Generic Evolutionary Quantum Universe

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We consider a Schrödinger quantum dynamics for the gravitational field associated to a generic cosmological model and then we solve the corresponding eigenvalue problem. We show that, from a phenomenological point of view, an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach.

1. Introduction

The canonical approach in Quantum Gravity is characterized by the so-called frozen formalism, i.e. the absence of a time evolution for the wave functional\(^1\,^2\). It has been proposed\(^3\,^4\) that such feature disappears as soon as the impossibility of a physical slicing without frame fixing is recognized for a quantum spacetime. In this work\(^5\) we start with a Schrödinger dynamics for the gravitational field using Planck mass particle as a “clock” for the system; and we will analyze the meaning of the corresponding spectrum (we deal with a new energy density) in the framework of a generic inhomogeneous Universe.

2. Evolutionary Quantum Gravity

In this section we briefly analyze the implication of a Schrödinger formulation of the quantum dynamics for the gravitational field. We require the theory to evolve along the spacetime slicing so that \(\Psi = \Psi(t, \{h_{ij}\})\); so the quantum evolution is governed by a smeared Schrödinger equation

\[
i\partial_t \Psi = \hat{\mathcal{H}} \Psi \equiv \int_{\Sigma_t} d^3 x \left(N \hat{H}\right) \Psi
\] (1)
being $\hat{H}$ the super-Hamiltonian operator and $N$ the lapse function. If we now take the right expansion for the wave functional, the Schrödinger dynamics is reduced to an eigenvalues problem of the form

$$\hat{H}\chi = \epsilon\chi, \quad \hat{H}_i\chi = 0,$$

which outlines the appearance of a non-zero super-Hamiltonian eigenvalue.

It is not difficult to show that the classical limit (in the sense of WKB approximation) of the above model is characterized by the appearance of a new matter contribution, which admits the following energy density:

$$\rho \equiv T_{00} = -\frac{\epsilon(x^i)}{2\sqrt{h}}, \quad h = \det h_{ij}. \quad (3)$$

The explicit form of (3) is that of a dust fluid co-moving with the slicing 3-hypersurfaces, i.e. we deal with $T_{\mu\nu} = \rho n_\mu n_\nu$.

### 3. The Generic Quantum Universe and its Spectrum

We now apply the Schrödinger approach to a Quantum Universe that has to be described by a generic inhomogeneous model, which has a dynamics summarized, asymptotically to the Big-Bang, by the following variational principle

$$\delta S = \delta \int_{\Sigma_T} dt d^3 x (p_\alpha \partial_t q^\alpha - NH) \quad (4)$$

where adopting Misner-like variables $R, \beta_\pm^8$ ($R$ is the scale factor and $\beta_\pm$ describes the anisotropies), the super-Hamiltonian has the structure

$$H(x^i) = \kappa \left[ -\frac{p_R^2}{R} + \frac{1}{R^3} \left( p^2_+ + p^2_- \right) + \frac{3}{8\pi} \frac{p^2_0}{R^3} V(\beta_\pm) + R^3(\rho_{ur} + \rho_{pg}) \right], \quad (5)$$

where $\kappa = 8\pi l_P^2$. We have added to the dynamics of the system an ultrarelativistic energy density ($\rho_{ur} = \mu^2/R^4$), a perfect gas contribution ($\rho_{pg} = \sigma^2/R^5$) and a scalar field $\phi$ (a free inflaton field).

Performing the canonical quantization of this model we obtain the following eigenvalue problem (2), with the right normal ordering:

$$\left\{ \kappa \left[ \partial_R \frac{1}{R} \partial_R - \frac{1}{R^3} \left( \partial^2_+ + \partial^2_- \right) \right] - \frac{3}{8\pi} \frac{p^2_0}{R^3} \partial_\phi - \frac{R^3}{4\kappa l_P^2} V(\beta_\pm) + R^3(\rho_{ur} + \rho_{pg}) \right\} \chi = \epsilon\chi. \quad (6)$$

The appropriate boundary condition for this problem are: i) $\chi(R = 0, \beta_\pm, \phi) < \infty$ that relies on the idea that the quantum Universe is singularity-free and ii) $\chi(R \to \infty, \beta_\pm, \phi) = 0$ that ensures a physical behavior at “large” scale factor.

In order to study the previous eigenvalue problem we expand the wave function as $\chi(R, \beta_\pm, \phi) = \int \theta_K(R) F_K(R, \beta_\pm, \phi) dK$, and then performing an adiabatic approximation ($|\partial_RF| \ll |\partial_R\theta|$) we obtain the following reduced problems:

$$\kappa \frac{d}{dR} \left( \frac{1}{R} \frac{d\theta}{dR} \right) + \left( \kappa \frac{K^2}{R^3} + R^3(\rho_{ur} + \rho_{pg}) - \epsilon \right) \theta = 0. \quad (7)$$
\[-(\partial_+^2 + \partial_-^2 + \frac{3}{8\pi\kappa}\partial_0^2)F + \frac{R_0^6}{4\kappa^2l_0^2}V(\beta_\pm)F = K^2(R)F.\]  

The function \(F\) is a plane wave as soon as we neglect the potential term in (8) for same \(R^* \ll 1\). The solution of (7) is a series in \(R\) multiplied by a Gaussian function peaked around \(R = \frac{\ell_P^2}{16\pi}\). Since we required the wave function to decay at large scale factor \(R\) we have to terminate the series and obtain the spectrum of the super-Hamiltonian:

\[\epsilon_{n,\gamma} = \frac{\sigma^2}{l_P^2(n + \gamma - 1/2)},\]

so that the ground state \(n = 0\) eigenvalue, for \(\gamma < 1/2\), is negative; therefore is associated via \(\mathbf{3}\) to a positive dust energy density.

4. Phenomenology of the Dust Fluid

In order to analyze the cosmological implication of this new matter contribution, we have to impose a cut-off length in our model, requiring that the Planck length \(l_P\) is the minimal physical length accessible by an observer \((l \geq l_P)\). So, from the thermodynamical relation for the perfect gas, we obtain a constraint on the \(\rho_{pg}\) and then on the super-Hamiltonian eigenvalue:

\[l^3 \equiv \frac{V}{N} = \frac{3}{2}\frac{l_P^3}{\rho_{pg}\lambda^2} \geq l_P^3 \Rightarrow \rho_{pg} \leq \mathcal{O}(1/l_P^4),\]

where \(l\) is the length per particle and \(\lambda\) the corresponding thermal length \((\lambda = l_P)\). Therefore we get \(\sigma^2 \leq \mathcal{O}(l_P)\) and so \(|\epsilon_0| \leq (1/l_P)\): the spectrum is limited by below.

The contribution of our dust fluid to the actual critical parameter is

\[\Omega_{dust} \sim \frac{\rho_{dust}}{\rho_{today}} \sim \mathcal{O} \left(10^{-60}\right) .\]

Such a parameter is much less then unity and so no phenomenology can came out (today) from our dust fluid. In other words an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach. Finally we face the question of the classical limit of the spectrum in the sense of large occupation numbers \(n \to \infty\). As we can see from \(\mathbf{3}\) the eigenvalue approaches zero as \(1/n\). Therefore for very large \(n\), our quantum dynamics would overlap the Wheeler-DeWitt approach.

References