares $|\varphi\rangle$ with the required accuracy. The Kolmogorov complexity of $|\varphi\rangle$ is then defined as the minimum of the classical Kolmogorov complexities of the strings $\omega(C^{B,\varepsilon}(|\varphi\rangle))$

$$K_{Net}^{B,\varepsilon}(|\varphi\rangle) = \min_{\omega(C^{B,\varepsilon}(|\varphi\rangle))} K(\omega(C^{B,\varepsilon}(|\varphi\rangle))).$$

Here, the minimum is taken over all circuits, $C^{B,\varepsilon}$, such that $|\langle\varphi|C^{B,\varepsilon}(|0\rangle^\otimes n)|^2 \geq 1 - \varepsilon$. Taking into account that one needs at most $2^n \log \frac{1}{\varepsilon}$ gates in order to prepare an $n$-qubit state, and that the length of $\omega^{B,\varepsilon}(|\varphi\rangle)$ is upper bounded by the number of gates, one obtains that $K_{Net}^{B,\varepsilon}(|\varphi\rangle)$ is upper bounded by $2^n \log \frac{1}{\varepsilon}$, up to terms of smaller order. From now on we will refer to this definition as “network” complexity.

Assuming that we restrict the choice of the set of basis gates to 1- and 2-qubit gates (differing from the original definition presented in [13]), one can show that the network complexity is independent (up to a constant) of the actual choice of basis. Keeping this in mind, we will thus simply write $K_{Net}^{C}(|\varphi\rangle)$.

It should be noted here that the network complexity has not only a clear physical meaning, since it gives a measure of how hard it is to describe a way to prepare the state, but furthermore it was also shown that there exists a connection between this definition of Kolmogorov complexity and entanglement [15]. In order for a state to have maximal complexity (that is, exponential in the number of qubits) it is in fact necessary that it is highly entangled: more precisely, a complex state must have maximum Schmidt measure [21]. On the contrary, a product state has complexity at most linear in the number of qubits.

Another possibility to describe a pure state classically is simply to describe the coefficients of the state in a certain basis, e.g. the computational basis. We will refer to this complexity as computational-basis-expansion complexity, or, more briefly, CBE complexity. The CBE complexity of a $n$-qubit state $|\Psi\rangle = \sum_{i_1,...,i_n} c_{i_1,...,i_n} |i_1,...,i_n\rangle$, with $i_j \in \{0,1\}$ and complex coefficients $c_{i_1,...,i_n}$ is then

$$K_{CBE}^{C}(|\Psi\rangle) = K_{Cl}(|c_0,\ldots,0,\ldots,c_1,\ldots,\ldots,1\rangle).$$

In general, the coefficients $c_{i_1,...,i_n}$ are computed only with a finite precision $\varepsilon$, i.e. each of them is approximated by a number $\tilde{c}_{i_1,...,i_n}$ such that $|c_{i_1,...,i_n} - \tilde{c}_{i_1,...,i_n}| \leq \varepsilon$. This implies that the state that has been described in this way, i.e. $|\Psi\rangle = \sum_{i_1,...,i_n} \tilde{c}_{i_1,...,i_n} |i_1,...,i_n\rangle$, fulfills the inequality $||\Psi\rangle - |\Psi\rangle||^2 \leq 2\varepsilon^2$. Thus, $1 - ||\Psi\rangle||^2 \leq 2^{-n-\varepsilon^2}$, which implies that the precision $\varepsilon$ must scale like $1/2^n \varepsilon^2$, similarly as the precision with which one must describe each gate occurring in the definition of the network complexity.

Yet another possibility is to use a measurement and describe the state by the outcome probability of this measurement. One might for instance choose an “algorithmically simple” measurement, i.e. a POVM that can be described with only a few bits, and describe the state by the outcome probability of this measurement.

As we have already mentioned, in classical information theory the different definitions of complexity are provably equivalent, while this does not hold in quantum theory. Indeed some of the definitions described here have qualitatively different behaviors, e.g. the scaling with $n$. In this paper we aim to shed new light on the notion of quantum Kolmogorov complexity and to better understand its physical meaning.
One of the main drawbacks of Kolmogorov complexity is the fact that it is not computable. Despite of this, though, it is possible to give upper bounds for it, and these are sufficient and instrumental for many applications. Furthermore, we recall that even in the case of computable quantities, such as, e.g., communication or computation complexities, one is often not interested in the exact value, since it might be very hard to compute. In these cases too, one is mainly concerned with the scaling of these quantities as a function of the input size, \( n \). Owing to this fact, the following notation is widely used. One says that a function, \( g(n) \), is \( \Omega(f(n)) \) if there exist two constants, \( c, n_0 \in \mathbb{N} \) such that \( g(n) \geq cf(n) \) for all \( n \geq n_0 \). That is, for sufficiently large \( n \), \( g \) is upper bounded by \( f \). Similarly one says that a function \( g(n) \) is \( \Omega(f(n)) \) if there exist two constants, \( c, n_0 \) such that \( g(n) \geq cf(n) \) for all \( n \geq n_0 \). If the behavior of two functions, \( g, f \), is asymptotically the same, up to a constant factor, i.e. if \( g \) is both \( \Omega(f(n)) \) and \( \Omega(f(n)) \) then we say that \( g \) is \( \Theta(f(n)) \).

From now on, we will call a bit string (a state) algorithmically simple when its Kolmogorov complexity is at most \( \mathcal{O}(\log(n)) \) (at most \( \mathcal{O}(n) \)).

### B. Communication complexity

The theory of communication complexity deals with the following type of problem. Two distant parties, Alice and Bob, hold each an input, \( x \) and \( y \) respectively (usually one has \( x, y \in \{0,1\}^n \)). The common goal of the parties is to compute the value \( f(x,y) \) for a given function \( f \) known to both parties. The communication complexity of a function \( f \), \( C_{cl}(f) \), is defined as the minimum number of bits that Alice and Bob need to communicate in order to evaluate \( f(x,y) \).

In the quantum setting, the communication between Alice and Bob takes place through a quantum channel. Analogous to the classical case, the quantum communication complexity of \( f \), \( C_Q(f) \), is the minimum number of qubits that Alice and Bob need to communicate to evaluate \( f(x,y) \).

A slightly different communication model is the so-called simultaneous message passing model, first introduced by Yao [22]. In this model there is a third party—the referee—in addition to Alice and Bob. As before, the common aim of the parties is to compute \( f(x,y) \), where \( f \) is a given function and \( x \) and \( y \) are the input strings held by Alice and Bob respectively. In this case, however, no direct communication between Alice and Bob is allowed. They can only send messages to the referee, who will then compute \( f(x,y) \) based on the information he has received.

There are many cases in which one does not require the function \( f \) to be computed with certainty, but allows a certain failure probability, \( \delta \). The conditions on this error, as well as other factors, distinguish different scenarios in which communication complexity problems are usually solved (see e.g. [23] for an overview). We will focus here on the so-called worst case scenario. In this case, one is interested in the amount of communication needed to compute \( f(x,y) \) correctly with probability larger or equal to \( 1 - \delta \), for any \( x \) and \( y \).

Of course, any function \( f \) can be exactly evaluated if both Alice and Bob send their full input to the referee, who can then compute \( f(x,y) \). It follows thus that \( 2n \) is an upper bound for any communication problem in the simultaneous message passing model [24]. To improve such a protocol, one might consider the case in which Alice and Bob send to the referee a compressed version of their input. In this case, the communication required depends on the Kolmogorov complexity of the two inputs. If one considers the worst-case scenario and does not allow any error, this does not help to improve the upper bound. By considering the case in which the inputs are incompressible \( n \)-bit strings, in fact, it is easy to see that \( 2n \) bits of communication are still needed [25]. However, if a certain failure probability is tolerated, one can show that only \( \mathcal{O}(\sqrt{n}) \) bits of communication are required (see below).

Even though Holevo’s Theorem states that by sending \( m \) qubits one cannot convey more than \( m \) classical bits of information, it has been shown that there are cases in which the quantum setting is significantly more powerful than the classical one (see e.g. [26] for an overview). In fact, there are functions for which the classical communication complexity is exponentially larger than the quantum one. One such instance, which we will consider in the following, is the equality problem. This is a worst–case simultaneous message passing – model where the function to be evaluated is

\[ EQ_n(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases} \]

with \( x, y \in \{0,1\}^n \).

It has been proved [24, 25] that, allowing a small probability of failure, only \( \mathcal{O}(\sqrt{n}) \) bits of classical communication are needed to compute \( EQ_n(x,y) \). Furthermore, this amount of communication has been shown to be both necessary and sufficient for the solution of the equality problem.

Let us now briefly recall one optimal classical protocol here. It is known that for every \( n \in \mathbb{N} \) and for a fixed \( c > 1 \), and \( 0 < \Delta < 1 \), there exists an error correcting code \( E : \{0,1\}^n \to \{0,1\}^m \), with \( m = cn \), such that the distance between code words \( E(x) \) and \( E(y) \) is at least \( (1 - \Delta)m \) [27]. Alice and Bob apply a given error correcting code (where the specification of the code is independent of \( n \)) to \( x, y \) respectively. Then they choose randomly an index \( i, j \in \{1, \ldots, m\} \), respectively. Alice (Bob) sends \( i \) and the \( i \)-th bit value of \( E(x) \), i.e. \( E_i(x) \), \((j, \text{and } E_j(y))\) to the referee. If \( i \neq j \) the referee cannot conclude anything and tells Alice and Bob to restart the protocol. However, if \( i = j \), the referee checks whether \( E_i(x) \) equals \( E_j(y) \). If this is the case
he concludes that \( x = y \) else he concludes \( x \neq y \). The probability of having a collision, i.e. that the random indices \( i, j \) coincide, is \( 1/m \). The probability of making an error is \( P_{\text{error}} = P(E_i(x) = E_i(y) \mid x \neq y) / m < \Delta \).

Thus, in order to make the error probability independent of \( m \) (i.e. independent of \( n \)), Alice and Bob have to send \( \mathcal{O}(\sqrt{n}) \) bits. Note that the error can be reduced to any \( \delta > |\mathbb{P} - \mathbb{P}| \).

The simultaneous message passing protocol for the Shannon-Kolmogorov complexity and communication complexity. To show now that this lower bound is exponential in the number of qubits.

Let \( K^\varepsilon_Q \) be an arbitrary definition of the quantum Kolmogorov complexity with the following physical meaning: given \( K^\varepsilon_Q(\varphi) \) classical bits one can prepare the state \( |\varphi\rangle \) with precision \( \varepsilon \). Now and in the following, we say that \( K^\varepsilon_Q(\varphi) \) bits of information allow to prepare \( |\varphi\rangle \) with precision \( \varepsilon \) if it contains all the information that is needed to prepare a state \( \varepsilon \)-close to \( |\varphi\rangle \) with a perfect apparatus. Or, equivalently, given \( K^\varepsilon_Q(\varphi) \) bits of information, we can write down a state, \( \varepsilon \)-close to \( |\varphi\rangle \). At the same time we imply that such a task is impossible with a smaller amount of information. We show now that for any such definition there exists a \( n \)-qubit state \( |\varphi\rangle \) such that \( K^\varepsilon_Q(\varphi) \) is exponential in \( n \), ruling out some of the definitions reviewed in Section II.

Let \( K^\varepsilon_Q(\langle x \rangle) \) and \( K^\varepsilon_Q(\langle y \rangle) \) be the Kolmogorov complexities of the two fingerprints used in the protocol. Then a possible classical communication protocol is the following:

1. Given \( x \) and \( y \), respectively, Alice and Bob send the description of the (log \( m + 1 \))-qubit states \( |x\rangle \) and \( |y\rangle \) to the referee, requiring \( K^\varepsilon_Q(\langle x \rangle) + K^\varepsilon_Q(\langle y \rangle) \) bits of communication.

2. In order to distinguish between the case where \( |x\rangle = |y\rangle \) or \( |x\rangle \neq |y\rangle \), the referee could e.g. simulate (on a classical computer) the circuit described in the quantum fingerprinting protocol.

With this test, the referee can determine the value of \( E_Q^\varepsilon(\langle x, y \rangle) \) with error probability \( (1 + \Delta^2)/2 \). Such a probability can be reduced to any \( \delta > 0 \) by considering the fingerprint \( |\varphi\rangle \), with \( k = \mathcal{O}(\log_2 \frac{1}{\delta}) \). In this case the length of each fingerprint is \( \mathcal{O}(\log_2 n \log_2 \frac{1}{\delta}) \).

### III. RELATION BETWEEN CLASSICAL COMMUNICATION COMPLEXITY AND QUANTUM KOLMOGOROV COMPLEXITY

In this section we establish a relation between Kolmogorov complexity and communication complexity. To this end, we consider a classical protocol that simulates the quantum fingerprinting protocol. That is, instead of sending quantum systems (prepared in the states \( |\varphi\rangle \), Alice and Bob send to the referee a classical description of these states. Since there exists a lower bound on the classical communication complexity for this problem, this consideration leads to a lower bound to the number of bits required to describe those states, i.e. to their quantum Kolmogorov complexity. We show now that this lower bound is exponential in the number of qubits.

Let \( K^\varepsilon_Q \) be an arbitrary definition of the quantum Kolmogorov complexity with the following physical meaning: given \( K^\varepsilon_Q(\varphi) \) classical bits one can prepare the state \( |\varphi\rangle \) with precision \( \varepsilon \). Now and in the following, we say that \( K^\varepsilon_Q(\varphi) \) bits of information allow to prepare \( |\varphi\rangle \) with precision \( \varepsilon \) if it contains all the information that is needed to prepare a state \( \varepsilon \)-close to \( |\varphi\rangle \) with a perfect apparatus. Or, equivalently, given \( K^\varepsilon_Q(\varphi) \) bits of information, we can write down a state, \( \varepsilon \)-close to \( |\varphi\rangle \). At the same time we imply that such a task is impossible with a smaller amount of information. We show now that for any such definition there exists a \( n \)-qubit state \( |\varphi\rangle \) such that \( K^\varepsilon_Q(\varphi) \) is exponential in \( n \), ruling out some of the definitions reviewed in Section II.

Let \( K^\varepsilon_Q(\langle x \rangle) \) and \( K^\varepsilon_Q(\langle y \rangle) \) be the Kolmogorov complexities of the two fingerprints used in the protocol. Then a possible classical communication protocol is the following:

1. Given \( x \) and \( y \), respectively, Alice and Bob send the description of the (log \( m + 1 \))-qubit states \( |x\rangle \) and \( |y\rangle \) to the referee, requiring \( K^\varepsilon_Q(\langle x \rangle) + K^\varepsilon_Q(\langle y \rangle) \) bits of communication.

2. In order to distinguish between the case where \( |x\rangle = |y\rangle \) or \( |x\rangle \neq |y\rangle \), the referee could e.g. simulate (on a classical computer) the circuit described in the quantum fingerprinting protocol.

The precision with which the referee is able to simulate the quantum circuit, and thus the probability with which the two quantum states are described, is \( \mathcal{O}(\sqrt{n}) \). The reduction of the number of classical bits required to prepare a state \( |\varphi\rangle \) with precision \( \varepsilon \) is exponential in \( n \).

The number of bits of communication of this protocol is \( K^\varepsilon_Q(\langle x \rangle) + K^\varepsilon_Q(\langle y \rangle) \). As we know that the optimal bound for the classical communication complexity of the equality problem, in the simultaneous passing model, is \( \sqrt{n} \), it follows that there must be at least a case for which \( K^\varepsilon_Q(\langle x \rangle) + K^\varepsilon_Q(\langle y \rangle) \geq \mathcal{O}(\sqrt{n}) \).

Considering that \( m = nc \), it follows that there must exist an \( M \)-qubit state \( |\Psi_M^\text{Compl} \rangle \), with \( M = \log_2 m \) such that \( K^\varepsilon_Q(\Psi_M^\text{Compl}) \geq \mathcal{O}(\sqrt{M^2/m}) \). This shows that any definition of the quantum Kolmogorov complexity that counts the number of classical bits required to prepare a state must grow exponentially with the number of qubits.

One of the definitions presented in Sec II that fulfills the necessary requirement of growing exponentially with the number of qubits is the network complexity. Note that also the CBE complexity fulfills this condition.
IV. IMPLICATIONS FOR CLASSICAL AND QUANTUM KOLMOGOROV COMPLEXITY

The considerations of the previous section allow us to derive some implications regarding the Kolmogorov complexity. The result, that the complexity of a pure \( n \)-qubit state can be \( 2^n \), holds for any definition of the quantum Kolmogorov complexity that measures the amount of classical bits required to prepare the state. Since a state can also be described by a outcome-probability distribution of local measurements, this bound leads also to a bound on a classical Kolmogorov complexity of a probability distribution. This consideration then guides us how to define the quantum Kolmogorov complexity for mixed state. Furthermore, we show that the Kolmogorov complexity of the states \(|h_\Psi\rangle\) is essentially equal to the classical Kolmogorov complexity of the classical bit string \( x \).

A. Classical Kolmogorov complexity of a probability distribution and quantum Kolmogorov complexity of mixed states

One possible description of a state is a complete measurement and the probability distribution of the outcomes of this measurement. Let us choose the measurement itself to be algorithmically simple, e.g. a local measurement. Thus, the POVM we consider consists of at most \( 2^{2n} \) elements, which can be easily described (the length of the description depends at most logarithmically on \( n \)). Since we consider a complete measurement, the complexity of the quantum state is upper bounded by the classical Kolmogorov complexity of its outcome probabilities, \( P_\Psi \), i.e. \( K^c_Q(|\Psi\rangle) \leq K(c)(P_\Psi) \), for \( \varepsilon \to 0 \). Note that the length of \( P_\Psi \) is upper bounded by \( 2^{2n} \). As we know that there exists a complex pure state, \(|\Psi\rangle\), this implies that there exists a complex probability distribution \( P_\Psi \), with \( \Omega(2^{2n}) \leq K(c)(P_\Psi) \leq \mathcal{O}(4^n) \). Furthermore, any probability distribution describing a complex state fulfills these inequalities.

We now find some constraints on the quantum Kolmogorov complexity of mixed states. Let us consider the pure state \(|\Psi\rangle = \sum_i \sqrt{p_i}|i, i\rangle\), with \( p_i \geq 0 \) and \( \sum_i p_i = 1 \) and where \(|i\rangle = |i_1, \ldots, i_n\rangle\) with \( i_j \in \{0, 1\} \) denotes the computational basis. This state is a so-called purification of the mixed state \( \rho = \sum_i p_i|i\rangle\langle i|\). \( \rho \) can be easily computed from \(|\Psi\rangle\) and vice versa, requiring only the additional information: ”\(|\Psi\rangle\) is a purification of \( \rho \), where the auxiliary system comprises the second set of qubits. Furthermore the Schmidt-basis is the same for both subsystems.” Thus, the quantum Kolmogorov complexity of these two states must be the same. One possible description of \( \rho \) (or \(|\Psi\rangle\)) is a description of the vector \( P = (p_1, \ldots, p_{2^n}) \), containing all the eigenvalues of \( \rho \) plus the information that \( \rho \) is diagonal in the computational basis. This implies the following relation between the Kolmogorov complexities:

\[
K^c_Q(|\Psi\rangle) = K^c_Q(\rho) \leq K(c)(P), \text{ for } \varepsilon \to 0
\]

where \( K^c_Q \) is any definition of the quantum Kolmogorov complexity which allows us to describe the states in such a way that they can be prepared.

Obviously, an arbitrary purification of a mixed state \( \rho \), plus the additional information of which systems are auxiliary systems allows one to describe \( \rho \). Therefore, for a general definition of the Kolmogorov complexity it must hold that

\[
K^c_Q(\rho) \leq \min_{|\psi\rangle} \{ K^c_Q(|\psi\rangle_{AR}) | Tr_R(|\psi\rangle_{AR}) = \rho \}.
\]

On the other hand, it is clear that there exists at least one purification of the state \( \rho \), whose complexity is the same as the one of \( \rho \), namely the state whose Schmidt-basis is the same for both subsystems and equals the eigenbasis of \( \rho \). These considerations strongly suggest that the generalization of the Kolmogorov complexity to mixed states should be

\[
K^c_Q(\rho) := \min_{|\psi\rangle} \{ K^c_Q(|\psi\rangle_{AR}) | Tr_R(|\psi\rangle_{AR}) = \rho \}.
\]

Let us show now that the precision with which the mixed state is described is exactly the same as the one with which its purification is described. If \( K^c_Q(|\psi\rangle_{AR}) \) bits describe the state \(|\psi\rangle_{AR}\), then \( \langle\psi|\psi\rangle^2 \geq 1 - \varepsilon \). Let us now use Ulhmann’s theorem, which states that \( F(\rho, \sigma) = \text{tr}[(\sqrt{\sigma} \sqrt{\rho})^{1/2}] = \max_{|\psi\rangle} |\langle \psi | \sigma | \psi \rangle| \), where the maximization is over all purifications \( |\psi\rangle \) of \( \rho \) and \( |\varphi\rangle \) of \( \sigma \). Therefore the reduced states \( \bar{\rho} = Tr_R(|\psi\rangle_{AR}\langle\psi|_{AR}) \) and \( \bar{\sigma} = Tr_R(|\varphi\rangle_{AR}\langle\varphi|_{AR}) \) fulfills the inequality \( F(\rho, \bar{\bar{\rho}})^2 \geq |\langle \psi | \bar{\bar{\rho}} | \varphi \rangle|^2 = 1 - \varepsilon \). Thus, if the description of the purification of the mixed state leads to a state which is \( \varepsilon \) close to the actual purification, then the corresponding reduced states are also \( \varepsilon \) close.

As an example we can consider the (very simple) case of the maximally mixed state of \( n \) qubits \( I_A \). We will choose the network complexity here. In this case it is easy to see that its complexity is at most logarithmic in the number of qubits. The \( 2n \)-qubit state \( (|00\rangle + |11\rangle)^{\otimes n} \) is a purification of the maximally mixed state on \( A \). To prepare such a state it is sufficient to divide the \( 2n \) qubits into couples, apply to the first qubit of each couple a Hadamard gate and then apply a C-NOT gate between the two. The description of the circuit is extremely easy, and depends on \( n \) only in the specification of the total number of qubits required. We have thus

\[
K^c_{\text{Net}}(\frac{I}{n}) \leq \log n + 2 \log \frac{1}{\varepsilon}.
\]

The term in \( \varepsilon \) is separated from that in \( n \) as there are only two different types of gates required.
B. Quantum complex states from algorithmically complex strings

From the analysis of the fingerprinting protocol, we have seen that there must exist a complex (and therefore highly entangled) state of the form $|h_x\rangle = \sum_{i=1}^m |i\rangle|E_i(x)\rangle$, where $x \in \{0,1\}^n$ and $E_i(x)$ is the $i$-th bit of the encoding $E(x) \in \{0,1\}^m$ of $x \in \{0,1\}^n$ using code $E$ (of the type described in section II.B). While it is not necessarily true in general, there are cases in which the codes $E$ have the additional property of being algorithmically simple, i.e. such that $K_{CH}(E) = \mathcal{O}(\log(n))$. This is true, e.g., for Justesen codes \[28\]. When such additional condition is satisfied, it is possible to prove an important relationship between the classical Kolmogorov complexities of $x$, $E(x)$, and the quantum (network) complexity of the states $|h_x\rangle$. More precisely, one finds that $K_{CH}(x) = \Theta(K_{CH}(E(x))) = \Theta(K_{Net}(|h_x\rangle))$. The proof of this statement consists of two parts. In the first part (Observation \[1\]) we show that the Kolmogorov complexity of the encoded word $E(x)$ coincides (up to a constant) with that of $x$. In the second part (Proposition \[2\]) we prove that the network complexity of the quantum state $|h_x\rangle$ is essentially the same as the classical Kolmogorov complexity of $E(x)$.

**Observation 1.** If $E$ is an algorithmically simple code, and $E(x) \in \{0,1\}^m$ is the encoding of $x \in \{0,1\}^n$, then $K_{CH}(E(x)) = \Theta(K_{CH}(x))$.

**Proof.** To prove the statement, we show that $K_{CH}(E(x)) = \mathcal{O}(K_{CH}(x))$ and $K_{CH}(x) = \mathcal{O}(K_{CH}(E(x)))$. The first of the two equalities follows immediately by the fact that the code $E$ is algorithmically simple. To describe $E(x)$, in fact, it is sufficient to give the program for $E$, and a description of $x$. Thus, $K_{CH}(E(x)) \leq K_{CH}(E) + K_{CH}(x) = \mathcal{O}(K_{CH}(x))$. In order to prove the reverse statement, we consider that $x$ can be described as the string whose encoding is $E(x)$. Thus it is possible to prepare $x$ by allowing a computer to generate all possible strings $E(y)$ for $y \in \{0,1\}^n$ and compare each of them with $E(x)$, halting when $E(y) = E(x)$, i.e., when $y = x$. This implies that $K_{CH}(x) \leq K_{CH}(E(x)) + K_{CH}(E) = \mathcal{O}(K_{CH}(E(x)))$. \[■\]

**Proposition 1.** Let $E$ be an algorithmically simple error-correcting code, such that $d(E(x), E(y)) \geq (1 - \Delta)m$, where $E(x), E(y) \in \{0,1\}^m$ are the encodings of $x, y \in \{0,1\}^n$ respectively. Then, the $(\log(m + 1)$-qubit state $|h_x\rangle = \sum_{i=1}^m |i\rangle|E_i(x)\rangle$, where $|ij\rangle = \delta_{ij}$, has complexity $K_{Net}(|h_x\rangle) = \Theta(K_{CH}(E(x)))$ for sufficiently small $\varepsilon$.

**Proof.** In order to prove the statement we will show that $K_{Net}(|h_x\rangle) = \mathcal{O}(K_{CH}(E(x)))$ and $K_{CH}(E(x)) = \mathcal{O}(K_{Net}(|h_x\rangle))$. Let us first prove that $K_{Net}(|h_x\rangle) = \mathcal{O}(K_{CH}(E(x)))$. Consider the following circuit, that acts on the initial state $|0\rangle^\otimes(m+1)$. Initially, a Hadamard gate is applied to each of the first log $m$ qubits, generating the state $((|0\rangle + |1\rangle)/2)^\otimes(m+1) \otimes |0\rangle$. Then, one applies the unitary $E_i$ (such that $E_i[0] = [E_i(x)]$) to the last qubit, conditional on the state of the first $m$ qubits being $|i\rangle$, i.e. the total operation being $|i\rangle|0\rangle \rightarrow |i\rangle|E_i(x)\rangle$. This operation can be performed by a circuit composed of single-qubit and CNOT gates, whose description requires at most $\mathcal{O}(\log(m)\log(m))$ bits \[39\]. The state generated by this circuit is $|h_x\rangle$. The complexity of the state $|h_x\rangle$ is of course bounded by the complexity of the circuit just described.

$$K_{Net}(|h_x\rangle) \leq 2\log\frac{1}{\varepsilon} + (\log m)^2 \log\frac{1}{\varepsilon} + K_{CH}(E(x)) = \mathcal{O}(K_{CH}(E(x))),$$

(12)

when regarded as a function of $m$. The first term takes into account the possibility that the Hadamard and the NOT gates might not be included in the gate basis $B$. Note that this contribution is constant in $m$. The second term takes into account the description of the subcircuit in which each gate $E_i$ is applied conditional on the state of the first log $m$ qubits. The term $K_{CH}(E(x))$ is the one that specifies when $E_i = \mathbb{I}$ and when, instead, $E_i = NOT$. Note that this proof remains valid even if the code $E$ is not algorithmically simple. In the following we see that this property of the code is needed in order to prove the reverse statement.

We show now that $K_{CH}(E(x)) = \mathcal{O}(K_{Net}(|h_x\rangle))$. Let $\omega_h$ be the classical description of the circuit that prepares $|h_x\rangle$, such that $K_{CH}(\omega_h) = K_{Net}(|h_x\rangle)$, and consider the following program.

1. Read $\omega_h$.
2. Simulate the circuit and compute the approximate state $|h_x\rangle$.
3. For $i$ from 1 to $m$, compute $|i\rangle|h_x\rangle$.

This is a purely classical program that, simulating the (exact) creation of the quantum state $|h_x\rangle$ and a number of operations on it, allows to prepare the classical bit string $E(x)$. The length of this program can be estimated as follows. The first step requires $\mathcal{O}(K_{CH}(\omega_h)) + \log K_{CH}(\omega_h)$ bits; the second contributes a quantity that is constant in $m$ and depends only on the precision with which we require the simulation to be effected. The last step gives a contribution that is $\mathcal{O}(\log m)$. As this will not in general be an optimal program, its length is an upper bound to the complexity of $E(x)$.

$$K_{CH}(E(x)) \leq \mathcal{O}(K_{CH}(\omega_h)) + \mathcal{O}(\log m) + \mathcal{O}(1) = \mathcal{O}(K_{CH}(\omega_h)) + \mathcal{O}(K_{Net}(|h_x\rangle)).$$

(13)

In the analysis of the program given above, we have not yet taken into account the effect of the error $\varepsilon$ in the preparation of the quantum state $|h_x\rangle$. We do so now, assuming that the circuit described by $\omega_h$ does not prepare $|h_x\rangle$, but rather some state $|\tilde{h}_x\rangle$ such that...
\[ |\langle h_x | h_x \rangle|^2 \geq 1 - \varepsilon. \] Without loss of generality we write 
\[ |\langle h_x | h_x \rangle | = (1/\sqrt{m}) \sum_{i=1}^{m} a_{ij} |E_i(x)\rangle. \] As a consequence, 
The program described above will yield as final output 
a string \( \tilde{E}(x) \), with 
\[ \langle \tilde{E}(x), E(x) \rangle \geq m\sqrt{1 - \varepsilon}, \]
where \( \langle \tilde{E}(x), E(x) \rangle \) is the scalar product between the two \( m \)-dimensional vectors whose coefficients (in an orthonormal basis) are given by \( E(x) \) and \( \tilde{E}(x) \) respectively.

Since \( E \) is chosen to be algorithmically simple, it is possible to verify whether or not the final string \( \tilde{E}(x) \) is part of the code. If it is not so, then it follows immediately that there has been an error somewhere, which, if \( \varepsilon \) is sufficiently small (depending on the error correcting code) can be corrected. The only case in which the program described above cannot correct itself is when the output word \( \tilde{E}(x) \) is a codeword, but different from \( E(x) \). We show now that this event can be excluded by choosing \( \varepsilon \) small enough. For any two words \( F, G \) we have \( m - (F, G) \geq d(F, G) \). If \( F, G \) are codewords we also have \( d(F, G) \geq (1 - \Delta)m \), which implies then that \( (F, G) \leq m\Delta. \) As we have seen earlier, 
\[ \langle E(x), \tilde{E}(x) \rangle \geq m\sqrt{1 - \varepsilon}. \]
Thus, choosing \( \varepsilon \leq 1 - \Delta^2 \) ensures that \( \tilde{E}(x) \) cannot be a codeword.

This property has two main consequences. The first is that the existence of a complex state of the form \( |h_x \rangle \) follows immediately from the existence of a complex string. The second is the fact that there exists a “recipe” that allows us to prepare complex states, or would allow us to do so if we could have a classical complex string.

**Corollary 1.** For any \( m \geq 1 \) there exists a \((\log m + 1)\)-qubit complex state of the form \( |h_x \rangle = \sum_{i=1}^{m} |i\rangle E_i(x) \), where \( |i\rangle = \delta_{ij} \), \( x_i \in \{0,1\} \), and \( E \) is an error-correcting code of the type introduced above.

**V. APPLICATIONS FOR QUANTUM KOLMOGOROV COMPLEXITY**

In the following we apply the results on quantum Kolmogorov complexity to prove some statements from the theory of communication and computation complexity. While some of the results are known, we underline that the proofs are simpler than those found in literature. Finally we see how the definition of network complexity can be applied to some aspects in thermodynamics, allowing us to generalize existing results.

**A. Communication complexity**

Let us first show that the exponential upper bound on the Kolmogorov complexity of quantum states implies that any quantum communication protocol in the SMP model, requiring \( q \) qubits of communication, can be simulated using at most \( O(2^q) \) classical bits of communication. Such result was already known, but here we can clearly see how it follows from basic considerations related to the known properties of Kolmogorov complexity.

**Proposition 2.** For any function \( f : \{0,1\}^n \rightarrow \{0,1\} \), the gain in communication complexity that one can obtain by using quantum instead of classical communication, if only one-way communication is allowed, is at most exponential, i.e.

\[ C_{CL}(f) \leq O(2^{C_Q(f)}). \] (14)

**Proof.** The idea of the proof is to simulate the optimal quantum protocol classically. Consider an optimal quantum communication protocol to evaluate \( f(x, y) \), and suppose that a single state of \( q \) qubits is sent. Such a state can be described by a classical sequence of at most \( O(2^q) \) bits, and any operation on the quantum state can be simulated. In this case, thus, \( C_{CL}(f) \leq O(2^{C_Q(f)}) \). If instead of a single \( q \)-qubit state, \( k \) states of \( q \) qubits are sent (with \( kq = q' \)), they can be described by classical sequences of at most \( O(2^q) \) qubits each, thus \( C_{CL}(f) \leq O(k2^q) \leq O(2^q) = O(2^{C_Q(f)}) \).

Note that a similar argument can be used to consider the more general case where also two-way communication between Alice and Bob is allowed. Similarly as before, one only has to consider the classical simulation of a quantum protocol. However, here it could be that Alice does not send all the qubits to Bob which she uses to solve the problem.

An exponential quantum/classical gap in a quantum communication problem, is obtained only when in Eq. \( 13 \) we have an equality (an exponential gap, in fact, requires \( C_Q(f) \leq O(q) \)). This can be obtained only when at least one of the states that appear in the quantum protocol is complex (that is, its classical description grows exponentially with the number of qubits). This implies that the power of quantum communication can be explained (mainly) by quantum entanglement. To see this, let us use for instance the network complexity (Eq. \( 8 \)). In \( 15 \) it has been proved that the entanglement of a quantum state, in terms of its Schmidt measure, is an upper bound for its complexity. This implies that in all communication complexity problems where one sees an exponential classical/quantum gap, maximally entangled states must be created. This seems to imply that the source of the quantum advantage is in fact entanglement.

**B. Computation complexity**

Quantum Kolmogorov complexity can also be used as a powerful tool to gain insight on some issues related to computation complexity theory. We use it here to determine some properties of the states that appear during the computation.

Let us describe a quantum algorithm by a sequence of single- and two-qubit operations, \( A_{i_t,j_t} \), where \( i_t \) and \( j_t \)
denote the two qubits on which the operation is acting during the \( t \)-th time step (if \( i_t = i_t \), \( A_{i_t,j_t} \) denotes a single qubit operation acting on qubit \( i_t \)). In the \( t \)-th step of the algorithm, the state \( |\Phi_t\rangle \) is transformed to \( |\Phi_t\rangle' = A_{i_t,j_t} |\Phi_t\rangle \), with \( |\Phi_0\rangle = |0\rangle^\otimes n \). We have the following

**Observation 2.** If \( A = (A_{i_1,j_1}, A_{i_2,j_2}, \ldots, A_{i_T,j_T}) \) is a quantum algorithm with running time \( T = O(\text{poly}(n)) \), then all the states \( |\Phi_t\rangle = A_{i_t,j_t} |\Phi_{t-1}\rangle \) must have network Kolmogorov complexity that grows at most polynomially with \( n \).

**Proof.** Since the quantum circuit model is universal, any algorithm with running time \( T \) can be simulated efficiently by a circuit. In particular this implies that, for each step \( t \), there exists a quantum circuit that prepares \( |\Phi_t\rangle \) requiring at most \( O(\text{poly}(t)) \) gates. As the network complexity of a state is trivially bounded from above by the size of the circuit, that is, the number of gates required to prepare it, and \( t \leq T = O(\text{poly}(n)) \), it follows that, for all \( t \), \( K_{\text{Net}}(|\Phi_t\rangle) \leq O(\text{poly}(n)) \).

On the other hand, one can show that if the complexities of the states occurring in the algorithm are independent of \( n \), then the quantum algorithm cannot have an exponential speed-up compared to a classical algorithm. Similarly to the classical case, we define \( K_\text{QC}^t(\langle \Psi \rangle \mid n) \) to be the number of bits required to prepare the \( n \)-qubit state \( |\Psi\rangle \) given the number \( n \).

**Proposition 3.** Let \( A = (A_{i_1,j_1}, A_{i_2,j_2}, \ldots, A_{i_T,j_T}) \) be a quantum algorithm with running time \( T = O(\text{poly}(n)) \), and let \( |\Phi_t\rangle = A_{i_t,j_t} |\Phi_{t-1}\rangle \) denote the state at the \( t \)-th step of the computation. If for all \( t \) \( K_\text{Net}^t(|\Phi_t\rangle \mid n) \leq r_t \), where \( r_t \) is independent of \( n \), then \( A \) can be simulated classically with only a polynomial overhead of operations.

**Proof.** If \( K_{\text{Net}}^t(|\Phi_t\rangle \mid n) \leq r_t \), then \( |\Phi_t\rangle = M_t |0\rangle^\otimes n \), where the dimension of the operator \( M_t \) is some function of \( r_t \), \( f_t(r_t) \), but independent of \( n \). Thus, the number of operations required to compute \( |\Phi_t\rangle \) classically is independent of \( n \). Therefore, the classical algorithm: compute for each \( t \) the state \( |\Phi_t\rangle \) and print it solves the same problem as the quantum algorithm and requires only \( RT = R O(\text{poly}(n)) \) computational steps, where \( R \) is the maximum of all \( f_t(r_t) \).

In [29], it was proved that in order for a quantum algorithm to be exponentially faster than a classical one that solves the same task, it is necessary that at least one of the states \( |\Phi_t\rangle \) must describe an entangled state of \( l \) qubits, where \( l \) grows with the input size \( n \). Together with the results above, this means that, in order to find algorithms that actually allow an exponential quantum/classical gap, we must look for states that are both (a) highly entangled and (b) have a non-constant Kolmogorov complexity that grows at most polynomially with the input size. We underline the relevance of this consideration which implies that, while entanglement is undoubtedly a fundamental resource, it is not sufficient: only particular types of entanglement are, in fact, useful.

### C. Kolmogorov complexity and thermodynamics

As a last application of the notion of quantum Kolmogorov complexity investigated in this paper let us reconsider some aspects of thermodynamics [10]. First of all we recall how the Kolmogorov complexity became important in the context of thermodynamics and review some results presented by Caves [11] in this context. Then we show how these results can be further developed by taking the notion of quantum Kolmogorov complexity into account.

The second law of thermodynamics states that the entropy of a closed system, defined as the logarithm of the number of microstates populated by a system, can never decrease. In his famous argument, Maxwell introduced a fictitious machine where this law seemed to be violated. This machine consists of a gas cylinder initially at equilibrium, and a hypothetical demon which could open and close a shutter between two parts of the cylinder. If the demon was able to separate the fast molecules from the slow ones, the entropy of the system could decrease, thus violating the second law of thermodynamics. The resolution to this paradox lies in the fact that the demon must measure the velocity of the molecules and, more importantly, that he must store this information in some memory. As this memory has to be finite, the demon must erase the measurement outcomes at some point in order to have space to store the new ones. It could be shown from a physical point of view that this erasure increases the entropy of the total system, ensuring that the second law of thermodynamics is valid.

Landauer formalized this fact in his famous principle, which states that the process of erasing \( n \) bits of information from a register increases the entropy of the environment by \( \log_2 n \) [31]. Thus, sequences with large Kolmogorov complexity cannot be erased, except with an irreversible process, which can provide the required energy. Somehow inverse to the process of erasure is the process of *randomization*. Whenever a computer makes a randomizing step (such as, e.g., tossing a coin) it is possible to increase the information contained in the computer register and thus, if used appropriately, a similar procedure can be used to lower the entropy of the environment.

Bennett [8] and later Zurek [11] showed that the Kolmogorov complexity of the microstates of a system is a fundamental quantity in the thermodynamics analysis of such problems. If the demon finds out a way to compress the bit string he wants to erase, he might use a much lower thermodynamic cost for the erasure. This deep connection between thermodynamic entropy and Kolmogorov complexity has been formalized by Zurek in [12], who has defined the total entropy \( \mathcal{S} \) of a system as the sum of the thermodynamic entropy and the Kolmogorov complexity.
Caves continued the investigation of the Maxwell’s demon in the quantum setting \[10,11\]. He studied the change of the total entropy, defined by Zurek as \( S = S + I \), where \( S \) denotes the statistical entropy and \( I \) the algorithmic information. Here, the algorithmic information is composed of two parts, the background information, which is sufficient to generate a list of all states (up to a certain precision), and the additional information, needed to describe a particular element of this list of states (see Section II). In his considerations, Caves chooses the background information to be negligible, implying that only the conditional algorithmic information contributes to the total entropy.

In order to demonstrate that it is indeed possible to conceive situations in which an intelligent demon could extract work from a system (i.e. increase the system’s total entropy) Caves considered the following situation. A demon, whose system is prepared in some standard product state denoted by \( |s\rangle = |s, \ldots, s\rangle \), is acting on a single photon, which is initially prepared in equilibrium, \( \rho_s = 1/2 \). The total entropy of the initial state is therefore \( S_\text{in} = S(\rho_s) + S(|s\rangle |s\rangle) = S(\rho_s) + I(|s\rangle |s\rangle) = 1 \) \[11\]. The main idea to increase the total entropy of the closed system is to project the state of the photon onto an algorithmically complex state. To this aim the demon randomizes his register (apart from one bit) to obtain a complex string, \( r \), of \( m \) bits. He associates to the bit string \( r \) the angle \( \theta \), where 0.0r is the binary representation (first \( m \) bits) of \( \theta/\pi \). The precision with which he can choose one of the \( 2^m \) angles, \( \{\theta_k = k\pi/2^m\}_{k=0}^{2^m-1} \) is therefore \( \varepsilon_\theta = 2^{-m}\pi \). For any choice of \( k \) (that is, \( r \)) he sets a polarizing beam splitter to measure the polarization of the photon in the basis \( \{ |\Psi_{\theta_k}\rangle = \cos(\theta_k) |0\rangle + \sin(\theta_k) |1\rangle, |\Psi_{\theta_k}\rangle = \sin(\theta_k) |0\rangle - \cos(\theta_k) |1\rangle \} \). Thus, the state describing the system and the demon is \( \rho = \sum_k (\rho_{\theta_k} \otimes |0\rangle \langle 0|) \otimes |\Psi_{\theta_k}\rangle \langle \Psi_{\theta_k}| + \rho_{\theta_k} \otimes |1\rangle \langle 1|) \langle 1 \rangle \langle 0| \otimes |\Psi_{\theta_k}\rangle \langle \Psi_{\theta_k}| \), where we have introduced an auxiliary system, \( s \) which keeps track of information about which arm the photon left the beam splitter and is available to the system. The auxiliary system \( d \) keeps track of all the possible measurement choices. Here, the states \( |e_k\rangle \) denote an orthonormal basis, and \( \rho_{\theta_k} = \langle \Psi_{\theta_k}| \rho_S |\Psi_{\theta_k}\rangle \) and \( \hat{\rho}_{\theta_k} = \langle \Psi_{\theta_k}| \rho_D |\Psi_{\theta_k}\rangle \) are not normalized. Thus, the final state describing the system is given by the reduced state of \( \rho \), i.e. by tracing over system \( D, d \). Observing the measurement outcome the demon stores this additional bit of information and thus holds the bit string 0.0r (or 0.1r) respectively. From the demons point of view, the system is after the measurement in a pure state, \( |\Psi_{\theta_k}\rangle \), or \( |\Psi_{\theta_k}\rangle \). The total entropy of the whole system (w.l.o.g. we assume that 0.0r has been measured) is therefore \( S_\text{fin} = S(|\Psi_{\theta_k}\rangle) + I(\theta_k = 0.0r) = I(|0.0r\rangle) = m + 1 \). Therefore, the total entropy is increased by \( \Delta S = m \), which implies that the work \( W^{(+)} = mk_B T \ln 2 \) has been extracted.

An important point in these considerations is the presence of the background information, which is the information required to fully describe the system \[20\]. In the example considered above, the background information is the list of states \( S = \{ \cos(\theta_k)|0\rangle + \sin(\theta_k)|1\rangle \mid \theta_k = k\pi/2^m \}_{k=0}^{2^m-1} \) together with the uniform probability distribution \[42\].

In the following we shall choose a different point of view, for two main reasons. First, while it is clear that a state can always be described with the background information and the conditional information, this might not always be the most convenient description. Second, we see that with this approach it is very hard to go from one system to composite systems. Consider the example studied above for the case where the demon tries to measure \( n \) photons. A possible generalization of the background information in the case of \( n \) systems is the set \( S^\otimes n \). In this case one would still be allowed to consider the background information negligible. At the same time, though, the demon does not have the possibility to choose an arbitrary measurement direction with arbitrary good precision, but is instead forced to project the global system onto product states. The only situation in which the demon could measure along arbitrary directions would be if the set of its available states increased with the dimension of the system. In fact, as seen before, only if his background information increased exponentially with the number of qubits the demon could indeed have the freedom to measure an arbitrary state. In this case it seems to be impossible to neglect the contribution due to the background information.

If the demon can only measure each single photon independently from all the others, the increase in total entropy is simply \( n \) times the one that we have in the case of the single photon, i.e. \( \Delta S = mn \). However, if the demon projects the system onto maximally complex (and thus necessarily fully entangled) states the increase of the total entropy can be exponentially larger. The maximum possible increase in the total entropy is given by \( \Delta S^{(\text{Max})} = I_\text{fin} - NH_\text{in} = 2^n \log(1/\varepsilon) - n \), where \( \varepsilon \) gives the precision with which the demon prepares its states.

Note that for a general definition of the quantum Kolmogorov complexity the background information is implicitly accounted for. This is why then it is no problem to consider the projection onto complex states. For instance using the network definition, the background information is a basis consisting of single-qubit and two-qubit gates, together with an initial tensor-product state. In this case the background information remains the same, and therefore negligible, when increasing the number of considered qubits. In the case of the CBE complexity, instead, the background information consists with the specification of the computational basis: which is indeed algorithmically simple. This is why e.g. the network, or the CBE, complexity of a state \( |\Psi\rangle \) is indeed the number of bits required to describe the state and no additional information is required.
VI. CONCLUSIONS

In this paper we investigated the generalization of the concept of Kolmogorov complexity to the quantum case. We have shown how this quantity can be related to other fundamental concepts of quantum information theory, such as communication and computation complexity. More precisely by analyzing a particular protocol of communication complexity we found a condition that allows us to discriminate among the different proposals for quantum Kolmogorov complexity. In particular we have proven that any definition of quantum Kolmogorov complexity that measures, in bits, the amount of information needed to prepare a \( n \)-qubit state must scale exponentially with \( n \). We have also shown how some of the existing definitions indeed satisfy such a condition. Furthermore, it has been possible to use properties of the quantum Kolmogorov complexity to prove statements about communication and computation complexity. While some of the results we have obtained were already known, the proofs based on quantum Kolmogorov complexity are extremely simple, and encourage us to look for other possible applications to related fields (such as query complexity and oracle models for computation). Referring to the concept of total entropy, first introduced by Zurek [9], and in particular to works by Caves [10, 11], we continued the study of the relation between quantum complexity and thermodynamics. In particular, by applying the concept of quantum Kolmogorov complexity, we were able to extend their analysis to a full quantum setting, including a first study of the effects of entanglement.

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[32] Analogously, any definition of the quantum Kolmogorov complexity of an \( n \)-qubit state will have a log \( n \) term that
specifies the number of qubits of the states and that can be removed by considering the conditional complexity.

[33] Note that this is usually the approach taken in classical theory, where the background information of $S$ is always small.

[34] As in [17], here and in the following we will say that circuit $C^{B,\varepsilon}_i$ prepares $|\varphi\rangle$ with precision (or accuracy) $\varepsilon$ if 
$$
|\langle \varphi | C^{B,\varepsilon}_i (|0\rangle \otimes \mathbb{1}^n) |^2 \geq 1 - \varepsilon
$$

[35] Note that the same argument holds when direct communication between Alice and Bob is allowed. In this case, though, the bound is $n$, and not $2n$, since only one of the parties need to communicate his/her input.

[36] Such a compression, though, might be useful if one is interested in other scenarios, or if the domain of $x$ and $y$ is restricted in some way.

[37] An example, for instance, is given by Justesen codes. In this case one may choose any $c > 2$ and $\Delta < 9/10 + 1/(15c)$.

[38] This is clear as all the states coded by the same word are almost parallel, while those who appear in the error correcting code are almost orthogonal.

[39] This second part of the circuit can be described as follows. For $i$ that goes from 0 to $m - 1$, repeat the following instructions. First apply a NOT gate to all the qubits whose position corresponds to a zero in the binary description of $i$. Second, apply the controlled-unitary $C^{\log m}(E_i)$ (as defined in [27]); this controlled unitary has a very simple description (see, e.g., [27]). Finally apply NOT gates to all the qubits that were modified in the first step (and reset all work qubits to zero).

[40] For a comprehensive analysis of the problem, and issues related to it, we refer to [30] and references therein.

[41] Here it is assumed, as mentioned above, that the algorithmic information of the equilibrium state is 0.

[42] Equivalently, one might consider as background information the two basis states $\{|0\rangle, |1\rangle\}$ and the set of angles $\{\theta_k = k \pi / m\}_k$, where one agrees that the specification of an angle $\theta_k$ implies preparing a state of the form $\cos \theta_k |0\rangle + \sin \theta_k |1\rangle$. 