Einstein-Maxwell gravitational instantons and five dimensional solitonic strings

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Abstract

We study various aspects of four dimensional Einstein-Maxwell multicentred gravitational instantons. These are half-BPS Riemannian backgrounds of minimal $\mathcal{N} = 2$ supergravity, asymptotic to $\mathbb{R}^4, \mathbb{R}^3 \times S^1$ or $AdS_2 \times S^2$. Unlike for the Gibbons-Hawking solutions, the topology is not restricted by boundary conditions. We discuss the classical metric on the instanton moduli space. One class of these solutions may be lifted to causal and regular multi ‘solitonic strings’, without horizons, of 4+1 dimensional $\mathcal{N} = 2$ supergravity, carrying null momentum.
1 Introduction

1.1 Motivation

Instantons play a key role in the nonperturbative dynamics of Yang-Mills theories, and indeed in a wide range of quantum mechanical systems. One useful property of instantons is that they can allow a semiclassical description where a full treatment is either difficult or even ill defined, as in the case of gravity. At the other extreme, in supersymmetric theories instantons are crucial in obtaining exact results.

Within the programme of Euclidean Quantum Gravity, multicentred gravitational instantons followed hotly on the tails of their Yang-Mills counterparts [1, 2]. However, while the Gibbons-Hawking metrics have found a surprising range of physical applications, their dynamical role within quantum gravity remains unclear. One reason for this is that if the instanton contains more than one centre, it is no longer Asymptotically Euclidean (∼\(\mathbb{R}^4\)) or Asymptotically Flat (∼\(\mathbb{R}^3 \times S^1\)). These are the most natural asymptotics for infinite volume quantum gravity at zero or finite temperature, respectively. In contrast, at constant large radius the multicentred Gibbons-Hawking spaces tend to \(S^1\) fibred over \(S^2\) with increasingly high Chern number.

Said differently, the boundary conditions determine the gravitational instanton topology. There is no sum over different spacetime topologies for a fixed asymptotics. In this sense, the Gibbons-Hawking spaces do not provide a semiclassical realisation of spacetime foam.

It is therefore of interest to study gravitational theories in which arbitrarily high instanton number is allowed with fixed asymptotics. One example of such a theory is conformal gravity, in which the Einstein-Hilbert term is replaced by the Weyl curvature squared [3, 4, 5]. Despite some rather attractive features of the gravitational instantons in this theory, the physical status of the theory itself is uncertain due to problems with higher derivative Lagrangians and unitarity.

In this paper we emphasise that Einstein-Maxwell theory also admits regular multicentred instantons with arbitrarily complicated topology for fixed asymptotics. These solutions have essentially appeared before in the literature [6, 7]. Various unsatisfactory aspects of these previous treatments, for instance we have preferred to use a Riemannian Maxwell field that is real, have lead us to carry out a systematic study de novo. We furthermore extend our understanding of Einstein-Maxwell gravitational instantons through discussions of uniqueness, supersymmetry, moduli space metrics and lifts to five dimensions. This final point may be of independent interest.
1.2 Summary

In Section 2 we present the instanton solutions. We detail the possible asymptotics: $\mathbb{R}^4, \mathbb{R}^3 \times S^1$ and $AdS_2 \times S^2$, and local versions thereof. We show that the solutions are half-BPS when embedded into minimal $\mathcal{N} = 2$ supergravity and that they are all the regular Riemannian half-BPS solutions. Finally, we evaluate the action of the solutions. The Asymptotically Euclidean case is found to only be well defined when a certain linear combination of the charge and potential is fixed at infinity.

In section 3 we discuss the moduli space metric on the Einstein-Maxwell instantons. We consider in some detail the ambiguities involved in finding an inner product on the space of metric fields. We show that there is a preferred inner product which is inherited from the action and for which zero modes are orthogonal to pure gauge modes.

Section 4 shows how the four dimensional instantons may be lifted to solitons of five dimensional Einstein-Maxwell-Chern-Simons theory, or minimal $\mathcal{N} = 2$ supergravity in five dimensions. Generically the lifted solutions are either singular or contain closed timelike curves. However, we find that one class of solutions lift to regular, causal plane fronted wave spacetimes with the fields localised in lumps orthogonal to the wave propagation. We call these ‘solitonic strings’ as they do not have an event horizon.

Section 5 briefly discusses the slow motion of the five dimensional solitons. Unlike in the case of the Gibbons-Hawking instantons and their lift to Kaluza-Klein monopoles, it seems that there is not a direct connection between the four dimensional instanton moduli space metric and the five dimensional soliton slow motion moduli space metric in our case.

We end with a discussion of possible physical applications of these multicentred Einstein-Maxwell instantons, and directions for future work.

2 The gravitational instantons

2.1 The solutions

The gravitational instantons on a four dimensional manifold $M_4$ are solutions to the Einstein-Maxwell equations with Riemannian signature

$$G_{ab} = 2F_a^\epsilon F_{b\epsilon} - \frac{1}{2}g_{ab}F^{cd}F_{cd},$$

$$\nabla_a F^{ab} = 0.$$
The metric is given by
\[ g^{(4)} = \frac{1}{UU} (d\tau + \omega)^2 + U\tilde{U}d\mathbf{x}^2, \]
where the functions \( U, \tilde{U} \) and the one form \( \omega \) depend on \( \mathbf{x} = (x, y, z) \) and satisfy
\[ \nabla^2 U = \nabla^2 \tilde{U} = 0, \]
\[ \nabla \times \omega = \tilde{U} \nabla U - U \nabla \tilde{U}. \]
We will work with four dimensional tangent space indices, \( a, b, \ldots \) and the vierbeins
\[ e^4 = \frac{1}{(UU)^{1/2}} (d\tau + \omega), \quad e^i = (U\tilde{U})^{1/2} dx^i. \]

The electromagnetic field strength may now be written
\[ F_{4i} = \frac{1}{2} \partial_i \left[ U^{-1} - \tilde{U}^{-1} \right], \]
\[ F_{ij} = \frac{1}{2} \varepsilon_{ijk} \partial_k \left[ U^{-1} + \tilde{U}^{-1} \right], \]
where the derivatives are partial derivatives with respect to the corresponding space-time indices. One can check that this field strength satisfies the Bianchi identities, and thus locally at least we can write \( F = dA \). Our expressions for the field strength in Riemannian signature differ slightly from others in the literature \([6, 7]\) which were not real. In particular the Riemannian Majumdar-Papapetrou metrics with \( U = \tilde{U} \) have purely magnetic field strength \( F = -2 \star_3 dU \).

These backgrounds were first found in the Lorentzian regime by Israel and Wilson \([8]\) and by Perjés \([9]\) as a stationary generalisation of the static Majumdar-Papapetrou multi black hole solutions. However, it was shown by Hartle and Hawking that all the non static solutions suffered from naked singularities \([10, 11]\).

With Riemannian signature however, regular solutions exist \([6, 7]\). We can take
\[ U = \frac{4\pi}{\beta} + \sum_{m=1}^{N} \frac{a_m}{|\mathbf{x} - \mathbf{x}_m|}, \quad \tilde{U} = \frac{4\pi}{\tilde{\beta}} + \sum_{n=1}^{N} \frac{\tilde{a}_n}{|\mathbf{x} - \tilde{\mathbf{x}}_n|}, \]
in these expressions \( \beta, \tilde{\beta}, a_m, \mathbf{x}_m, \tilde{a}_n, \tilde{\mathbf{x}}_n, N, \tilde{N} \) are constants. For the signature to remain \((+, +, +, +)\) throughout we can require \( U, \tilde{U} > 0 \) which in turn requires \( a_m, \tilde{a}_n > 0 \). From the explicit forms for \( U \) and \( \tilde{U} \) in \((6)\) we can write down explicit expressions for the one forms \( \omega \) and \( A \), which so far we have only defined implicitly. These are given in Appendix A.
If there is at least one non coincident centre, \( x_m \neq \tilde{x}_n \), regularity requires that \( \tau \) is identified with period \( 4\pi \) and that the constants satisfy the following constraints at all the non-coincident centres

\[
U(\tilde{x}_n)\tilde{a}_n = 1, \quad \tilde{U}(x_m)a_m = 1, \quad \forall m, n.
\]  

(7)

Given the locations of the centres \( \{x_n, \tilde{x}_m\} \), these constraints may be solved uniquely for the \( \{a_n, \tilde{a}_m\} \) \[^7\]. When \( \frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} = 0 \) the solution is only unique up to the overall scaling

\[
U \rightarrow e^s U, \quad \tilde{U} \rightarrow e^{-s} \tilde{U}.
\]  

(8)

In general this scaling leaves the metric invariant and induces a linear duality transformation on the Maxwell field mapping solutions to solutions

\[
E \rightarrow \cosh s E + \sinh s B, \quad B \rightarrow \sinh s E + \cosh s B.
\]  

(9)

The rescaling does not leave the action and other properties of the solutions invariant.

The constants \( \beta \) and \( \tilde{\beta} \) determine the asymptotics of the solution. There are three possibilities:

- The case \( \frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} \neq 0 \) gives an Asymptotically Locally Flat metric, tending to an \( S^1 \) bundle over \( S^2 \) at infinity, with first Chern number \( N - \tilde{N} \). Without loss of generality we have rescaled the harmonic functions using \(^8\) so that \( \beta = \tilde{\beta} \). Equations \(^7\) now imply that \( \sum a_m - N = \sum \tilde{a}_n - \tilde{N} \). If \( N = \tilde{N} \) the asymptotic bundle is trivial and we obtain Asymptotically Flat \((\sim R^3 \times S^1)\) solutions.

- The case \( \frac{4\pi}{\beta} = 0, \frac{4\pi}{\tilde{\beta}} = 1 \) gives an Asymptotically Locally Euclidean metric, tending to \( R^4 / Z_{|N - \tilde{N}|} \). We have used the rescaling \(^8\) to set \( \frac{4\pi}{\beta} = 1 \) without loss of generality. In this case the constraints \(^7\) require that \( \sum a_m = N - \tilde{N} \). Of course we can reverse the roles of \( \beta \) and \( \tilde{\beta} \). If \( N = \tilde{N} + 1 \) the solution is Asymptotically Euclidean \((\sim R^4)\).

- The case \( \frac{4\pi}{\beta} = \frac{4\pi}{\tilde{\beta}} = 0 \) leads to an Asymptotically Locally Robinson-Bertotti metric, tending to \( AdS_2 \times S^2 \) or \( AdS_2 / Z \times S^2 \). The former case only arises if all of the centres are coincident, so that \( U = \tilde{U} \), and \( \tau \) need not be made periodic. For both these asymptotics, the constraints \(^7\) require that \( N = \tilde{N} \). We may further use the rescaling \(^8\) to set \( \sum a_m = \sum \tilde{a}_n \).

As Riemannian solutions, the backgrounds are naturally thought of as generalisations of the Gibbons-Hawking multicentre metrics which in fact they include as the special case \( \tilde{U} = 1 \), albeit with an additional antiselfdual Maxwell field. A crucial new
aspect of the Asymptotically Locally Euclidean (ALE) and Asymptotically Locally Flat (ALF) Israel-Wilson-Perjés solutions is that when

$$ N = \tilde{N} \pm 1 \text{ (for ALE) or } N = \tilde{N} \text{ (for ALF)}, \quad (10) $$

the fibration of the $\tau$ circle over $S^2$ at infinity is trivial and the metrics do not require the $\mathbb{Z}_N$ identifications at infinity that are needed in the Gibbons-Hawking case. The spacetimes are therefore strictly Asymptotically Euclidean and Asymptotically Flat respectively in these cases. The Euler number is given by $\chi = N + \tilde{N} - 1$ in the ALF and ALE cases \cite{7}. Thus the spaces admit arbitrarily complicated topology, not restricted by the asymptotic topology, and provide a semiclassical realisation of spacetime foam in quantum Einstein-Maxwell theory.

The metric \cite{2} has vanishing scalar curvature. If $U$ or $\tilde{U}$ is constant then \cite{2} is Ricci flat, and hyperKähler. It is natural to ask whether any other special choices of harmonic functions $U$ and $\tilde{U}$ lead to scalar flat Kähler metrics with a symmetry $\partial/\partial \tau$ preserving the Kähler structure. Such metrics would be conformally anti–self–dual and thus interesting in twistor theory. The answer is negative. From \cite{4} any such metric is of the form

$$ g^{(4)} = \frac{1}{\mathcal{W}} (d\tau + \omega)^2 + \mathcal{W} h^{(3)}, \quad (11) $$

where the metric $h^{(3)}$ on the three dimensional orbit space of $\partial/\partial \tau$, and the function $\mathcal{W}$ on this space satisfies a coupled nonlinear system of PDEs. In the case that $h^{(3)}$ is flat the equations reduce to

$$ \nabla \times \omega = \nabla \mathcal{W}. \quad (12) $$

Therefore $\mathcal{W} = U \tilde{U}$ is harmonic, and then \cite{3} implies that $\tilde{U}$ is a constant.

### 2.2 Killing spinors

The solutions have the further important property of admitting two complex Killing spinors. These satisfy

$$ e^{\mu}_a \partial_{\mu} \varepsilon + \frac{1}{4} \left[ \omega^{bc}_a \Gamma_{bc} + iF^{bc} \Gamma_{bc} \Gamma_a \right] \varepsilon = 0, \quad (13) $$

where $\omega^{bc}_a$ are the components of the the spin connection one form $\omega^{bc}$ defined by $de^a = \omega^{bc} \wedge e^c$. We use Greek letters $\mu, \nu, ...$ to denote Euclidean spacetime indices. Our gamma matrix conventions are given in Appendix B, as is the spin connection for the background. With these conventions one may solve the equation \cite{13} to find

$$ \varepsilon = \begin{pmatrix} U^{-1/2} \varepsilon_0 \\ i\tilde{U}^{-1/2} \varepsilon_0 \end{pmatrix}, \quad (14) $$
where $\varepsilon_0$ is a constant two-component complex spinor: $\partial_\mu \varepsilon_0 = 0$.

Within Einstein-Maxwell theory, the Killing spinors imply that the solutions saturate a Bogomolny bound [12]. It is also natural to view the solutions as half-BPS states of four dimensional $\mathcal{N} = 2$ supergravity [13]. This theory has a complex spin-$\frac{3}{2}$ Rarita-Schwinger field as well as the graviton and photon. In fact, in a paper that anticipated current interest in classifying supersymmetric solutions, Tod has shown that the Lorentzian version of these solutions are all the supersymmetric solutions to $\mathcal{N} = 2$ supergravity with a timelike Killing spinor [14].

In the following subsection we shall repeat Tod’s analysis in the Riemannian case. As well as recovering the local form of the metric, it will find that $|\nabla U^{-1}|$ and $|\nabla \tilde{U}^{-1}|$ are both bounded\footnote{This is stronger than the Lorentzian result of [11] where the separate bounds cannot be established.}. Combined with a result from analysis [15], it will follow that (2) together with (6) is the most general regular supersymmetric solution to minimal $\mathcal{N} = 2$ supergravity. To put it differently, only harmonic functions with a finite number of point sources lead to regular metrics.

As usual, given Killing spinors $\varepsilon$ and $\eta$ we can build differential forms. In particular, we have the one forms

$$V = \frac{1}{2} \bar{\eta} \Gamma_a e^a, \quad K = \frac{1}{2} \bar{\eta} \Gamma_5 \Gamma_a e^a,$$

and the two form

$$\Omega = -\frac{i}{2} \bar{\eta} \Gamma_{ab} e^a \wedge e^b.$$

In our representation of the Clifford algebra, given in the appendix, all the gamma matrices are hermitian and therefore bar simply denotes complex conjugation. From the Killing spinor condition (13) we have that

$$d\Omega = -2 V \wedge F, \quad dV = 0, \quad \nabla (a K_b) = 0.$$

With a little more work one can also show that

$$\nabla_a \Omega_{bc} = 2 V_a F_{bc} - 4 F_{a[b} V_{c]} + 4 V^d F_{d[b} g_{c]a},$$

$$\nabla_a V_b = \frac{1}{4} F^{cd} \Omega_{cd} g_{ab} + F^c_{(a} \Omega_{b)c}.$$

In fact there is more structure. The two form $\Omega$ can be split into self dual and anti-self dual parts: $\Omega = \Omega^+ + \Omega^-$. One can then show that $\Omega^+$ and $\Omega^-$ separately satisfy the first equation in (18) with $F$ replaced by its self dual, $F^+$, and anti-self dual, $F^-$, parts respectively.
Three important cases giving real forms are when \( \eta = \varepsilon = \varepsilon^I \), for \( I = 1, 2, 3 \), which are defined by \( \varepsilon_0 \) in (14) satisfying \( \bar{\varepsilon}_I \varepsilon^I_0 = \delta^I_J \) and \( \varepsilon_0^I \varepsilon_I^J = \delta^I_J \). For these cases we find

\[
V^I = dx^I, \quad K = \frac{1}{UU} (d\tau + \omega),
\]

and

\[
\Omega^I = \left( U^{-1} - \tilde{U}^{-1} \right) e^A \wedge e^I + \frac{1}{2} \left( U^{-1} + \tilde{U}^{-1} \right) e^{Ijk} e^j \wedge e^k.
\]

Raising the index, the Killing vector is \( K = \partial/\partial \tau \) as we should expect.

### 2.3 Uniqueness of the solutions

Here we show that the solution (2), (5) with the harmonic functions described by (6) and satisfying the constraints (7) is the most general regular Einstein-Maxwell instanton with a complex Killing spinor.

In this section it will be convenient to write the Dirac spinor \( \varepsilon = (\alpha^A, \beta_{A'}) \) as a pair of complex two-component spinors. When dealing with these spinors we use the conventions given in Appendix C. With positive signature, spinor conjugation preserves the type of spinors. Thus if \( \alpha_A = (p, q) \) we can define \( \hat{\alpha}_A = (\overline{q}, -\overline{p}) \) so that \( \hat{\alpha}_A = -\alpha_A \). This hermitian conjugation induces a positive inner product

\[
\alpha_A \hat{\alpha}^A = \epsilon_{AB} \alpha^B \hat{\alpha}^A = |p|^2 + |q|^2.
\]

We define the inner product on the primed spinors in the same way. Here \( \epsilon_{AB} \) and \( \epsilon_{A'B'} \) are covariantly constant symplectic forms with \( \epsilon_{01} = \epsilon_{0'1'} = 1 \). These are used to raise and lower spinor indices according to \( \alpha_B = \epsilon_{AB} \alpha^A, \beta_{A'} = \epsilon_{A'B'} \beta^{A'} \), and similarly for primed spinors. In terms of our gamma matrices, \( \hat{\varepsilon} = \Gamma^{31} \varepsilon \).

The Killing spinor equation (13) becomes

\[
\nabla_A \alpha_B - i\sqrt{2} \phi_{AB} \beta_A = 0, \quad \nabla_A \beta_{B'} + i\sqrt{2} \bar{\phi}_{A'B'} \alpha_A = 0,
\]

where the spinors \( \phi \) and \( \bar{\phi} \) are symmetric in their respective indices and give the anti-self dual and self dual parts of the electromagnetic field

\[
F_{ab} = \phi_{AB} \epsilon_{A'B'} + \bar{\phi}_{A'B'} \epsilon_{AB}.
\]

Suppose that \( \varepsilon = (\alpha^A, \beta_{A'}) \) solves the Killing spinor equation (22). Now we can reconstruct the spacetime metric and Maxwell field.

Define

\[
U = (\alpha_A \hat{\alpha}^A)^{-1}, \quad \tilde{U} = (\beta_{A'} \hat{\beta}^{A'})^{-1}.
\]
In our positive definite case, these two inverted functions do not vanish unless \( \alpha \) or \( \beta \) vanish. In the Lorentzian case their possible vanishing leads to plane wave spacetimes \cite{14}. If \( \alpha \) or \( \beta \) vanish identically, we recover the Gibbons-Hawking solutions. Now define a (complex) null tetrad

\[
X_a = \alpha_A \beta_{A'}, \quad \overline{X}_a = \hat{\alpha}_A \hat{\beta}_{A'}, \quad Y_a = \alpha_A \hat{\beta}_{A'}, \quad \overline{Y}_a = -\hat{\alpha}_A \beta_{A'}. \tag{25}
\]

We can check that \( \hat{\varepsilon} \) is also a solution to the Killing spinor equation (22). It therefore follows from (22) that \( X_a, \overline{X}_a, Y_a - \overline{Y}_a \) are gradients and that \( K_a = Y_a + \overline{Y}_a \) is a Killing vector. Now define local coordinates \((x, y, z, \tau)\) by

\[
X = \frac{1}{\sqrt{2}} (dx + idy), \quad (Y - \overline{Y}) = i\sqrt{2}dz, \quad K^a \nabla_a = \sqrt{2} \frac{\partial}{\partial \tau}, \tag{26}
\]

where the form \( X = X_a e^a = X_{AA'} e^{AA'} \) and similarly for \( Y, \overline{Y} \). The vector \( K \) Lie derives the spinors \((\alpha_A, \beta_{A'})\), implying that \( U \) and \( \tilde{U} \) are independent of \( \tau \).

The metric is now given by \( ds^2 = \epsilon_{AB} \epsilon_{A'B'} e^{AA'} e^{BB'} \). This expression may be evaluated by noting that from (24) we have \( \epsilon_{AB} = U(\alpha_A \hat{\beta}_B - \alpha_B \hat{\beta}_A) \) and similarly for \( \epsilon_{A'B'} \). Using the fact that from the above definitions \( K_a K^a = 2(U - \tilde{U}) \), we find that the metric takes the form (2) for some one form \( \omega \). The next step is to find \( \omega \).

The definitions of \( U, \tilde{U} \) and \( K \) together with (22) imply

\[
\nabla_a K_b = i\sqrt{2} \left[ \tilde{U}^{-1} \phi_{AB} \epsilon_{A'B'} + U^{-1} \tilde{\phi}_{A'B'} \epsilon_{AB} \right], \tag{27}
\]

and

\[
\nabla_a U^{-1} = i\sqrt{2} \phi_{AB} K_{A}^{B'}, \quad \nabla_{a} \tilde{U}^{-1} = -i\sqrt{2} \tilde{\phi}_{A'B'} K_{A}^{B'}. \tag{28}
\]

The formulae in (28) may be inverted to find expressions for \( \phi_{AB} \) and \( \tilde{\phi}_{A'B'} \), using \( K_{A}^{A'} K_{B'}^{C'} = \frac{1}{2} \epsilon^{A'C'} K_{D'E'} K_{D'E'} \). Substituting the result into (27) yields the expression (3) for \( \nabla \times \omega \).

Finally, differentiating the relations (22) shows that the energy momentum tensor is that of Einstein–Maxwell theory: \( T_{ab} = 2\phi_{AB} \tilde{\phi}_{A'B'} \). The Maxwell equations

\[
\nabla^{AA'} \phi_{AB} = 0, \quad \nabla^{AA'} \phi_{A'B'} = 0, \tag{29}
\]

now imply that \( U \) and \( \tilde{U} \) are harmonic on \( \mathbb{R}^3 \). This completes the local reconstruction of the solution from the Killing spinors.

So far everything has proceeded as in \cite{14} with minor differences in the reality conditions. The main difference arises in global regularity considerations which lead us to consider the invariant

\[
F_{ab} F^{ab} = 2(\phi_{AB} \phi^{AB} + \tilde{\phi}_{A'B'} \tilde{\phi}^{A'B'}) = |\nabla U^{-1}|^2 + |\nabla \tilde{U}^{-1}|^2, \tag{30}
\]

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where the norm of the gradients is taken with respect to the flat metric on $\mathbb{R}^3$, and we have used (28). Regularity requires this invariant be bounded. Therefore both $|\nabla U^{-1}|$ and $|\nabla \tilde{U}^{-1}|$ must be bounded. The various boundary conditions we have described imply that $U$ and $\tilde{U}$ are regular as $|x| \to \infty$. In particular, they are both regular outside a ball $B_R$ of sufficiently large radius $R$ in $\mathbb{R}^3$.

The coordinates $\{x, \tau\}$ cover $\mathbb{R} \times (\mathbb{R}^3 \setminus S)$, where $S$ is the compact subset of $B_R$ on which $U$ or $\tilde{U}$ blow up. A theorem from [15] can now be applied separately to both harmonic functions to prove that $S$ consists of a finite number of points. In fact
\[
\# S < \max\{|\nabla U^{-1}|, |\nabla \tilde{U}^{-1}|\} |U(p) + \tilde{U}(p)| R + 1,
\]
where $p$ is any point in $B_R$ which does not belong to $S$. This combined with the maximum principle shows that (6) are the most general harmonic functions leading to regular metrics. It also follows from (24) and the positivity of the spinor inner product that $a_m$ and $\tilde{a}_n$ in (6) are all non negative.

The spinors $\alpha_A, \beta_{A'}$ and their conjugates give a preferred basis for the space $\Lambda^2(M)$ of two forms. The anti-self dual two forms are given in terms of $\alpha_A$ by
\[
\text{Re}(\alpha_A \alpha_B e_{A'B'}), \quad \text{Im}(\alpha_A \alpha_{B'} e_{A'B'}) , \quad i\alpha_\Lambda \hat{\alpha}_B e_{A'B'},
\]
and the self dual two forms are given in terms of $\beta_{A'}$ by analogous expressions. The three two forms (20) can be expressed in this basis as
\[
\Omega^1 + i\Omega^2 = - (\alpha_A \alpha_B e_{A'B'} + \beta_{A'} \beta_B e_{AB}) e^{AA'} \wedge e^{BB'},
\]
\[
\Omega^3 = i(\beta_{A'} \hat{\beta}_{B'} e_{AB} - \alpha_\Lambda \hat{\alpha}_B e_{A'B'}) e^{AA'} \wedge e^{BB'}.
\]
The spinor expressions for (18) can now be easily derived using (22).

### 2.4 Action of the instantons

The contribution of instantons to physical processes is of course weighted by their actions. Therefore it is important to evaluate the actions of the spacetimes we are considering. Previous computations on this subject should be approached with caution: there are computational errors in [6] leading to unphysical results such as an action unbounded from below, while in [7] the Maxwell contribution to the action is not considered. Both of these papers also work with imaginary electric fields which leads to some undesirable properties of the actions.

The Riemannian Einstein-Maxwell action, including the Gibbons-Hawking boundary term, is
\[
S = - \int_{M_4} d^4 x \sqrt{g^{(4)}} \left[R^{(4)} - F_{ab} F^{ab}\right] - 2 \int_{\partial M_4} d^3 x \sqrt{g} K,
\]
where $g^{(4)}$, $R^{(4)}$, and $K$ are the Riemannian 4-dimensional metric, scalar curvature, and 3-dimensional metric, respectively.
where $\gamma$ is the induced metric on the boundary and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary.

Evaluated on the Einstein-Maxwell instantons we are considering, one finds

$$S = -2\pi \lim_{r \to \infty} \int_{S^2} d\Omega^2 r^2 \left[ \frac{(U + \tilde{U})^2 \partial_r (U \tilde{U})}{(U \tilde{U})^2} + \frac{8}{r} \right].$$  \hspace{1cm} (35)

Here we have introduced spherical polar coordinates $dx^2 = dr^2 + r^2 d\Omega^2$. The expression (35) is divergent and needs to be regularised by subtracting off the action of a reference geometry. This must be done separately for the Asymptotically Locally Flat, Euclidean and Robinson-Bertotti cases. We have assumed in (35) that $\tau$ is identified with period $4\pi$.

The easiest case is Asymptotic Local Flatness, with $\beta = \tilde{\beta} \neq 0$. Here the background has simply $U = \tilde{U} = \frac{4\pi}{\beta}$, giving flat $S^1 \times \mathbb{R}^3$ and a vanishing Maxwell field. One finds

$$\Delta S_{\text{ALF}} = 8\pi \beta \left( \sum a_m + \sum \tilde{a}_n \right).$$  \hspace{1cm} (36)

Recall that furthermore $\sum a_m = \sum \tilde{a}_n + N - \tilde{N}$ in this case.

The Asymptotically Locally Robinson-Bertotti case is also straightforward. Here the background is the Robinson-Bertotti spacetime with $\tau$ identified, $\text{AdS}_2 / \mathbb{Z} \times S^2$, supported by magnetic flux, that is $U = \tilde{U} = \sum a_m r = \sum \tilde{a}_n r$. The regularised action turns out to vanish

$$\Delta S_{\text{ALRB}} = 0.$$  \hspace{1cm} (37)

Now consider the Asymptotically Locally Euclidean case, with $\frac{4\pi}{\beta} = 0$ and $\frac{4\pi}{\beta} = 1$. The required background is Euclidean space with anti self dual Maxwell field, that is $U = \sum \frac{a_m}{r}$ and $\tilde{U} = 1$. Subtracting this background regularises the gravitational action, but it does not remove all the divergences from the Maxwell action. The divergence of the regularised action tells us that we have not imposed the correct boundary conditions for the Maxwell field with these asymptotics.

The standard action (34) is appropriate for fixing the potential at infinity: $\delta A_a = 0$. Different boundary conditions may be implemented by adding a boundary term to the action. To obtain a finite action for ALE asymptotics we need to add a boundary term that entirely cancels the bulk Maxwell action when evaluated on solutions. The required term is

$$S_{\text{ALE}}\big|_{\text{bdy.}} = 2 \int_{\partial M_4} d^3x \sqrt{\gamma} A^a F_{ab} n^b,$$  \hspace{1cm} (38)

where $n^b$ is a unit normal vector to the boundary. The resulting boundary condition is

$$A_a \delta (F^{ab} n_b) = \delta A_a F^{ab} n_b \quad \text{on} \quad \partial M_4.$$  \hspace{1cm} (39)
Physically this equation corresponds to keeping a certain linear combination of the charge and potential fixed at infinity.

With the boundary term \( \text{(38)} \) added, the action is found to be given by

\[
\Delta S_{\text{ALE}} = 16\pi^2 \sum \tilde{a}_n .
\]

(40)

At this moment, we do not have a physical understanding of why the ALE instantons only contribute to processes in which the particular boundary condition \( \text{(39)} \) is imposed.

3 Instanton moduli space metric

The analysis done in section \( \text{(2.3)} \) has demonstrated that the Einstein-Maxwell gravitational instantons with a Killing spinor have \( 3(N + \tilde{N}) \) free parameters or moduli. The Euclidean group in three dimensions can be used to fix six of these, except in the case when \( N + \tilde{N} = 2 \), in which case it only fixes five, due to the axisymmetry. To obtain the moduli space one should also quotient by the symmetric group \( S_N \times S_{\tilde{N}} \) acting on the centres. Note that fixing the action then adds a further constraint on the centres in the Asymptotically Locally Flat and Euclidean cases.

While computation of the measure and metric on the moduli space of Yang-Mills instantons is by now a highly developed field, the case of gravitational instantons in four dimensions appears to have been less systematically treated in the literature. In two dimensions of course the measure plays a fundamental role in string theory. Reflecting this state of affairs, we now give a fairly general exposition of the formalism needed to compute moduli space metrics for gravitational instantons in pure gravity and Einstein-Maxwell theory.

3.1 Inner products

Let us recall the Yang-Mills procedure, but work with just the \( U(1) \) Maxwell case both for simplicity and because this is what we need anyhow. One begins by writing down a natural ultralocal inner product on the space of field perturbations. Strictly speaking it is an inner product on the tangent bundle to the space of fields

\[
\langle \delta A, \delta A' \rangle = 2 \int_{M_4} d^4x \sqrt{g} g^\mu_\nu \delta A_\mu \delta A'_\nu .
\]

(41)

In this section it is appropriate to work with spacetime indices \( \mu, \nu \ldots \). One now restricts to considering only perturbations that are orthogonal to pure gauge transfor-
mations. Thus one requires
\[
0 = \langle \delta A, d\Omega \rangle = -2 \int_{M_4} d^4x \sqrt{g} g^{\mu\nu} \Omega \nabla_\mu \delta A_\nu ,
\] (42)
for all \( \Omega \). Therefore, perturbations must be considered in Lorenz gauge
\[
\nabla_\mu \delta A^\mu = 0 .
\] (43)

Given this gauge, we can note that the inner product (41) should be thought of as coming from the quadratic terms in the action. In particular, this determines the normalisation. The quadratic action is
\[
S^{(2)}_{\delta A} = 2 \int_{M_4} d^4x \sqrt{g} \left( \nabla^\mu \delta A^\rho \nabla_\mu \delta A_\rho - \nabla^\mu \delta A^\rho \nabla_\rho \delta A_\mu \right)
\rightarrow -2 \int_{M_4} d^4x \sqrt{g} \rho^{\rho\sigma} \delta A_\rho \nabla^2 \delta A_\sigma + \text{non-derivative terms} .
\] (44)

Where the arrow denotes imposition of the Lorenz gauge. We can see that the index structure of the gauge field is now that of the inner product (41). That is to say, the term in the last line of (44) is just \(-\langle \delta A, \nabla^2 \delta A \rangle\), where \(\nabla^2\) should be regarded as an operator on \(M_4\). In this way the inner product is inherited from the action. The metric on the moduli space is obtained by restricting the inner product (41) to zero modes. To summarise the logic: the metric on the moduli space is inherited from the quadratic kinetic terms in the action written in a specific gauge. However, that gauge must simultaneously imply that field fluctuations are orthogonal to pure gauge transformations.

We should note at this point that imposing orthogonality to gauge transformations, with a consequent choice of gauge imposed, is not completely essential. However, it does greatly simplify instanton computations and gives a clear physical meaning to the moduli space metric itself.

For the case of metric fluctuations, there is not a unique ultralocal inner product with the correct symmetries. Rather we have the family of de Witt metrics parametrised by \(\lambda \in \mathbb{R}\)
\[
\langle \delta g, \delta g' \rangle_\lambda = \int_{M_4} d^4x G^{\mu\nu\rho\sigma}_{(\lambda)} \delta g_{\mu\nu} \delta g'_{\rho\sigma} ,
\] (45)
where
\[
G^{\mu\nu\rho\sigma}_{(\lambda)} = \frac{1}{8} \sqrt{g} \left[ g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} - 2\lambda g^{\mu\nu} g^{\rho\sigma} \right] .
\] (46)

Thus \(\lambda\) parametrises the possible inner products. The metric is positive definite for \(\lambda < 1/4\) and non-degenerate for \(\lambda \neq 1/4\). In Appendix D we demonstrate that different values of \(\lambda\) indeed give non-equivalent inner products on moduli space.
The de Witt metric with $\lambda = 1$ also appears in Hamiltonian treatments of gravity. This is not what we are doing here; the metric we want is on four dimensional Riemannian geometries. In the case of pure gravity there is a connection, as the four dimensional Euclidean theory can be lifted to $4+1$ Einstein theory. The gravitational instantons become Kaluza-Klein monopoles in five dimensions. In this context the moduli space on the multicentred Gibbons-Hawking spaces has been computed as the slow motion moduli space metric of the Kaluza-Klein monopoles \[16\]. We will describe a lift of our solutions in a later section, but for the moment we are pursuing a four dimensional treatment.

The ambiguity in the inner product translates into a choice of gauge. Imposing orthogonality to pure gauge transformations now requires

$$0 = \langle \delta g, \mathcal{L}_\xi g \rangle_\lambda = -\frac{1}{2} \int_{M_4} d^4 x \sqrt{g} \xi^\mu \left[ \nabla^\nu \delta g_{\mu \nu} - \lambda \nabla_\mu \delta g^\nu \right]. \quad (47)$$

Here $\mathcal{L}$ is the Lie derivative. Therefore, metric fluctuations must be considered in the gauge

$$\nabla^\nu \delta g_{\mu \nu} = \lambda \nabla_\mu \delta g^\nu. \quad (48)$$

In Appendix D we discuss the extent to which the different choices of $\lambda$ lead to isometric inner products. The result will certainly not depend on $\lambda$ if all fluctuations are trace free. All the gauges are equivalent in that case. Indeed, for noncompact gravitational instantons, all normalisable zero modes are trace free. This is not true for the compact gravitational instanton, K3. However, we now need to check compatibility with the quadratic kinetic terms in the action. The quadratic action, only keeping track of derivative terms, is

$$S_{\delta g}^2 = \frac{1}{4} \int_{M_4} d^4 x \sqrt{g} \left( \nabla^\mu \delta g^{\rho \sigma} \nabla_\mu \delta g_{\rho \sigma} - \nabla^\mu \delta g^\rho \nabla_\mu \delta g_\sigma - 2 \nabla^\mu \delta g^\rho \nabla_\rho \delta g_\sigma + 2 \nabla^\mu \delta g^\rho \nabla_\sigma \delta g_{\mu \rho} \right) \delta g_{\mu \nu} \nabla^2 \delta g_{\rho \sigma}, \quad (49)$$

where arrow denotes imposition of the gauge \[18\]. Generically, this does not correspond to the de Witt \[16\] inner product which we started with. For consistency, we now need to impose $2\lambda^2 - 3\lambda + 1 = 0$. The two solutions to this equation are $\lambda = 1$ and $\lambda = \frac{1}{2}$. These are in fact rather interesting values. The first is that obtained from viewing the instanton moduli space as the slow motion moduli space of 4+1 dimensional Kaluza-Klein monopoles \[16\]. The second corresponds to de Donder gauge, perhaps the most natural gauge for the theory, and was considered recently because gradient flow on the space of metrics with this inner product is Ricci flow \[17\].
It follows from the previous few paragraphs that for gravitational instantons there are two preferred gauges, which correspond to taking \( \lambda = 1 \) or \( \lambda = \frac{1}{2} \) in the de Witt metric. However, we are interested in Einstein-Maxwell theory, so we furthermore need to take into account the fact that the Maxwell field also transforms under infinitessimal diffeomorphisms \( A \to A + \mathcal{L}_\xi A \). Orthogonality to such diffeomorphisms therefore requires
\[
\langle \delta g, \mathcal{L}_\xi g \rangle + \langle \delta A, \mathcal{L}_\xi A \rangle = 0. \tag{50}
\]
Using the Lorenz condition on the gauge field perturbation (43) one finds that the orthogonality condition (50) requires that the following gauge be implemented for the moduli
\[
\nabla^\nu \delta g_{\mu \nu} - \lambda \nabla_\mu \delta g^{\nu} = -4 \delta A^\nu F_{\nu \mu}. \tag{51}
\]
Once again, we need to substitute this gauge choice into the quadratic term of the action. This is similar to the case of pure gravity (49) except that there are two extra terms due to the right hand side of the gauge condition (51). One of these does not involve any derivatives of \( \delta A_\mu \) or \( \delta g_{\mu \nu} \) and so does not contribute to the quadratic terms. However, the other term involves a single derivative. This latter term is always present unless \( \lambda = \frac{1}{2} \), suggesting that this is the preferred gauge for Einstein-Maxwell instantons.

### 3.2 Towards the moduli space metric

To find the moduli space metric we need to find the general solution to the linearised Einstein-Maxwell equations satisfying the gauge conditions (43) and (51). Once we have the solution, we should then evaluate the norm of the fluctuations using the results of the previous section. Given that we have the general solution at a nonlinear level, we can easily solve the linearised Einstein-Maxwell system by perturbing the full solutions. However, these solutions will not be in the required gauge. Finding a gauge transformation to map the solution into the correct gauge does not appear easy.

An alternative and more elegant approach is that employed in [16] to find the moduli space metric on the Gibbons-Hawking gravitational instantons. This uses the existence of \( N \) closed self dual two forms on the background, \( F^J \), as well as the three self dual Kähler forms \( \Omega^i \) to write the metric fluctuation
\[
\delta g_{\mu \nu} = \Omega^i (_{\mu} F^J _{\nu \rho}) \delta \Omega^i. \tag{52}
\]
This perturbation solves the linearised Einstein equations. Furthermore, it is transverse and tracefree and therefore solves the gauge condition required for pure Einstein gravity.

Note that this approach combines supersymmetry, which provides the three Kähler forms, and the topology of solution, which has \( b_2^+ = N \) and hence implies the existence
of the closed self dual forms $F^J$. Using these modes, \cite{16} shows that the moduli space metric is given in terms of intersection matrix of the Gibbons-Hawking background and is flat.

So far, we have not been able to adapt this argument to the Einstein-Maxwell case in a way consistent with the gauge condition \cite{51}. We hope that the framework presented in this section will be a useful starting point for future work on the moduli space metric.

4 Lift to five dimensions

4.1 Lifting the solutions

Recall the following feature of field theory instantons: instantons in $D$ dimensions may be viewed as solitons in $(D+1)$ dimensions. Furthermore, the $L^2$ instanton metric coincides with a natural Riemannian metric on the moduli space of solitons that is induced from the kinetic term in the $(D+1)$ dimensional action. This is interesting given the differing interpretations of the metrics in each case. The metric is relevant at the classical level in $(D+1)$ dimensions, as its geodesic motion approximates the soliton dynamics in the nonrelativistic limit \cite{18}. However, in $D$ dimensions the metric is only important in quantum field theory, where measures on solution spaces are needed.

This procedure can also be applied to the 4 dimensional Einstein-Maxwell gravitational instantons \cite{2} if it is possible to lift them to Lorentzian metrics which are solitons of some theory in higher dimensions. Of course the resulting moduli space metric could depend on the choice of higher dimensional theory. In this section we study one possible theory in $(4+1)$ dimensions. The five dimensional metrics resulting from the lift are interesting in their own right, and we clarify some of their properties in this section. In the following section \cite{} we shall discuss the metric on the slow motion moduli space of these solitons.

Einstein-Maxwell theory without a dilaton cannot be consistently lifted to pure gravity in five dimensions\footnote{The need for a dynamical scalar field was not originally appreciated in the 1920s by Kaluza and Klein who set it to a constant. This mistake was corrected more than 20 years latter by Jordan and Thiry.}. However, Einstein-Maxwell configurations may be lifted to solutions of five dimensional Einstein-Maxwell theory with a Chern-Simons term. This lift is the bosonic sector of the lift from $\mathcal{N} = 2$ supergravity in four dimensions to $\mathcal{N} = 2$ supergravity in five dimensions \cite{19,20}. We are interested in lifting the four dimensional Riemannian theory to a Lorentzian theory on a five dimensional manifold.
$M_5$. The four dimensional action is

$$S_4 = \int d^4x \sqrt{g^{(4)}} \left[ R^{(4)} - F_{ab}F^{ab} \right], \quad (53)$$

with equations of motion given by (1). The five dimensional action is

$$S_5 = \int d^5x \sqrt{-g^{(5)}} \left[ R^{(5)} - H_{\alpha\beta}H^{\alpha\beta} \right] - \frac{8}{3\sqrt{3}} \int H \wedge H \wedge W, \quad (54)$$

where $H = dW$ is the five dimensional Maxwell field. We use greek indices ranging from 0 to 4 in five dimensions. The equations of motion in five dimensions are

$$G_{\alpha\beta} = 2H_{\alpha\gamma}H_{\beta\gamma} - \frac{1}{2}g^{(5)}_{\alpha\beta}H^{\gamma\delta}H_{\gamma\delta},$$

$$d \ast_5 H = -\frac{2}{\sqrt{3}} H \wedge H. \quad (55)$$

Given a solution, $g^{(4)}$ and $F = dA$, to the four dimensional equations (1), we may lift the solution to five dimensions as follows:

$$g^{(5)} = g^{(4)} - (dt + \Phi)^2,$$

$$W = \frac{\sqrt{3}}{2} A, \quad (56)$$

where $\Phi$ is a one form determined by $g^{(4)}$ and $F$ through

$$d\Phi = \ast_4 F. \quad (57)$$

One may then check that the five dimensional configuration (56) solves the equations of motion (55). Note that solutions to (57) exist because $d\ast_4 F = 0$ on shell. In our case we may solve for $\Phi$ explicitly to find

$$\Phi = -\frac{1}{2} \left( U^{-1} + \tilde{U}^{-1} \right) (d\tau + \omega) + \chi, \quad (58)$$

where $\chi$ satisfies

$$\nabla \times \chi = \frac{1}{2} \nabla \left( U - \tilde{U} \right). \quad (59)$$

The supersymmetric solutions of $\mathcal{N} = 2$ supergravity in five dimensions have been classified [21]. For the case of a timelike Killing spinor the general solution is given as a $U(1)$ fibration over a four real dimensional hyperKähler manifold. It was shown in [21] how the lift of the Lorentzian Israel-Wilson-Perjés solutions to five dimensions could be expressed as a fibration over the multicentred Gibbons-Hawking metrics [2].
It turns out that the lift of the Riemannian Israel-Wilson-Perjés solutions we are considering may also be expressed as a fibration over the multicentred Gibbons-Hawking metrics. The five dimensional metric (56) can be written as follows

\[ g^{(5)} = -f^2 (d\tau + \omega')^2 + f^{-1}g^{GH}, \]  

where the Gibbons-Hawking metric is

\[ g^{GH} = V^{-1} (dt + \chi)^2 + V dx^2, \]

with harmonic function

\[ V = \frac{1}{2} \left( U - \bar{U} \right). \]

The remaining functions in the metric (60) are

\[ f = \frac{V}{UU}, \]

and

\[ \omega' = \omega - \frac{1}{2f^2} \left( U^{-1} + \bar{U}^{-1} \right) (dt + \chi). \]

Note that the hyperKähler base itself is in general not regular, even changing signature at points where \( U = \bar{U} \). This is perfectly compatible with regularity of the five dimensional spacetime.

The case \( U = \bar{U} \) is exceptional and cannot be written in the form (60). Instead, these metrics have null supersymmetry in five dimensions. The metric is

\[ g^{(5)} = \frac{2dt d\tau}{U} - dt^2 + U^2 dx^2. \]

### 4.2 Regularity and causality

The interesting points in the five dimensional metric are the centres where \( U \to \infty \) or \( \bar{U} \to \infty \). In the four dimensional Riemannian Israel-Wilson-Perjés solutions these can always be made to be regular points [6, 7] as we reviewed above. We need to re-examine the regularity of the metric around these points and also check for the possible occurrence of closed timelike curves.

---

3Writing the spacetime in the form (60) locates the five dimensional solution in the classification of [21]. In section 3.7 of that paper the general supersymmetric fibration over a Gibbons-Hawking base with \( \partial/\partial t \) a Killing vector is given in terms of three harmonic functions. For our solution these correspond to \( L = 2\bar{U} \), \( K = -U \) and \( M = -2U \).

4The metric (65) falls within the classification of [21] for spacetimes with null supersymmetry by setting their functions \( H = -F = U \) and \( a = 0 \).
Before zooming in on the centres note the following. Firstly, that
\[ g^{(5)}_{\tau\tau} \equiv g^{(5)} \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau} \right) = -\frac{(U - \widetilde{U})^2}{(2UU)^2} < 0, \]  
(66)
if \( U \neq \widetilde{U} \). Therefore, to avoid closed timelike curves throughout the five dimensional spacetime we must not identify \( \tau \). Secondly, possible candidates for the location of horizons are where the metric becomes degenerate
\[ 0 = g^{(5)}_{tt}g^{(5)}_{\tau\tau} - [g^{(5)}_{\tau\tau}]^2 = -\frac{1}{UU}. \]  
(67)
This occurs at the centres where \( U \) or \( \widetilde{U} \) diverge.

In order to understand the geometry near the centres, there are three different cases we need to consider separately. The first is that \( U \to \infty \) while \( \widetilde{U} \) remains finite. Using polar coordinates \( (r = \rho^2/4, \theta, \phi) \) centred on the point \( x_m \) and requiring that \( a_m\widetilde{U}(x_m) = 1 \), the metric becomes
\[ ds^2 = d\rho^2 + \frac{\rho^2}{4} \left[ (d\tau + \cos \theta d\phi)^2 + d\Omega^2_{S^2} \right] - (dt - a_m d\tau / 2)^2 \]  
(68)
as \( \rho \to 0 \), with \( d\Omega^2_{S^2} = d\theta^2 + \sin^2 \theta d\phi^2 \). The metric may be made regular about this point if we identify \( \tau \) with period \( 4\pi \). Unfortunately this introduces closed timelike curves as we discussed. If we choose not to identify \( \tau \) we are left with timelike naked singularities at the centres. We see that there is no horizon at these points, but rather a (singular) origin of polar coordinates. Therefore, metrics with this behaviour at the centres cannot lift to causal, regular solitons in five dimensions.

The remaining two possibilities involve coincident centres where both \( U \) and \( \widetilde{U} \) go to infinity, so that \( x_m = \tilde{x}_m \). One needs to treat separately the cases where \( a_m = \tilde{a}_m \) and where \( a_m \neq \tilde{a}_m \). In the latter case we again find regularity at the expense of closed timelike curves going out to infinity, or alternatively naked singularities. This leaves only the former case with \( a_m = \tilde{a}_m \) for all \( m \). That is, \( \widetilde{U} = U + k \), with \( k \) some constant.

By considering the asymptotic regime, one can see that in order to obtain a sensible asymptotic geometry without closed timelike curves, one requires that either both \( U \) and \( \widetilde{U} \) go to a constant at infinity or they both go to zero. Rescaling the harmonic functions and performing a duality rotation on the Maxwell field, as we discussed in four dimensions above, implies that without loss of generality \( U = \widetilde{U} \). We consider this case in the following subsection.
4.3 Multi solitonic strings

The only lift that leads to a globally regular and causal five dimensional spacetime is the case $U = \tilde{U}$, which corresponds to the Euclidean Majumdar-Papapetrou metric in four dimensions. The metric is \((65)\), with a null Killing spinor. Away from the centres, the spacetimes approach either $\mathbb{R}^{1,4}$ or $AdS_3 \times S^2$, with $U$ going to a constant or zero at infinity, respectively.

With a rescaling of coordinates, the geometry near the centres where $U \rightarrow \infty$ may be written
\[
d s^2 = a_m^2 \left[ \frac{d\tau^2}{r^2} + 2r d\tau d\tau - d\tau^2 + d\Omega^2_{S^2} \right].
\]
Calculating the curvature shows that this metric locally describes $AdS_3 \times S^2$. One might be tempted to conclude that this represents the near horizon geometry of an extremal black string in five dimensions. However, the coordinates \((69)\) are a little unusual, the sign of $d\tau^2$ differing from the metric of an extremal BTZ black hole \([22]\). In particular, the Killing vector $\partial/\partial \tau$ is everywhere regular and timelike. This remains true in the full spacetime \((65)\). There is no horizon and the degeneration of the metric at the centres is analogous to an origin of polar coordinates.

The coordinates in \((69)\) may be mapped to Poincaré coordinates as follows
\[
Y = \frac{1}{r^{1/2} \cos \frac{t}{2}},
\]
\[
X = \frac{\tau}{2} - \frac{1}{2} \left[ 1 \frac{1}{r} - 1 \right] \tan \frac{t}{2},
\]
\[
T = \frac{\tau}{2} - \frac{1}{2} \left[ 1 \frac{1}{r} + 1 \right] \tan \frac{t}{2},
\]
so that the metric becomes
\[
d s^2 = \frac{4a_m^2}{Y^2} \left( -dT^2 + dX^2 + dY^2 \right) + a_m^2 d\Omega^2_{S^2}.
\]
There is no singularity at $t = \pm \pi$ as may be checked by writing down the embedding of $AdS_3$ as a quadric in $\mathbb{R}^{2,2}$ in terms of these coordinates. The map \((70)\) is periodic in $t$. Taking $t$ with infinite range corresponds to passing to the (causal) universal cover of $AdS_3$. There is no need to identify $\tau$ and therefore the spacetime is causal.

The metrics \((65)\) give causal, regular solutions to the five dimensional theory with an everywhere defined timelike Killing vector. Writing the metric in the form
\[
g^{(5)} = -(dt - d\tau/U)^2 + \frac{d\tau^2}{U^2} + U^2 d\mathbf{x}^2,
\]

20
suggests that the spacetimes should be thought of as containing $N$ parallel ‘solitonic strings’. The strings have worldvolumes in the $t - \tau$ plane. There is a plane fronted wave \cite{21} carrying momentum along the $\partial/\partial\tau$ direction of the string. We call these plane fronted waves solitonic strings to emphasise that the fields are localised along strings and there are no horizons. The strings are magnetic sources for the two form field strength

$$H = -\sqrt{3} \ast_3 dU.$$ \hfill (73)

This is possible because of the topologically nontrivial $S^2$ at each centre \cite{69}.

We end this subsection by remarking that any solution to Einstein-Maxwell-Chern-Simons theory \cite{55} in 4 + 1 dimensions can be lifted to a solution to 11 dimensional supergravity given by the product metric of $g^{(5)}$ and a flat metric on the six torus. The eleven dimensional four form is given by $H \wedge \Omega_T$, where $\Omega_T$ is the Kähler form on the torus. We have not pursued here an M theory interpretation of these solutions.

5 Slow motion in five dimensions

An interesting feature of BPS solitons is the cancelation between forces which makes static multi-soliton configurations possible. This is clear for the 3+1 dimensional Majumdar-Papapetrou multi black holes, where the electrostatic repulsion is balanced by gravitational attraction. These black holes are in this sense analogous to a nonrelativistic system of massive charged particles, with the charge-to-mass ratio chosen to balance the Newtonian attraction and Coulomb repulsion.

The nature of the forces in the, stationary but not static, 4+1 dimensional solution \cite{56} is presumably more complicated. We shall not study this problem here, and instead focus on the scattering of slowly moving solitons. The question we are interested in is whether there is a direct connection between the metric on the moduli space of four dimensional instantons and the metric on the moduli space describing slow motion of the 4+1 dimensional solitons. The metrics do coincide for pure gravity instantons \cite{16}.

One can follow Manton’s method for truncating the infinite number of degrees of freedom of the gravitational field to the finite dimensional moduli space $\mathcal{M}$ of solitons\footnote{In this section we will refer to any of the solutions \cite{55} as solitons, even if they are singular or contain closed timelike curves. Part of our motivation is to compare with the moduli space metric of four dimensional gravitational instantons \cite{12} where everything is regular, even if $U \neq \tilde{U}$.}. This means that we shall be neglecting both gravitational and electromagnetic radiation, and consider only velocity dependent forces which perturbed solitons induce on each other. As for the four dimensional instantons, the space $\mathcal{M}$ is not the whole of
To obtain $\mathcal{M}$ we need to quotient by the permutation group $S_N \times S_{\tilde{N}}$, and the Euclidean group in three dimensions.

By considering the slow motion approximation to the initial value formulation of 4+1 dimensional Einstein-Maxwell-Chern-Simons (EMCS) theory, one can find the moduli space metric from the effective action where the field degrees of freedom have been integrated out. In the moduli space approximation the centres become functions of $t$ and geodesic curves $\{x_m(t), \tilde{x}_n(t)\}$ correspond to slow motion of a multi solitonic string configuration.

The initial data for EMCS theory consists of a four dimensional manifold $\Sigma$ together with a Riemannian metric $\gamma_{\mu\nu}$, a symmetric tensor $K_{\mu\nu}$, a two form $B$ and a one form $E$. Given a metric $g^{(5)}$ and a one form potential $W$ on $M_5$ we can perform a 4+1 decomposition if there exist a function $t$ whose gradient is everywhere timelike. In this case $\Sigma$ is a level set of $t$, and we choose adapted local coordinates $(t, x^a)$ such that the normal to $\Sigma$ takes the form

$$N = N^{-1} (\partial_t - N^\mu \partial_\mu), \quad (74)$$

where $N$ and $N^\mu$ are the lapse function and the shift vector. The spatial metric $\gamma_{\mu\nu}$ and the second fundamental form $K_{\mu\nu}$ can now be read off from the formulae

$$g^{(5)} = -N^2 dt^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dt)(dx^\nu + N^\nu dt),$$

$$K_{\mu\nu} = \frac{1}{2} N^{-1} (\partial_t \gamma_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu), \quad (75)$$

where $D$ is the covariant derivative compatible with $\gamma$ on $\Sigma$. We also decompose the one form $W$ and two form $H = dW$ as

$$W = W_0 N dt + W_\mu dx^\mu, \quad H = E \wedge N dt + B. \quad (76)$$

This last formula implies expressions for $E$ and $B$ as exterior derivatives of the potentials $W_0$ and $W_\mu$.

The next step is to implement the 4+1 decomposition at the level of the action. After neglecting a total derivative term, the following action is obtained from substituting (75) and (76) into the EMCS action (54)

$$S_{4+1} = \int d^4 x dt N \sqrt{\gamma} \left[ R^\gamma + K_{\mu\nu} K^{\mu\nu} - K^2 \right] + \int d^4 x dt N \sqrt{\gamma} \left[ 2 E_\mu E^\mu - B_{\mu\nu} B^{\mu\nu} + 2 B_{\mu\nu} B^{\rho\nu} \frac{N^\mu N^\rho}{N^2} \right] - \frac{8}{3 \sqrt{3}} \int [W_0 B \wedge B - 2 B \wedge E \wedge W_\mu dx^\mu] \wedge N dt. \quad (77)$$
Here $R^\gamma$ is the Ricci scalar of $\gamma$, $K = \gamma^{\mu\nu}K_{\mu\nu}$, and all contractions use the metric $\gamma$. The three lines come from the Einstein-Hilbert, Maxwell and Chern-Simons terms in the action (54), respectively. If we think of the expression (171) as an action for the fields $\{\gamma_{\mu\nu}, W_\mu, W_0, N_\mu, N\}$, then we see that the last three of these appear without time derivatives. They are Lagrange multipliers and impose the constraints of conservation of energy, momentum and charge

$$\frac{\delta S_{4+1}}{\delta N} = \frac{\delta S_{4+1}}{\delta N_\mu} = \frac{\delta S_{4+1}}{\delta NW_0} = 0.$$  \hspace{1cm} (78)

Arbitrary initial data will not evolve to a solution of the EMCS theory. One needs to impose the constraint equations (78).

To consider the slow motion dynamics of a perturbed stationary solution, we allow the moduli to become time dependent and work to first order in the velocities

$$v^J = \frac{dx^J}{dt},$$ \hspace{1cm} (79)

where we have used $x^J$ to denote a general modulus. This induces a time dependence in the solution which to first order can be written

$$\frac{d\gamma_{\mu\nu}}{dt} = \delta\gamma^J_{\mu\nu}v^J, \quad \frac{dW_\mu}{dt} = \sqrt{3} \delta A^J_\mu v^J,$$ \hspace{1cm} (80)

where $\delta\gamma^J_{\mu\nu}, \delta A^J_\mu$ is the zero mode corresponding to the modulus $x^J$. In general, simply allowing the moduli to depend on time will not give a spacetime that solves the constraint equations, even to first order in the velocities. Instead, it will be necessary to add extra terms linear in the velocities to the original solution. An early example of this technique in gravity is the slow motion of Majumdar-Papetrou black holes [23].

For the case of the Kaluza-Klein monopole lift of the Gibbons-Hawking solutions, it turns out that it is sufficient to simply promote the moduli to time dependent fields. The constraint equations are automatically solved to first order in velocities [16]. This lies behind the simple identification of the moduli space metrics in four and five dimensions. Let us see whether the constraint equations are solved in our case.

To first order in velocities, the charge conservation and momentum conservation constraints become

$$D^\mu(\delta A^J_\mu/N) = 0, \quad D^\mu(\delta\gamma^J_{\mu\nu}/N) = D_\nu(\delta\gamma^J_{\mu\nu}/N).$$ \hspace{1cm} (81)

Here we used (80). It is interesting to see that these two constraints take the form of gauge conditions. They may be imposed on the moduli fields and no extra terms are necessary. Although these gauge conditions look similar to those encountered in
section 3.1, they are quite different. The choice of time slicing is not the same. By comparing (56) and (75) we see that
\[ \gamma_{\mu\nu} = g_{\mu\nu} - \Phi_\mu \Phi_\nu. \]
Working through the changes to the covariant derivative shows that the charge conservation constraint in (81), for instance, becomes
\[ \nabla_\mu \left( \left[ (1 - \Phi^2) g^{\mu\nu} + \Phi_\mu \Phi_\nu \right] \delta A^I_\nu \right) = 0. \quad (82) \]
Deriving this expression uses \( 1/N^2 = 1 - \Phi^2 \). Here \( \Phi^2 \) is contracted with \( g_{\mu\nu} \). A similar expression exists for the momentum constraint. It is clear that this is not the Lorenz gauge that we used for the instanton moduli space. As discussed, the instanton moduli space metric is gauge dependent. This is the first indication that there is not a direct connection between the instanton and soliton moduli space metrics for our solutions.

A more significant problem arises from the Hamiltonian constraint. To first order in velocities the constraint is
\[ \delta g^{I\mu}_{\mu} D^\nu N^\nu - \delta g^{J\mu}_{\mu} D^\nu N^\nu = \frac{N}{\sqrt{7}} \varepsilon^{\mu\rho\sigma\tau} F_{\mu\nu} \delta A^I_\rho A_\sigma. \quad (83) \]
This is an algebraic relation between the various metric and Maxwell field moduli. We might hope that (83) is solved for all moduli for \( \lambda = 1 \). Unfortunately, it is clear that this will not work. Notice that the Hamiltonian constraint (83) involves a symmetric derivative of \( N^\mu \). This translates into a symmetrised derivative of \( \Phi^\mu \). However, only the antisymmetrised derivative of \( \Phi^\mu \) can be expressed in terms of the four dimensional fields via (57). The Hamiltonian constraint will require extra modes to be turned on for a consistent time dependent solution.

The upshot of this section is therefore that, unlike in case of Yang-Mills instantons or pure gravitational instantons, the slow motion moduli space metric of the five dimensional soliton cannot be directly reduced to the four dimensional instanton moduli space metric. A full blooded computation of the backreaction of the moduli velocities onto the spacetime is necessary.

6 Discussion

In this paper we have discussed various properties of multi-instanton solutions of Euclidean Einstein-Maxwell theory. We have also shown how these solutions may be lifted to ‘solitonic string’ solutions of five dimensional Lorentzian Einstein-Maxwell-Chern-Simons theory. There are roughly three types of application for the solutions we have discussed. We hope that the present work has provided a solid base for future investigations.
Firstly and perhaps most interestingly, given that the instantons only involve fields that are observed to exist in nature, would be to understand the physical effects mediated by these solutions. A well known example of the physical effect of Euclidean Einstein-Maxwell theory is the bounce that describes the pair creation of charged black holes in a sufficiently strong electromagnetic field \[24\]. One possible direction of study would be to ask whether the instantons tell us anything about the structure of the vacuum of Einstein-Maxwell theory, say as a function of temperature.

Secondly, it would be of interest to understand the role of these solutions as supersymmetric building blocks within string and M theory. Either as higher dimensional supergravity instantons \[25\], or as a component of Lorentzian compactification or brane solutions. This would be analogous to the ubiquitous appearance of the Gibbons-Hawking metrics in higher dimensions.

Thirdly, there are various mathematical aspects that we have not developed completely. Some of these have physical consequences. It is important to understand the index theory associated with the zero modes of the instantons. This will determine which correlators the instantons contribute to and also their effect on topological terms in the Lagrangian. Furthermore, we have not discussed determinants of quadratic fluctuations about the solutions. An interesting question is whether supersymmetry is sufficient in this case to force the one loop determinants to cancel.

On a slightly different note, a completely distinct set of Einstein-Maxwell instantons may be constructed. LeBrun has found explicit multicentred scalar flat Kähler metrics \[4\]. These give solutions to Einstein-Maxwell theory with the field strength given by half the Kähler form plus the Ricci form. It would interesting to study these solutions in more depth and elucidate their relation, if any, with the solutions that we have discussed.

Acknowledgements

We would like to thank David Berman, Roberto Emparan, Gary Gibbons, Jan Gutowski, Matt Headrick, Hari Kunduri, James Lucietti, David Mateos, Malcolm Perry, Simon Ross, Paul Tod and David Tong for helpful comments at various points during this work.

This project was begun while SAH was supported by a research fellowship from Clare College Cambridge. His research was supported in part by the National Science Foundation under Grant No. PHY99-07949.
A Expressions for the potentials

An explicit formula for $\omega$ may be obtained from integrating (3). There is a choice of
gauge involved as $\omega$ is only defined up to gradient. We can see that the contributions
to $\omega$ will come from cross terms in the sums defining $U$ and $\tilde{U}$ (6). Therefore we can
write

$$\omega = \sum_{mn} \omega_{mn} - \frac{4\pi}{\beta} \sum_n \tilde{\omega}_n + \frac{4\pi}{\beta} \sum_m \omega_m.$$  

(84)

A possible form for the first term $\omega_{mn}$ is

$$\omega_{mn} = -a_m \tilde{a}_n \frac{\langle x - x_m \rangle \cdot \langle x - \tilde{x}_n \rangle}{|x - x_m| |x - \tilde{x}_n|} \frac{(x_m - \tilde{x}_n) \times (x - (x_m + \tilde{x}_n)/2)}{|(x_m - \tilde{x}_n) \times (x - (x_m + \tilde{x}_n)/2)|^2} \left\{ \tan^{-1} \frac{|x_m \times (x_{mn} \times (x_m - x_m - x_n))|}{|x_m| |x_{mn} \times (x_m - x_m - x_n)|} \right\}.$$  

(85)

In the second expression $x_{mn} = x_m - \tilde{x}_n$, $x_m = x - x_m$, $x_n = x - \tilde{x}_n$ and $k$ is an
arbitrary constant vector. This breaking of symmetry is the price we need to pay for
expressing part of the term as a gradient.

A possible expression for the remaining terms, writing $\omega$ as a form for ease of
notation, is

$$\omega_m = a_m \bar{a}_n \frac{(z - z_m) (-y - y_m) dx + (x - x_m) dy}{|x - x_m| [(x - x_m)^2 + (y - y_m)^2]}.$$  

(86)

The $\tilde{\omega}_n$ are given by the same expression but with $a_m \rightarrow \tilde{a}_n$ and $x_m \rightarrow \tilde{x}_n$. As is usual,
the choice of gauge for (86) necessarily breaks the rotational symmetry and has Dirac
strings.

Both of the previous two formulae are more naturally given in polar coordinates.
However, the angles would depend on the centres or pairs of centres in question. If we
want coordinates that are valid for all the $\omega_{mn}$ and $\omega_m$ at once then we need to use
Cartesian coordinates.

The gauge for $\omega$ that we have chosen in (85) and (86) satisfies $\nabla \cdot \omega = 0$. In fact,
the expression in curly brackets in (85) is a harmonic function.

We may also integrate the field strength (5) to obtain an explicit potential. This is
defined up to a gradient. A possible expression is

$$A = A_4 (d\tau + \omega) + \mathbf{A},$$  

(87)

with

$$A_4 = \frac{U - \tilde{U}}{2UU} \quad \text{and} \quad \mathbf{A} = -\frac{1}{2} \left[ \sum_m \omega_m + \sum_n \tilde{\omega}_n \right].$$  

(88)

Where the $\omega_m$, $\tilde{\omega}_n$ are as given in (86). More invariantly, $\nabla \times \mathbf{A} = -\frac{1}{2} (U + \tilde{U})$. Note
that with this choice of gauge, $\nabla \cdot \mathbf{A} = 0$. 

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B  Gamma matrix conventions and spin connection

We work with a chiral representation of the Euclidean gamma matrices

\[ \Gamma^a = \begin{pmatrix} 0 & -i\sigma^a \\ i\bar{\sigma}^a & 0 \end{pmatrix}, \quad (89) \]

where \( \sigma^a = (i, \tau) \) and \( \bar{\sigma}^a = (-i, \tau) \). Here \( \tau \) are the Pauli matrices. The gamma matrices satisfy \( \{\Gamma^a, \Gamma^b\} = 2\delta^{ab} \). We define

\[ \Gamma^{ab} = \frac{1}{2} [\Gamma^a, \Gamma^b] = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}, \quad (90) \]

where \( \sigma^{ab} = \frac{1}{2} [\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a] \) and \( \bar{\sigma}^{ab} = \frac{1}{2} [\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a] \). As two forms, \( \sigma^{ab} \) is anti-self dual whilst \( \bar{\sigma}^{ab} \) is self dual. Finally, let \( \Gamma^5 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \).

In computing the Killing spinor, one needs to know the self dual and anti-self dual parts of the spin connection and field strength. For the field strength these are

\[ F^{ab} \sigma_{ab} = \frac{-2i\tau \cdot \nabla U}{U^2}, \]
\[ F^{ab} \bar{\sigma}_{ab} = \frac{-2i\tau \cdot \nabla \bar{U}}{\bar{U}^2}. \quad (91) \]

Whilst for the spin connection we have

\[ \omega^{ab} \sigma_{ab} = \frac{-2i}{(UU)^{1/2}} \left[ \frac{\tau \cdot \nabla U e^0}{U} + \frac{\tau \times \nabla U \cdot e}{U} \right], \]
\[ \omega^{ab} \bar{\sigma}_{ab} = \frac{-2i}{(U\bar{U})^{1/2}} \left[ -\frac{\tau \cdot \nabla \bar{U} e^0}{\bar{U}} + \frac{\tau \times \nabla \bar{U} \cdot e}{\bar{U}} \right]. \quad (92) \]

C Two component spinor conventions

We can use the matrices of Appendix B to relate vectors in four component notation to two component spinor notation

\[ X^{AA'} = \frac{-i\sigma^{AA'}}{\sqrt{2}} X^a. \quad (93) \]

Because \( a \) is a Euclidean signature tangent space index, raising and lowering this index does not have any effect. The inverse to this relation is

\[ X^a = \frac{i\sigma_a^{BB'}}{\sqrt{2}} \bar{X}^{BB'}, \quad (94) \]
which works because
\[ \sigma^A_{\alpha} \tilde{\sigma}^B_{\beta} = 2\delta^A_{\beta} \delta^B_{\alpha}. \] (95)

Another useful relation is
\[ \tilde{\sigma}^A_{\alpha} \tilde{\sigma}^B_{\beta} = -2\epsilon_{\alpha\beta} \epsilon_{A'B'}, \] (96)

and similarly for the \( \sigma^A_{\alpha} \).

## D Equivalence and inequivalence of inner products

Suppose a metric perturbation satisfies the gauge condition
\[ \nabla^\mu \delta g_{\mu\nu} = \lambda \nabla_\mu \delta g^\nu_\nu. \] (97)

Consider the same perturbation in a different gauge,
\[ \nabla^\mu \delta \hat{g}_{\mu\nu} = \hat{\lambda} \nabla_\mu \delta \hat{g}^\nu_\nu. \] (98)

The two perturbations are thus related by
\[ \delta \hat{g}_{\mu\nu} = \delta g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \] (99)

and the two gauge conditions (97) and (98) require that \( \xi \) satisfy
\[ 2\nabla^\mu \nabla_\nu (\mu_\nu \xi_\nu) = (\hat{\lambda} - \lambda) \nabla_\nu \delta g^\mu_\mu + 2\hat{\lambda} \nabla_\nu \nabla^\mu \xi_\mu. \] (100)

Using this relation it is straightforward to show that
\[ \langle \delta \hat{g}, \delta \hat{g} \rangle_{\hat{\lambda}} = \langle \delta g, \delta g \rangle_{\lambda} + 2(\lambda - \hat{\lambda}) \int d^4x \sqrt{\hat{g}} \delta g^\mu_\mu \delta \hat{g}^\nu_\nu. \] (101)

Therefore, if \( \lambda \neq \hat{\lambda} \) the two inner products are generically inequivalent. They are equivalent if all the modes are trace free in one of the gauges. This is consistent with the observation in the main text that the inner products are manifestly equivalent on trace free modes. We also noted in the text that on noncompact gravitational instantons all normalisable zero modes are indeed trace free, as follows from the fact that these satisfy \( \nabla^2 \delta g^\mu_\mu = 0 \). However, even in this noncompact pure gravity case, nonzero modes will of course generically have a trace component.

## References


