QCD traveling waves beyond leading logarithms

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We derive the asymptotic traveling-wave solutions of the nonlinear 1-dimensional Balitsky-Kovchegov QCD equation for rapidity evolution in momentum-space, with 1-loop running coupling constant and equipped with the Balitsky-Kovchegov-Kuraev-Lipatov kernel at next-to-leading logarithmic accuracy, conveniently regularized by different resummation schemes. Traveling waves allow to define "universality classes" of asymptotic solutions, i.e. independent of initial conditions and of the nonlinear damping. A dependence on the resummation scheme remains, which is analyzed in terms of geometric scaling properties.

I. INTRODUCTION

In the large-$N_c$ and "mean-field" approximation of high energy (high density) QCD, the density of gluons with transverse momenta $k$ in a target evolves with rapidity $Y$ according to the nonlinear Balitsky-Kovchegov (BK) equation. This equation is supposed to capture essential features of saturation effects. In the 1-dimensional approximation, and within leading logarithmic (LL) accuracy, it reads \[ \partial_Y N(L,Y) = \bar{\alpha} \chi(-\partial_L) N(L,Y) - \bar{\alpha} N^2(L,Y), \] where $L = \log(k^2/k_0^2)$, with $k_0^2$ being an arbitrary constant. In the LL approximation the characteristic function of the kernel has the standard Balitsky-Kovchegov-Kuraev-Lipatov (BFKL) form \[ \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \] and the coupling $\bar{\alpha} = \alpha_s N_c/\pi$ is kept fixed.

The goal of the present paper is to extend the known traveling-wave method \[ \text{[7, 8, 9]} \] for obtaining asymptotic solutions of the nonlinear equation \[ \text{[1]} \] to the case where one considers the equation at the next-to-leading logarithmic (NLL) accuracy. In that case, the QCD coupling constant is running, i.e.

\[ \bar{\alpha}(L) = \frac{1}{bL}, \quad b = \frac{11N_c - 2N_f}{12N_c}, \] and the 1-dimensional BK equation which we consider reads

\[ bL \partial_Y N(L,Y) = \chi(-\partial_L, \partial_Y, \bar{\alpha}) N(L,Y) - N^2(L,Y). \] As we shall recall later on, $\chi(-\partial_L, \partial_Y, \bar{\alpha})$ is a "renormalization-group improved" NLL kernel. It follows from specific resummation scheme implying higher order contributions while keeping the known expression \[ \text{[10, 11]} \] of the NLL term in the kernel. Note that there could be also NLL corrections to the nonlinear term, but, as we shall see from the traveling-wave properties, they do not change the asymptotic solutions.

To summarize the present situation of the QCD traveling waves for our purpose, it has been shown \[ \text{[2, 8, 9]} \] that the BK equation with fixed coupling $\bar{\alpha}$ and in the LL approximation belongs to the same universality class as the Fisher and Kolmogorov-Petrovsky-Piscounov (F-KPP) equation \[ \text{[12, 13]} \]. This means in particular that the BK equation admits solutions in the form of traveling waves $N(L - v_g \bar{\alpha} Y)$. $L$ has the interpretation of a space variable while $t = \bar{\alpha} Y$, interpreted as time, is an increasing function of rapidity $Y$. $v_g$ is the critical velocity of the wave, defined in this case as the minimum of the phase velocity. From the point of view of QCD the traveling-wave solution for
the quantity $N$ translates into the property of geometric scaling [14] which yields $N(k^2/Q^2_s(Y))$, where the function $Q^2_s(Y) \propto \exp(\gamma Y)$ is called the saturation scale. Hence the asymptotic traveling-wave solutions of the BK equation satisfy geometric scaling (up to sub-dominant scaling violations which we will not analyze in the present work).

The existence of this solution depends on the general form of the initial condition $N(L, Y_0)$ for large values of $L$. It is however universal, i.e. independent on the details of the kernel or the specific form of the nonlinear term (e.g. independent of NLL nonlinear terms) provided that it plays the rôle of a damping saturation term. This unique condition is that $N(L, Y_0)$ decreases at large $L$ at least as rapidly as an exponential $\exp(-\gamma_0 Y)$ with $\gamma_0 > \gamma_c$, where $\gamma_c$ is the critical anomalous dimension, solution of the equation

$$\chi(\gamma_c) = \gamma_c \chi'(\gamma_c).$$

The corresponding value for $\gamma_c$ obtained with the LL kernel [2] is $\gamma_c = 0.6275$ while the value for the critical velocity $v_g = \chi'(\gamma_c) = 4.883$. The above condition is fulfilled for high energy QCD due to the “color transparency” property of the gluon density, i.e. $N(L, Y = Y_0) \sim e^{-L}$, valid at large $L$. Hence, $\gamma_0 > \gamma_c$.

Within the framework of the traveling-wave approach the results for the gluon density $N(L, Y)$ and the saturation scale $Q^2_s(Y)$ were obtained in the LL approximation [8, 9]. Moreover, in [8], the method was also extended to the case of the equation with the LL kernel and 1-loop QCD running coupling constant [3]. In fact one considered the equation similar to (4), but keeping the LL kernel [2]. In this case, the traveling-wave method leads to a different universality class of solutions than the F-KPP one, with, in particular, geometric scaling property in $\sqrt{Y}$ instead of $Y$.

One would like to extend the above procedure the for BFKL kernel considered at the NLL accuracy and find the analytic solution in the framework of the traveling-wave approach for the corresponding BK equation [14]. This has been done for the case of the fixed coupling $\bar{\alpha}$ [15]. The case of the running coupling, which has not been considered so far, is the subject of this paper.

As we know now, although the NLL corrections were calculated for the BFKL kernel [10, 11] they turned out to be negative and so large that they are of no use for $\alpha_s$ if it is not extremely small. However, several equivalent ways to cure this pathological behavior were proposed. They are based on the observation that the problem of the NLL corrections comes from the existence of spurious collinear singularities. This singularities may be canceled by resummation of the collinear terms at all orders satisfying, at the same time, the renormalization group constraints.

In practice, various resummation schemes were developed from which we will use the S3 and S4 schemes [16] and CCS scheme [17]. It is important for further use (and possible generalization to other schemes), to distinguish two types of resummation among the kernels. In the CCS “implicit” scheme [17], the higher order resummation appears through the dependence of the kernel on two variables only $\chi(\gamma = -\partial_L, \omega = \partial_Y)$, where $\omega = O(\bar{\alpha})$ drives the higher-order corrections. In the S3 and S4 schemes [16], an “explicit” dependence on the coupling constant appears, leading to a triple-variable dependence $\chi(\gamma = -\partial_L, \omega = \partial_Y, \bar{\alpha})$. As we shall see later on, this introduces significant analytical (and eventually phenomenological) differences in the traveling wave solutions.

The aim of this paper is to apply the traveling-waves method with running coupling at the NLL level. As an outcome we obtain the result for the gluon density and the saturation scale valid at large $Y$ which is universal and can be used with any resummed NLL kernel. The specific cases of the S3, S4 and CCS kernels are studied in more detail. We also compare the traveling-wave solutions with that obtained with a method of the linear BFKL evolution in the presence of absorbing boundary conditions [18], which has been applied using the CCS scheme [19].

The plan of the paper is the following. In Sec. II we present in detail the calculation which leads to the solution of the BK equation with the NLL BFKL kernel in the case of running coupling. We arrive at the analytic asymptotic expressions for the saturation scale and the gluon density. This allows us to define in which universality class the traveling-wave solutions lie, depending on the type of the resummation scheme. In Sec. III we specifically analyze the dependence of the logarithmic derivatives of the saturation scale on the resummation scheme used. We check the consistency and generality of our approach by comparison with the previously known results for the CCS scheme. Finally, the conclusions and outlook are given in Sec. IV.

II. BK EQUATION WITH NLL BFKL KERNEL AND RUNNING COUPLING

Following the general method [8] we first write the solution to the linearized version of the BK equation at NLL [14]. It has the form of double Mellin transform [17]

$$N(L, Y) = \int \frac{d\gamma}{2\pi i} \int \frac{d\omega}{2\pi i} N_0(\gamma, \omega) \exp \left( -\gamma L + \omega Y + \frac{1}{b\omega} X(\gamma, \omega) \right),$$

(6)
with
\[ X(\gamma, \omega) = \int_{\gamma}^{\gamma} d\gamma' \chi(\gamma', \omega), \] (7)
and \( \dot{\gamma} \) being an unspecified constant. Indeed, using the saddle-point method for the integration over \( \gamma \) at large enough \( L \), one gets the saddle-point equation
\[ -L + \frac{1}{k\omega} \chi(\gamma, \omega) = 0, \] (8)
or equivalently in operator form
\[ bL \partial Y N(L, Y) = \chi(-\partial_L, \partial_Y) N(L, Y), \] (9)
which is nothing else than the restriction of the BK equation (4) to its linear part.

Here again \( \chi \) as well as \( X \) may depend on \( \alpha \). In order to solve Eqs. (6) and (7), we shall make the approximation of fixing \( \alpha \) in the kernel at some, phenomenologically motivated, value, knowing that it applies only to the higher-order corrections. We shall come back to this approximation later on. In order to simplify the notation we do not write the dependence on \( \alpha \) explicitly. Let us remind that the CCS kernel, as well as all other “implicit” schemes do not include this dependence on the coupling constant.

As a next step we perform the saddle point integration over \( \omega \) in the limit of large \( Y \) and obtain
\[ N(L, Y) = \int \frac{d\gamma}{2\pi i} N_0(\gamma) \exp(-\gamma L + F(\omega_s) Y), \] (10)
where \( F(\omega_s) = \frac{1}{Y b} \omega_s \left( 2X(\gamma, \omega_s) - \omega_s \dot{X}(\gamma, \omega_s) \right) \) and the condition for the saddle point \( \omega_s \) is given by the implicit equation
\[ Y b \omega_s^2 - X(\gamma, \omega_s) + \omega_s \dot{X}(\gamma, \omega_s) = 0. \] (11)

We introduce the notation in which the “dot” means the derivative with respect to \( \omega \) whereas the “prime” means the derivative with respect to \( \gamma \). Let us now expand the integral of the kernel (7) near \( \omega = 0 \)
\[ X(\gamma, \omega) = \sum_{p=0}^{\infty} \frac{X^{(p)}(\gamma, 0)}{p!} \omega^p. \] (12)
Using a similar expansion for its derivative \( \dot{X} \) with respect to \( \omega \) and substituting both quantities into Eq. (11) we have
\[ \left[ Y b + \frac{1}{2} \dot{X}(\gamma, 0) \right] \omega_s^2 = X(\gamma, 0) - \left\{ \sum_{p=3}^{\infty} \frac{1}{p!} X^{(p)}(\gamma, 0) \omega_s^p \right\}, \] (13)
by collecting terms by powers of \( \omega_s \). It is clear from Eq. (13) that for asymptotic \( Y, \omega_s \sim Y^{-1/2} \) while in the convergence domain of the series the remaining terms between braces are at most of order \( Y^{-3/2} \). As we shall check later on, they may contribute only to non-universal sub-asymptotic terms. We also checked this by looking for an iterative solution where the kernel \( \chi(\gamma, \omega) \) is expanded around \( \omega_0 \) and truncated at the order \( P \). The universal terms are shown not to depend on either \( \omega_0 \) or the truncation.

### A. Traveling-wave critical parameters

Considering then (13) up to the order two, we obtain
\[ \omega_s = \sqrt{\frac{X(\gamma, 0)}{Y b + \frac{1}{2} \dot{X}(\gamma, 0)}}. \] (14)
The saddle point value behaves like \( \omega_s \sim Y^{-1/2} \). With this form of \( \omega_s \) the gluon density is given by
\[ N(L, Y) = \int \frac{d\gamma}{2\pi i} N_0(\gamma) \exp(-\gamma L + \Omega(\gamma)t), \] (15)
where time is interpreted as
\[ t = \sqrt{Y + Y_0}, \] (16)
with \( Y_0 = \ddot{X}(\gamma, 0)/2b \). Interestingly, this non-universal term in formula (14) absorbs the arbitrary constant \( \hat{c} \) in \( \ddot{X}(\gamma, 0) \). The dispersion relation has the form
\[ \Omega(\gamma) = \frac{\sqrt{4bX(\gamma, 0)}}{\gamma_c}. \] (17)
In analogy to the LL case, following [8], a critical group velocity (defined as the minimum of the phase velocity in the wave language) is obtained as
\[ v_g = \frac{\Omega(\gamma_c)}{\gamma_c} = \frac{\Omega'(\gamma_c)}{\gamma_c}. \] (18)
However, \( \gamma_c \) determined in such a way still depends on the arbitrary constant \( \hat{c} \). Thus, requiring \( v_g \) to be independent on the choice of \( \hat{c} \) means imposing \( dv_g(\gamma)/d\gamma = 0 = dv_g(\gamma_c)/d\gamma_c \). This is because the dependence of the velocity on \( \hat{c} \) comes through \( \gamma_c \) only. Applying this condition to Eq. (17) one gets
\[ dv_g(\gamma)/d\gamma = 0 = \frac{2\chi(\gamma_c, 0)}{b\gamma_c} \Rightarrow v_g = \sqrt{\frac{2\chi(\gamma_c, 0)}{b\gamma_c}}, \] (19)
eliminating all dependence on the arbitrariness in the definition of \( X \) in Eq. (7). This is to be contrasted with the situation for the linear problem with NLL kernels.

The value of \( \gamma_c \) at the NLL level in general depends on the resummation scheme. This is clearly shown in graphical form in Fig. 1. Geometrically, the value of \( \gamma_c \) is given by the tangent to the characteristic function of the kernel, different for each NLL scheme. Note also that the curve corresponding to the CCS scheme at \( \omega = 0 \) is nothing else than the LL curve given by Eq. (2), and thus the critical parameters are the same in this case. This holds for any other “implicit” scheme which recovers the LL kernel at \( \omega = 0 \) like, e.g., those proposed in [20, 21].
The linearized version of the BK equation \( \text{[4]} \) with the kernel expanded around \( \omega_0 \) up to the second order and the rapidity variable \( Y \) changed to time variable \( t = \sqrt{Y + Y_0} \) is given by

\[
\frac{bL}{2t} \partial_t \mathcal{N} = \left\{ \begin{array}{ll}
\text{"LL"} & \left\{ -b v_g^2 \partial_L + \frac{1}{2} \chi'' (\partial_L^2 + 2 \gamma_c \partial_L + \gamma_c^2) \\
& + \frac{1}{2t} \tilde{\chi} \partial_t - \frac{1}{2t} \tilde{\chi}' \partial_L \partial_t - \frac{1}{2t} \tilde{\chi}' \gamma_c \partial_L + \frac{1}{8 t^2} \tilde{\chi} (\partial_L^2 - \frac{1}{t} \partial_t) \right\} \mathcal{N}.
\end{array} \right.
\]  

(20)

We singled out two parts in the above equation. The part denoted as “LL” has already been present in the LL case (cf. Eq. (33) from [3]). The remaining part called “NLL” contains new terms which originate from the dependence of the NLL BFKL kernel on \( \omega \). In accordance with the approach developed in [3] we take the ansatz for the solution in the form \( \text{[22]} \)

\[
\mathcal{N}(z, t) = t^\alpha \, G(z) \exp \left( -\gamma_c z t^\alpha \right), \quad z = \frac{L - v_g t + c(t)}{t^\alpha},
\]  

(21)

where we assume \( \dot{c}(t) = \beta t^{k-1} \) and the constants \( \alpha, \beta \) and \( k \) are to be determined. In order to recover the LL equation in the limit \( \tilde{\chi}, \tilde{\chi}', \tilde{\chi} \to 0 \), we set \( \alpha = \frac{1}{4} \) and \( k = \frac{3}{4} \), as in [3]. In fact this ansatz remains valid beyond the leading order. It is easy to check the consistency of this choice by looking at the time dependence of the new terms generated at higher order in Eq. (20). When all derivatives which appear in “NLL” part are calculated it turns out that their leading contributions are proportional to \( t^\frac{3}{4} \). This means however that the terms in “NLL” part contribute only at the order \( t^{-\frac{3}{2}} \) since each derivative is multiplied by at least the factor \( t^{-1} \). Thus, the BK NLL linearized equation in this approach has exactly the same form as in the LL case and reduces to the Airy equation

\[
G''(z) = \frac{b \gamma_c v_g}{\chi''(\gamma_c, 0)} (z - 4 \beta) G(z).
\]  

(22)

The condition \( G(z) \sim z \) as \( z \to 0 \) allows to fix the constant \( \beta \) to

\[
\beta = -\frac{1}{4} \left( \frac{\chi''(\gamma_c, 0)}{\gamma_c v_g b} \right)^{\frac{1}{2}} \xi_1,
\]  

(23)

where \( \xi_1 = -2.338 \) is the zero of the Airy function. Finally the result for the gluon density is given by

\[
\mathcal{N}(L, t) = \text{const.} \cdot t^{\frac{3}{4}} \cdot \text{Ai} \left( \left( \frac{\sqrt{2 \gamma_c b \chi(\gamma_c, 0)}}{\chi''(\gamma_c, 0)} \right)^{\frac{1}{4}} \ln \frac{k^2}{Q_s^2(t)} t^{-\frac{3}{4}} + \xi_1 \right) \cdot \left( \frac{k^2}{Q_s^2(t)} \right)^{-\gamma_c},
\]  

(24)

and the saturation scale up to a multiplicative constant has the form

\[
Q_s^2(t) = Q_0^2 \exp \left( \frac{2 \chi(\gamma_c, 0)}{b \gamma_c} t + \frac{3}{4} \left( \frac{\chi''(\gamma_c, 0)}{\sqrt{2 \gamma_c b \chi(\gamma_c, 0)}} \right)^{\frac{1}{2}} \xi_1 t^{\frac{3}{4}} \right).
\]  

(25)

Here \( \chi(\gamma_c, 0) \) is the NLL BFKL kernel resummed in a given scheme (and possibly taken at some value of \( \bar{\alpha} \), see Sec. I). Hence, the solution of the BK equation with the resummed NLL kernel and running coupling has the same functional form as the solution for the LL kernel given in [3]. In particular, in the saturation scale the leading exponential term proportional to the time variable \( t \) is supplemented by a universal term in \( t^{\frac{3}{4}} \), sub-leading by order \( t^{-\frac{3}{2}} \). Note that the higher order contributions to the saddle-point Eq. (12), being sub-leading by the order \( Y^{-\frac{3}{4}} \sim t^{-1} \), do not interfere with this analysis and are expected to be non-universal.

Despite this formal similarity with the LL case with running coupling, the critical exponents are not the same for LL and NLL case if \( \chi(\gamma, \omega = 0, \bar{\alpha}) \neq \chi(\gamma) \), given in [2]. This is due to the fact that the NLL result depends on the resummation scheme used for the kernel \( \chi(\gamma, \omega = 0, \bar{\alpha}) \). However, the traveling-wave solutions are not sensitive to the values of the kernels for \( \omega \neq 0 \), contrary to the applications of the NLL BFKL linear equation without saturation (see, e.g. [23, 24]).
III. SATURATION SCALES BEYOND LEADING ORDER

As we have seen, the time variable scales like $t \sim Y^{1/2}$. For a quantitative analysis of the solution, one defines the logarithmic time-derivative of the saturation scale (or intercept) as

$$\lambda_t(Y) = \frac{d \log(Q_s^2(Y)/\Lambda_{QCD}^2)}{dt}. \tag{26}$$

In order to compare with the usual definition of geometric scaling implying a linear rapidity dependence of the saturation scale, one may also consider an “effective” intercept

$$\lambda_{t,eff}(Y) = \frac{d \log(Q_s^2(Y)/\Lambda_{QCD}^2)}{dY}, \tag{27}$$

so the relation between them is $\lambda_t(Y) = 2t \lambda_{t,eff}(Y)$.

From Eq. (26) we obtain

$$\lambda_t(Y) = \sqrt{\frac{2\chi(\gamma_c, 0)}{b_{\gamma_c}}} + \frac{1}{4} \left( \frac{\chi''(\gamma_c, 0)}{\sqrt{2\gamma_c b\chi(\gamma_c, 0)}} \right)^{\frac{3}{2}} \xi_1 t^{-\frac{3}{2}}, \tag{28}$$

where the time $t$ is defined by Eq. (16). In Fig. 2 we show $\lambda_t(Y)$ from Eq. (28) as a function of $Y^{1/2}$ for the three different resummation schemes S3, S4 and CCS and the value of the coupling $\bar{\alpha} = 0.15$. The result depends on the scheme used. For $Y^{1/2} \rightarrow \infty$ the logarithmic derivative $\lambda_t$ goes to its asymptotic value equal to the group velocity $v_g$, which is also scheme-dependent.

A. CCS scheme

In fact, there appears a difference between the scheme CCS and the schemes S3 and S4. In the first case, since the higher order effects in the kernel are all given as a function of $\omega$, one has identically $\chi(\gamma, \omega = 0) \equiv \chi(\omega)$, i.e. the LL kernel of (2). In mathematical terms, this means that the CCS scheme falls into the same universality class of solutions as the equation with a LL kernel and running coupling constant.

In order to check this result, we compared the traveling-waves solution with that obtained with the different method applied to the CCS scheme. In this method, one considers only the linear evolution term supplemented by the absorbing boundary conditions. It has been confirmed in [7, 8, 9] that the direct traveling-waves approach applied with the LL kernel leads to the same asymptotics (the traveling-wave method allows to add the third universal term for the fixed coupling case while an eventual third term for the running case remains an open problem).
Extracting the asymptotic analytic form of the solution found in [19] (cf. Eqs. (58) and (59) therein), and after changing it to our notations, one obtains

$$\lambda_t(Y) = \sqrt{\frac{2\chi(\gamma_c,0)}{b\gamma_c}} + \frac{1}{4} \left( \frac{\chi''(\gamma_c,0)}{\sqrt{2\gamma_c b\chi(\gamma_c,0)}} \right)^{\frac{1}{2}} \xi_1 \left( t - \sqrt{\frac{2b}{\gamma_c \chi(\gamma_c,0)}} \right)^{\frac{1}{2}} .$$

(29)

We see that the two results (28) and (29) are consistent up to the corrections of the order $t^{-\frac{5}{2}}$. However, the corrections at this order are not expected to be universal. The detailed comparison of the two results for $\lambda_t$ given by Eqs. (28) and (29) is shown in Fig. 3. We observe that they converge for the asymptotic values of $Y$.

For a quantitative comparison at lower rapidities, we can also confront our result for $\lambda^{eff}_t$ with that given in [19]. This is presented in Fig. 4. We see that $\lambda^{eff}_t$ based on the definition of Eq. (27) with appropriate $Y_0$ can successfully mimic the results from [19]. In our analysis, varying $Y_0$ plays the role of parameterizing typical non-universal terms, i.e. terms which depend on the initial conditions, details of the kernel, or of the method used for extracting the
asymptotic behavior. This is a hint for the compatibility of the results of [19] with the universality class defined by the LL kernel with running coupling constant. The analysis of non-universal terms is beyond the scope of the present paper and deserves further study, since these terms may be phenomenologically important.

B. S3 and S4 schemes

FIG. 5: Dependence of the logarithmic derivative $\lambda^t_s$ obtained in the traveling-waves approach in the S3 and S4 schemes as a function of $\bar{\alpha}$. Top: S3 scheme; Bottom: S4 scheme. The curves go up when the value of $\bar{\alpha}$ decreases. The fixed curve corresponding to the CCS scheme, i.e. the same as for LL kernel, is also indicated in both cases.

The schemes S3 and S4, by contrast, are not expected to lie in the same universality class as the previous one. Indeed, the resummed kernels depend explicitly on the the value of the coupling constant. This is depicted in Fig. 5 where one can see how the time-derivative of the saturation scale varies with $\bar{\alpha}$. In order to characterize the universality class of these schemes one would have to go one step beyond the approximation of a fixed $\bar{\alpha}$ in the kernel that we used in the beginning. In fact one would have to first solve exactly the linear operator equation

$$bL \partial_Y N(L, Y) = \chi(-\partial_L, \partial_Y, \bar{\alpha}(L) = 1/bL) N(L, Y),$$

and then apply the general traveling-waves method to find the critical velocity and asymptotic solution. We postpone this detailed study for future work. What, however, is indicated by our study is that the “universality class” of the solution may be different from the CCS case. It remains to be found which kind of geometric scaling property will be satisfied.

Hence we obtain the new result from the QCD traveling-waves approach that the specific asymptotic solutions of the BK equation at NLL accuracy depend on the resummation schemes. In particular they vary between the schemes...
which include an explicit or implicit dependence on the running coupling constant in the definition of the NLL kernel. This introduces a theoretical distinction between NLL effects. It would be an interesting issue to know whether this distinction may appear at physically reachable rapidities. This deserves certainly more study in the future.

IV. CONCLUSIONS

We considered the 1-dimensional Balitsky-Kovchegov (BK) equation in momentum space with running coupling constant. The Balitsky-Kovchegov-Kuraev-Lipatov (BFKL) kernel was taken at the next-to-leading logarithmic (NLL) accuracy with higher orders following known resummation schemes and satisfying renormalization-group (RG) constraints.

Using mathematical properties of nonlinear equations, we derived the traveling-wave solutions, which are valid in the limit of large rapidities \( Y \) and obey universality properties, i.e. are independent on the specific form of the initial conditions, the detailed form of the kernel and the nonlinearities.

We found that the results for the gluon density and the saturation scale acquire dependence on the resummation scheme. For a scheme with an implicit higher order dependence, i.e. where the NLL kernel is a function of \( \gamma = -\partial L \) and \( \omega = \partial_Y \) only, one is expected to fall into the same universality class as in the LL, running coupling case. This is explicitly derived for the CCS scheme [17].

For schemes with an explicit higher order dependence, such as S3 and S4 of [16], where in addition the NLL kernel is an explicit function of \( \tilde{\alpha} \), the asymptotic solution does change as a function of the value of \( \tilde{\alpha} \). The precise determination of the universality class when \( \tilde{\alpha} \) is also considered as running in the NLL kernel is an interesting challenge for the future work.

It is interesting to address further questions about the properties of QCD saturation beyond leading logarithms. For instance, it will be useful to ask which is the universality class of the BK equation (or whether they are modified) when different ways of implementing the running coupling constant are used, since there remain an ambiguity concerning this problem (see e.g. [22, 26]). On a more phenomenological ground, one would like to say something about the physically reachable rapidity region where non-universal terms may be important. For instance, a way was proposed in [27, 28] to take into account the actual form of nonlinearities beyond the asymptotics where their specific form does not play a big role (except in the selection of the critical velocity by setting the unitarity limit). Finally, there remains the problem of a more complete QCD solution with NLL accuracy going beyond the mean-field approximation. We postpone these studies for further work.

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