KO-HOMOLOGY AND TYPE I STRING THEORY

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ABSTRACT. We study the classification of D-branes and Ramond-Ramond fields in Type I string theory by developing a geometric description of KO-homology. We define an analytic version of KO-homology using KK-theory of real $C^*$-algebras, and construct explicitly the isomorphism between geometric and analytic KO-homology. The construction involves recasting the $\text{Cl}_n$-index theorem and a certain geometric invariant into a homological framework which is used, along with a definition of the real Chern character in KO-homology, to derive cohomological index formulas. We show that this invariant also naturally assigns torsion charges to non-BPS states in Type I string theory, in the construction of classes of D-branes in terms of topological KO-cycles. The formalism naturally captures the coupling of Ramond-Ramond fields to background D-branes which cancel global anomalies in the string theory path integral. We show that this is related to a physical interpretation of bivariant KK-theory in terms of decay processes on spacetime-filling branes. We also provide a construction of the holonomies of Ramond-Ramond fields in terms of topological KO-chains.

INTRODUCTION

This paper continues the development and applications of the topological classification of D-branes in string theory using generalized homology theories. As explained by [46, 59, 35, 28, 47, 27], and reviewed in [50, 20, 60], D-brane charges and Ramond-Ramond fluxes are necessarily classified by the K-theory of spacetime in order to explain certain dynamical processes that cannot be accounted for by ordinary cohomology theory alone. However, as emphasized by [51, 44, 31, 57, 1, 58], a much more natural description of D-branes is provided by K-homology which at the analytic level links them to Fredholm modules and spectral triples. This point of view was exploited in great detail in [52] to provide a rigorous geometric description of D-branes in Type II string theory using the Baum-Douglas construction of K-homology [10, 11]. In this paper we extend this description to D-branes and Ramond-Ramond fields in Type I string theory. The classification using KO-theory is explored extensively in [59, 13, 49, 50, 47, 14, 2, 43]. We use this and Jakob's approach [36] to construct a geometric realization of KO-homology as the homology theory dual to KO-theory, and describe various implications for the classification of Type I Ramond-Ramond charges and fluxes. As in [52], we simplify our treatment by dealing only with topologically trivial $B$-fields, and by ignoring the square-root of the Atiyah-Hirzebruch genus which naturally appears in the cohomological formula for D-brane charge [46, 50, 22]. Throughout we will compare and contrast with the complex case of Type II D-branes.

We will also develop the analytic description of KO-homology. We define this using Kasparov’s KK-theory for real $C^*$-algebras, which also encompasses the analytic KR-homology theories appropriate to D-branes in orientifold backgrounds. Generally, there is a description of the KK-theory group $\text{KK}(A, B)$ in terms of an additive category whose objects are separable $C^*$-algebras and whose morphisms $A \to B$ are precisely the elements of $\text{KK}(A, B)$, with the intersection product given by composition of morphisms. This category may be viewed as a certain completion of the stable homotopy category of separable $C^*$-algebras [16]. We use this description to provide a physical interpretation of KK-theory in terms of what we call “generalized D9-brane decay”, which unifies the description of charges in terms of tachyon condensation with
the description of fluxes in terms of holonomies over anomaly-cancelling background D-branes. In particular, we find a certain bound state obstruction to measuring the KO-theory class of a Ramond-Ramond field analogous to that found recently in [29]. Our physical interpretation of Kasparov’s theory is different from the proposal of [1, 2] (see also [51]) and is better suited to the global constructions of D-branes that we present. The use of KK-theory in string theory has also been exploited in context of string and other dualities in [22].

One of the main technical achievements of this paper is an explicit, detailed proof of the equivalence between the topological and analytic definitions of KO-homology, an ingredient missing from the original Baum-Douglas construction. In the course of working out the details, we came across the unpublished recent preprint [12] in which a proof is also given. While having some overlap with the present work, our proof is fundamentally different, although we have borrowed some results from [12] to shorten our argument somewhat. Our approach is more tailored to the physical applications that we have in mind, as it employs the construction of a certain geometric invariant which is related later on to D-brane charges and Ramond-Ramond fluxes. This invariant gives a rigorous definition to the $\mathbb{Z}_2$ Wilson lines which are used in physical constructions of Type I D-branes with torsion charges through tachyon condensation [55, 59, 14], and it is related to the mod 2 index that appears in the phase of the Type IIA partition function [17, 24, 48]. It is also related to the homological invariants that we construct is our description of fluxes as holonomies over background D-branes. Mathematically, our technique leads to a straightforward derivation of index formulas in the real case, whose proof is also missing from [10, 11] and which we provide in detail here. On the other hand, in contrast to our approach, the method of proof given in [12] has the virtue of being applicable to a potentially wider class of generalized homology theories.

The first four sections of this paper present most of the technical details of the construction of KO-homology and its applications, a lot of which have not appeared in completeness anywhere in the literature and contain mathematical results of independent interest. Our exposition begins in Section 1 with a self-contained description of analytic KO-homology using Kasparov’s KK-theory for real $C^*$-algebras. Section 2 details a Baum-Douglas type construction of geometric KO-homology. Using the approach of [36] we prove that this theory is equivalent to the usual definition provided by the loop spectrum of KO-theory, and thereby establish that the geometric definition really is a generalized homology theory. The content of Section 3 is the crux of our mathematical results, the detailed proof of the isomorphism between geometric and analytic KO-homology. This is done by recasting the $\mathcal{Cl}_n$-index theorem into a homological setting and thereby obtaining the associated homological invariant. (This is where our proof differs from that of [12].) In Section 4 we construct the Chern character in KO-homology and use it to derive cohomological formulas for the topological index (in the appropriate dimensionalities).

The final two sections of the paper turn to more physical applications of the geometric KO-homology framework. In Section 5 we explain the virtues of classifying Type I D-branes within the homological framework, and adapt some of the results of [24, 52] concerning the stability of brane constructions to the real case. The precise definitions involving D-branes, along with further motivation and results from the geometric K-homology formalism, can be found in [52] and will not be repeated here. We also examine in detail the problem of constructing torsion D-branes. In the real case this turns out to be much more involved than in the complex case, but we nevertheless formally give the constructions using the invariant built in Section 3 and suspension techniques. In the final Section 6 we demonstrate that the topological classification of Ramond-Ramond fields in Type I string theory is also much more natural within the context of geometric KO-homology. We show that the pertinent differential KO-theory group, which normally classifies fluxes, naturally describes the holonomies on background D-branes which are used to cancel the topological anomaly in the string theory path integral. This relation may be tied to the generalized D9-brane decay which lends a physical interpretation to KK-theory.
We then provide a construction of the holonomies in terms of a geometric invariant defined on KO-chains representing the background D-branes, and describe some of their properties.

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1. Analytic KO-Homology

In this section we will give a detailed overview of the definition of KO-homology in terms of Kasparov’s KK-theory for real $C^*$-algebras \[38\], and describe various properties that we will need in subsequent sections of this paper.

1.1. Real $C^*$-Algebras. We begin with an overview of the theory of real $C^*$-algebras. The main references are \[30\] [40].

Definition 1.1. A real algebra is a ring $A$ which is also an $\mathbb{R}$-vector space such that $\lambda (x y) = (\lambda x) y = x (\lambda y)$ for all $\lambda \in \mathbb{R}$ and all $x, y \in A$. A real $*$-algebra is a real algebra $A$ equipped with a linear involution $*: A \rightarrow A$ such that $(x y)^* = y^* x^*$ for all $x, y \in A$. A real Banach algebra is a real algebra $A$ equipped with a norm $\|\cdot\| : A \rightarrow \mathbb{R}$ such that $\|x y\| \leq \|x\| \|y\|$ and such that $A$ is complete in the norm topology. If $A$ is a unital algebra then we assume $\|1\| = 1$. A real Banach $*$-algebra is a real Banach algebra which is also a real $*$-algebra. A real $C^*$-algebra is a real Banach $*$-algebra such that

(i) The involution is an isometry, i.e. $\|x^*\| = \|x\|$ for all $x \in A$; and

(ii) $1 + x^* x$ is invertible in $A$ for all $x \in A$.

Remark 1.2. Although in the complex case invertibility of $1 + x^* x$ for all $x \in A$ would follow immediately from the $C^*$-algebra structure, in the real case this is no longer true. For example, consider the real Banach $*$-algebra $\mathbb{C}$ with involution given by the identity map. Then $1 + i^* i$ is not invertible, where $i := \sqrt{-1}$. This invertibility condition is fundamental to obtaining the usual representation theorem below for $C^*$-algebras in terms of bounded self-adjoint operators on a real Hilbert space. However, $\mathbb{C}$ with involution given by complex conjugation is a real $C^*$-algebra. Since the only $\mathbb{R}$-linear involutions of $\mathbb{C}$ are the identity and complex conjugation, when we consider $\mathbb{C}$ as a real $C^*$-algebra the involution will always be implicitly assumed to be complex conjugation. More generally any complex $C^*$-algebra, regarded as a real vector space and with the same operations, is a real $C^*$-algebra.

Let us now give a number of examples of real $C^*$-algebras, some of which we will use later on in representation theorems.

Example 1.3. Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space. Then the set of bounded linear operators $\mathcal{B}(\mathcal{H}_\mathbb{R})$ with the usual operations is a real $C^*$-algebra. Any closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H}_\mathbb{R})$ is also a real $C^*$-algebra. More generally, any closed self-adjoint subalgebra of a real $C^*$-algebra is always a real $C^*$-algebra.

Example 1.4. Let $X$ be a locally compact Hausdorff space and $C_0(X, \mathbb{R})$ the space of real-valued continuous functions vanishing at infinity. Then $C_0(X, \mathbb{R})$ with pointwise operations, the supremum norm and involution given by the identity map is a real $C^*$-algebra. As in the complex case, $C_0(X, \mathbb{R})$ is unital if and only if $X$ is compact.

Example 1.5. With $X$ as in Example 1.4 above, let $Y$ be a closed subspace of $X$ and $C_0(X, Y; \mathbb{R})$ the subspace of $C_0(X, \mathbb{C})$ consisting of maps $f : X \rightarrow \mathbb{C}$ such that $f(Y) \subset \mathbb{R}$. Then with the operations inherited from $C_0(X, \mathbb{C})$, the subspace $C_0(X, Y; \mathbb{R})$ is a real $C^*$-algebra.
Example 1.6. Let $X$ be a locally compact Hausdorff space with involution $\tau : X \to X$, i.e. a homomorphism such that $\tau \circ \tau = \text{id}_X$, and consider the subset $C_0(X, \tau)$ of $C_0(X, \mathbb{C})$ consisting of maps $f$ such that $f \circ \tau = f^* = \overline{f}$. Then $C_0(X, \tau)$, with the operations inherited from $C_0(X, \mathbb{C})$, is a real $C^*$-algebra. If $\tau = \text{id}_X$ then $C_0(X, \tau) = C_0(X, \mathbb{R})$. If $X$ is compact and $Y$ is a closed subspace of $X$, then there is a compact Hausdorff space $Z$ with an involution $\tau$ such that $C(X, Y; \mathbb{R}) \cong C(Z, \tau)$. However, the converse does not hold in general.

Example 1.7. Let $V$ be a real vector space equipped with a quadratic form $Q$, and consider the associated real Clifford algebra $Cl(V, Q)$. Assume, without loss of generality, that $Q(v) = \langle v, \phi(v) \rangle$ for all $v \in V$ with respect to an inner product on $V$, where the linear operator $\phi \in \mathcal{L}(V)$ is symmetric and orthogonal. We can then define an involution on $Cl(V, Q)$ by $(v_1 \cdots v_k)^* = \phi(v_k) \cdots \phi(v_1)$, i.e. if $v \in V$ then $v^* = \phi(v)$. The isomorphism $\Phi : Cl(V \oplus V, Q \oplus -Q) \to \mathcal{L}(\Lambda^2 V)$ induces a norm on $Cl(V, Q)$ by pullback of the operator norm on $\mathcal{L}(\Lambda^2 V)$, and the inclusion $Cl(V, Q) \hookrightarrow Cl(V, Q) \oplus Cl(V, -Q) \cong Cl(V \oplus V, Q \oplus -Q)$ given by $x \mapsto x \oplus 1$ thereby induces a norm on $Cl(V, Q)$. Then $Cl(V, Q)$ with its algebra structure, this involution and norm is a real $C^*$-algebra.

If $A, B$ are real $*$-algebras then a real $**$-algebra homomorphism is a real algebra map $\phi : A \to B$, i.e. an $\mathbb{R}$-linear ring homomorphism, such that $\phi(x^*) = \phi(x)^*$ for all $x \in A$. The homomorphism is assumed to be unital if both algebras are unital. We now come to the most general representation theorems for real $C^*$-algebras. If $A$ is an algebra, we denote by $M_n(A)$ the algebra of $n \times n$ matrices with entries in $A$.

**Theorem 1.8.** Let $A$ be a finite-dimensional real $C^*$-algebra. Then there exist $k, n_1, \ldots, n_k \in \mathbb{N}$ such that $A \cong M_{n_1}(A_1) \times \cdots \times M_{n_k}(A_k)$ as real $C^*$-algebras with $A_1, \ldots, A_k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

**Proof.** Let $x \in A$. If $x^* x = x x^*$ and $x^n = 0$ for some $n \in \mathbb{N}$, then $x = 0$. This implies that $A$ has no non-zero nilpotent two-sided ideals. Wedderburn’s theorem on the representation of finite-dimensional real algebras states that any real algebra with no non-zero nilpotent two-sided ideals is isomorphic (as a real algebra) to a finite direct product of $\mathbb{R}$-algebras of the form $M_k(D)$, with $k \in \mathbb{N}$ and $D$ a finite-dimensional division algebra over $\mathbb{R}$. The only finite-dimensional division $\mathbb{R}$-algebras are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. The direct product, with direct product operations, supremum norm

$$\|(a_{ij})\| = \sup_{i,j} \|a_{ij}\|$$

and involution $(a_{ij})^* = (a_{ij}^*)$, is a real $C^*$-algebra. One then shows, as in the complex case, that these two algebras are isomorphic as real $C^*$-algebras.

Analogously to the complex case, one also has the following result.

**Theorem 1.9.** Let $A$ be any real $C^*$-algebra. Then there exists a real Hilbert space $\mathcal{H}_R$ such that $A$ is isomorphic as a real $C^*$-algebra to a closed self-adjoint subalgebra of $B(\mathcal{H}_R)$.

Let $A$ be a real $C^*$-algebra. We denote by $A_{\mathbb{C}} := A \otimes \mathbb{C}$ the complexification of $A$, which is a complex algebra containing $A$ as a real algebra. We can define a map $J_A : A_{\mathbb{C}} \to A_{\mathbb{C}}$ by $J_A(x + iy) = x - iy$ for all $x, y \in A$. The map $J_A$ is a conjugate linear $*$-isomorphism of the complex $C^*$-algebra $A_{\mathbb{C}}$. If $\phi : A \to A$ is a continuous $*$-homomorphism, then the map $J_A(\phi) : A_{\mathbb{C}} \to A_{\mathbb{C}}$ defined by $J_A(\phi)(x + iy) = \phi(x) + i\phi(y)$ is a continuous $*$-homomorphism such that $J_A \circ J_A(\phi) = J_A(\phi) \circ J_A$. Conversely, if $J$ is a conjugate linear $*$-isomorphism of a complex $C^*$-algebra $B$, then $A = \{x \in B \mid J(x) = x\}$ is a real $C^*$-algebra. This implies the following result.

**Proposition 1.10.** Let $\mathcal{C}^*_R$ be the category of real $C^*$-algebras and continuous $*$-algebra homomorphisms. Let $\mathcal{C}^*_{\mathbb{C}, \text{cl}}$ be the category of pairs $(A, J)$, where $A$ is a complex $C^*$-algebra and $J$ is
a conjugate linear $*$-isomorphism of $A$, and continuous $*$-homomorphisms commuting with $J$. Then the assignments $A \mapsto (A_C, J_A)$, $\phi \mapsto J_A(\phi)$ define a functor

$$\mathcal{J} : C_\mathbb{R} \rightarrow C_{\mathbb{C}, \text{cl}}$$

which is an equivalence of categories.

1.2. Commutative Real $C^*$-Algebras. We will now specialize to the case of commutative algebras. As with complex Banach algebras, a maximal two-sided ideal in a real Banach algebra $A$ is closed in $A$. If $M$ is a maximal two-sided ideal of a real Banach algebra $A$, then $A/M$ is isomorphic to one of $\mathbb{R}$ or $\mathbb{C}$ as real algebras. A character on a real algebra $A$ is a non-zero real algebra map $\chi : A \rightarrow \mathbb{C}$, assumed unital if $A$ is unital. Let $\Omega_A$ be the space of characters of $A$. This can be given, as in the complex case, a locally compact Hausdorff space topology such that $\Omega_A$ is homeomorphic to $\Omega_{A_C}$. Furthermore, $A$ is unital if and only if $\Omega_A$ is compact.

Given $x \in A$, evaluation at $x$ gives a continuous map $\Gamma(x) : \Omega_A \rightarrow \mathbb{C}$ called the Gel’fand transform of $x$. From this we obtain the Gel’fand transform of $A$, $\Gamma : A \rightarrow C_0(\Omega_A, \mathbb{C})$, which is a continuous real algebra homomorphism of unit norm. If $A$ is a real $*$-algebra, then $\Gamma$ is a $*$-algebra homomorphism. The most important results on the representation of commutative real $C^*$-algebras are the following.

**Theorem 1.11.** Let $A$ be a commutative real $C^*$-algebra. Then:

(i) The map $\tau : \Omega_A \rightarrow \Omega_A$ defined by $\tau(\chi) = \overline{\chi}$ is an involution; and

(ii) The Gel’fand transform $\Gamma : A \rightarrow C_0(\Omega_A, \tau)$ is a real $C^*$-algebra isomorphism.

**Proof.** (i) The map $\tau$ is a bijection. The collection of sets

$$U_{x, V} = \{ \chi \in \Omega_A \mid \chi(x) \in V \}$$

for every $x \in A$ and $V$ open in $\mathbb{C}$ is a sub-basis for the topology of $\Omega_A$. The complex conjugate $\overline{V}$ of $V$ is an open set and $\tau^{-1}(U_{x, V}) = U_{x, \overline{V}}$. Thus $\tau$ is continuous.

(ii) The map $\Gamma$ is a real $*$-algebra map with $\|\Gamma(x)\| = \|x\|$. One also has

$$\Gamma(x) \circ \tau(\chi) = \Gamma(x)(\overline{\chi}) = \overline{\chi(x)} = \Gamma(x)^*(\chi),$$

and so $\Gamma(x) \circ \tau = \Gamma(x)^*$ and $\Gamma(A) \subset C_0(\Omega_A, \tau)$. Let $\theta : A \rightarrow A_C$ be the $C^*$-algebra embedding of $A$ into its complexification. The map $\vartheta : \Omega_{A_C} \rightarrow \Omega_A$ given by $\vartheta(f) = f \circ \theta$ is a homeomorphism and there is a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\Gamma} & C_0(\Omega_A, \mathbb{C}) \\
\theta \downarrow & & \downarrow \vartheta^* \\
A_C & \xrightarrow{\Gamma} & C_0(\Omega_{A_C}, \mathbb{C}).
\end{array}$$

Using this one then shows that $\Gamma(A) = C_0(\Omega_A, \tau)$.

**Corollary 1.12.** Let $A$ be a commutative real $C^*$-algebra with trivial involution. Then $A$ is $*$-isomorphic to $C_0(\Omega_A, \mathbb{R})$.

1.3. Hilbert Modules. We will now start presenting an overview of KK-theory for real $C^*$-algebras. The basic references are [51, 10]. We begin by generalizing the notion of Hilbert space.

**Definition 1.13.** Let $A$ be a (not necessarily commutative) real $C^*$-algebra. A pre-Hilbert module over $A$ is a (right) $A$-module $E$ equipped with an $A$-valued inner product, i.e. a bilinear map $(-, -) : E \times E \rightarrow A$ such that
(i) \((x,x) \geq 0\) for all \(x \in \mathcal{E}\) and \((x,x) = 0\) if and only if \(x = 0\);
(ii) \((x,y) = (y,x)^{\ast}\) for all \(x,y \in \mathcal{E}\); and
(iii) \((x,ya) = (x,y)a\) for all \(x,y \in \mathcal{E}, a \in A\).

For \(x \in \mathcal{E}\) we define \(\|x\|_{\mathcal{E}} := \|(x,x)\|^{1/2}\). This defines a norm on \(\mathcal{E}\) satisfying the Cauchy-Schwartz inequality. If \(\mathcal{E}\) is complete under this norm, then it is called a Hilbert module over \(A\).

We can define tensor products of \(C^{\ast}\)-algebras and Hilbert modules in the usual way (see [16, 54] for the constructions). If \(\mathcal{E}\) is a pre-Hilbert module over the real \(C^{\ast}\)-algebra \(A\), we assume that the complexification \(\mathcal{E} \otimes \mathbb{C}\) is a pre-Hilbert module over \(A_{\mathbb{C}}\). This means that the \(A\)-valued inner product extends to a sesquilinear map. We assume that sesquilinear maps are linear in the second variable.

Let \(\mathcal{E}, \mathcal{F}\) be Hilbert \(A\)-modules and \(T : \mathcal{E} \to \mathcal{F}\) an \(A\)-linear map. We call a map \(T^{\ast} : \mathcal{F} \to \mathcal{E}\) such that \((Tx,y)_{\mathcal{F}} = (x,T^{\ast}y)_{\mathcal{E}}\) for all \(x \in \mathcal{E}, y \in \mathcal{F}\) the adjoint of \(T\). If it exists the adjoint is unique by Definition 1.13(i). Not every \(A\)-linear map between Hilbert \(A\)-modules has an adjoint. We denote the set of all \(A\)-linear maps \(T : \mathcal{E} \to \mathcal{F}\) admitting an adjoint by \(\mathcal{L}(\mathcal{E}, \mathcal{F})\). Elements of \(\mathcal{L}(\mathcal{E}, \mathcal{F})\) are bounded \(A\)-linear maps and \(\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})\) is a \(C^{\ast}\)-algebra with the operator norm and involution given by the adjoint. Given \(x \in \mathcal{F}, y \in \mathcal{E}\) we define an operator \(\theta_{x,y} \in \mathcal{L}(\mathcal{E}, \mathcal{F})\) by \(\theta_{x,y}(z) = x(y,z)_{\mathcal{E}}\). These operators generate an \(\mathcal{L}(\mathcal{E}) - \mathcal{L}(\mathcal{F})\)-bimodule whose norm closure in \(\mathcal{L}(\mathcal{E}, \mathcal{F})\) is denoted \(\mathcal{K}(\mathcal{E}, \mathcal{F})\). Elements of \(\mathcal{K}(\mathcal{E}, \mathcal{F})\) are called generalized compact operators.

If \(\mathcal{E} = \mathcal{H}_{\mathbb{R}}\) is a real Hilbert space, then \(\mathcal{L}(\mathcal{E})\) is the usual space of bounded linear operators and \(\mathcal{K}(\mathcal{E})\) is the space of compact operators. If \(n \in \mathbb{N} \cup \{\infty\}\), then \(A^{n}\) with inner product

\[\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_{i}^{\ast} y_{i}\]

for all \(\vec{x} = (x_{i})_{1 \leq i \leq n}, \vec{y} = (y_{i})_{1 \leq i \leq n}\) is a Hilbert module. One has \(\mathcal{K}(A) \cong A\) and \(\mathcal{K}(A^{\infty}) \cong A \otimes \mathcal{K}_{\mathbb{R}}\) where \(\mathcal{K}_{\mathbb{R}} := \mathcal{K}(\mathcal{H}_{\mathbb{R}})\).

**Definition 1.14.** Let \(A\) be a real \(C^{\ast}\)-algebra. The multiplier algebra of \(A\), \(M(A)\), is the maximal \(C^{\ast}\)-algebra containing \(A\) as an essential ideal. Equivalently, by representing \(A \subset \mathcal{L}(\mathcal{H}_{\mathbb{R}})\) one has

\[M(A) = \{ T \in \mathcal{L}(\mathcal{H}_{\mathbb{R}}) \mid TS, ST \in A \text{ for all } S \in A \} \, .\]

The multiplier algebra \(M(A)\) is a \(C^{\ast}\)-algebra which is \(*\)-isomorphic to the \(C^{\ast}\)-algebra of double centralizers, i.e. pairs \((T_{1}, T_{2}) \in \mathcal{L}(A) \times \mathcal{L}(A)\) such that \(aT_{1}(b) = T_{2}(a)T_{1}(b), T_{1}(a)T_{2}(b) = T_{1}(a)T_{2}(b), T_{2}(a)b = aT_{2}(b)\) for all \(a, b \in A\). If \(A\) is unital, then \(M(A) = A\). Furthermore, \(M(\mathcal{K}_{\mathbb{R}}) = \mathcal{L}(\mathcal{H}_{\mathbb{R}})\), and \(M(C_{0}(X, \mathbb{R})) = C_{b}(X, \mathbb{R})\) is the \(C^{\ast}\)-algebra of real-valued bounded continuous functions on a locally compact Hausdorff space \(X\).

**Proposition 1.15.** Let \(\mathcal{E}\) be a Hilbert \(A\)-module. Then there is an isomorphism

\[\mathcal{L}(\mathcal{E}) \cong M(\mathcal{K}(\mathcal{E})) \, .\]

1.4. **KKO-Theory.** We will now define the KKO-theory groups using Kasparov’s approach [38]. A useful survey of Kasparov’s theory can be found in [32]. We assume that a real \(C^{\ast}\)-algebra \(A\) is separable and a real \(C^{\ast}\)-algebra \(B\) is \(\sigma\)-unital.

**Definition 1.16.** A (Kasparov) \((A, B)\)-module is a triple \((\mathcal{E}, \rho, T)\), where \(\mathcal{E}\) is a countably generated Hilbert \(B\)-module, \(\rho : A \to \mathcal{L}(\mathcal{E})\) is a \(*\)-homomorphism and \(T \in \mathcal{L}(\mathcal{E})\) such that

\[T - T^{\ast} (\rho(a), (T^{2} - 1) \rho(a), [T, \rho(a)] \in \mathcal{K}(\mathcal{E})\]

for all \(a \in A\). A Kasparov module \((\mathcal{E}, \rho, T)\) is called degenerate if all operators in \(\mathbb{1}_{\mathcal{E}}\) are zero. Two Kasparov modules \((\mathcal{E}_{i}, \rho_{i}, T_{i})\), \(i = 1, 2\) are said to be orthogonally equivalent if there is an isometric isomorphism \(U \in \mathcal{L}(\mathcal{E}_{1}, \mathcal{E}_{2})\) such that \(T_{1} = U^{\ast}T_{2}U\) and \(\rho_{1}(a) = U^{\ast}\rho_{2}(a)U\) for all \(a \in A\).
Orthogonal equivalence is an equivalence relation on the set of Kasparov modules. We denote the set of equivalence classes by \( E(A, B) \). The subset containing degenerate modules is denoted \( D(A, B) \). Direct sum makes \( E(A, B) \) and \( D(A, B) \) into monoids.

**Definition 1.17.** Let \( (E_i, \rho_i, T_i) \in E(A, B) \) for \( i = 0, 1 \), \( (E, \rho, T) \in E(A, B \otimes C([0, 1], \mathbb{R})) \), and let \( f_t : B \otimes C([0, 1], \mathbb{R}) \to B \) be the evaluation map \( f_t(g) = g(t) \). Then \( (E_0, \rho_0, T_0) \) and \( (E_1, \rho_1, T_1) \) are said to be homotopic and \( (E, \rho, T) \) is called a homotopy if \( (E \otimes f_t, f_t \circ \rho, f_{ts}(T)) \) is orthogonally equivalent to \( (E_i, \rho_i, T_i) \) for \( i = 0, 1 \), where \( f_{ts}(T)(a) := f_t(T(a)) \).

Homotopy is an equivalence relation on \( E(A, B) \) and we denote the equivalence classes by \([E, \rho, T]\). It is useful to consider special kinds of homotopy. If \( E = C([0, 1], \mathcal{E}_0) \), \( \mathcal{E}_0 = \mathcal{E}_1 \) and the induced maps \( t \mapsto T_t \), \( t \mapsto \rho_t(a) \) for all \( a \in A \) are strongly *-continuous, then we call \( (E, \rho, T) \) a standard homotopy. If in addition \( \rho_t = \rho \) is constant and \( T_t \) is norm continuous, then \( (E, \rho, T) \) is called an operator homotopy. Any degenerate module is homotopic to the zero module. The quotient \( \Omega(E) := \mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \) is a generalization of the Calkin algebra. If \( \rho(a) [T_1, T_2] \rho(a) = 0 \) in \( \Omega(E) \), then \( (E, \rho, T_1) \) and \( (E, \rho, T_2) \) are operator homotopic.

**Definition 1.18.** The set of equivalence classes in \( E(A, B) \) with respect to homotopy of \((A, B)\)-modules is denoted \( \text{KKO}(A, B) \) or \( \text{KKO}_0(A, B) \). For \( p, q \geq 0 \) we define

\[
\text{KKO}_{p,q}(A, B) = \text{KKO}(A, B \otimes C\ell_{p,q}),
\]

where \( C\ell_{p,q} := C(\mathbb{R}^{p,q}) \) is the real Clifford algebra of the vector space \( \mathbb{R}^{p+q} \) with quadratic form of signature \((p, q)\).

The equivalence relation allows us to simplify the \((A, B)\)-modules required to define \( \text{KKO}(A, B) \). We need only consider modules of the form \((B^\infty, \rho, T)\) with \( T = T^* \). If \( A \) is unital, we can further assume that \( \|T\| \leq 1 \) and \( T^2 - 1 \in \mathcal{K}(B^\infty) \).

There is another equivalence relation that we can define on \( E(A, B) \). We say that two \((A, B)\)-modules \((E_i, \rho_i, T_i), i = 0, 1\) are stably operator homotopic, \( (E_0, \rho_0, T_0) \simeq_{oh} (E_1, \rho_1, T_1) \), if there exist \((E'_i, \rho'_i, T'_i) \in D(A, B)\) such that \((E_0 \oplus E'_0, \rho_0 \oplus \rho'_0, T_0 \oplus T'_0)\) and \((E_1 \oplus E'_1, \rho_1 \oplus \rho'_1, T_1 \oplus T'_1)\) are operator homotopic up to orthogonal equivalence. The set of equivalence classes with respect to \( \simeq_{oh} \) coincides with the set \( \text{KKO}(A, B) \) defined above.

**Proposition 1.19.** The set \( \text{KKO}(A, B) \) enjoys the following properties:

1. \( \text{KKO}(A, B) \) is an abelian group.
2. \( \text{KKO}(-, -) \) is a covariant bifunctor from the category of separable \( C^* \)-algebras into the category of abelian groups which is additive:
   \[
   \text{KKO}(A_1 \oplus A_2, B) = \text{KKO}(A_1, B) \oplus \text{KKO}(A_2, B),
   \]
   \[
   \text{KKO}(A, B_1 \oplus B_2) = \text{KKO}(A, B_1) \oplus \text{KKO}(A, B_2).
   \]
3. Any two \( * \)-homomorphisms \( f : A_2 \to A_1 \) and \( g : B_1 \to B_2 \) induce group homomorphisms
   \[
   f^* : \text{KKO}(A_1, B) \to \text{KKO}(A_2, B),
   \]
   \[
   g_* : \text{KKO}(A, B_1) \to \text{KKO}(A, B_2)
   \]
   defined by
   \[
   f^*[E, \rho, T] = [E, \rho \circ f, T],
   \]
   \[
   g_*[E, \rho, T] = [E \otimes g, B_2, \rho \otimes 1, T \otimes 1]
   \]
   and
4. Any two homotopies \( f_t : A_2 \to A_1 \) and \( g_t : B_1 \to B_2 \) induce the same homomorphism for all \( t \in [0, 1] \), i.e. \( f_t^* = f_0^* \) and \( g_{ts} = g_{0s} \).
If we assume $B$ unital, then we can identify $\mathcal{L}(B^{\infty}) \cong \mathbb{M}_2(M(B \otimes \mathbb{K}_R))$ where $M(B \otimes \mathbb{K}_R)$ is the multiplier algebra of $B \otimes \mathbb{K}_R$. Thus we can give $\rho$ and $T$ the form

$$\rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & T^* \\ T' & 0 \end{pmatrix}$$

with $\rho_0(a), \rho_1(a), T' \in M(B \otimes \mathbb{K}_R) \cong \mathcal{L}(B^{\infty}), \|T'\| \leq 1$, and

$$T'^* T' - 1, \quad T' T'^* - 1, \quad T' \rho_1(a) - \rho_0(a) T' \in B \otimes \mathbb{K}_R$$

for all $a \in A$.

1.5. **Analytic KO-Homology.** Specializing all of our constructions to the case $A = \mathbb{R}$ and $B$ unital we get the KO-theory groups $\text{KKO}(\mathbb{R}, B) \cong \text{KO}_0(B)$ and $\text{KKO}_{p,q}(\mathbb{R}, B) \cong \text{KO}_{p,q}(B)$. In particular, $\text{KKO}(\mathbb{R}, C(X, \mathbb{R})) \cong \text{KO}_0(C(X, \mathbb{R})) \cong \text{KO}^0(X)$ for any compact Hausdorff space $X$.

On the other hand, using the Gel’fand transform the contravariant functor $(X, \tau) \mapsto \text{C}(X, \tau)$ induces an equivalence of categories between the category of compact Hausdorff spaces with involution and the category of commutative real $C^*$-algebras. Since $\text{KKO}_2(-, \mathbb{R})$ is also a contravariant functor, it follows that their composition $(X, \tau) \mapsto \text{KKO}_2(C(X, \tau), \mathbb{R})$ is a covariant functor.

**Definition 1.20.** Let $(X, \tau)$ be a compact Hausdorff space with involution. The *analytic KO-homology groups* of $(X, \tau)$ are defined by

$$\text{KO}_n^a(X, \tau) = \text{KKO}_{n,0}(C(X, \tau), \mathbb{R}) = \text{KKO}(C(X, \tau), C\ell_n)$$

where $C\ell_n := C\ell_{n,0} = C\ell(\mathbb{R}^n)$.

It will be helpful in some of our later analysis to have a closer look at our definition of $\text{KO}_n(A) = \text{KKO}_{n,0}(A, \mathbb{R}) = \text{KKO}(A, C\ell_n)$, the KO-homology of a real $C^*$-algebra $A$. Following through the definitions, this is based on triples $(\mathcal{H}_R, \rho, T)$ which are defined by the data:

(i) $\mathcal{H}_R$ is a separable real Hilbert space;

(ii) $\rho : A \to \mathcal{L}(\mathcal{H}_R)$ is a unital representation of $A$; and

(iii) $T$ is a bounded linear operator on $\mathcal{H}_R$.

These are assumed to satisfy the following conditions:

(i) $\mathcal{H}_R$ is equipped with a $\mathbb{Z}_2$-grading such that $\rho(a)$ is even for all $a \in A$ and $T$ is odd;

(ii) For all $a \in A$ one has

$$(T^2 - 1) \rho(a), \quad (T - T^*) \rho(a), \quad T \rho(a) - \rho(a) T \in \mathcal{K}_R$$

and

(iii) There are odd $\mathbb{R}$-linear operators $\varepsilon_1, \ldots, \varepsilon_n$ on $\mathcal{H}_R$ with the $C\ell_n$ algebra relations

$$\varepsilon_i = \varepsilon_i^2, \quad \varepsilon_i^2 = -1, \quad \varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$$

for $i \neq j$ such that $T$ and $\rho(a)$ commute with each $\varepsilon_i$.

From [152] it follows that $T$ may be taken to be a Fredholm operator without loss of generality (see [39], Lemma 5.1), and we shall refer to the triple $(\mathcal{H}_R, \rho, T)$ as an *n-graded Fredholm module*.

Let us denote by $\text{GO}_n(A)$ the set of all $n$-graded Fredholm modules over $A$. Consider the equivalence relation $\sim$ on $\text{GO}_n(A)$ generated by the relations:

- **Orthogonal equivalence:** $(\mathcal{H}_R, \rho, T) \sim (\mathcal{H}_R', \rho', T')$ if and only if there exists an isometric degree-preserving linear operator $U : \mathcal{H}_R \to \mathcal{H}_R'$ such that $U \rho(a) = \rho'(a) U$ for all $a \in A$, $U T = T' U$, and $U \varepsilon_i = \varepsilon_i' U$; and

- **Homotopy equivalence:** $(\mathcal{H}_R, \rho, T) \sim (\mathcal{H}_R, \rho, T')$ if and only if there exists a norm continuous function $t \mapsto T_t$ such that $(\mathcal{H}_R, \rho, T_t)$ is a Fredholm module for all $t \in [0, 1]$ with $T_0 = T$, $T_1 = T'$.
We define the direct sum of two Fredholm modules \((\mathcal{H}_R, \rho, T)\) and \((\mathcal{H}_R', \rho', T')\) to be the Fredholm module \((\mathcal{H}_R \oplus \mathcal{H}_R', \rho \oplus \rho', T \oplus T')\).

We may now define \(\text{KO}_n(A)\) as the free abelian group generated by elements in \(\text{GO}_n(A)/\sim\) and quotiented by the ideal generated by the set \(\{[x_0 \oplus x_1] - [x_0] - [x_1] | x_0, x_1 \in \text{GO}_n(A)/\sim\}\). In \(\text{KO}_n(A)\) the inverse of a class represented by the module \((\mathcal{H}_R, \rho, T)\) is given by \((\mathcal{H}_R^0, \rho, T)\), where \(\mathcal{H}_R^0\) is the Hilbert space \(\mathcal{H}_R\) with the opposite \(\mathbb{Z}_2\)-grading and where the operators \(\varepsilon_i\) reverse their signs. For a compact Hausdorff space \(X\) we define

\[ \text{KO}_n^a(X) := \text{KO}_n^a(C(X, \mathbb{R})) = \text{KKO}(C(X, \mathbb{R}), C\ell_n). \]

Of course, this construction is exactly the one given before, only spelled out in more detail here. For further details and properties of this construction in the complex case, see [12].

### 1.6. The Intersection Product

Let \(D\) be a real \(C^*\)-algebra. Then there is a natural homomorphism

\[ \tau_D : \text{KKO}(A, B) \rightarrow \text{KKO}(A \otimes D, B \otimes D) \]

defined by

\[ \tau_D[B^\infty, \rho, T] = [B^\infty \otimes D, \rho \otimes 1, T \otimes 1]. \]

We can define in KKO-theory a product

\[ \otimes_D : \text{KKO}(A, D) \times \text{KKO}(B, D) \rightarrow \text{KKO}(A, B) \]

called the intersection product by

\[ [\varepsilon_1, \rho_1, T_1] \otimes_D [\varepsilon_2, \rho_2, T_2] = [\varepsilon_1 \otimes_{\rho_2} \varepsilon_2, \rho_1 \otimes \rho_2 1, T_1 \# T_2], \]

where \(T_1 \# T_2 \in \mathcal{L}(\varepsilon_1 \otimes_{\rho_2} \varepsilon_2)\) is a suitably defined operator [32]. If all \(C^*\)-algebras involved are separable, then the intersection product extends to a bilinear map

\[ \otimes_D : \text{KKO}(A_1, B_1 \otimes D) \times \text{KKO}(D \otimes A_2, B_2) \rightarrow \text{KKO}(A_1 \otimes A_2, B_1 \otimes B_2) \]

given by

\[ x \otimes_D y = \tau_{A_2}(x) \otimes_{B_1 \otimes D} \otimes_{A_2} \tau_{B_1}(y) \]

for all \((x, y)\).

#### Proposition 1.21

Let \(A\) be a separable \(C^*\)-algebra and \(B, D_1, D_2\) \(\sigma\)-unital algebras. Suppose there exist \(\alpha \in \text{KKO}(D_1, D_2)\) and \(\beta \in \text{KKO}(D_2, D_1)\) with \(\alpha \otimes_{D_2} \beta = 1_{D_1}\) and \(\beta \otimes_{D_1} \alpha = 1_{D_2}\). Then there are isomorphisms

\[ \otimes_{D_1} \alpha : \text{KKO}(A, B \otimes D_1) \rightarrow \text{KKO}(A, B \otimes D_2), \]

\[ \otimes_{D_2} \beta : \text{KKO}(A, B \otimes D_2) \rightarrow \text{KKO}(A, B \otimes D_1). \]

If \(D_1, D_2\) are separable, then one has isomorphisms

\[ \alpha \otimes_{D_2} : \text{KKO}(A \otimes D_2, B) \rightarrow \text{KKO}(A \otimes D_1, B), \]

\[ \beta \otimes_{D_1} : \text{KKO}(A \otimes D_1, B) \rightarrow \text{KKO}(A \otimes D_2, B). \]

If in addition there exist \(\alpha' \in \text{KKO}(D_1 \otimes D_2, \mathbb{R})\) and \(\beta' \in \text{KKO}(\mathbb{R}, D_1 \otimes D_2)\) such that \(\beta' \otimes_{D_1} \alpha' = 1_{D_2}\) and \(\beta' \otimes_{D_2} \alpha' = 1_{D_1}\), then there are isomorphisms

\[ \otimes_{D_1} \alpha' : \text{KKO}(A, B \otimes D_1) \rightarrow \text{KKO}(A, B \otimes D_2), \]

\[ \otimes_{D_2} \alpha' : \text{KKO}(A, B \otimes D_2) \rightarrow \text{KKO}(A, B \otimes D_1), \]

\[ \beta' \otimes_{D_1} : \text{KKO}(A \otimes D_1, B) \rightarrow \text{KKO}(A, B \otimes D_2), \]

\[ \beta' \otimes_{D_2} : \text{KKO}(A \otimes D_2, B) \rightarrow \text{KKO}(A, B \otimes D_1). \]
The last result in Proposition 1.21 allows us to conclude that the KKO-groups are stable, i.e.
there are isomorphisms
\[
\text{KKO}(A \otimes \mathcal{K}_R, B) \cong \text{KKO}(A, B) \cong \text{KKO}(A, B \otimes \mathcal{K}_R).
\]
One also has the isomorphisms
\[
\text{KKO}(A \otimes C_{p,q}, B \otimes C_{r,s}) \cong \text{KKO}(A \otimes C_{p,q} \otimes C_{r,s}, B) \cong \text{KKO}(A \otimes C_{p-q+s-r,0}, B)
\]
along with symmetric isomorphisms. Since KKO_n(\mathbb{R}, A) is the operator algebraic KO-theory of
A, these isomorphisms and the periodicity of real Clifford algebras immediately imply mod 8
real Bott periodicity. Analogously, we obtain from the symmetric isomorphism Bott periodicity
in analytic KO-homology.

2. Geometric KO-Homology

We will now define geometric KO-homology, analogously to the Baum-Douglas construction
of K-homology [10, 11, 52], and describe the basic properties of the topological KO-homology
groups of a topological space that we will need later on. We will prove directly that this is a
homology theory by comparing it with other formulations of KO-homology as the dual theory
to KO-theory. In particular, in the next section we will show that this homology theory is
equivalent to the analytic homology theory of the previous section.

2.1. Spin Bordism. Throughout X will denote a finite CW-complex.

Definition 2.1. A KO-cycle on X is a triple \((M, E, \phi)\) where
(i) \(M\) is a compact spin manifold without boundary;
(ii) \(E\) is a real vector bundle over \(M\); and
(iii) \(\phi : M \to X\) is a continuous map.

There are no connectedness requirements made upon \(M\), and hence the bundle \(E\) can have
different fibre dimensions on the different connected components of \(M\). It follows that disjoint
union
\[
(M_1, E_1, \phi_1) \amalg (M_2, E_2, \phi_2) := (M_1 \amalg M_2, E_1 \amalg E_2, \phi_1 \amalg \phi_2)
\]
is a well-defined operation on the set of KO-cycles on \(X\).

Definition 2.2. Two KO-cycles \((M_1, E_1, \phi_1)\) and \((M_2, E_2, \phi_2)\) on \(X\) are
isomorphic if there exists a diffeomorphism \(h : M_1 \to M_2\) such that
(i) \(h\) preserves the spin structures;
(ii) \(h^*E_2 \cong E_1\) as real vector bundles; and
(iii) The diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{h} & M_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
X
\end{array}
\]

commutes.

The set of isomorphism classes of KO-cycles on \(X\) is denoted \(\Gamma_0(X)\).

Definition 2.3. Two KO-cycles \((M_1, E_1, \phi_1)\) and \((M_2, E_2, \phi_2)\) on \(X\) are
spin bordant if there exists a compact spin manifold \(W\) with boundary, a real vector bundle \(E \to W\), and a continuous
map \(\phi : W \to X\) such that the two KO-cycles
\[
(\partial W, E|_{\partial W}, \phi|_{\partial W}) , \quad (M_1 \amalg (-M_2), E_1 \amalg E_2, \phi_1 \amalg \phi_2)
\]
are isomorphic, where \(-M_2\) denotes \(M_2\) with the spin structure on its tangent bundle \(TM_2\)
reversed. The triple \((W, E, \phi)\) is called a spin bordism of KO-cycles.
2.2. Real Vector Bundle Modification. Let $M$ be a spin manifold and $F \to M$ a $C^\infty$ real spin vector bundle with fibres of dimension $n := \dim_{\mathbb{R}} F_p \equiv 0 \mod 8$ for $p \in M$. Let $\mathbb{R}_M := M \times \mathbb{R}$ denote the trivial real line bundle over $M$. Then $F \otimes \mathbb{R}_M$ is a real vector bundle over $M$ with fibres of dimension $n + 1$ and projection map $\lambda$. By choosing a $C^\infty$ metric on it, we may define the unit sphere bundle
\begin{equation}
\widehat{M} = S(F \otimes \mathbb{R}_M)
\end{equation}
by restricting the set of fibre vectors of $F \otimes \mathbb{R}_M$ to those which have unit norm. The tangent bundle of $F \otimes \mathbb{R}_M$ fits into an exact sequence of bundles given by
\begin{equation}
0 \to \lambda^*(F \otimes \mathbb{R}_M) \to T(F \otimes \mathbb{R}_M) \to \lambda^*(TM) \to 0.
\end{equation}
Upon choosing a splitting, the spin structures on $TM$ and $F$ induce a spin structure on $T\widehat{M}$, and hence $\widehat{M}$ is a compact spin manifold. By construction, $\widehat{M}$ is a sphere bundle over $M$ with $n$-dimensional spheres $S^n$ as fibres. We denote the bundle projection by
\begin{equation}
\pi : \widehat{M} \to M.
\end{equation}
We may regard the total space $\widehat{M}$ as consisting of two copies $\mathbb{B}^\pm(F)$, with opposite spin structures, of the unit ball bundle $\mathbb{B}(F)$ of $F$ glued together by the identity map $\text{id}_{\mathbb{B}(F)}$ on its boundary so that
\begin{equation}
\widehat{M} = \mathbb{B}^+(F) \cup_{\mathbb{S}(F)} \mathbb{B}^-(F).
\end{equation}

Since $n \equiv 0 \mod 8$, the group $\text{Spin}(n)$ has two irreducible real half-spin representations. The spin structure on $F$ associates to these representations real vector bundles $S_0(F)$ and $S_1(F)$ of equal rank $2n/2$ over $M$. Their Whitney sum $S(F) = S_0(F) \oplus S_1(F)$ is a bundle of real Clifford modules over $TM$ such that $\text{Cl}(F) \cong \text{End} S(F)$, where $\text{Cl}(F)$ is the real Clifford algebra bundle of $F$. Let $\mathbb{S}^+(F)$ and $\mathbb{S}^-(F)$ be the real spinor bundles over $F$ obtained from pullbacks to $\widehat{M}$ by the bundle projection $F \to M$ of $S_0(F)$ and $S_1(F)$, respectively. Clifford multiplication induces a bundle map $F \otimes S_0(F) \to S_1(F)$ that defines a vector bundle map $\sigma : \mathbb{S}^+(F) \to \mathbb{S}^-(F)$ covering $\text{id}_F$ which is an isomorphism outside the zero section of $F$. Since the ball bundle $\mathbb{B}(F)$ is a sub-bundle of $F$, we may form real spinor bundles over $\mathbb{B}^\pm(F)$ as the restriction bundles $\Delta^\pm(F) = \mathbb{S}^\pm(F)|_{\mathbb{B}^\pm(F)}$. We can then glue $\Delta^+(F)$ and $\Delta^-(F)$ along $\mathbb{S}(F) = \partial \mathbb{B}(F)$ by the Clifford multiplication map $\sigma$ giving a real vector bundle over $\widehat{M}$ defined by
\begin{equation}
H(F) = \Delta^+(F) \cup_{\sigma} \Delta^-(F).
\end{equation}
For each $p \in M$, the bundle $H(F)|_{\pi^{-1}(p)}$ is the real Bott generator vector bundle over the $n$-dimensional sphere $\pi^{-1}(p)$ [10].

**Definition 2.4.** Let $(M, E, \phi)$ be a KO-cycle on $X$ and $F$ a $C^\infty$ real spin vector bundle over $M$ with fibres of dimension $\dim_{\mathbb{R}} F_p \equiv 0 \mod 8$ for $p \in M$. Then the process of obtaining the KO-cycle $(\widehat{M}, H(F) \otimes \pi^*(E), \phi \circ \pi)$ from $(M, E, \phi)$ is called real vector bundle modification.

2.3. Topological KO-Homology. We are now ready to define the topological KO-homology groups of the space $X$.

**Definition 2.5.** The **topological KO-homology group** of $X$ is the abelian group obtained from quotienting $\Gamma O(X)$ by the equivalence relation $\sim$ generated by the relations of

(i) spin bordism;
(ii) direct sum: if $E = E_1 \oplus E_2$, then $(M, E, \phi) \sim (M, E_1, \phi) \sqcup (M, E_2, \phi)$; and
(iii) real vector bundle modification.

The group operation is induced by disjoint union of KO-cycles. We denote this group by $\text{KO}_0^t(X) := \Gamma O(X) / \sim$, and the homology class of the KO-cycle $(M, E, \phi)$ by $[M, E, \phi] \in \text{KO}_0^t(X)$.
Since the equivalence relation on $\text{FO}(X)$ preserves the dimension of $M$ mod 8 in $\text{KO}$-cycles $(M, E, \phi)$, one can define the subgroups $\text{KO}_t(X)$ consisting of classes of $\text{KO}$-cycles $(M, E, \phi)$ for which all connected components $M_i$ of $M$ are of dimension $\dim M_i \equiv n \mod 8$. Then

$$\text{(2.5)} \quad \text{KO}_t^i(X) = \bigoplus_{n=0}^7 \text{KO}_n^i(X)$$

has a natural $\mathbb{Z}_8$-grading.

The geometric construction of $\text{KO}$-homology is functorial. If $f : X \to Y$ is a continuous map, then the induced homomorphism

$$f_* : \text{KO}_t^i(X) \longrightarrow \text{KO}_t^i(Y)$$

of $\mathbb{Z}_8$-graded abelian groups is given on classes of $\text{KO}$-cycles $[M, E, \phi] \in \text{KO}_t^i(X)$ by

$$f_*[M, E, \phi] := [M, E, f \circ \phi].$$

One has $(\text{id}_X)_* = \text{id}_{\text{KO}_t^i(X)}$ and $(f \circ g)_* = f_* \circ g_*$. Since real vector bundles over $M$ extend to real vector bundles over $M \times [0, 1]$, it follows by spin bordism that induced homomorphisms depend only on their homotopy classes.

If $\text{pt}$ denotes a one-point topological space, then the collapsing map $\zeta : X \to \text{pt}$ induces an epimorphism

$$\text{(2.6)} \quad \zeta_* : \text{KO}_t^i(X) \longrightarrow \text{KO}_t^i(\text{pt}).$$

The reduced topological $\text{KO}$-homology group of $X$ is

$$\text{(2.7)} \quad \tilde{\text{KO}}_t^i(X) := \ker \zeta_*.$$

Since the map $\text{(2.6)}$ is an epimorphism with left inverse induced by the inclusion of a point $i : \text{pt} \hookrightarrow X$, one has $\text{KO}_t^{i}(X) \cong \text{KO}_t^{i}(\text{pt}) \oplus \tilde{\text{KO}}_t^{i}(X)$ for any space $X$. As in the complex case $\text{[52]}$, one has the following basic calculational tools for computing the geometric $\text{KO}$-homology groups.

**Proposition 2.6.** The abelian group $\text{KO}_t^i(X)$ enjoys the following properties:

(i) $\text{KO}_t^i(X)$ is generated by classes of $\text{KO}$-cycles $[M, E, \phi]$ where $M$ is connected.

(ii) If $\{X_j\}_{j \in J}$ is the set of connected components of $X$ then

$$\text{KO}_t^i(X) = \bigoplus_{j \in J} \text{KO}_t^i(X_j).$$

(iii) The homology class of a $\text{KO}$-cycle $(M, E, \phi)$ on $X$ depends only on the $\text{KO}$-theory class of $E$ in $\text{KO}_0^0(M)$; and

(iv) The homology class of a $\text{KO}$-cycle $(M, E, \phi)$ on $X$ depends only on the homotopy class of $\phi$ in $[M, X]$.

**2.4. Homological Properties.** We have not yet established that the geometric definition of $\text{KO}$-homology above is actually a (generalized) homology theory. Defining $\text{KO}_{t+8k}^i(X) := \text{KO}_t^i(X)$ for all $k \in \mathbb{Z}$, $0 \leq i \leq 7$, we will now show that $\text{KO}_t^i(X)$ is an 8-periodic unreduced homology theory. We know that $\text{KO}$-theory is an 8-periodic cohomology theory which can be defined in terms of its spectrum $\text{KO}^\infty$. For $n \geq 1$, let $\mathcal{H}_R$ be a real $\mathbb{Z}_2$-graded separable Hilbert space which is a $*$-module for the real Clifford algebra $\text{Cl}(\mathbb{R}^{n-1})$ as in Section 1.5. Let $\text{Fred}_n$ be the space of all Fredholm operators on $\mathcal{H}_R$ which are odd, $\text{Cl}(\mathbb{R}^{n-1})$-linear and self-adjoint. Then $\text{Fred}_n$ is the classifying space for $\text{KO}_t^i$. For $n \leq 0$, we choose $k \in \mathbb{N}$ such that $8k+n \geq 1$ and define $\text{Fred}_n := \text{Fred}_{8k+n}$. One then has $\text{KO}_t^i = \{\text{Fred}_n\}_{n \in \mathbb{Z}}$, and so we can define $\text{KO}_t^i$ a homology theory related to $\text{KO}^i$ by the inductive limit

$$\text{(2.8)} \quad \text{KO}_t^i(X, Y) := \lim_{n \to \infty} \pi_{n+i}((X/Y) \wedge \text{Fred}_n).$$
for all \( i \in \mathbb{Z} \), where \( Y \) is a closed subspace of the topological space \( X \) and \( \wedge \) denotes the smash product. Bott periodicity then implies that this is an 8-periodic homology theory.

One can give a definition of relative KO-homology groups \( KO^i_{t}(X, Y) \) in such a way that there is a map \( \mu^i : KO^i_{t}(X, Y) \to KO^i_{t}(X, Y) \) which defines a natural equivalence between functors on the category of topological spaces having the homotopy type of finite CW-pairs \((X, Y)\), where \( KO^i_{t}(X, Y) \) is Jakob's realization of KO-homology \([36]\). The building blocks of \( KO^i_{t}(X) \) are triples \((M, x, \phi)\) as in Definition 2.4 but now \( x \in KO^{0}(M) \) is a KO-theory class over \( M \) such that \( \text{dim} M + n \equiv i \mod 8 \). The equivalence relations are as in Definition 2.5 with real vector bundle modification modified from Definition 2.4 as follows. The nowhere zero section

\[
\Sigma^F : M \to F \oplus \mathbb{R}^n_M
\]

defined by

\[
\Sigma^F(p) = 0_p \oplus 1
\]

for \( p \in M \) induces an embedding

\[
\Sigma^F : M \to \hat{M}.
\]

Then real vector bundle modification is replaced by the relation

\[
(M, x, \phi) \sim (\hat{M}, \Sigma^F(x), \phi \circ \pi),
\]

where the functorial homomorphism \( \Sigma^F : KO^0(M) \to KO^0(\hat{M}) \) is the Gysin map induced by the embedding (2.9). On stable isomorphism classes of real vector bundles \([E] \in KO^0(M)\) one has

\[
\Sigma^F[E] = [H(F) \otimes \pi^*(E)].
\]

In the present category, \( KO^i_{t}(X, Y) \) is naturally equivalent to \( KO^i_{s}(X, Y) \).

One can give a spin bordism description of \( KO^i_{s}(X, Y) \) as follows. We consider the set \( \Gamma O(X, Y) \) of isomorphism classes of triples \((M, E, \phi)\) where

(i) \( M \) is a compact spin manifold with (possibly empty) boundary;
(ii) \( E \) is a real vector bundle over \( M \); and
(iii) \( \phi : M \to X \) is a continuous map with \( \phi(\partial M) \subset Y \).

The set \( \Gamma O(X, Y) \) is then quotiented by relations of relative spin bordism, which is modified from Definition 2.5 by the requirement that \( M_1 \Pi(-M_2) \subset \partial W \) is a regularly embedded submanifold of codimension 0 with \( \phi(\partial W \setminus M_1 \Pi(-M_2)) \subset Y \), direct sum, and real vector bundle modification, which is applicable in this case since \( S(F \oplus \mathbb{R}^n_M) \) is a compact spin manifold with boundary \( S(F \oplus \mathbb{R}^n_M)_{\partial M} \). The collection of equivalence classes is a \( \mathbb{Z}_8 \)-graded abelian group with operation induced by disjoint union of relative KO-cycles. One has \( KO^i_{s}(X, \emptyset) = KO^i_{t}(X) \).

**Theorem 2.7.** The map

\[
\mu^i : KO^i_{s}(X, Y) \to KO^i_{t}(X, Y)
\]

defined on classes of KO-cycles by

\[
\mu^i[M, E, \phi]_s = [M, [E], \phi]_t
\]

is an isomorphism of abelian groups which is natural with respect to continuous maps of pairs.

**Proof.** Taking into account the equivalence relations on \( \Gamma O(X, Y) \) used to define both KO-homology groups, the map \( \mu^i \) is well-defined and a group homomorphism. Let \([M, x, \phi]_s \in KO^i_{s}(X, Y)\) with \( m := \text{dim} M \). We may assume that \( M \) is connected and \( x \) is non-zero in \( KO^i(M) \). Then \( m - i \equiv n \mod 8 \). Consider the trivial spin vector bundle \( F = M \times \mathbb{R}^{n+7m+1} \) over \( M \). In this case the sphere bundle (2.9) is \( \hat{M} = M \times S^{n+7m+1} \) and the associated Gysin homomorphism in KO-theory is a map

\[
\Sigma^F_i : KO^i(M) \to KO^{i+7m+n}(\hat{M}).
\]
Since \( i + 7m + n \equiv (i + 7m + m - i) \mod 8 \equiv 0 \mod 8 \), one has \( \text{KO}^{i+7m+p}(\tilde{M}) \cong \text{KO}^0(\tilde{M}) \).

It follows that there are real vector bundles \( E, H \rightarrow \tilde{M} \) such that \( \Sigma^F_i(x) = [E] - [H] \), and so by real vector bundle modification one has \( [M, x, \phi]_s = [\tilde{M}, [E], \phi \circ \pi]_s - [\tilde{M}, [H], \phi \circ \pi]_s \) in \( \text{KO}'(X,Y) \). Therefore \( \mu^s([\tilde{M}, E, \phi \circ \pi]_t - [\tilde{M}, H, \phi \circ \pi]_t) = [M, x, \phi]_s \), and we conclude that \( \mu^s \) is an epimorphism.

Now suppose that \( \mu^s[M_1, E_1, \phi_1]_s = \mu^s[M_2, E_2, \phi_2]_s \) are identified in \( \text{KO}'(X,Y) \) through real vector bundle modification. Then, for instance, there is a real spin vector bundle \( F \rightarrow M_1 \) such that \( M_2 = M_1 \) and \( [E_2] = \Sigma^F_i[E_1] \). This implies that the Gysin homomorphism is a map

\[
\Sigma^F_i : \text{KO}^0(M_1) \rightarrow \text{KO}^0(M_1) \cap \text{KO}^r(\tilde{M}_1)
\]

where \( r = \dim F_p \) for \( p \in M_1 \). Since \( \text{KO}^0(M_1) \cap \text{KO}^r(\tilde{M}_1) \neq \{0\} \) in this case, we have \( r \equiv 0 \mod 8 \) which implies that these two homology classes are also identified in \( \text{KO}'(X,Y) \) through real vector bundle modification. As this is the only relation in \( \text{KO}'(X,Y) \) that might identify these classes without identifying them as KO-cycles, we conclude that \( \mu^s \) is a monomorphism and therefore an isomorphism.

Remark 2.8. Theorem 2.7 establishes the existence of a natural equivalence between covariant functors \( \text{KO}^t \cong \text{KO}' \). Since \( \text{KO}' \) is a homological realization of the homology theory associated with \( \text{KO}^t \)-theory, it follows that the same is true of \( \text{KO}' \). We have thus constructed an unreduced 8-periodic geometric homology theory dual to \( \text{KO}^t \)-theory. It is the periodicity mod 8 of the fibre dimensions of the spin vector bundle \( F \) used for real vector bundle modification in \( \text{KO}^t \) that accounts for the isomorphism \( \text{KO}^t \cong \text{KO}' \).

Having established that \( \text{KO} \)-homology is a generalized homology theory, we may throughout exploit standard homological properties (see [56] for example). In particular, there is a long exact homology sequence for any pair \( (X,Y) \). Because \( \text{KO}^t \) is an 8-periodic theory, this sequence truncates to a 24-term exact sequence. In the spin bordism description, the connecting homomorphism

\[
\partial : \text{KO}^h(X,Y) \rightarrow \text{KO}^{h-1}(Y)
\]

is given by the boundary map

\[
(2.11) \quad \partial[M, E, \phi] := [\partial M, E|_{\partial M}, \phi|_{\partial M}]
\]

on classes of KO-cycles and extended by linearity. \( \partial \) is natural and commutes with induced homomorphisms.

Other homological properties are direct translations of those of the complex case provided by [129], where a more extensive treatment can be found. For example, one has the usual excision property. If \( U \subset Y \) is a subspace whose closure lies in the interior of \( Y \), then the inclusion \( \varsigma^U : (X \setminus U, Y \setminus U) \hookrightarrow (X,Y) \) induces an isomorphism

\[
\varsigma^U_* : \text{KO}^t_j(X \setminus U, Y \setminus U) \cong \text{KO}^t_j(X,Y)
\]

of \( \mathbb{Z}_8 \)-graded abelian groups.

2.5. Products. There are two important products that can be defined on topological KO-homology groups. The cap product is the \( \mathbb{Z}_8 \)-degree preserving bilinear pairing

\[
\lrcorner : \text{KO}^0(X) \otimes \text{KO}^t_j(X) \rightarrow \text{KO}^t_j(X)
\]

given for any real vector bundle \( F \rightarrow X \) and KO-cycle class \( [M, E, \phi] \in \text{KO}^t_j(X) \) by

\[
[F] \lrcorner [M, E, \phi] := [M, \phi^*F \otimes E, \phi]
\]

and extended linearly. It makes \( \text{KO}^t_j(X) \) into a module over the ring \( \text{KO}^0(X) \). As in the complex case, this product can be extended to a bilinear form

\[
\lrcorner : \text{KO}^i(X) \otimes \text{KO}^j(X) \rightarrow \text{KO}^t_{i+j}(X)
\]
The construction utilizes Bott periodicity and the isomorphism $\text{KO}^{-n}(X) \cong \text{KO}^{0}(\Sigma^{n}X)$, where $\Sigma^{n}X = S^{n} \wedge X$ is the $n$-th iterated reduced suspension of the space $X$. The product $\cup : \text{KO}^{n}(X) \otimes \text{KO}^{l}(X) \rightarrow \text{KO}^{n+l}(X)$ is given by the pairing $\cup : \text{KO}^{0}(\Sigma^{n}X) \otimes \text{KO}^{l}_{i-n}(\Sigma^{n}X) \rightarrow \text{KO}^{l}_{i-n}(\Sigma^{n}X)$.

If $X$ and $Y$ are spaces, then the exterior product

$$
\times : \text{KO}^{i}_{1}(X) \otimes \text{KO}^{j}_{1}(Y) \rightarrow \text{KO}^{i+j}_{1}(X \times Y)
$$

given for classes of KO-cycles $[M, E, \phi] \in \text{KO}^{i}_{1}(X)$ and $[N, F, \psi] \in \text{KO}^{j}_{1}(Y)$ by

$$
[M, E, \phi] \times [N, F, \psi] := [M \times N, E \boxtimes F, (\phi, \psi)],
$$

where $M \times N$ has the product spin structure uniquely induced by the spin structures on $M$ and $N$, and $E \boxtimes F$ is the real vector bundle over $M \times N$ with fibres $(E \boxtimes F)_{(p,q)} = E_{p} \boxtimes F_{q}$ for $(p, q) \in M \times N$. This product is natural with respect to continuous maps. Unfortunately, in contrast to the complex case, we don’t have a version of the Künneth theorem for KO-homology.

Indeed, should such a formula exist, one could use it to show that $\text{KO}_{2}(pt) \otimes \text{KO}_{2}(pt)$ has to be a tensor product as modules over the ring $\text{KO}^{2}(pt)$. But this does not work correctly as pointed out by Atiyah in [3]. Moreover, for $A = B = \mathbb{C}$ considered as a real $C^{*}$-algebra, one has that the map

$$
\text{K}_{2}(A) \otimes \text{K}_{2}(B) \rightarrow \text{K}_{2}(A \otimes B)
$$

is not surjective. The correct framework for Künneth formula for real K-theory is united K-theory [21, 18], which is a machinery that involves real K-theory, complex K-theory, and self-conjugate K-theory, and has the property that its homological algebra behaves better. We will return to this point in Section 6.2.

2.6. The Thom Isomorphism. Let $X$ be an $n$-dimensional compact manifold with (possibly empty) boundary, and $\mathbb{B}(TX) \rightarrow X$ and $S(TX) \rightarrow X$ the unit ball and sphere bundles of $X$. An element $\tau \in \text{KO}^{n}(\mathbb{B}(TX), S(TX))$ is called a Thom class or an orientation for $X$ if $\tau|_{(\mathbb{B}(TX)_{x}, S(TX)_{x})} \in \text{KO}^{n}(\mathbb{B}(TX)_{x}, S(TX)_{x}) \cong \text{KO}^{0}(pt)$ is a generator for all $x \in X$ [37]. The manifold $X$ is said to be $KO$-orientable if it has a Thom class. In that case the usual cup product on the topological KO-theory ring yields the Thom isomorphism

$$
\mathfrak{T}_{X} : \text{KO}^{i}(X) \cong \text{KO}^{i+n}(\mathbb{B}(TX), S(TX))
$$

given for $i = 0, 1, \ldots, 7$ and $\xi \in \text{KO}^{i}(X)$ by

$$
\mathfrak{T}_{X}(\xi) := \pi_{\mathbb{B}(TX)}^{*}(\xi) \cup \tau,
$$

where $\pi_{\mathbb{B}(TX)} : \mathbb{B}(TX) \rightarrow X$ is the bundle projection. This construction also works by replacing the tangent bundle of $X$ with any O($r$) vector bundle $V \rightarrow X$, defining a Thom isomorphism

$$
\mathfrak{T}_{X,V} : \text{KO}^{i}(X) \cong \text{KO}^{i+r}(\mathbb{B}(V), S(V))
$$

given by

$$
(2.12) \quad \mathfrak{T}_{X,V}(\xi) := \pi_{\mathbb{B}(V)}^{*}(\xi) \cup \tau_{V},
$$

where the element $\tau_{V} \in \text{KO}^{r}(\mathbb{B}(V), S(V))$ is called the Thom class of $V$.

Any KO-oriented manifold $X$ of dimension $n$ has a uniquely determined fundamental class $[X]_{s} \in \text{KO}^{n}_{s}(X, \partial X)$, which is represented by the element $[X, \mathbb{B}^{r}, \text{id}_{X}]$ in $\text{KO}^{r}_{s}(X, \partial X)$. One then has the Poincaré duality isomorphism

$$
\Phi_{X} : \text{KO}^{i}(X) \cong \text{KO}^{n-i}_{s}(X, \partial X)
$$

given for $i = 0, 1, \ldots, 7$ and $\xi \in \text{KO}^{i}(X)$ by taking the cap product

$$
(2.13) \quad \Phi_{X}(\xi) := \xi \smile [X]_{s}.
$$
In particular, if \( X \) is a compact spin manifold of dimension \( n \) without boundary, then \( X \) is KO-oriented and so in this case we have a Poincaré duality isomorphism \([36, 52, 56]\) giving
\[
\text{KO}^i(X) \cong \text{KO}^{n-i}(X).
\]

The isomorphism \([2.14]\) may be compared with the universal coefficient theorem for KO-theory \([61, 29]\), which asserts that there is an exact sequence
\[
0 \longrightarrow \text{Ext}(\text{KO}^{i-1}(X), \mathbb{Z}) \longrightarrow \text{KO}^{i+1}(X) \longrightarrow \text{Hom}(\text{KO}^i(X), \mathbb{Z}) \longrightarrow 0
\]
for all \( i \in \mathbb{Z} \). The degree shift by 4 arises from the fact that \( \text{KO}^{-3}(\text{pt}) = 0 \) and that there is a cup product pairing \( \text{KO}^{-4}(\text{pt}) \otimes \text{KO}^{-i}(\text{pt}) \to \text{KO}^{-4}(\text{pt}) \cong \mathbb{Z} \). Under the same conditions as above, one then also has the Thom isomorphism in KO-homology
\[
(2.16)
\mathbb{T}^{X,V} : \text{KO}^i(X) \xrightarrow{\cong} \text{KO}^i(B(V), S(V)).
\]

3. The Isomorphism

One of the main results of this paper is an explicit realization of the isomorphism between topological and analytic KO-homology. The primary goal of this section is to prove the following result.

**Theorem 3.1.** There is a natural equivalence
\[
\mu^a : \text{KO}^1 \xrightarrow{\cong} \text{KO}^a
\]
between the topological and analytic KO-homology functors.

As for any (generalized) homology theory, the proof of this theorem is tantamount to proving that the map
\[
\mu^a : \text{KO}^1_n(\text{pt}) \longrightarrow \text{KO}^a_n(\text{pt})
\]
is an isomorphism for \( n = 0, 1, \ldots, 7 \). From the realization \([2.8]\) it follows that
\[
\text{KO}_n^1(\text{pt}) \cong \lim_k \pi_{n+8k}(\text{Fred}_0)
\cong \pi_n(\text{Fred}_0) \cong \overline{\text{KO}}^0(S^n).
\]

The main idea behind our proof is to show that there exist surjective “index” homomorphisms \( \text{ind}^t_n \) and \( \text{ind}^a_n \) such that the diagram
\[
(3.1)
\begin{array}{ccc}
\text{KO}^1_n(\text{pt}) & \xrightarrow{\mu^a} & \text{KO}^a_n(\text{pt}) \\
\downarrow \text{ind}^t_n & & \downarrow \text{ind}^a_n \\
\text{KO}^{-n}(\text{pt}) & & \\
\end{array}
\]
commutes for every \( n \). The KO-theory groups \( \text{KO}^{-n}(\text{pt}) \) appear here because they are the coefficient groups of the \( \text{KO}^t_n \) and \( \text{KO}^a_n \) homology theories. This setup is motivated by the fact \([10]\) that the map \( \mu^a \) and the commutativity of the diagram \([3.1]\) are intimately related to an index theorem, as we demonstrate explicitly in Section \([4.3]\) and hence the motivation behind our terminology above. Since the groups \( \text{KO}^{-n}(\text{pt}) \) are equal to either 0, \( \mathbb{Z} \) or \( \mathbb{Z}_2 \) depending on the particular value of \( n \), the commutativity of the diagram \([3.1]\) along with surjectivity of the index maps are sufficient to prove that \( \mu^a \) is an isomorphism. For clarity and later use, we will divide the proof into four parts. We will first give the constructions of the three maps in \([3.1]\) each in turn, and then present the proof of commutativity of the diagram.
3.1. The Map $\mu^a$. Let $(M, E, \phi)$ be a topological KO-cycle on $X$ with $\dim M = n$. We construct a corresponding class in $\text{KO}_n^a(X)$ as follows. Consider the Clifford bundle

$$\mathcal{G}(M) := P_{\text{Spin}}(M) \times_{\lambda_n} C_{\ell_n}$$

where $C_{\ell_n} = C\ell(\mathbb{R}^n)$, $\lambda_n : \text{Spin}(n) \to \text{End}(C_{\ell_n})$ is given by left multiplication with $\text{Spin}(n) \subset C\ell_n \subset C\ell_n$, and $P_{\text{Spin}}(M)$ is the principal $\text{Spin}(n)$-bundle over $M$ associated to the spin structure on the tangent bundle $TM$. Since $C_{\ell_n} = C\ell_0 \oplus C\ell_1$ is a $\mathbb{Z}_2$-graded algebra, it follows that

$$\mathcal{G}(M) = \mathcal{G}^0(M) \oplus \mathcal{G}^1(M)$$

is a $\mathbb{Z}_2$-graded real vector bundle over $M$ with respect to the $C\ell(TM)$-action. The Clifford algebra $C_{\ell_n}$ acts by right multiplication on the fibres whilst preserving the bundle grading (3.2).

Choose a $C^\infty$ Riemannian metric $g^M$ on $TM$. Let $\mathfrak{D}^M : C^\infty(M, \mathcal{G}(M)) \to C^\infty(M, \mathcal{G}(M))$ be the canonical Atiyah-Singer operator $[4]$ defined locally by

$$\mathfrak{D}^M = \sum_{i=1}^{n} e_i \cdot \nabla e_i^M,$$

where $\{e_i\}_{1 \leq i \leq n}$ is a local basis of sections of the tangent bundle $TM$, $\nabla e_i^M$ are the corresponding components of the spin connection $\nabla^M$, and the dot denotes Clifford multiplication. The operator $\mathfrak{D}^M$ is a $C_{\ell_n}$-operator $[39]$, i.e. one has

$$\mathfrak{D}^M(\Psi \cdot \varphi) = \mathfrak{D}^M(\Psi) \cdot \varphi$$

for all $\Psi \in C^\infty(M, \mathcal{G}(M))$ and all $\varphi \in C_{\ell_n}$, where $\cdot \varphi$ denotes right multiplication by $\varphi$. Since $\mathfrak{D}^M$ commutes with the $C_{\ell_n}$-action, the vector space $\ker \mathfrak{D}^M$ is a $C_{\ell_n}$-module.

We now construct a triple $(\mathcal{H}^M_E, \rho^M_E, T^M_E)$ comprising the following data:

(i) The separable real Hilbert space $\mathcal{H}^M_E := L^2(M, \mathcal{G}(M) \otimes E; dg^M)$;

(ii) The $*$-homomorphism $\rho^M_E : C(M, \mathbb{R}) \to \mathcal{L}(\mathcal{H}^M_E)$ defined by

$$\rho^M_E(f)(\Psi) = f(p) \Psi(p)$$

for $f \in C(M, \mathbb{R})$, $\Psi \in C^\infty(M, \mathcal{G}(M) \otimes E)$ and $p \in M$; and

(iii) The bounded Fredholm operator

$$T^M_E := \frac{\mathfrak{D}^M_E}{\sqrt{1 + (\mathfrak{D}^M_E)^2}}$$

acting on $\mathcal{H}^M_E$, where $\mathfrak{D}^M_E$ is the Atiyah-Singer operator $[39]$ twisted by the real vector bundle $E \to M$.

This triple satisfies the following properties:

(i) $\mathcal{H}^M_E$ is $\mathbb{Z}_2$-graded according to the splitting $[39]$ of the Clifford bundle;

(ii) $\rho^M_E(f)$ is an even operator on $\mathcal{H}^M_E$ for all $f \in C(M, \mathbb{R})$;

(iii) Since $M$ is compact, $T^M_E$ is an odd Fredholm operator which obeys the compactness conditions $[12]$ with $\rho^M_E(f)$; and

(iv) There are odd operators $\varepsilon_i$, $i = 1, \ldots, n$ commuting with both $\rho^M_E(f)$ and $T^M_E$ which generate a $C_{\ell_n}$-action on $\mathcal{H}^M_E$ as in $[13]$, and which are given explicitly as right multiplication by elements $\varepsilon_i$ of a basis of the vector space $\mathbb{R}^n$.

It follows that $(\mathcal{H}^M_E, \rho^M_E, T^M_E)$ is a well-defined $n$-graded Fredholm module over the real $C^*$-algebra $C(M, \mathbb{R})$.

We now define the map $\mu^a$ in $(3.1)$ by

$$\mu^a(M, E, \phi) := \phi_*(\mathcal{H}^M_E, \rho^M_E, T^M_E) = (\mathcal{H}^M_E, \rho^M_E \circ \phi^*, T^M_E),$$

where $\phi : (M, E, \phi) \to (M', E', \phi')$ is a bundle isomorphism that preserves the $C^*$-algebra $C(M, \mathbb{R})$. This defines the map $\mu^a$ for the $n$-graded Fredholm module $(\mathcal{H}^M_E, \rho^M_E, T^M_E)$ associated to the $n$-graded Clifford bundle.
where $\phi^* : C(X, \mathbb{R}) \to C(M, \mathbb{R})$ is the real $C^*$-algebra homomorphism induced by the map $\phi$. At this stage the map $\mu^a$ is only defined on $\text{KO}$-cycles. By adopting the proof of Theorem 6.1 in [12] for the complex case, we can straightforwardly arrive at the following result.

**Proposition 3.2.** The map $\mu^a : \text{KO}^n(X) \to \text{KO}^n_n(X)$ induced by ([3.2]) is a well-defined homomorphism of abelian groups for any $n \in \mathbb{N}$.

### 3.2. The Map $\text{ind}^a_n$

Let $(\mathcal{H}_\mathbb{R}, \rho, T)$ be an $n$-graded Fredholm module over the real $C^*$-algebra $C(X, \mathbb{R})$. Since the Fredholm operator $T$ commutes with $\varepsilon_i$ for $i = 1, \ldots, n$, the kernel $\ker T \subseteq \mathcal{H}_\mathbb{R}$ is a real $\ell_n$-module with $\mathbb{Z}_2$-grading induced by the grading of $\mathcal{H}_\mathbb{R}$. Thus we can define

$$\text{ind}^a_n(T) := [\ker T] \in \left( \widetilde{\mathcal{M}}_n/\iota^* \widetilde{\mathcal{M}}_{n+1} \right) \cong \text{KO}^{-n}(pt),$$

where $\widetilde{\mathcal{M}}_n$ is the Grothendieck group of real graded $\ell_n$ representations and $\iota^*$ is induced by the natural inclusion $\iota : \ell_n \hookrightarrow \ell_{n+1}$. We will call ([3.3]) the analytic or Clifford index of the Fredholm operator $T$. An important property of this definition is the following result [39].

**Theorem 3.3.** The analytic index

$$\text{ind}^a_n : \text{Fred}_n \longrightarrow \text{KO}^{-n}(pt)$$

is constant on the connected components of $\text{Fred}_n$.

Given two Fredholm modules $(\mathcal{H}_\mathbb{R}, \rho, T)$ and $(\mathcal{H}_\mathbb{R}, \rho, T')$ over a real $C^*$-algebra $A$, we will say that $T$ is a compact perturbation of $T'$ if $(T - T') \rho(a) \in \mathcal{K}_\mathbb{R}$ for all $a \in A$. We then have the following elementary result.

**Lemma 3.4.** If $T$ is a compact perturbation of $T'$, then the Fredholm modules $(\mathcal{H}_\mathbb{R}, \rho, T)$ and $(\mathcal{H}_\mathbb{R}, \rho, T')$ are operator homotopic over $A$.

**Proof.** Consider the path $T_t = (1 - t)T + tT'$ for $t \in [0, 1]$. Then the map $t \mapsto T_t$ is norm continuous. We will show that for any $t \in [0, 1]$, the triple $(\mathcal{H}_\mathbb{R}, \rho, T_t)$ is a Fredholm module over $A$, i.e. that the operator $T_t$ satisfies

$$\begin{align*}
(T_t^2 - 1) \rho(a), & \quad (T_t - T_t^*) \rho(a), & \quad T_t \rho(a) - \rho(a) T_t \in \mathcal{K}_\mathbb{R}
\end{align*}$$

for all $a \in A$. The last two inclusions in ([3.4]) are easily proven because the path $T_t$ is “linear” in the operators $T$ and $T'$. To establish the first one, for any $t \in [0, 1]$ and $a \in A$ we compute

$$\begin{align*}
(T_t^2 - 1) \rho(a) &= \left[ (T^2 - 1) + t^2 (T - T')^2 - t (T^2 - 1) - t (T - T')^2 + t (T'^2 - 1) \right] \rho(a).
\end{align*}$$

By using the fact that $(\mathcal{H}_\mathbb{R}, \rho, T)$ and $(\mathcal{H}_\mathbb{R}, \rho, T')$ are Fredholm modules, that $T$ is a compact perturbation of $T'$, and that $\mathcal{K}_\mathbb{R}$ is an ideal in $\mathcal{L}(\mathcal{H}_\mathbb{R})$, one easily verifies that the right-hand side of ([3.5]) is a compact operator. This implies that $(\mathcal{H}_\mathbb{R}, \rho, T_t)$ is a well-defined family of Fredholm modules over $A$. \hfill $\Box$

**Proposition 3.5.** The induced map

$$\text{ind}^a_n : \text{KO}^n_n(X) \longrightarrow \text{KO}^{-n}(pt)$$

given on classes of $n$-graded Fredholm modules by

$$\text{ind}^a_n[\mathcal{H}_\mathbb{R}, \rho, T] = [\ker T]$$

is a well-defined surjective homomorphism for any $n \in \mathbb{N}$.

**Proof.** We first show that to the direct sum of two Fredholm modules $(\mathcal{H}_\mathbb{R}, \rho, T)$ and $(\mathcal{H}'_\mathbb{R}, \rho', T')$ over $A = C(X, \mathbb{R})$, the map $\text{ind}^a_n$ associates the class $[\ker T] + [\ker T'] \in \widetilde{\mathcal{M}}_n/\iota^* \widetilde{\mathcal{M}}_{n+1} \cong \text{KO}^{-n}(pt)$. The kernel

$$\ker(T \oplus T') = \ker(T) \oplus \ker(T')$$
is a real graded $\text{C}\ell_n$-module. By the definition of the group $\hat{\mathfrak{m}}_n$ and of its quotient by $i^*\hat{\mathfrak{m}}_{n+1}$, one thus has $\text{ind}^n(T \oplus T') = [\ker T] + [\ker T']$ and so the map $\text{ind}^n$ respects the algebraic structure on $\Gamma \text{O}_n(A)$.

Consider now two Fredholm modules $(\mathfrak{H}_{\mathbb{R}}, \rho, T)$ and $(\mathfrak{H}'_{\mathbb{R}}, \rho', T')$ which are orthogonally equivalent. Then there exists an even isometry $U : \mathfrak{H}_{\mathbb{R}} \to \mathfrak{H}'_{\mathbb{R}}$ such that

$$T' = U T U^*, \quad \varepsilon'_i = U \varepsilon_i U^*.$$

This implies that $\ker T' = U(\ker T)$, and that the graded $\text{C}\ell_n$ representations given respectively by $\varepsilon'_i$ and $\varepsilon_i$ are equivalent. In particular, they represent the same class in $\hat{\mathfrak{m}}_n/i^*\hat{\mathfrak{m}}_{n+1}$.

Finally, consider two homotopic $n$-graded Fredholm modules $(\mathfrak{H}_{\mathbb{R}}, \rho, T)$ and $(\mathfrak{H}'_{\mathbb{R}}, \rho, T')$ over $A$. In general, $T$ and $T'$ are not elements of $\text{Fred}_n$ because they need not be self-adjoint. However, one can always perform a compact perturbation to obtain an equivalent Fredholm module whose operator is self-adjoint by simply replacing $T$ with $\hat{T} := \frac{1}{2} (T + T^*)$. By Lemma 3.4, compact perturbation implies operator homotopy and so there is no loss of generality in considering only homotopy of “self-adjoint” Fredholm modules. Then the function $t \mapsto T_t$ gives a homotopy $\hat{T}_t = \frac{1}{2} (T_t + T_t^*)$ in $\text{Fred}_n$ connecting $T$ and $T'$. The proposition now follows from Theorem 3.3 □

### 3.3. The Map $\text{ind}^n_A$

Given a KO-cycle $(M, E, \phi)$ on $X$ with $M$ an $n$-dimensional compact spin manifold, we can assign to it the Atiyah-Milnor-Singer (AMS) invariant $[\mathcal{A}]$ defined by

$$\mathcal{A}_E(M) = \beta \circ i^* \circ \zeta^*(\tau_E) \in \text{KO}^{-n}(pt)$$

where:

(i) $\nu$ is the normal bundle $N(S^{n+8k}/M)$, with projection $\varpi : \nu \to M$, identified with a tubular neighbourhood of an embedding

$$f : M \hookrightarrow S^{n+8k}$$

for some $k \in \mathbb{N}$ sufficiently large;

(ii) $\tau_E = \tau_\nu \sim [\varpi^* E] \in \text{KO}^0(\nu, \nu \setminus M)$ where

$$\tau_\nu := \left[\varpi^* S^+(\nu), \varpi^* S^-(\nu); \sigma\right] \in \text{KO}^0(\nu, \nu \setminus M)$$

is the Thom class of $\nu$, with $\sigma : \varpi^* S^+(\nu) \to \varpi^* S^-(\nu)$ given by Clifford multiplication;

(iii) $\zeta^* : \text{KO}^0(\nu, \nu \setminus M) \to \text{KO}^0(S^{n+8k}/S^{n+8k} \setminus M)$ is given by the excision theorem;

(iv) $i^* : \text{KO}^0(S^{n+8k}, S^{n+8k} \setminus M) \to \text{KO}^0(S^{n+8k})$ is given by the inclusion $i : (S^{n+8k}, pt) \hookrightarrow (S^{n+8k}, M)$; and

(v) $\beta : \overline{\text{KO}}^0(S^{n+8k}) \to \overline{\text{KO}}^0(S^n) := \text{KO}^{-n}(pt)$ is given by Bott periodicity.

This definition does not depend on the embedding $f$ nor on the integer $k$. We define

$$\text{ind}^n(M, E, \phi) := \mathcal{A}_E(M).$$

### Proposition 3.6

The map

$$\text{ind}^n : \text{KO}_{h}^n(X) \to \text{KO}^{-n}(pt)$$

induced by (3.10) is a well-defined surjective homomorphism for any $n \in \mathbb{N}$.

**Proof.** We first prove that the map $\text{ind}^n$ respects the algebraic structure on the abelian group $\text{KO}^n(X)$. Given two $n$-dimensional compact spin manifolds $M_1$ and $M_2$, let $M = M_1 \amalg M_2$. Embed $M$ in the sphere $S^{n+8k}$ for some $k$ sufficiently large as in (3.10). Then the normal bundle $\nu$ of this embedding is given by $N(S^{n+8k}/M_1) \amalg N(S^{n+8k}/M_2)$. Identify $\nu$ with a tubular
neighbourhood of the embedding given by $\nu_1 \amalg \nu_2$, with projection $\pi = \pi_1 \amalg \pi_2 : \nu_1 \amalg \nu_2 \to M_1 \amalg M_2$. The Thom class of $\nu$ is given by

$$\tau_\nu := [\pi^* s^+(\nu), \pi^* s^-(\nu); \sigma]$$

$$= [\pi_1^* s^+(\nu_1) \amalg \pi_2^* s^+(\nu_2), \pi_1^* s^-(\nu_1) \amalg \pi_2^* s^-(\nu_2); \sigma_1 \amalg \sigma_2]$$

$$= \tau_{\nu_1} \amalg \tau_{\nu_2} \in \text{KO}^0(\nu, \nu \setminus M) \cong \text{KO}^0(\nu_1, \nu_1 \setminus M_1) \amalg \text{KO}^0(\nu_2, \nu_2 \setminus M_2).$$

Let $E_1$ and $E_2$ be real vector bundles over $M_1$ and $M_2$, respectively, and let $E = E_1 \amalg E_2$. Then in $\text{KO}^0(\nu, \nu \setminus M)$ one has

$$\tau_\nu(E) = \tau_\nu \circ [\pi^* E] = \tau_{\nu_1} \amalg \tau_{\nu_2} \circ [\pi_1^* E_1 \amalg \pi_2^* E_2] = \tau_{\nu_1}(E_1) + \tau_{\nu_2}(E_2).$$

Using the fact that the maps $\varsigma^*, \iota^*$ and $\beta$ are group homomorphisms, one then finds

$$\text{ind}_n^\iota((M_1, E_1, \phi_1) \amalg (M_2, E_2, \phi_2)) := \overset{\wedge}{A}_E(M)$$

$$= \overset{\wedge}{A}_{E_1}(M_1) + \overset{\wedge}{A}_{E_2}(M_2)$$

$$= \text{ind}_n^\iota(M_1, E_1, \phi_1) + \text{ind}_n^\iota(M_2, E_2, \phi_2) \in \text{KO}^{-n}(pt),$$

showing that $\text{ind}_n^\iota$ is a homomorphism of abelian groups.

Next we have to check that the map $\text{ind}_n^\iota$ is independent of the choice of representative of a homology class in $\text{KO}_n^\iota(X)$. The independence of the direct sum relation follows from the discussion above, while spin bordism independence is guaranteed by the property that the AMS invariant $\overset{\wedge}{A}_E(M)$ is a spin cobordism invariant [4]. Finally, we have to verify that the map $\text{ind}_n^\iota$ does not depend on real vector bundle modification. We will give a fairly detailed proof of this result, as we believe it is instructive.

Let $M$ be a smooth $n$-dimensional compact spin manifold and let $E \to M$ be a smooth real vector bundle. Given an embedding (3.10), the AMS invariant of the pair $(M, E)$ may be written as [38]

$$\overset{\wedge}{A}_E(M) = \beta \circ f_1[E],$$

where

$$f_1 : \text{KO}^0(M) \longrightarrow \text{KO}^0(\mathbb{S}^{n+8l})$$

is the Gysin homomorphism of the embedding $f$. Let $F$ be a real spin vector bundle over $M$ with fibres of real dimension $8l$ for some $l \in \mathbb{N}$. Consider the corresponding sphere bundle [21] with projection (2.2). As discussed in Section 2.4 (see (2.9) and (2.10)), real vector bundle modification of a KO-cycle $(M, E, \phi)$ on $X$ induced by $F$ produces the KO-cycle $(\overset{\wedge}{M}, \overset{\wedge}{E}, \phi \circ \pi)$, where $\overset{\wedge}{E} = H(F) \otimes \pi^*(E)$ is the real vector bundle over $\overset{\wedge}{M}$ such that

$$[\overset{\wedge}{E}] = \Sigma^F[E]$$

with $[E] \in \text{KO}^0(M)$ and $[\overset{\wedge}{E}] \in \text{KO}^0(\overset{\wedge}{M})$. We may compute the AMS invariant for the pair $(\overset{\wedge}{M}, \overset{\wedge}{E})$ by choosing an embedding

$$\overset{\wedge}{f} : \overset{\wedge}{M} \hookrightarrow \mathbb{S}^{n+8k+8l}$$

so that

$$\overset{\wedge}{A}_{\overset{\wedge}{E}}(\overset{\wedge}{M}) = \beta \circ \overset{\wedge}{f}_1([\overset{\wedge}{E}])$$

$$= \beta \circ \overset{\wedge}{f}_1 \circ \Sigma^F[E] = \beta \circ (\overset{\wedge}{f} \circ \Sigma^F)[E],$$

where in the last equality we have used functoriality of the Gysin map. Notice that

$$\overset{\wedge}{f} \circ \Sigma^F : M \hookrightarrow \mathbb{S}^{n+8k+8l} = \mathbb{S}^{n+8m}$$
is an embedding of $M$ into a “large enough” sphere. Since $\hat{A}_E(M)$ is independent of the embedding and the integer $m$, we have

$$\hat{A}_E(\hat{M}) = \hat{A}_E(M)$$

as required. □

3.4. The Isomorphism Theorem. We can now assemble the constructions of Sections 3.1–3.3 above to finally establish our main result. Notice first of all that since $\ker \mathcal{D}^M_M E \cong \ker T^M_E$, one has

$$(3.12) \quad \ind_n^a \circ \mu^a(M, E, \phi) = \ind_n^a(\mathcal{D}^M_E)$$

for any KO-cycle $(M, E, \phi)$ on $X$ with $\dim M = n$. At this point we can use an important result from spin geometry called the $\Cl_n$-index theorem [39].

**Theorem 3.7.** Let $M$ be a compact spin manifold of dimension $n$ and let $E$ be a real vector bundle over $M$. Let

$$\mathcal{D}^M_E : C^\infty(M, \mathcal{S}(M) \otimes E) \to C^\infty(M, \mathcal{S}(M) \otimes E)$$

be the $\Cl_n$-linear Atiyah-Singer operator with coefficients in $E$. Then

$$\ind_n^a(\mathcal{D}^M_E) = \hat{A}_E(M).$$

The proof of Theorem 3.1 is now completed once we establish the following result.

**Proposition 3.8.** The map

$$\mu^a : \KO_n^k(\text{pt}) \to \KO_n^a(\text{pt})$$

is an isomorphism for any $n \in \mathbb{N}$.

**Proof.** As noticed at the beginning of this section, it suffices to establish the commutativity of the diagram (3.1), i.e. that

$$\ind_n^k = \ind_n^a \circ \mu^a.$$ 

Let $[M, E, \phi]$ be the class of a KO-cycle over pt with $\dim M = n$. Using Theorem 3.7 and (3.12) we have

$$\ind_n^k[M, E, \phi] := \hat{A}_E(M)$$

$$= \ind_n^a(\mathcal{D}^M_E) = \ind_n^a \circ \mu^a[M, E, \phi]$$

as required. □

4. The Real Chern Character

In this section we will describe the natural complexification map from geometric KO-homology to geometric K-homology and use it to define the Chern character homomorphism in topological KO-homology. We describe various properties of this homomorphism, most notably its intimate connection with the AMS invariant which was the crux of the isomorphism of the previous section.
4.1. The Complexification Homomorphism. Let $X$ be a compact topological space. Consider the topological, generalized homology groups $K^i_0(X)$ and $KO^i_0(X)$, along with the corresponding $K$-theory and $KO$-theory groups. The complexification of a real vector bundle over $X$ is a complex vector bundle over $X$ which is isomorphic to its own conjugate vector bundle. The complexification map is compatible with stable isomorphism of real and complex vector bundles, and thus defines a homomorphism from stable equivalence classes of real vector bundles to stable equivalence classes of complex vector bundles. It thereby induces a natural transformation of cohomology theories

$$(\otimes \mathbb{C}^*)_t : KO^t(X) \longrightarrow K^t(X)$$

given by

$$[E] - [F] \mapsto [E] - [F]$$

where $E_\mathbb{C} := E \otimes \mathbb{C}$ is the complexification of the real vector bundle $E \to X$.

We can also define a complexification morphism relating the homology theories

(4.1)

$$(\otimes \mathbb{C})_* : KO^i_0(X) \longrightarrow K^i_0(X)$$

by

$$[M, E, \phi] \otimes \mathbb{C} := [M, E_\mathbb{C}, \phi]$$

and extended by linearity, where on the right-hand side we regard $M$ endowed with the spin$^c$ structure induced by its spin structure as a $KO$-cycle. One can easily see that

(4.2)

$$[M, E, \phi] \otimes \mathbb{C} = \phi_*([E_\mathbb{C}] - [M]_K)$$

where $[M]_K \in K^2(M)$ denotes the $K$-theory fundamental class of $M$. Thus in the case when $X$ is $KO$-oriented (and therefore $K$-oriented), i.e. $X$ is a compact spin manifold, the homomorphism $(\otimes \mathbb{C})_*$ is just the Poincaré dual of $(\otimes \mathbb{C})^*$. This is clearly a natural transformation of homology theories.

A related natural transformation between cohomology theories is the realification morphism

$$(\mathbb{R}^*)_t : K^t(X) \longrightarrow KO^t(X)$$

induced by assigning to a complex vector bundle over $X$ the underlying real vector bundle over $X$. Because a spin$^c$ manifold is not necessarily spin, we cannot implement this transformation in the homological setting in general. Rather, we must assume that $X$ is a compact spin manifold. In this case the $K$-homology group $K^i_0(X)$ has generators $[2]$ $[X \times S^n, E_i, \text{pr}_1] - [X \times S^n, F_i, \text{pr}_1]$, $0 \leq n \leq 7$, where $\text{pr}_1 : X \times S^n \to X$ is the projection onto the first factor. We can then define the morphism

$$(\mathbb{R}^*)_t : K^i_0(X) \longrightarrow KO^i_0(X)$$

by

$$([X \times S^n, E_i, \text{pr}_1] - [X \times S^n, F_i, \text{pr}_1]) \mathbb{R} := [X \times S^n, E_i \mathbb{R}, \text{pr}_1] - [X \times S^n, F_i \mathbb{R}, \text{pr}_1]$$

and extending by linearity. Since this definition depends on a choice of generators for $K^i_0(X)$, the transformation is not natural. As for the complexification morphism, the morphism $(\otimes \mathbb{R})_*$ thus defined is Poincaré dual to $(\otimes \mathbb{R})^*$. It follows that the composition $(\otimes \mathbb{R})_* \circ (\otimes \mathbb{C})_*$ is multiplication by 2.

4.2. Chern Character in KO-Homology. We can use the natural transformation provided by the complexification homomorphism $[4, 6]$ to define a real homological Chern character

(4.3)

$$\text{ch}_\mathbb{R}^t : KO^t_0(X) \longrightarrow H^t_2(X, \mathbb{Q})$$

by

$$\text{ch}_\mathbb{R}^t(\xi) = \text{ch}_* (\xi \otimes \mathbb{C})$$
for \( \xi \in \text{KO}^j(X) \), where on the right-hand side we use the K-homology Chern character \( \text{ch}_E : \text{K}^j(X) \to H^j_\mathbb{Z}(X, \mathbb{Q}) \). Tensoring with \( \mathbb{Q} \) gives a map

\[
\text{ch}_E \otimes \text{id}_\mathbb{Q} = (\text{ch}_E \otimes \text{id}_\mathbb{Q}) \circ ((\otimes C)_* \otimes \text{id}_\mathbb{Q}) : \text{KO}^j(X) \otimes \mathbb{Q} \to H^j(X, \mathbb{Q})
\]

with \( \text{ch}_E \otimes \text{id}_\mathbb{Q} : \text{K}^j(X) \otimes \mathbb{Q} \to H^j(X, \mathbb{Q}) \) an isomorphism. The real Chern character is a natural transformation of homology theories.

An important point here is that the real Chern character requires a somewhat finer analysis than the usual Chern character. Although it detects all the homology classes, there can be KO-homology elements which have the same image under it because of the complexification map and the different periodicities of K-theory and KO-theory. For example, consider the KO-cycles \([pt, \mathbb{R}^2_{pt}, \text{id}_{pt}]\) and \([S^4, \mathbb{R}^2_{S^4}, \zeta]\) over \(pt\). They have the same image through \(\text{ch}^E_X\), namely the generator of \(H_0(pt, \mathbb{Q})\). But since they belong to different subgroups \(\text{KO}^j(pt)\) with respect to the grading of \(\text{KO}^j(pt)\), we conclude that these are the generators of the lattice \(\Lambda_{\text{KO}^j(pt)} := \text{KO}^j(pt) / \text{tor}_{\text{KO}^j(pt)}\). This fact will be important when we study brane constructions in the next section.

We can give a characteristic class description of \(\text{ch}^E_X\) as follows. Let \(\tau^E_{\text{KO}}\) be the KO-theory Thom class and \(\tau^H\) the cohomology Thom class of a real spin vector bundle \(E\) over \(X\). Let \(\text{ch}^* : \text{K}^2(X) \to H^2(X, \mathbb{Q})\) be the usual cohomology Chern character which is a multiplicative \(\mathbb{Z}_2\)-degree preserving natural transformation of cohomology theories. Denote by \(\tilde{A}(E) \in H^\text{even}(X, \mathbb{Q})\) the Atiyah-Hirzebruch class of \(E\). By using the analysis of natural transformations given in [36], along with the Hirzebruch formulation of the Riemann-Roch formula

\[
\text{ch}^* ((\tau^E_{\text{KO}}) \otimes C) = \tau^H \sim \tilde{A}(E)^{-1}
\]

and (1.2), one then has

(4.4) \[
\text{ch}^E_X(M, E, \phi) = \phi_* (\text{ch}^*(E_C) \sim \tilde{A}(TM) \sim [M])
\]

where \([M] \in H_0(M, \mathbb{Z})\) is the orientation cycle of the compact spin manifold \(M\). Since \(E_C \cong \overline{E_C}\) for any real vector bundle \(E \to X\), one has \(\text{ch}^* (E_C) = \text{ch}^* (\overline{E_C})\). Thus all components of the cohomology Chern character in the formula (1.4) of degree \(4j + 2\) vanish.

4.3. \(\text{Cl}_{n}\)-Index Theorems. We will now explore the relation between the homological real Chern character and the topological index defined in [3.11]. We first show that up to Poincaré duality the topological index is the homological morphism induced by the collapsing map. Recall that up to isomorphism, the AMS invariant is given by

\[
\tilde{A}_E(M) = \xi^\text{KO}_E[E]
\]

where \(M\) is a compact spin manifold of dimension \(n\), \(E\) is a real vector bundle over \(M\), \(\xi : M \to pt\) is the collapsing map on \(M\), and \(\xi^\text{KO}_M\) is the corresponding Gysin homomorphism on KO-theory. In this case we have

\[
\tilde{\xi}^\text{KO}_E = \Phi_M \circ \tilde{\xi}^\text{KO} \circ \Phi^{-1}_{E_{pt}}
\]

where \(\tilde{\xi}^\text{KO}_E\) is the induced morphism on \(\text{KO}^j(X, pt)\), and \(\Phi_{pt}\) and \(\Phi_M\) are the Poincaré duality isomorphisms on \(pt\) and \(M\), respectively. It then follows that

(4.5) \[
\Phi_{pt} \circ \text{ind}^1_n(M, E, \phi) = \Phi_{pt} \circ \tilde{\xi}^\text{KO}_E [E] = \Phi_{pt} \circ \tilde{\xi}^\text{KO} \circ \Phi_M^{-1}(M, E, \text{id}_M) = \tilde{\xi}^\text{KO}_E [M, E, \text{id}_M] = [M, E, \tilde{\xi}] = \xi^\text{KO}_E[M, E, \phi]
\]

where \(\xi : X \to pt\) is the collapsing map on \(X\) with \(\tilde{\xi} = \xi \circ \phi\).
We will next describe how the real Chern character can be used to give a characteristic class description of the map $\text{ind}_n^l$ in the torsion-free cases. Consider first the case $n \equiv 4 \mod 8$. We begin by showing that there is a commutative diagram

\[
\begin{array}{ccc}
\text{KO}_4^l(X) & \xrightarrow{\zeta_{\text{KO}}} & \text{KO}_4^l(pt) \\
\zeta^H \circ \text{ch}_* & \downarrow & \downarrow \text{ch}_* \\
H_0(pt, \mathbb{Q}) & & \\
\end{array}
\]

where $\zeta^H_*$ is the induced morphism on homology. Recall that $\text{ch}_* = \phi_* (\otimes \mathbb{C})_*$, where $(\otimes \mathbb{C})_*$ is the complexification map \([4]\). Then one has

\[
\zeta^H_* \circ \text{ch}_*(M, E, \phi) = \zeta^H_* \circ \phi_*(\text{ch}^*(E) \sim \widehat{A}(TM) \sim [M])
\]

\[
= \langle (\zeta \circ \phi)_* (\text{ch}^*(E) \sim \widehat{A}(TM) \sim [M]) \rangle
\]

\[
= \text{ch}^R(M, E, \zeta) = \text{ch}^R \circ \zeta_* \text{KO}(M, E, \phi).
\]

Now recall from Section 4.2 above that the map $\text{ch}^R_* : \text{KO}_0^l(pt) \to H_0(pt, \mathbb{Q})$ sends $\mathbb{Z} \to 2\mathbb{Z} \subset \mathbb{Q}$. On its image, the homomorphism $\text{ch}^R_*$ is thus invertible and its inverse is given as division by $2$. An explicit realization is gotten by noticing that

\[
\zeta^H_* \circ \text{ch}^R_*(M, E, \phi) = \zeta^H_* \circ \Phi_M (\text{ch}^*(E_C) \sim \widehat{A}(TM))
\]

\[
(4.7) = \Phi_{pt} \circ \zeta^H_*(\text{ch}^*(E_C) \sim \widehat{A}(TM)) = \langle \text{ch}^*(E_C) \sim \widehat{A}(TM) \rangle, [M] \rangle,
\]

where $\langle -,- \rangle : H^2(M, \mathbb{Q}) \times H^2(M, \mathbb{Q}) \to \mathbb{Q}$ is the canonical dual pairing between cohomology and homology. In \([17]\) we have used the fact that $\Phi_{pt}$ is the identity on $H_0(pt, \mathbb{Q}) \cong \mathbb{Q}$, and the proof of the last equality uses the Atiyah-Hirzebruch version of the Grothendieck-Riemann-Roch theorem and can be found in Section V.4.20 of \([37]\). Recall that for a spin manifold $M$ of dimension $4k + 8$, one has $\langle \text{ch}^*(E_C) \sim \widehat{A}(TM), [M] \rangle \in 2\mathbb{Z}$. After using the isomorphism $\text{KO}_4(pt) \cong \mathbb{Z}$, we thus deduce that $\zeta_* \text{KO}(M, E, \phi) = \frac{1}{2} \langle \text{ch}^*(E_C) \sim \widehat{A}(TM), [M] \rangle$, and from \((4.5)\) we arrive finally at

\[
\text{ind}_n^l(M, E, \phi) = \frac{1}{2} \langle \text{ch}^*(E_C) \sim \widehat{A}(TM), [M] \rangle.
\]

When $n \equiv 0 \mod 8$, one obtains a similar result but now without the factor $\frac{1}{2}$, since in that case $\text{ch}^R_* : \text{KO}_0^l(pt) \to H_0(pt, \mathbb{Q})$ is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. In the remaining non-trivial cases $n \equiv 1, 2 \mod 8$ the homological Chern character is of no use, as $\text{KO}^{-n}(pt)$ is the pure torsion group $\mathbb{Z}_2$, and there is no cohomological formula for the AMS invariant in these instances. However, by using Theorem 4.7 one still has an interesting mod 2 index formula for the topological index in these cases as well \([4]\). We can summarize our homological derivations of these index formulas as follows.

**Theorem 4.1.** Let $[M, E, \phi] \in \text{KO}_4^l(X)$, and let $\mathcal{D}_E^M$ be the Atiyah-Singer operator on $M$ with coefficients in $E$. Let $F_E^M := \ker \mathcal{D}_E^M$ denote the vector space of real harmonic $E$-valued spinors on $M$. Then one has the $\text{Cl}_n$-index formulas

\[
\text{ind}_n^l(M, E, \phi) = \left\{ \begin{array}{ll}
\frac{1}{2} \langle \text{ch}^*(E_C) \sim \widehat{A}(TM), [M] \rangle, & n \equiv 0 \mod 8, \\
\dim \mathbb{C} F_E^M \mod 2, & n \equiv 1 \mod 8, \\
\dim \mathbb{R} F_E^M \mod 2, & n \equiv 2 \mod 8, \\
0, & \text{otherwise}.
\end{array} \right.
\]
KO-HOMOLOGY AND TYPE I STRING THEORY

5. Brane Constructions in Type I String Theory

In Type I superstring theory with topologically trivial B-field, a D-brane in an oriented ten-dimensional spin manifold \( X \) is usually described by a spin submanifold \( W \hookrightarrow X \), together with a Chan-Paton bundle which is equipped with a superconnection and defined by an element \( \xi \in \text{KO}^0(X) \) (see [52] for a more precise treatment). In this section we will apply the mathematical formalism developed thus far to the classification and construction of Type I D-branes in topological KO-homology. The main new impetus that we will emphasize is the role of the AMS geometric invariant, which was the crucial ingredient in the proof of Section [3]. It will provide a precise, rigorous framework for certain physical aspects of Type I brane constructions.

5.1. Classification of Type I D-Branes. We begin with some general remarks concerning how KO-homology classifies D-brane charges in Type I superstring theory. Let \( X \) be a locally compact spin manifold of dimension 10. Let \( W \) be a spin submanifold of \( X \) of dimension \( p + 1 \). As in the complex case [52], KO-cycle representatives of classes \([M, E, \phi] \in \text{KO}^0_\text{cpt}(X)\) are naturally interpreted as Type I D-branes in the spacetime \( X \). Such a D-brane is said to ‘wrap’ \( W \) if \( \text{dim } M = p + 1 \) and \( \phi(M) \subset W \). Determining explicit representatives for the wrapped Type I D-branes in \( X \) is one of the main goals of the present section.

Let \( \nu_W \) be the normal bundle \( N(X/W) \to W \) of \( W \subset X \), equipped with the spin structure induced by the spin structure on \( W \) and on \( X \). By the Sen-Witten construction [59], the group of topological charges of \( D_p \)-branes wrapping \( W \) is the compactly supported KO-theory group

\[
\text{KO}^0_{\text{cpt}}(\nu_W) := \text{KO}^0 (\mathbb{B}(\nu_W), \mathbb{S}(\nu_W))
\]

where \( \mathbb{B}(\nu_W) \) and \( \mathbb{S}(\nu_W) \) are respectively the unit ball and sphere bundle of the normal bundle. To translate this into a statement in KO-homology, we use Poincaré duality to get

\[
\text{KO}^0_{\text{cpt}}(\nu_W) \cong \text{KO}^0_{\text{cpt}}(\nu_W) := \text{KO}^0_{10} (\mathbb{B}(\nu_W), \mathbb{S}(\nu_W))
\]

Let \( W' \) be a tubular neighbourhood of \( W \) in \( X \), which we can identify with the interior of the ball bundle \( \mathbb{B}(\nu_W) \setminus \mathbb{S}(\nu_W) \). It follows that the elements of the KO-homology group \( \text{KO}^0_{\text{cpt}}(\nu_W) \) can be naturally interpreted as spacetime-filling \( D_9 \)-branes (or \( D_9 \) brane-antibrane pairs) in \( X \). This requires extending the Chan-Paton bundles over \( W' \cong \mathbb{B}(\nu_W) \) to \( X \setminus W' \) in the standard way [6] [52] [50] [52], possibly by stabilizing with the addition of extra brane-antibrane pairs. In the following, we will always assume that this has been implicitly done and identify the normal bundle \( \nu_W \) with spacetime \( X \) itself.

These unstable \( D_9 \)-branes will have \( D_p \)-branes wrapping \( W \) as their decay products. To see this, we apply the Thom isomorphism \( (2.16) \) to the real spin vector bundle \( N(X/W) \to W \) of rank \( 9 - p \) to get

\[
\text{KO}^0_{\text{cpt}}(\nu_W) \cong \text{KO}^{10}_{p+1}(W)
\]

The elements of the group \( \text{KO}^{10}_{p+1}(W) \) are interpreted as (classes of equally charged) \( D_p \)-branes \([M, E, \phi] \) wrapping the worldvolume \( W \) with given Chan-Paton bundle \( E \to M \). Contrary to what happens in the KO-theory classification, the full graded KO-homology group \( \text{KO}^n_{\text{cpt}}(W) \) appears here.

Remark 5.1. The same interpretation holds in Type II superstring theory, when one works instead with K-homology. In particular, in the case of Type IIB strings on the spacetime \( X = \mathbb{R}^{10} \), one can show in a completely analogous way, along with Bott periodicity, that \( \text{K}^n_{p+1}(\mathbb{R}^{p+1}) \cong \text{K}^n_0(\mathbb{R}^{10}) \) for \( p \) odd, reproducing the usual statement [52] that the charge group of \( D_p \)-branes in Type IIB superstring theory on \( \mathbb{R}^{10} \) is \( \text{K}^n_0(\mathbb{R}^{10}) \). Here \( \text{K}^n_0(\mathbb{R}^n) \) denotes the reduced K-homology group \( \text{K}^n_0(\mathbb{S}^n) \).

The K-homology Thom isomorphism generally constitutes one of the main advantages of using K-homology instead of K-theory for the classification of D-branes. In particular, in Type I
string theory, the construction of (the class of) a Dp-brane in a system of D9-brane-antibrane pairs is somewhat tricky, and must be done for each value of p on a case-by-case basis \[59, 14\]. This is due to the degree shift and mod 8 periodicity of KO-theory, and because higher-degree KO-theory groups have no natural interpretations in term of D-branes, i.e. as vector bundles defined over the spacetime. (Higher KO-groups in fact arise through the chain of orientifolds one encounters when taking T-duals of the Type I theory and require the use of KR-theory \[15, 49\].) On the contrary, in KO-homology the degree of the KO-group naturally corresponds to the dimensionality of the brane, allowing one to use the Thom isomorphism \[5.3\] as an algorithmic technique for the construction above. The difference between the two classifications is that the group \[5.1\] measures charges of D-branes as defined by the behaviour of the Ramond-Ramond fields that they couple to. These fields are differential forms on spacetime X which should be classified by a (generalized) cohomology theory, such as KO-theory. The isomorphism KO\[^{0}\;\text{cpl}(nu_W) \rightarrow KO\[^{1}\;p+1(W)\] constructed above will be reinterpreted in the next section in terms of the “jump” in Ramond-Ramond fields across a D-brane \[17\].

The topological charges of the D-branes arising in this way are provided by the AMS invariant \[3.9\], or equivalently by the topological index as computed in the Cl\(_{n}\)-index theorems \[3.7\] and \[14\]. This naturally links the D-brane charge to a fermionic field theory on the brane world-volume, as \(\mathcal{D}_E^M\) is the Atiyah-Singer operator defined on sections of the irreducible spinor bundle over M coupled to the real vector bundle \(E \rightarrow M\). The precise form of the charge in Theorem \[4.1\] is dictated by whether the corresponding spinor representations are real, complex or pseudo-real. Most noteworthy are the (non-BPS) torsion charges. The AMS invariant in these instances gives a precise realization to the notion of a “\(\mathbb{Z}_2\) Wilson line” which is usually used in the physics literature for the construction of torsion D-branes in Type I string theory \[50, 59, 14\]. It is defined as a non-trivial element in the set of \(\mathbb{R}/\mathbb{Z}\)-valued gauge holonomies on \(M\) which are invariant under the involution which sends a complex vector bundle \(V\) to its complex conjugate \(\overline{V}\). Within our framework, it is determined instead by the coupling of the branes to the worldvolume fermions \(\psi\), valued in \(E\), which are solutions of the harmonic equation \(\mathcal{D}_E^M \psi = 0\). This provides a rigorous framework for describing the torsion charges, and moreover identifies the bundles used in tachyon condensation processes as the usual spinor bundles coupled to the Chan-Paton bundle \(E\). We will see some explicit examples in Section 5.3 below.

5.2. Wrapped Branes. Let us now make some of these constructions more explicit. Given the real Chern character, we can mimick some (but not all) of the constructions of Type II D-branes in complex K-homology. However, in light of the remarks made in Section 4.1 special care must be taken as the Chern character in the real case is not a rational injection. With this in mind, we have the following adaptation of Theorem 2.1 from \[52\].

**Theorem 5.2.** Let \(X\) be a compact connected finite CW-complex of dimension \(n\) whose rational homology can be presented as

\[
H_\ell(X, \mathbb{Q}) = \bigoplus_{p=0}^{n} \bigoplus_{i=1}^{m_p} [M^p_i] \mathbb{Q},
\]

where \(M^p_i\) is a \(p\)-dimensional compact connected spin submanifold of \(X\) without boundary and with orientation cycle \([M^p_i]\) given by the spin structure. Suppose that the canonical inclusion map \(i^p_i : M^p_i \hookrightarrow X\) induces, for each \(i, p\), a homomorphism \((i^p_i)_* : H_p(M^p_i, \mathbb{Q}) \rightarrow H_p(X, \mathbb{Q}) \cong \mathbb{Q}^{m_p}\) with the property

\[
(i^p_i)_* [M^p_i] = \kappa_{ip} [M^p_i]
\]

for some \(\kappa_{ip} \in \mathbb{Q} \setminus \{0\}\). Then the KO-homology lattice \(\Lambda_{KO^*_M(X)} = KO^*_M(X) / \text{tor}_{KO^*_M(X)}\) contains a set of linearly independent elements given by the classes of KO-cycles

\[
[M^p_i, \mathbb{I}^P_M, i^p_i], \quad 0 \leq p \leq n, \quad 1 \leq i \leq m_p.
\]
Proof. By [52] the cycles \([M^p_i, \mathbb{I}^R_{M^p_i}, v^p_i]\) form a basis for the lattice \(\Lambda_{K^i_4(X)} := K^i_4(X) / \text{tor}_{K^i_4(X)}\) in K-homology. The conclusion follows from the fact that

\[
[M^p_i, \mathbb{I}^R_{M^p_i}, v^p_i] \otimes \mathbb{C} = [M^p_i, \mathbb{I}^C_{M^p_i}, v^p_i],
\]
i.e. that the elements \(\text{ch}^R(M^p_i, \mathbb{I}^R_{M^p_i}, v^p_i)\) form a set of generators of \(H^i_2(X, \mathbb{Q})\).

Theorem 5.2 provides sufficient combinatorial criteria on the rational homology of \(X\) which ensure that torsion-free D-branes can wrap non-trivial spin cycles of the spacetime \(X\). As in the complex case, this is related to an analogous problem for the spin bordism group \(MSpin_4(X)\), which can also be defined in terms of a spectrum \(MSpin^\infty\). Just as in K-theory, the Atiyah-Bott-Shapiro (ABS) orientation map \(\mathbb{R} \xrightarrow{\text{MSpin}^\infty} \mathbb{KO}^\infty\) induces an \(MSpin_4(pt)\)-module structure on \(KO^i_4(pt)\). Then analogously to the complex case we have the following result [33].

**Theorem 5.3.** The map

\[
MSpin_4(X) \otimes_{MSpin_4(pt)} KO^i_4(pt) \longrightarrow KO^i_4(X), \quad [M, \phi] \longmapsto [M, \mathbb{I}^R_M, \phi]
\]

induced by the ABS orientation is a natural isomorphism of \(KO^i_4(pt)\)-modules for any finite CW-complex \(X\).

This immediately implies the following result, reducing the problem of calculating the KO-homology generators to the analogous problem in spin bordism.

**Theorem 5.4.** Let \(X\) be a finite CW-complex. Suppose that \([M_i, \phi_i], 1 \leq i \leq m\) are the generators of \(MSpin_4(X)\) as an \(MSpin_4(pt)\)-module. Then \([M_i, \mathbb{I}^R_{M_i}, \phi_i], 1 \leq i \leq m\) generate \(KO^i_4(X)\) as a \(KO^i_4(pt)\)-module.

In other words, for each \(n = 0, 1, \ldots, 7\) the group \(KO^i_n(X)\) is generated by elements \([M_i, \mathbb{I}^R_{M_i}, \phi_i], 1 \leq i \leq m\) with \(\dim M_i = n\).

### 5.3. Torsion Branes

We now describe a geometrical approach to the computation of torsion KO-cycle generators, thus elucidating the role of the AMS invariant in the construction of torsion D-branes. The general problem in KO-homology turns out to be much more involved than in the complex case. We discuss this further in Section 5.3 below. For now we will content ourselves with finding explicit representatives for the generators of the non-trivial groups \(KO^i_n(pt)\) with \(n = 0, 1, 2, 4\). This entails instructive exercises in the computations of topological indices which aid in better understanding the origins of Type I torsion D-brane charges. Recall from Section 5.2 that for the non-torsion cases \(n = 0\) and \(n = 4\), using the real Chern character one finds that the classes \([pt, \mathbb{I}^R_{pt}, \text{id}_{pt}]\) and \([S^4, \mathbb{I}^R_{S^4}, \zeta]\) are generators of the groups \(KO^0_0(pt) \cong \mathbb{Z}\) and \(KO^4_4(pt) \cong \mathbb{Z}\), respectively.

We begin with the group \(KO^4_1(pt)\). Consider the circle \(S^1\) and assign to it a Riemannian metric. Since there is only one unit tangent vector at any point of \(S^1\), one has \(P_{SO}(S^1) \cong S^1\). A spin structure on \(S^1\) is thus given by a double covering

\[
P_{\text{Spin}}(S^1) \longrightarrow S^1
\]

and by the fibration

\[
\mathbb{Z}_2 \longrightarrow P_{\text{Spin}}(S^1),
\]

\[
\mathbb{Z}_2 \longrightarrow P_{\text{Spin}}(S^1) \longrightarrow S^1
\]

There are only two double coverings of the circle, one disconnected and the other connected, given respectively by

\[
S^1 \times \mathbb{Z}_2 \longrightarrow S^1, \quad S^1_M \longrightarrow S^1
\]
where $S^1_M$ is the total space of the principal $\mathbb{Z}_2$-bundle associated to the Möbius strip. We will call these two spin structures the “interesting” and the “uninteresting” spin structures, respectively.

Corresponding to these two spin structures (labelled ‘i’ and ‘u’, respectively), we construct classes in $KO^i_1(pt)$ given by $[S^1, \mathbb{R}_{S^1}, \zeta]$ and $[S^1, \mathbb{R}_{S^1}, \zeta]$ where $\zeta : S^1 \to pt$ is as usual the collapsing map. We will now compute the topological indices in detail, finding the AMS invariants \[39\]

\[\hat{A}_{\mathbb{R}_{S^1}}(S^1) = 1, \quad \hat{A}_{\mathbb{R}_{S^1}}(S^1) = 0\]

in $KO^{-1}(pt) \cong \mathbb{Z}_2$. Hence the two classes above represent the elements of $KO^i_1(pt) \cong \mathbb{Z}_2$. In particular, $[S^1, \mathbb{R}_{S^1}, \zeta]$ is a generator, analogous to the non-BPS Type I D-particle that arises from tachyon condensation on the Type I D1 brane-antibrane system with a Wilson line [55 [59 [14].

Let us first consider the circle with the interesting spin structure. Since $C\ell_1 \cong \mathbb{C}$, one has $\Theta(S^1) := P_{Spin}(S^1) \times_{\mathbb{Z}_2} C\ell_1 \cong S^1 \times \mathbb{C}$. By decomposing $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, one has the identifications $\Theta^0(S^1) = S^1 \times \mathbb{R}$ and $\Theta^1(S^1) = S^1 \times i\mathbb{R}$. As the Clifford bundle is trivial, its space of sections is given by $C^\infty(S^1, \Theta(S^1)) = C^\infty(S^1, \mathbb{C})$. By coordinatizing the circle $S^1$ with arc length $s$, the Atiyah-Singer operator \[33\] can be expressed as

\[\hat{D}^S = i \frac{d}{ds}\]

where $e_1 = i$ is a generator of the Clifford algebra $C\ell_1$. To compute the topological index $\hat{A}_{\mathbb{R}_{S^1}}(S^1)$, we use the $C\ell_1$-index Theorem [3.7] and hence determine the vector space $\ker(\hat{D}^S)^0$, or equivalently the chiral subspace $\ker(\hat{D}^S)^0$. Since $C^\infty(S^1, \Theta^0) = C^\infty(S^1, \mathbb{R})$, the kernel of the chiral Atiyah-Singer operator $(\hat{D}^S)^0 : C^\infty(S^1, \Theta^0) \to C^\infty(S^1, \Theta^1)$ is given by the space of real-valued constant functions on $S^1$. The dimension of this vector space, as a module over $C\ell_1^0 \cong \mathbb{R}$, is 1 and hence

\[\text{ind}^1(S^1, \mathbb{R}_{S^1}, \zeta) = [\ker(\hat{D}^S)^0] = 1\]

in $\mathcal{M}_0/\ast\mathcal{M}_1 \cong KO^{-1}(pt) \cong \mathbb{Z}_2$. (Note that here we are using ungraded Clifford modules.)

We now turn to the uninteresting spin structure on $S^1$. This time the bundle $\Theta(S^1)$ is the (infinite complex) Möbius bundle. It can be described by a trivialization made of three charts $U_1$, $U_2$ and $U_3$ with $\mathbb{Z}_2$-valued transition functions $g_{12} = 1$, $g_{23} = 1$ and $g_{31} = -1$. In this case, the vector space $\ker(\hat{D}^S)^0$ consists of locally constant real-valued functions $\psi_i$ defined on $U_i$ which satisfy $\psi_j = g_{ij} \psi_i$ on the intersections $U_i \cap U_j \neq \emptyset$. Because of the non-trivial transition function $g_{31}$, there are no non-zero solutions $\psi$ to the equation $(\hat{D}^S)^0 \psi = 0$. The kernel $\ker(\hat{D}^S)^0$ is thus trivial, and so

\[\text{ind}^1(S^1, \mathbb{R}_{S^1}, \zeta) = 0\].

Let us now consider the structure of the group $KO^i_2(pt)$. Analogously to the construction above, one can equip the torus $T^2 = S^1 \times S^1$ with an “interesting” spin structure and show that

\[\hat{A}_{\mathbb{R}_{T^2}}(T^2) = 1, \quad \hat{A}_{\mathbb{R}_{T^2}}(S^2) = 0\]

in $KO^{-2}(pt) \cong \mathbb{Z}_2$. It follows that the classes $[T^2, \mathbb{R}_{T^2}, \zeta]$ and $[S^2, \mathbb{R}_{S^2}, \zeta]$ represent the elements of the group $KO^2_2(pt) \cong \mathbb{Z}_2$. In particular, $[T^2, \mathbb{R}_{T^2}, \zeta]$ is a generator, and it is analogous to the Type I non-BPS D-instanton which is usually constructed as the $\Omega$-projection of the Type IIB D(−1) brane-antibrane system [59 [14]. We will now give some details of these results.

Equip $T^2$ with the flat metric $d\theta_1 \otimes d\theta_1 + d\theta_2 \otimes d\theta_2$, where $(\theta_1, \theta_2)$ are angular coordinates on $S^1 \times S^1$. Since $T^2$ is a Lie group, its tangent bundle is trivializable, and hence the oriented
orthonormal frame bundle is canonically given by $P_{SO}(T^2) = T^2 \times S^1$. Consider the spin structure on $T^2$ given by

$$P_{Spin}(T^2) = T^2 \times S^1 \xrightarrow{id_{T^2} \times z^2} T^2 \times S^1.$$ 

Since $\mathcal{C}\ell_2 \cong \mathbb{H}$ and $\mathcal{C}\ell_0^0 \cong \mathbb{C}$, the corresponding Clifford bundles are $\mathfrak{S}(T^2) = T^2 \times \mathbb{H}$ and $\mathfrak{S}^0(T^2) = T^2 \times \mathbb{C}$. In the riemannian coordinates $(\theta_1, \theta_2)$, the Atiyah-Singer operator (3.3) can be expressed as

$$\mathfrak{S}^{T^2} = \sigma_1 \frac{\partial}{\partial \theta_1} + \sigma_2 \frac{\partial}{\partial \theta_2}$$

where the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

represent the generators $e_1$, $e_2$ of $\mathcal{C}\ell_2$, acting by left multiplication. The chiral operator $(\mathfrak{S}^{T^2})^0$ is locally the Cauchy-Riemann operator, and hence its kernel consists of holomorphic sections of the chiral Clifford bundle $\mathfrak{S}^0(T^2)$. These are simply the complex-valued constant functions on $T^2$, as the torus is a compact complex manifold. As a module over $\mathcal{C}\ell_0^0$, this vector space is one-dimensional and so

$$\text{ind}_{T^2}^1(\mathbb{H}, \mathbb{C}) = [\ker(\mathfrak{S}^{T^2})^0] = 1$$

in $M_1/\mathbb{C} \cong KO^{-2}(pt) \cong \mathbb{Z}_2$.

Consider now the two-sphere $S^2$ as a riemannian manifold. It is not difficult to see that

$$P_{SO}(S^2) = SO(3) \twoheadrightarrow SO(3)/SO(2) \cong S^2$$

is the oriented orthonormal frame bundle over $S^2$. The (unique) spin structure on $S^2$ is thus given by

$$P_{Spin}(S^2) \cong SU(2) \xrightarrow{h} P_{SO}(S^2) \cong SO(3) \twoheadrightarrow \mathbb{CP}^1 \cong S^2$$

with $h : SU(2) \to SO(3)$ the usual double covering, and by

$$U(1) \longrightarrow P_{Spin}(S^2) \xrightarrow{\text{}} \mathbb{CP}^1 \cong S^2$$

which is the Hopf fibration of $S^2$. Recall that the group $\text{Spin}(2) \cong U(1) \cong SO(2)$ acts on $\mathcal{C}\ell_2 \cong \mathbb{H}$ as multiplication by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

If one gives the sphere $S^2$ the structure of the complex projective line $\mathbb{CP}^1$, then there are isomorphisms $\mathfrak{S}^0(S^2) = P_{Spin}(S^2) \times_{U(1)} \mathbb{C} \cong \mathbb{T}^{1,1}\mathbb{CP}^1$ since the bundle $\mathfrak{S}^0(S^2)$ has the same transition functions as the Hopf fibration. In other words, $\mathfrak{S}^0(S^2)$ is isomorphic to the canonical line bundle $L_{\mathcal{C}}$ over $\mathbb{CP}^1$. The vector space $\ker(\mathfrak{S}^{S^2})^0$ thus consists of the holomorphic sections of $L_{\mathcal{C}}$. The only such section on $\mathbb{CP}^1$ is the zero section, and we finally find

$$\text{ind}_{S^2}^1(\mathbb{C}, \mathbb{C}, \zeta) = [\ker(\mathfrak{S}^{S^2})^0] = 0$$

in $M_1/\mathbb{C} \cong \mathbb{Z}_2$. 
5.4. General Constructions. The analysis of Section 5.3 above shows that the problem of finding generators of the geometric KO-homology groups of a space $X$, representing the Type I D-branes in $X$, becomes increasingly involved at a very rapid rate. Even in the case of spherical D-branes, we have not been able to find a nice explicit solution in the same way that can be done in the complex case [52]. Nevertheless, at least in these cases we can find a formal solution as follows, which also illustrates the generic problems at hand.

Suppose that we want to construct generating branes for the group $KO^k_k(\mathbb{S}^n)$ for some $n > 0$. Poincaré duality gives the map

$$KO^{n-k}(\mathbb{S}^n) \rightarrow KO^k_k(\mathbb{S}^n), \quad \xi \mapsto \Sigma \xi \in [\mathbb{S}^n, \mathbb{R}^{\mathbb{S}^n}, id_{\mathbb{S}^n}].$$

As Poincaré duality is a group isomorphism, picking a generator in $KO^{n-k}(\mathbb{S}^n)$ will give a generator in $KO^k_k(\mathbb{S}^n)$. But the problem is that the class $\xi$ is not a (virtual or stable) vector bundle over $\mathbb{S}^n$ in the cases of interest $k < n$. To this end, we rewrite the cap product in (5.6) by using the suspension isomorphism $\Sigma$ and the desuspension $\Sigma^{-1}$ to get

$$\xi \in [\mathbb{S}^n, \mathbb{R}^{\mathbb{S}^n}, id_{\mathbb{S}^n}] = \Sigma^{-1}\left(\Sigma(\xi) \in \Sigma[\mathbb{S}^n, \mathbb{R}^{\mathbb{S}^n}, id_{\mathbb{S}^n}]\right).$$

As we are interested only in generators, we can substitute $\Sigma(\xi)$ with the generators of the KO-theory group $KO^0(\mathbb{S}^{n-k}\mathbb{S}^n) = KO^0(\mathbb{S}^{2n-k})$. The generators of the latter groups are given by [37] the canonical line bundle $L_F$ over the projective line $\mathbb{P}^1$, with $\mathbb{F}$ the reals $\mathbb{R}$ for $k = 2n - 1$, the complex numbers $\mathbb{C}$ for $k = 2n - 2$, the quaternions $\mathbb{H}$ for $k = 2n - 4$ and the octonions $\mathbb{O}$ for $k = 2n - 8$ (the remaining groups are trivial up to Bott periodicity).

6. Fluxes

In this final section we shall explore the classification of Type I Ramond-Ramond (RR) fields, in the absence of D-branes, using the language of topological KO-homology. As in Section 5.3 most of what we say has a direct counterpart in complex K-theory for Type II RR fields, but we shall stick to the real case to further emphasize the role of the homology gradings. We will find again a crucial role played by a certain invariant, analogous to the AMS invariant but this time determined by the holonomy of RR-fields over background D-branes. We will see that these holonomies find their most natural interpretation within the context of geometric KO-homology. Along the way we will also propose a physical interpretation of KK-theory.

6.1. Classification of Type I Ramond-Ramond Fields. We will start with a description of how the Ramond-Ramond fluxes in Type I string theory naturally fit into the framework of topological KO-homology, and then propose in Section 6.2 below a unified description of the couplings of D-branes to RR fields using the bivariant KO-theory constructed in Section 1.4. The Type I Ramond-Ramond fields (or ‘$B$-fields’) are classified by a local formulation of KO-theory called “differential KO-theory”, a specific instance of a generalized differential cohomology theory which provides a characterization in terms of bundles with connection [27, 34, 29]. Consider the short exact sequence of coefficient groups given by

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 1,$$

and use it to define the KO-theory groups $KO^i\mathbb{R}/\mathbb{Z}(X)$ of a space $X$ with coefficients in the circle group $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ as the KO$^i$(pt)-module theory which fits into the corresponding long exact sequence

$$\cdots \rightarrow KO^i(X) \rightarrow KO^i(X) \otimes \mathbb{R} \rightarrow KO^i\mathbb{R}/\mathbb{Z}(X) \rightarrow KO^{i+1}(X) \rightarrow \cdots.$$

Then the RR fluxes in Type I string theory on $X$, in the absence of D-branes, are classified by the group $KO^{-1}_{\mathbb{R}/\mathbb{Z}}(X)$ [47].
If $X$ is a finite-dimensional smooth spin manifold of dimension ten, then by using the real cohomology Chern character the RR phases are described by the short exact sequence

$$
(6.1) \quad 0 \rightarrow H^{\text{odd}}(X, \mathbb{R})/\Lambda_{\text{KO}}^{-1}(X) \rightarrow \text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \rightarrow \text{tor}_{\text{KO}^0(X)} \rightarrow 0
$$

where $\beta$ is the Bockstein homomorphism. Thus the identity component of the circle coefficient KO-theory group $\text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X)$ is the torus $H^{\text{odd}}(X, \mathbb{R})/\Lambda_{\text{KO}}^{-1}(X)$. The cohomology class of an element in this component is determined by the real Chern character $\text{ch}^\ast_{\mathbb{R}}$, which is an epimorphism on $\Lambda_{\text{KO}}^{-1}(X) \rightarrow H^{\text{odd}}(X, \mathbb{R})$. On the other hand, the group of components of $\text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X)$ is given by the torsion classes in $\text{KO}^0(X)$, and the corresponding RR fluxes are represented by real (virtual) vector bundles over $X$. Thus the flat RR fluxes in $\text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X)$ coincide with the torsion Type I D-brane charges $\text{tor}_{\text{KO}^0(X)}$.

A torsion RR flux $\xi \in \text{KO}^0(X)$ gives an additional phase factor to a D-brane in the string theory path integral, which we will realize in Section 6.3 below in terms of an analog of the AMS invariant $\tilde{A}_\xi(M)$ introduced in Section 3.3 [17]. Generally, the origin of these phases can be understood from the topological classification of the physical coupling between Type I D-branes and RR fields. For this, we exploit Pontrjagin duality of KO-theory [29].

**Proposition 6.1.** There is a natural isomorphism

$$
\text{KO}^i_{\mathbb{R}/\mathbb{Z}}(X) \cong \text{Hom}(\text{KO}^i_{\mathbb{R}(-4)}(X), \mathbb{R}/\mathbb{Z})
$$

for all $i \in \mathbb{Z}$.

**Proof.** Apply the universal coefficient theorem (2.15), and use the fact that the circle group $\mathbb{R}/\mathbb{Z}$ is divisible which implies $\text{Ext}(\text{KO}_j^i(X), \mathbb{R}/\mathbb{Z}) = 0$ for all $j \in \mathbb{Z}$. \qed

**Remark 6.2.** The surjection in (2.15) is given by the index pairing determined by the analytic index constructed in Section 5.3. Thus the isomorphism provided by Proposition 6.1 above is determined by an index map involving the Atiyah-Singer operator (3.3), and hence the AMS invariant (3.3). Let $W$ be a compact spin submanifold of $X$ of dimension $p + 1$. We could then identify the spacetime $X$ with the normal bundle $\nu_W$ as in Section 5.1. However, for the present discussion it is more convenient to work with compact closed manifolds $X$, so we replace $\nu_W$ with its sphere bundle $S(\nu_W)$. Thus in the following spacetime is regarded as a spin fibration $\pi : X \rightarrow W$ whose fibres are spheres $X/W \cong S^{p+1}$.

By Proposition 6.1 and the Thom isomorphism (2.16), the group of RR fluxes is given by

$$
\text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \cong \text{Hom}(\text{KO}_5^1(X), \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(\text{KO}^1_{p-14}(W), \mathbb{R}/\mathbb{Z})
$$

and by Bott periodicity we have finally

$$
(6.2) \quad \text{KO}^{-1}_{\mathbb{R}/\mathbb{Z}}(X) \cong \text{Hom}(\text{KO}_{p+2}^1(W), \mathbb{R}/\mathbb{Z})
$$

The KO-homology group $\text{KO}_{p+2}^1(W)$ consists of wrapped Type I D-branes $[M, E, \phi]$ with the properties $\dim M = p + 2$ and $\phi(M) \subset W$. The dimension shift is related to the topological anomaly in the worldvolume fermion path integral [17], as we now explain.

Consider a one-parameter family of $p + 1$-dimensional Type I brane worldvolumes specified by a circle bundle $U \rightarrow W$ whose total space $U$ is generically a $p + 2$-dimensional submanifold of spacetime $X$ with the topology of $W \times S^1$. Real vector bundles $E_g$ of rank $n$ over generic fibres $U/W \cong S^1$ are determined by elements $g \in \text{SO}(n)$ by the clutching construction (analogously to Section 2.2). Thus the family of self-adjoint twisted Atiyah-Singer operators $D^{-1}_{E_g}$ determined by $g$ is parametrized by the group $\text{SO}(n)$. The anomaly [25] arises as the real determinant line bundle of this family, which is essentially defined as the highest exterior power of the kernel of the family. This defines a non-trivial real line bundle on the group $\text{SO}(n)$ called...
the Pfaffian line bundle, which has the property that its lift to Spin$(n)$ is the trivial real line bundle. One can also construct a connection and holonomy of the Pfaffian line bundle [28]. As in Section 5.1 $U$ is wrapped by D-branes in $K^0_{(p+2)}(U)$. One can now restrict to the subgroup $K^0_{p+2}(W) \subset K^0_{p+2}(U)$ by keeping only those D-branes which are wrapped on the embedding $W \hookrightarrow U$ by the zero section of $U \to W$. The isomorphism [82] reflects the fact that the topological anomaly is cancelled by coupling D-branes to the RR fields through the RR phase factors. This cancellation necessitates that the worldvolume $W$ be a spin manifold [59, 28].

6.2. Generalized D9-Brane Decay. The couplings described in Section 6.1 above are intimately related to a topological classification of the D9-brane decay described in Section 5.1, which lends a physical interpretation to the bivariant KO-theory groups introduced in Section 1.4. Let us explain this first for the simpler case of Type II D-branes and complex KK-theory. Consider the KK-theory groups $KK_i(X,W) := KK_{i,0}(C(X,\mathbb{C}), C(W,\mathbb{C}))$. By the Rosenberg-Shochet universal coefficient theorem [53], one then has a split short exact sequence of abelian groups given by

$$0 \to Ext^i_Z(K^{i+1}(X), K^i(W)) \to KK_i(X,W) \to \text{Hom}_Z(K^i(X), K^i(W)) \to 0$$

for all $i \in \mathbb{Z}$. Composition of group morphisms with Poincaré duality $K^i(W) \cong K^i_{p+1-i}(W)$ gives

$$0 \to Ext^i_Z(K^{i+1}(X), K^i_{p+1-i}(W)) \to KK_i(X,W) \to \text{Hom}_Z(K^i(X), K^i_{p+1-i}(W)) \to 0$$

(6.3)

For $i = 0$ the sequence (6.3) expresses the fact that the elements of the free part of the abelian group $KK_0(X,W)$ correspond to classes of morphisms $K^{1}_{10}(X) \cong K^0(X) \to K^0_{p+1}(W)$, generalizing the brane decay described by the analog of isomorphism (5.3) in K-Homology. We may thus interpret $KK_0(X,W)$ as the group of “generalized D9-brane decays”. An example of such a generalized decay can be straightforwardly given in the case $p \equiv 1 \mod 2$ (for $p = 1$, $W$ is the worldvolume of a D-string). Moreover, suppose that $W$ is a spin manifold. Then there is a direct image map on K-theory

$$\pi_! : K^0(X) \to K^0(W)$$

(6.4)

given by taking the intersection product of Section 1.6 (see Proposition 1.21) by the longitudinal element in $KK_0(X,W)$ [28], defined by the fibrewise Atiyah-Singer operator on the spin fibration $\pi : X \to W$ as follows. Fix a spin structure and a Riemannian metric $g^{X/W}$ on the relative tangent bundle $T(X/W)$. This determines a bundle $\mathcal{S}(X/W) \to X$ of Clifford algebras. Let $H_X$ be a horizontal distribution of planes on $X$, so that $H_X \oplus T(X/W) = TX$, which together with the metric determines a spin connection $\nabla^{X/W}$ on $T(X/W) \to X$. For any $w \in W$ we let $\mathcal{P}_w^{X/W}$ be the corresponding Atiyah-Singer operator [33] along the fibre $\pi^{-1}(w) \cong S^2$ acting on $C^\infty(X/W, \mathcal{S}(X/W))$. Define the corresponding closure $T^{X/W}_w$ analogously to (3.4). This defines a continuous family $\{T^{X/W}_w\}_{w \in W}$ of bounded Fredholm operators over $W$ acting on an infinite-dimensional Hilbert bundle $\mathfrak{H}^{X/W} \to W$, whose fibre at $w \in W$ is $\mathfrak{H}_w^{X/W} = L^2(X/W, \mathcal{S}(X/W); dg^{X/W})$. By the Atiyah-Singer index theorem [5], the topological index $\pi_!(\xi)$ is equal to the analytic index of the family of Atiyah-Singer operators $\{\mathcal{P}_w^{X/W}\}_{w \in W}$ on $X/W$ appropriately twisted by $\xi \in K^0(X)$.

On the other hand, for $i = -1$ one sees from (5.3) that torsion-free elements of the group $KK_{-1}(X,W)$ correspond to classes of morphisms $K^{-1}(X) \to K^i_{p+2}(W)$ linking RR fields to anomaly cancelling D-branes. Any such morphism gives an element of $KK_{-1}(X,W)$, but not conversely. The obstruction consists of classes of group extensions of $K^i_{p+2}(W)$ by $K^i(X)$, which we may interpret as bound states of anomaly cancelling D-branes wrapping the worldvolume $W$ and D9-branes wrapped on spacetime $X$. This property seems to reflect the fact [29] that
flux operators which correspond to torsion elements of K-theory do not all commute among themselves, as a result of the torsion link pairing provided by Pontrjagin duality. In this way the KK-theory group KK_{-1}(X,W) naturally captures the correct topological classification of RR fluxes after quantization. Note that if we disregard the ambient spacetime by setting X = pt, then we recover the group KK_{-1,0}(C(C(W,C)) \cong K^{-1}(W) \cong K_{p+2}(W) which relates to anomaly cancelling D-branes wrapped on the worldvolume W.

As in Section 2.5, the Type I case is more subtle. Indeed, the universal coefficient theorem proven in [53] is not valid in the case of real C*-algebras, due to obstructions that lie in the homological algebra [3]. One still has the homomorphism

\[ \text{KKO}_i(X,W) \rightarrow \text{Hom}_\mathbb{Z}(\text{KO}^i(X), \text{KO}^{i+1}(W)) \]

but this is no longer surjective. Again, a universal coefficient theorem exists in united KK-theory [19], giving rise to a homomorphism

\[ \text{KKO}_0(X,W) \rightarrow [\text{K}^{\text{crt}}(X), \text{K}^{\text{crt}}(W)] \]

where K^{crt} is united K-theory and \([-,-]\) is given by all CRT-module homomorphisms of degree 0. Most probably, this can have an interpretation in term of generalized D9-brane decay in Type I string theory, though we have not investigated the details of this.

6.3. Holonomy over Type I D-Branes. To make the discussion at the end of Section 6.1 above more precise, we need to refine our analysis by considering a larger collection of triples and finding an appropriate invariant substituting the AMS invariant. This is necessary to take into account the particular role played by the RR fluxes in the string theory path integral. To give a homological description of the coupling of D-branes to RR fields, we must first of all remember that the topological classification given in Section 6.1 above is valid only for Type I RR fields in homological description of the coupling of D-branes to RR fields, we must first of all remember that the topological classification given in Section 6.1 above is valid only for Type I RR fields in spacetime which are not sourced by D-branes. Thus given a KO-cycle \((M, E, \phi)\) on X wrapping W, instead of considering the one-parameter family \(U \rightarrow W\) of brane worldvolumes above, we will assume the existence of a compact smooth spin manifold \(\tilde{M}\) with boundary \(\partial \tilde{M} = M\) and dimension \(n+1\) when \(\dim M = n\). Suppose in addition that there exists a real vector bundle \(\tilde{E} \rightarrow \tilde{M}\) with \(\tilde{E}|_{\partial \tilde{M}} \cong E\), and a continuous map \(\tilde{\phi} : \tilde{M} \rightarrow X\) such that \(\tilde{\phi}|_{\partial \tilde{M}} = \phi\). Then \((M, E, \phi)\) is spin bordant to the trivial KO-cycle \((\emptyset, \emptyset, \emptyset)\), and so \([M, E, \phi] = 0\) in \(\text{KO}^2(X)\). The charge of this D-brane thus vanishes and so it cannot source any RR fields, as required. We call such a triple \((M, E, \phi)\) a “background D-brane”, because it should be regarded as equivalent to the closed string vacuum. Any neighbourhood of the boundary in \(\tilde{M}\) looks like a product \(M \times I\), with \(I = [0,1]\) the unit interval, and so locally the extension of \(M\) mimics the fibrations \(U \rightarrow W\) considered previously.

By (6.1), the holonomy of flat RR fields over such a brane can be represented in terms of a virtual flat real vector bundle \(\xi = [E_0] - [E_1] \in \text{KO}^0(X)\) of rank 0, restricted to \(\tilde{M}\) as follows. Fix a spin structure and Riemannian metric on \(\tilde{M}\) which coincide with those of the product \(M \times I\) in a neighbourhood of the boundary. Let \(\mathcal{M}^M_E : C^\infty(\tilde{M}, \mathcal{S}(\tilde{M}) \otimes \tilde{E}) \rightarrow C^\infty(\tilde{M}, \mathcal{S}(\tilde{M}) \otimes \tilde{E})\) be the canonical Atiyah-Singer operator of \(\tilde{M}\) with coefficients in \(\tilde{E}\), defined with respect to the global Szegö boundary conditions considered by Atiyah-Patodi-Singer (APS) [7]. Then the restriction of the Clifford algebra bundle \(\mathcal{S}(\tilde{M})\) to \(M\) may be identified with \(\mathcal{S}(M)\). Near the boundary, in \(M \times I\), we have

\[ \mathcal{S}^M_E = \sigma \cdot (\frac{\partial}{\partial u} + \mathcal{M}^M_E) \]

where \(u\) is the inward normal coordinate and \(\sigma\cdot\) is Clifford multiplication by the unit inward normal vector.

Let \(\text{spec}^0(T^M_E)\) denote the spectrum of the closure (6.4) of the twisted Atiyah-Singer operator on the chiral Hilbert space \((3\mathcal{K}^E)^0 = L^2_\mathbb{R}(M, \mathcal{S}^0(M) \otimes E; dg^M)\). It is a discrete unbounded subset of \(\mathbb{R}\) with no accumulation points such that the eigenspaces are finite-dimensional subspaces of
The above construction generalizes, to the case of the geometric homology of background

\[ M, \hat{\kappa} \]

for \( M, \hat{\kappa} \), define classes \( \[ \] \) above. Given the flat RR-flux \( \xi = [E_0] - [E_1] \) in \( KO_{\mathbb{R}/\mathbb{Z}}(X) \), we can define classes \( [M, \xi, \phi] := [M, F_0, \phi] - [M, F_1, \phi] \) in the KO-homology of \( W \) where \( F_i := \phi^* \pi_1(E_i) \) for \( i = 0, 1 \). The corresponding invariant

\[ \Omega(\tilde{M}, \tilde{\xi}, \phi) = \exp \left[ 2\pi i \left( \Xi(\tilde{M}, \tilde{F}_0, \phi) - \Xi(\tilde{M}, \tilde{F}_1, \phi) \right) \right] \]

is then the holonomy over the D-brane background with the given virtual Chan-Paton bundle. The above construction generalizes, to the case of the geometric homology of background

\[ \mathcal{H}_E^M \]. An eigenvalue \( \lambda \) is repeated in \( \text{spec}^0(T_E^M) \) according to its multiplicity. For \( s \in \mathbb{C} \) with \( \text{Re}(s) \gg 0 \), define the absolutely convergent series

\[ \eta(s, \mathcal{D}_E^M) = \sum_{\lambda \in \text{spec}^0(T_E^M) \setminus \{0\}} \lambda |\lambda|^{-s-1}. \]

Let \( \eta(\mathcal{D}_E^M) \) be the value of the meromorphic continuation of \( \eta(s, \mathcal{D}_E^M) \) at \( s = 0 \). This is called the APS eta-invariant \( \Xi(X, X \xi, \phi) \) and it is a measure of the spectral asymmetry of the Atiyah-Singer operator \( \mathcal{D}_E^M \).

The reduced eta-invariant is the geometric invariant defined by \( \Xi(X, X \xi, \phi) \)

\[
\Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) = \frac{\dim_{\mathbb{R}} \mathcal{H}_E^M + \eta(\mathcal{D}_E^M)}{2} \mod \mathbb{Z},
\]

where \( \mathcal{H}_E^M \) is the vector space of harmonic \( E \)-valued spinors on \( M \) as in Theorem \( 4.4 \). Under an operator homotopy \( t \mapsto \{T_E^M\}_t \); the quantity \( \eta(\mathcal{D}_E^M) \) is not a continuous function of \( t \) but its jumps are due to eigenvalues \( \lambda \) changing sign as they cross zero, and so it has only integer jump discontinuities. As a consequence, \( \Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) \) takes values in \( \mathbb{R}/\mathbb{Z} \). By exponentiating we obtain a geometric invariant valued in the unit circle group \( U(1) \subset \mathbb{C} \) defined by

\[ \Omega(\tilde{M}, \tilde{E}, \tilde{\phi}) = \exp \left( 2\pi i \Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) \right). \]

Consider the collection of \( KO \)-chains \( (\tilde{M}, \tilde{E}, \tilde{\phi}) \), which are defined as the triples as in Definition \( 2.1 \) but now the spin manifolds \( \tilde{M} \) can have boundary. The boundary of a \( KO \)-chain is defined as in (2.11), i.e. \( \partial(\tilde{M}, \tilde{E}, \tilde{\phi}) = (M, E, \phi) \) in the notation above. The difference here from our definition in Section 2.4 of relative KO-cycles \( \Gamma_0(X, Y) \) is that the background D-branes are free to live anywhere in \( X \), i.e. \( \tilde{\phi}(\tilde{M}) \subset X \). In other words, we take \( Y = X \) and define \( KO \)-chains to be the relative KO-cycles \( \Gamma_0(X, X) \). Two isomorphic \( KO \)-chains \( (\tilde{M}_1, \tilde{E}_1, \tilde{\phi}_1) \) and \( (\tilde{M}_2, \tilde{E}_2, \tilde{\phi}_2) \) yield conjugate Atiyah-Singer operators, and so \( \Xi \) is well-defined on the set of isomorphism classes \( \Gamma_0(X, X) \). By adapting the proofs of Lemmas 1 and 2 in \( 13 \), one can now straightforwardly establish the behaviour of \( \Xi \) under the equivalence relations on \( KO \)-chains described in Section 2.4.

**Proposition 6.3.** The map

\[ \Xi : \Gamma_0(X, X) \longrightarrow \mathbb{R}/\mathbb{Z} \]

induced by (6.6) respects:

(i) **Algebraic operation:**

\[ \Xi( (\tilde{M}_1, \tilde{E}_1, \tilde{\phi}_1) \sqcup (\tilde{M}_2, \tilde{E}_2, \tilde{\phi}_2) ) = \Xi(\tilde{M}_1, \tilde{E}_1, \tilde{\phi}_1) + \Xi(\tilde{M}_2, \tilde{E}_2, \tilde{\phi}_2) ; \]

(ii) **Direct sum:**

\[ \Xi(\tilde{M}, \tilde{E}, \tilde{\phi} \circ \tilde{\pi}) = \Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) + \Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) ; \] and

(iii) **Real vector bundle modification:**

\[ \Xi(\tilde{M}, \tilde{E}, \tilde{\phi} \circ \tilde{\pi}) = \Xi(\tilde{M}, \tilde{E}, \tilde{\phi}) \]

Note that we do not say anything about the spin bordism relation in Proposition 6.3 and in fact the eta-invariant \( \eta(\mathcal{D}_E^M) \) is not a spin cobordism invariant \( 8 \). In fact, taking the quotient of \( \Gamma_0(X, X) \) by the spin bordism relation along with the relations of Proposition 6.3 gives the trivial KO-homology group \( KO_1^f(X, X) = 0 \), consistent with the assumptions made on the D-brane background \( [M, E, \phi] \) above. Given the flat RR-flux \( \xi = [E_0] - [E_1] \) in \( KO_{\mathbb{R}/\mathbb{Z}}^1(X) \), we can define classes \( [M, \xi, \phi] := [M, F_0, \phi] - [M, F_1, \phi] \) in the KO-homology of \( W \) where \( F_i := \phi^* \pi_1(E_i) \) for \( i = 0, 1 \). The corresponding invariant

\[ \Omega(\tilde{M}, \tilde{\xi}, \tilde{\phi}) = \exp \left[ 2\pi i \left( \Xi(\tilde{M}, \tilde{F}_0, \tilde{\phi}) - \Xi(\tilde{M}, \tilde{F}_1, \tilde{\phi}) \right) \right] \]
D-branes in Type I string theory, the usual couplings that are inserted into the Type II string theory path integral \[ \text{[17].} \]

**Remark 6.4.** Just as we arrived at the $\zeta_n$-Index Theorem \[\text{[3,7]}\] by using the complexification homomorphism of Section \[\text{[4,4]}\] it is possible to extract a KO-homology version of the APS index theorem in certain dimensionalities. For example, if $n \equiv 0 \mod{8}$, then one has the Fredholm index formula

\[
\text{(6.9)} \quad \text{ind}_n^a(\widetilde{M}, \tilde{E}, \tilde{\phi}) = \langle \text{ch}^s(\tilde{E}_C) \sim \hat{A}(T\widetilde{M}), [\widetilde{M}] \rangle - \Xi(\tilde{M}, E, \phi).
\]

Reducing mod $\mathbb{Z}$ the differences of \[\text{[6,9]}\] evaluated on bundles $E_0$ and $E_1$ then yields the same holonomy \[\text{[6,8]}\]. This is essentially a KO-theory version \[\text{[26, 28]}\] of the index theorem for flat bundles \[\text{[9, 11]}\], which provides a topological formula for differences of the reduced eta-invariants \[\text{[6,6]}\] in terms of the direct image of the collapsing map $\tilde{\zeta}_i : \text{KO}_{R/\mathbb{Z}}^{-1}(W) \to \mathbb{R}/\mathbb{Z}$. In particular, in these dimensions $\Xi(\tilde{M}, \tilde{\phi}, \tilde{\phi})$ is a spin cobordism invariant. For example, for $n \equiv 3 \mod{8}$ the flat index takes values in $\text{KO}_{R/\mathbb{Z}}^{-1}(\text{pt}) \cong \mathbb{R}/\mathbb{Z}$. However, in contrast to the complex case of Type II RR-fields, this is not possible in generic dimensions due to the mod 2 indices appearing in Theorem \[\text{[3,7]}\]. Moreover, it is not clear how to use these couplings to cancel the worldvolume anomalies in the path integral, which arise in the low-energy effective field theory on the D-brane. In this regime the D-branes are genuinely described as spin submanifolds of the spacetime $X$. On the other hand, the geometric KO-homology formalism includes non-representable D-branes, which do not wrap homology cycles of spacetime represented by non-singular spin submanifolds \[\text{[52, 25]}\], and thereby provides a description of the D-brane physics deeper into the stringy regime.

**References**


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