On First Order Generalized Maxwell Equations

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Abstract

The generalized Maxwell equations including an additional scalar field are considered in the first order formalism. The gauge invariance of the Lagrangian and equations is broken resulting the appearance of a scalar field. We find the canonical and symmetric energy momentum tensors. It is shown that the trace of the symmetric energy-momentum tensor is not equal to zero in the theory considered. The matrix Hamilton form of equations is obtained after the exclusion of the non-dynamical components. The canonical quantization is performed and the propagator of the fields was found in the first order formalism.

1 Introduction

In [1] we found and investigated the matrix (first order) formulation of the generalized Maxwell equations, and solutions for a free particle in the form of the projection matrix-dyads. The matrices of an equation obey the generalized Duffin-Kemmer algebra. The generalized Maxwell equations can be obtained by breaking the gauge invariance and refusing the Lorentz condition, $\partial_{\mu}A_{\mu} \neq 0$. As a result, a scalar field $\varphi = \partial_{\mu}A_{\mu}$ appears because the vector field realizes the $(1/2, 1/2)$ representation of the Lorentz group and possesses four degrees of freedom (three of them describe spin one and another - spin zero). The scalar field here is a ghost field which can be removed by the path integration formulation of electrodynamics [2]. The reason, why we leave this ghost field, is: scalar fields play the important role in Standard Model (Higgs field) and in astrophysics (inflation field). It should be noted that in some scenarios of Universe inflation ghost scalar fields (k-essence,
Phantom) are insensitively explored [3]. An interesting scenario of inflation deals with phenomenon of the ghost condensation [4].

In this work we find the energy-momentum tensor, and study the matrix Hamilton form of equations, and the canonical quantization.

The paper is organized as follows. In Sec. 2, we formulate the generalized Maxwell equations in the matrix form and obtain the canonical and symmetric energy momentum tensors. The quantum-mechanical Hamiltonian is obtained in Sec. 3. In Sec. 4, the canonical quantization of fields is performed and the matrix propagator of fields is obtained. We make a conclusion in Sec. 5. The system of units $\hbar = c = 1$ is chosen.

## 2 Matrix form of equations

The generalized Maxwell equations may be cast as follows [1]:

\[
\begin{align*}
\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 &= 0, \\
\partial_\nu \psi_\mu - \partial_\mu \psi_\nu + \kappa \psi_{[\mu\nu]} &= 0, \\
\partial_\mu \psi_\mu + \kappa \psi_0 &= 0.
\end{align*}
\]

(1)

The mass parameter $\kappa$ is introduced to have fields $\psi_A (A = 0, \mu, [\mu\nu])$ with the same dimension. Physical values do not depend on $\kappa$. It follows from Eq.(1) that $\partial_\mu^2 \psi_A(x) = 0$. The gradient term $j_\mu = -\partial_\mu \psi_0$ plays the role of the four-current and it is not independent external electric current but depends on the vector-potential $A_\mu = \psi_\mu / \kappa$. In such electrodynamics the vector-potential $A_\mu$ is a “physical” variable as well as strengths of electric, $E_\mu = i \psi_{[\mu 4]}$, and magnetic, $H_k = (1/2) \varepsilon_{kmn} \psi_{[mn]}$, fields.

Eq.(1) can be written in the $11 \times 11$-matrix form [1]:

\[
(\alpha_\nu \partial_\nu + \kappa P) \Psi(x) = 0,
\]

(2)

where

\[
\alpha_\nu = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu} + \varepsilon^{\mu,0} + \varepsilon^{0,\nu}, \quad P = \varepsilon^{0,0} + \frac{1}{2} \varepsilon_{[\mu\nu],[\mu\nu]}, \quad \Psi(x) = \{ \psi_A(x) \},
\]

(3)

and elements of the entire matrix algebra $\varepsilon^{A,B}$ obey equations: $(\varepsilon^{A,B})_{CD} = \delta_{AC}\delta_{BD}$, $\varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}$. Matrices $\alpha_\nu$, $P$ are Hermitian matrices, and the $P$ is the projection operator, $P^2 = P$. 

2
The $11 \times 11$-matrices $\alpha_\mu$ satisfy the algebra as follows [1]:

$$\alpha_\mu \alpha_\rho \alpha_\alpha + \alpha_\alpha \alpha_\rho \alpha_\mu + \alpha_\mu \alpha_\alpha \alpha_\rho + \alpha_\alpha \alpha_\rho \alpha_\mu + \alpha_\rho \alpha_\alpha \alpha_\mu + \alpha_\mu \alpha_\rho \alpha_\alpha = 2(\delta_{\mu\nu} \alpha_\alpha + \delta_{\alpha\nu} \alpha_\mu + \delta_{\mu\alpha} \alpha_\nu).$$  

(4)

The Lagrangian of the theory suggested is given by

$$\mathcal{L} = -\overline{\Psi}(x)(\alpha_\mu \partial_\mu + \kappa P) \Psi(x),$$

(5)

where $\overline{\Psi}(x)\Psi(x) = \Psi^+(x)\eta \Psi(x)$, and the Hermitianizing matrix $\eta$ is

$$\eta = -\varepsilon^{0,0} + \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2}\varepsilon^{[mn],[mn]}.$$  

(6)

The $\eta$ is the Hermitian matrix, $\eta^+ = \eta$, and the Lagrangian (5) is the real function that can be verified with the help of relations: $\eta \alpha_\mu = -\alpha_\mu^+ \eta^+$ ($m=1,2,3$), $\eta \alpha_4 = \alpha_4^+ \eta^+$. In terms of fields $\psi_A$, the Lagrangian (5) reads

$$\mathcal{L} = \psi_0 \partial_\mu \psi_\mu - \psi_\mu \partial_\mu \psi_0 - \psi_\rho \partial_\mu \psi_{\rho[m]} + \psi_{[\rho m]} \partial_\mu \psi_\rho + \kappa \left(\psi_0^2 + \frac{1}{2}\psi_{[\rho m]}^2\right).$$  

(7)

It is easy to verify that Euler-Lagrange equations $\partial \mathcal{L}/\partial \psi_A - \partial_\mu (\partial \mathcal{L}/\partial \partial_\mu \psi_A) = 0$ lead to Eq.(1). For fields $\psi_A$ obeying Eq.(1) the Lagrangian (7) vanishes.

The canonical energy-momentum tensor can be found with the help of the standard procedure, and is given by

$$T_{\mu\nu} = \left(\partial_\nu \overline{\Psi}(x)\right) \alpha_\mu \Psi(x).$$

(8)

With the help of Eq.(3) it becomes

$$T_{\mu\nu} = \psi_0 \partial_\nu \psi_\mu - \psi_\mu \partial_\nu \psi_0 - \psi_\rho \partial_\mu \psi_{\rho[m]} + \psi_{[\rho m]} \partial_\mu \psi_\rho.$$  

(9)

One can verify, using Eq.(1), that the energy-momentum tensor (9) (or (8)) is conserved tensor, $\partial_\nu T_{\mu\nu} = 0$, but it is not the symmetric tensor, $T_{\mu\nu} \neq T_{\nu\mu}$. The symmetric energy-momentum tensor can be found from the relation [5]

$$T^\text{sym}_{\mu\nu} = T_{\mu\nu} + \Lambda_{\mu\nu},$$

(10)

where the function $\Lambda_{\mu\nu}$ has to satisfy the equation $\partial_\nu \Lambda_{\mu\nu} = 0$ for the fields $\psi_A$ obeying Eq.(1) in such a way that $T^\text{sym}_{\mu\nu} = T^\text{sym}_{\nu\mu}$. We obtain

$$\Lambda_{\mu\nu} = \psi_0 \partial_\nu \psi_\mu - \psi_\nu \partial_\mu \psi_0 - \psi_\rho \partial_\mu \psi_{\rho[m]} + \psi_{[\rho m]} \partial_\nu \psi_\rho.$$  

(11)
It is interesting that contrary to the classical electrodynamics, the trace of the symmetric energy-momentum tensor (11) is not equal to zero:

\[ T^{\text{sym}}_{\mu\mu} = -\kappa \left(2\psi_0^2 + \psi_{[\mu\nu]}^2\right). \]  

(12)

So, at the classical level, the anomaly exists in generalized electrodynamics considered.

3 Quantum-mechanical Hamiltonian

The quantum-mechanical Hamiltonian may be obtained from Eq.(2). From Eq.(2), we arrive at

\[ i\alpha_4 \partial_t \Psi(x) = (\alpha_a \partial_a + \kappa P) \Psi(x). \]  

(13)

One obtains from algebra (4) the relation as follows:

\[ \alpha_4 \left(\alpha_4^2 - 1\right) = 0. \]  

(14)

Eq.(14) indicates that eigenvalues of the matrix \( \alpha_4 \) are one and zero. As a result, the 11-dimensional function \( \Psi(x) \) possesses as dynamical as well as non-dynamical components.

To separate the dynamical and non-dynamical components of the wave function, we introduce projection operators:

\[ \Lambda \equiv \alpha_4^2 = \varepsilon^{0,0} + \varepsilon^{\mu,\mu} + \varepsilon^{[m_4],[m_4]}, \quad \Pi \equiv 1 - \Lambda = \frac{1}{2} \varepsilon^{[mn],[mn]}, \]  

(15)

so that \( \Lambda = \Lambda^2, \Pi^2 = \Pi, \Lambda \Pi = 0 \). These operators extract dynamical, \( \phi(x) \), and non-dynamical \( \chi(x) \) components:

\[ \phi(x) = \Lambda \Psi(x), \quad \chi(x) = \Pi \Psi(x). \]  

(16)

Multiplying Eq.(13) by the matrix \( \alpha_4 \) and \( \Pi \), one obtains the system of equations

\[ i\partial_t \phi(x) = \alpha_4 \left(\alpha_a \partial_a + \kappa P\right) (\phi(x) + \chi(x)), \]  

(17)

\[ 0 = \left(\alpha_4^2 - 1\right) \left(\alpha_a \partial_a + \kappa P\right) (\phi(x) + \chi(x)). \]  

(18)

With the help of equation \( \Pi \Pi = \Pi \), one obtains from Eq.(18), the expression:

\[ \chi(x) = \frac{1}{\kappa} \left(\alpha_4^2 - 1\right) \alpha_a \partial_a \phi(x). \]  

(19)
Replacing the $\chi(x)$ from Eq.(19) into Eq.(17), with the aid of the equality $\alpha_4 \alpha_a \partial_a \Pi \alpha_b \partial_b \Pi = 0$, we find the Hamiltonian form:

$$i \partial_t \phi(x) = \mathcal{H} \phi(x),$$

$$\mathcal{H} = \alpha_4 (\alpha_a \partial_a + \kappa P).$$

The 8-component Eq.(20) describes four spin states (spin one and zero) with positive and negative energies. In the component form Eq.(20) leads to equations as follows:

$$i \partial_t \psi_0 = \partial_m \psi_{[4m]}, \quad i \partial_t \psi_m = -\partial_m \psi_4 + \kappa \psi_{[m4]},$$

$$i \partial_t \psi_4 = \partial_m \psi_m + \kappa \psi_0, \quad i \partial_t \psi_{[m4]} = \partial_m \psi_0 + \partial_n \psi_{[mn]}.$$

Eq. (21) can also be obtained from Eq.(1) retaining only components with time derivatives. Hamiltonian Eq.(20) describes the evolution of the wave function $\phi(x)$ in time. Eq.(19) is equivalent to equation $\kappa \psi_{[mn]} = \partial_m \psi_n - \partial_n \psi_m$ which does not contain the time derivative.

### 4 Field Quantization

We obtain from Eq.(5) the momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \Psi(x))} = i \bar{\Psi} \alpha_4.$$

With the help of the relationship $[\Psi_M(x, t), \pi_N(y, t)] = i \delta_{MN} \delta(x - y)$, one arrives from Eq.(22) at simultaneous quantum commutators

$$[\Psi_M(x, t), (\bar{\Psi}(y, t) \alpha_4)_N] = \delta_{MN} \delta(x - y).$$

From Eq.(23), we obtain non-zero commutators of fields $\psi_A(x)$:

$$[\psi_0(x, t), \psi_4(y, t)] = \delta(x - y), \quad [\psi_{[m4]}(x, t), \psi_n(y, t)] = \delta_{mn} \delta(x - y).$$

Let us consider solutions of Eq.(2) with definite energy and momentum in the form of plane waves:

$$\Psi^{(\pm)}(x) = \sqrt{\frac{1}{2k_0 V}} \psi_\lambda(\pm k) \exp(\pm ikx),$$
where $V$ is the normalization volume, $k^2 = k_0^2 - k^2 = 0$, and $\lambda$ is the spin index ($\lambda = 1, 2, 3, 4$). The 11-dimensional function $v_\lambda(\pm k)$ obeys the equation as follows:

$$ \left( \hat{k} \pm \kappa P \right) v_\lambda(\pm k) = 0, \quad (26) $$

where $\hat{k} = \alpha_\mu k_\mu$. We use the normalization conditions

$$ \int_V \Psi_\lambda^{(\pm)}(x) \alpha_4 \Psi_\lambda^{(\mp)}(x) d^3 x = \pm \delta_{\lambda\lambda'}, \quad \int_V \overline{\Psi}_\lambda^{(\pm)}(x) \alpha_4 \Psi_\lambda^{(\mp)}(x) d^3 x = 0, \quad (27) $$

where $\overline{\Psi}_\lambda^{(\pm)}(x) = \left( \Psi_\lambda^{(\pm)}(x) \right)^\dagger \eta$. From normalization conditions (27), one finds relations for functions $v_\lambda(\pm k)$:

$$ v_\lambda(\pm k) \alpha_4 v_\lambda'(\pm k) = 0, \quad v_\lambda(\pm k) \alpha_4 v_\lambda'(\mp k) = 0. \quad (28) $$

Multiplying the first equation in (28) by $k_\mu$, one obtains

$$ \nu_\lambda(\pm k) \hat{k} v_\lambda'(\pm k) = 0, \quad \nu_\lambda(\pm k) P v_\lambda'(\mp k) = 0. \quad (29) $$

The Hamiltonian density (energy density) is given by the equation

$$ E = \pi(x) \partial_0 \Psi(x) - \mathcal{L} = i \overline{\Psi}(x) \alpha_4 \partial_0 \Psi(x). \quad (30) $$

In the second quantized theory, the field operators may be represented as

$$ \Psi(x) = \sum_{k,\lambda} \left[ a_{k,\lambda} \Psi_\lambda^{(+)}(x) + a_{k,\lambda}^+ \Psi_\lambda^{(-)}(x) \right], $$

$$ \overline{\Psi}(x) = \sum_{k,\lambda} \left[ a_{k,\lambda}^+ \overline{\Psi}_\lambda^{(+)}(x) + a_{k,\lambda} \overline{\Psi}_\lambda^{(-)}(x) \right], \quad (31) $$

where positive and negative parts of the wave function are given by Eq.(25). The creation and annihilation operators of particles, $a_{k,\lambda}^+$, $a_{k,\lambda}$ obey the commutation relations as follows:

$$ [a_{k,\lambda}, a_{k',\lambda'}^+] = \delta_{\lambda\lambda'} \delta_{kk'}, \quad [a_{k,\lambda}, a_{k',\lambda'}] = [a_{k,\lambda}^+, a_{k',\lambda'}^+] = 0, \quad (32) $$

With the aid of Eq.(30)-(32), and normalization condition (28), one obtains the Hamiltonian

$$ H = \int E d^3 x = \sum_{k,\lambda} k_0 \left( a_{k,\lambda}^+ a_{k,\lambda} + a_{k,\lambda} a_{k,\lambda}^+ \right). \quad (33) $$
From Eq. (30)-(33), it is not difficult to find commutation relations for different times:
\[ [\Psi_M(x), \Psi_N(x')] = [\overline{\Psi}_M(x), \overline{\Psi}_N(x')] = 0, \]
\[ [\Psi_M(x), \overline{\Psi}_N(x')] = N_{MN}(x, x'), \]  
\[ N_{MN}(x, x') = N_{MN}^+(x, x') - N_{MN}^-(x, x'), \]
\[ N_{MN}^+(x, x') = \sum_{k, \lambda} \left( \Psi_\lambda^{(+)}(x) \right)_M \left( \overline{\Psi}_\lambda^{(+)}(x') \right)_N, \]
\[ N_{MN}^-(x, x') = \sum_{k, \lambda} \left( \Psi_\lambda^{(-)}(x) \right)_M \left( \overline{\Psi}_\lambda^{(-)}(x') \right)_N. \]

We obtain from Eq. (36):
\[ N_{MN}^{\pm}(x, x') = \sum_{k, \lambda} \frac{1}{2k_0 V} (v_{\lambda}(\pm k))_M (\overline{v}_{\lambda}(\pm k))_N \exp[\pm ik(x - x')]. \]

In [1], we obtained matrices-dyad for spin-0 state
\[ v_0^{(0)}(k) \cdot \overline{v}_0^{(0)}(k) = \left( 1 - \frac{\sigma^2}{2} \right) \left( \frac{i\hat{k} - \kappa\overline{P}}{\kappa} \right)^2, \]
and for spin-1 states\(^2\)
\[ v_{\pm 1}(k) \cdot \overline{v}_{\pm 1}(k) = \frac{1}{2} \sigma_k \left( \sigma_k \pm 1 \right) \frac{\sigma^2}{2} \left( \frac{i\hat{k} - \kappa\overline{P}}{\kappa} \right)^2, \]
\[ v_0(k) \cdot \overline{v}_0(k) = \left( 1 - \frac{\sigma^2}{2} \right) \frac{\sigma^2}{2} \left( \frac{i\hat{k} - \kappa\overline{P}}{\kappa} \right)^2, \]
where \( \sigma_k \) is the operator of the spin projection on the direction of the momentum \( k \), \( \sigma^2 \) is the square spin operator, \( \overline{P} = 1 - P \) is the projection operator, and \( (v \cdot \overline{v})_{MN} = v_M \overline{v}_N \). From Eq. (38), (39), one obtains
\[ \sum_{\lambda} (v\lambda(\pm k))_M (\overline{v}\lambda(\pm k))_N = \left( \frac{\pm i\hat{k} - \kappa\overline{P}}{\kappa} \right)^2_{MN}. \]

\(^2\)In [1] we wrote out only two solutions for spin-1 states (for helicity \( \pm 1 \))
Taking into account Eq.(40), one obtains from Eq. (37):

\[ N_{MN}^\pm(x, x') = \sum_k \frac{1}{2k_0V} \left( \frac{\pm ik - \kappa \bar{P}}{\kappa} \right)_M^N \exp[\pm ik(x - x')] \]

\[ = \left( \frac{\alpha_\mu \partial_\mu - \kappa \bar{P}}{\kappa} \right)_M^N D_\pm(x), \]

where the singular functions are given by [6]

\[ D_+(x) = \sum_k \frac{1}{2k_0V} \exp(ikx), \quad D_-(x) = \sum_k \frac{1}{2k_0V} \exp(-ikx), \]

With the help of the function [6]

\[ D_0(x) = i(D_+(x) - D_-(x)), \]

from Eq.(36), (41), we obtain

\[ N_{MN}(x, x') = -i \left( \frac{\alpha_\mu \partial_\mu - \kappa \bar{P}}{\kappa} \right)_M^N D_0(x - x'), \]

It easy to verify, using Eq.(3), that the equation

\[ (\alpha_\mu \partial_\mu + \kappa P) \left( \frac{\alpha_\nu \partial_\nu - \kappa \bar{P}}{\kappa} \right) \partial_\alpha^2 \]

is valid. As a result, we arrive at

\[ (\alpha_\mu \partial_\mu + \kappa P) N^\pm(x, x') = 0. \]

The vacuum expectation of chronological pairing of operators (propagator)

is defined by the equation

\[ \langle T\Psi_M(x)\bar{\Psi}_N(y) \rangle_0 = N^\pm_{MN}(x - y) \]

\[ = \theta(x_0 - y_0) N^+_M(x - y) + \theta(y_0 - x_0) N^-_M(x - y), \]

where \( \theta(x) \) is the theta-function. One obtains from Eq.(47):

\[ \langle T\Psi_M(x)\bar{\Psi}_N(y) \rangle_0 = \left( \frac{\alpha_\mu \partial_\mu - \kappa \bar{P}}{\kappa} \right)_M^N D_c(x - y), \]
where the function $D_c(x - y)$ is given by

$$D_c(x - y) = \theta (x_0 - y_0) D_+(x - y) + \theta (y_0 - x_0) D_-(x - y).$$

(49)

Expressions (38), (39), (48) can be used for calculating different electrodynamics processes involving polarized massless particles. Taking into consideration the equation $\Box \, \partial_\mu^2 D_c(x) = i\delta(x)$, one finds

$$(\alpha_\mu \partial_\mu + \kappa P) \langle T \Psi(x) \cdot \overline{\Psi}(y) \rangle_0 = \frac{i}{\kappa^2} \left( \alpha_\mu \partial_\mu - \kappa P \right) \delta(x - y).$$

(50)

It should be noted that propagator (48) includes the contribution of the spin-0 state.

5 Conclusion

We have considered the generalized Maxwell equations which describe massless fields including spin-0 state (ghost). The reason for this is possible cosmological applications or further investigations of quantum field theory with indefinite metric. Ghost gives the negative contribution to the Hamiltonian which requires to introduce an indefinite metric [7]. Quantization of fields and propagator obtained allow us to make some necessary calculations in the quantum theory considered.

One can arrive at the classical Maxwell equations by imposing the constraint $\psi_0(x) = 0$ to eliminate ghost. As a result, the gauge invariance will be recovered, longitudinal states eliminated, and one has only two spin states with helicity $\pm 1$.

References


