Is quantum mechanics based on an invariance principle?

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Non-relativistic quantum mechanics for a free particle is shown to emerge from classical mechanics through the requirement of an invariance principle under transformations that preserve the Heisenberg position-momentum inequality. These transformations acting on the position and momentum uncertainties are induced by isotropic space dilatations. This invariance imposes a change in the laws of classical mechanics that exactly corresponds to the transition to quantum mechanics. Space-time geometry is affected with possible consequences for quantum gravity.

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Quantization of classical mechanics is generally not considered as deriving from an invariance principle. While relativity requires the invariance of the laws of nature under space-time transformations, quantum mechanics is usually presented as deriving from prescriptions relating classical quantities to Hermitian operators acting on Hilbert space. The former theory is deeply rooted in space-time geometry, the latter is not. This deep difference is perhaps one of the main obstacles hampering the construction of a coherent theoretical framework for quantum gravity.

In contrast with this view, quantum mechanics is shown here to derive from an invariance principle under transformations that are induced by isotropic and homogeneous space coordinates dilatations. These transformations act on the uncertainties of position and momentum measurements made by observers. In the approach presented here, observers or frames of references are characterized not only by the origins of their space and time coordinates, the direction of their axis and relative velocities but also by the accuracy or resolution of their measurements. Precision of measurements is, consequently, embodied in geometry and the laws of nature must be invariant under precision scale transformations.

An explanation of quantum mechanics based on scale relativity has not often been contemplated in modern physics, a notorious exception being, to our knowledge, the audacious work of L.Nottale [1],[2]. The approach reported in the present article, however, fundamentally differs from that developed by Nottale. In his theory, space-time is supposed to have a fractal geometry, while in ours no such assumption is needed. His fractality postulate leads Nottale to introduce scale transformations that are based on the notion of fractal dimension. As a consequence, these scale transformations involve the logarithm of the coordinates and are fundamentally different from the usual space coordinates dilatations appearing in our work.

Stronger convergence can be found between our approach and the work of M.J.W. Hall and M. Reginatto [11], [12]. These authors assume, as we do, that uncertainty can be considered as the essential property in which quantum and classical mechanics differ. This point of view leads them to introduce non-classical fluctuations in the momentum of a physical system. By postulating that these fluctuations are entirely determined by the position probability density function, they are able to derive the quantum dynamical law from the classical mechanics of a non-relativistic particle. To do so, they need two supplementary postulates that are causality and the additivity of the energy of N non-interacting particles. Both their theory and ours allocate a fundamental importance to the Heisenberg uncertainty principle. Yet, the difference between the approach of Hall and Reginatto and ours resides in the fact that they have to postulate the existence of non-classical fluctuations and to assume that their statistical covariance depends only on the position probability density. In contrast, in our work, these postulates are derived from an invariance principle under a group of scale transformations affecting the position and momentum uncertainties and preserving the Heisenberg inequality. These differences and similarities will be analyzed more deeply in the course of the present report.

In this article, we first introduce transformations rules for the position and momentum uncertainties that preserve the Heisenberg inequality. We then postulate their fundamental status. Next, we show that the classical mechanical definition of the momentum uncertainty is incompatible with these transformations. We are, thus, led to modify the classical definition of the momentum uncertainty in order to satisfy the imposed transformation rules. The modification is obtained by adding a new term to the classical quadratic momentum uncertainty. The transformation requirement implies that this new term should be proportional to the inverse of a measure of the quadratic position uncertainty. As a consequence, this new contribution only depends on the position probability density.

The full determination of the functional form of the new term is then completed by imposing the Hall-Reginatto conditions of causality and additivity of the kinetic energy of a system of non-interacting particles. This leads to a complete specification of the functional dependance of this term. The latter turns out to be proportional to the
quantum potential [4]. The passage from classical to quantum mechanics is, thus, explained as the Schrödinger equation is a simple consequence of this result. Finally, several important consequences of this theory are examined.

Let us consider a non-relativistic free particle in the 3-dimensional Euclidean space. In that space, observers are supposed to perform position and momentum measurements with instruments of limited precision. Hence, an observer is characterized by parameters denoting the statistical position and momentum uncertainties of its instruments. These parameters, let us call them $\Delta x$ and $\Delta p$, are not uniquely defined as there exist many statistical measures of fluctuations for a given probability distribution. For example, $\Delta x^2$ could be defined as the trace of the covariance matrix associated to a given position probability density $\rho(x)$ or as the inverse of the trace of the Fisher matrix [4] associated to the same probability density $\rho(x)$. We limit our definitions of uncertainty to scalar parameters as we suppose isotropy of measurements. However, this restriction could be lifted without changing the major conclusions of our work.

In this picture, observers with different values of their uncertainty parameters are related by space dilatations. Our main postulate is the following: Under dilatations of space coordinates, the parameters $\Delta x$ and $\Delta p$ must transform in such a way that the Heisenberg position-momentum inequality

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

is kept invariant. In other words, the Heisenberg inequality is a fundamental invariant for the changes of precision relating all the observers, and precision becomes part of the geometrical description of the physical space. The transformations on $\Delta x$ and $\Delta p$ we are referring to in the above postulate are the following

$$\Delta x^2 = e^{-\alpha} \Delta x^2$$  \hspace{1cm}  (1)

$$\Delta p^2 = e^{-\alpha} \Delta p^2 + \frac{\hbar^2}{4} \left( e^\alpha - e^{-\alpha} \right) \frac{1}{\Delta x^2}$$  \hspace{1cm}  (2)

where the parameter $\alpha$ is any real number. The group property of these transformations is easily established.

As already said, the definitions of $\Delta x$ and of $\Delta p$ are still unknown. The functional forms of these two quantities are derived in the sequel by the condition that the above transformations should be induced by dilatations of the space coordinates. The complementary and natural requirements of causality of the motion and of additivity of the kinetic energy of a system of $N$ non-interacting particles are also needed in order to get complete functional specification of these two statistical parameters.

In order to clarify the relation between the above transformations and the Heisenberg inequality, let us multiply equation (1) by equation (2). One gets

$$\Delta x^2 \Delta p^2 = e^{-2\alpha} \Delta x^2 \Delta p^2 + \frac{\hbar^2}{4} \left( 1 - e^{-2\alpha} \right)$$  \hspace{1cm}  (3)

The asymptotic behaviors under these transformations are readily obtained. When $\alpha \to +\infty$ one has $\Delta x^2 \Delta p^2 \to \frac{\hbar^2}{4}$. If $\Delta x^2 \Delta p^2$ is already equal to $\frac{\hbar^2}{4}$ then the product $\Delta x^2 \Delta p^2$ keeps the value $\frac{\hbar^2}{4}$ for any value of $\alpha$. For $\alpha \to -\infty$, $\Delta x^2 \Delta p^2 \to +\infty$ for any value of $\Delta x^2 \Delta p^2 \geq \frac{\hbar^2}{4}$.

Hence, transformations (1) and (2), as required, preserve the Heisenberg inequality. We would like to stress that, at this stage of our work, we are unable to prove that these transformations are unique. Nevertheless, a reason for preferring these transformations to others that would also keep invariant the Heisenberg inequality is that, as shown below, they give a simple and complete explanation of the passage from classical mechanics to quantum mechanics for a free particle. Another argument is that equation (3) is the simplest that reduces to the classical dilatation law for $\Delta x \Delta p$ when $\hbar$ tends to zero. These arguments are, however, too weak to guarantee the uniqueness of these transformations and more reflection on this aspect is needed.

These above remarkable properties bear some similarities with the Lorentz transformations. In analogy with the fact that the velocity of light constitutes an upper limit for the velocities of material bodies, the parameter $\frac{\hbar^2}{4}$ represents a lower limit for the product of uncertainties $\Delta x^2 \Delta p^2$. Latter in the article, the analogy will appear even more striking.

We now show that our postulate of the fundamental role of transformations (1), (2) imposes a radical modification of the laws of dynamics that precisely corresponds to the passage from classical to quantum mechanics.

To do so, let us start from the classical mechanical description of a free non-relativistic particle of mass $m$ in the 3-D Euclidean space. In order to take into account from the beginning the finite precision of the observer, we introduce an ensemble of initial positions characterized by the probability density $\rho(x)$. This function together with the classical action of the particle, $s(x)$, are the basic variables of the formalism. They are fields and due to this classical mechanics
appears here as a field canonical theory \[\text{[5]}\]. Let us stress that by assuming an initial position probability density we introduce only classical fluctuations of the position variable in the theory.

The time evolution of any functional of type

\[ A = \int d^3x F(x, \rho, \nabla \rho, \nabla \nabla \rho, \ldots, s, \nabla s, \nabla \nabla s, \ldots) \]  

of the two variables \(\rho\) and \(s\) and their spatial derivatives, that is at least once functionally differentiable in terms of \(\rho\) and \(s\) is given by

\[ \partial_t A = \{A, \mathcal{H}_\text{cl}\} \]  

where

\[ \mathcal{H}_\text{cl} = \int d^3 x \frac{\rho |\nabla s|^2}{2m} \]  

is the classical Hamiltonian functional and

\[ \{A, B\} = \int d^3 x \left[ \frac{\delta A}{\delta \rho(x)} \frac{\delta B}{\delta s(x)} - \frac{\delta B}{\delta \rho(x)} \frac{\delta A}{\delta s(x)} \right] \]  

where \(\frac{\delta}{\delta \rho(x)}\) and \(\frac{\delta}{\delta s(x)}\) are functional derivatives. The above functional Poisson bracket endows the set of functionals of type \(4\) with an infinite Lie algebra structure \(\mathcal{G}\).

Any functional belonging to \(\mathcal{G}\), and \(\mathcal{H}_\text{cl}\) is one of them, generates a one-parameter continuous group of transformations. The time transformations are generated by \(\mathcal{H}_\text{cl}\). Equation \(5\) when applied to \(\rho(x)\) and \(s(x)\) respectively, yields the continuity equation and the Hamilton-Jacobi equation

\[ \partial_t \rho = -\nabla \cdot \left( \frac{\rho \nabla s}{m} \right) \]  

\[ \partial_t s = -\frac{|\nabla s|^2}{2m} \]  

where the gradient \(\nabla s\) is the classical momentum of the particle. It is a random variable over the ensemble of initial conditions corresponding to \(\rho(x)\).

Now let us consider the group of space dilatations \(x \rightarrow e^{-\frac{\alpha}{2}} x\) and its action on \(\rho\) and \(s\)

\[ \rho'(x) = e^{\frac{3\alpha}{2}} \rho(e^{\frac{-\alpha}{2}} x), \quad s'(x) = e^{-\frac{\alpha}{2}} s(e^{\frac{-\alpha}{2}} x) \]  

where \(\alpha\) is any real number. Note that these transformations preserve the normalization of the probability density \(\rho(x)\) \[\text{[3]}\]. Clearly, they also keep the dynamical equations \(8\) and \(9\) invariant.

Let us assume that the average momentum of the particle is vanishing. This corresponds to a particular choice of a "comoving" frame of reference but, by no means, reduces the generality of our results. The general results can, indeed, be retrieved by performing an arbitrary Galilean transformation. In this particular frame, the classical definition of the scalar quadratic momentum uncertainty, i.e. the trace of the statistical covariance matrix, is given by

\[ \Delta p^2 = \int d^3 x \rho |\nabla s|^2 = 2m \mathcal{H}_\text{cl} \]  

and under transformations \(10\) becomes

\[ \Delta p'^2 = e^{-\alpha} \Delta p^2 \]  

Also, any definition of the scalar quadratic position uncertainty measuring the dispersion of \(\rho(x), \Delta x^2\), transforms as

\[ \Delta x^2 = e^{-\alpha} \Delta x^2 \]  

Here, as in equation \(1\), the scalar quantity \(\Delta x^2\) still remains undefined. Not surprisingly, equation \(12\) shows that the classical quadratic momentum uncertainty does not obey the prescribed transformation rule \(2\). Notice, however,
that the transformation law (12) corresponds to the first term in the right hand side of equation (2). As a consequence, the requirement of transformations (1) and (2) to be fundamental compels us to modify the definition (11) of $\Delta p_{cl}^2$ in order to get a quantity whose variance satisfies equation (2). The latter involves the constant $\hbar^2$. Moreover, the new definition of $\Delta p^2$ should reduce to the classical one when $\hbar$ tends to zero. Indeed, in classical mechanics the changes in the accuracy of position and momentum measurements are not constrained by the Heisenberg inequality. This is clear when considering the product of equations (12) and (13). The desired modification to the definition (11) should, thus, consist in adding a supplementary term proportional to $\hbar^2$. Indeed, the new quantity $\Delta p^2$ must transform under mapping (10) in such a way that, at least, the first term of the right hand side of equation (3) is retrieved.

Let us translate this constraint by adding a new term to the above definition of $\Delta p_{cl}^2$ and get a new expression for

$$\Delta p_q^2 = \int d^4 x \rho(x) |\nabla s(x)|^2 + \hbar^2 Q$$

(14)

As we now prove, the condition that under the dilatations (10) the quantity $\Delta p_q^2$ defined above should transform as prescribed by equations (1), (2) and (3) reduces the set of possible functional forms of $Q$. First, note that following equation (11) the classical quadratic momentum uncertainty is proportional to the energy functional. This represents the fact that in our particular comoving inertial reference frame, the quadratic momentum uncertainty is proportional to the average kinetic energy. It is natural to consider that this proportionality is preserved for the new definition of the quadratic momentum uncertainty $\Delta p_q^2$ we are looking for. It is also reasonable to assume that the energy functional should belong to the Lie algebra $G$. Hence, the new term $Q$ must also be a functional belonging to the Lie algebra $G$, that is, it must be of the form (4).

Let us now apply the space dilatation (10) to the definition (14) of $\Delta p_q^2$. This leads to

$$\Delta p'_q^2 = e^{- \alpha} \Delta p_{cl}^2 + \hbar^2 Q'$$

(15)

where $Q'$ is the transform of $Q$ under the transformation (10). Adding and subtracting an appropriate term, $e^{- \alpha} \hbar^2 Q$, to the right hand side of equation (15) we get

$$\Delta p'_q^2 = e^{- \alpha} \Delta p_q^2 + \hbar^2 (Q' - e^{- \alpha} Q)$$

(16)

The identification of this equation with equation (2) imposes

$$Q' - e^{- \alpha} Q = \frac{1}{4\Delta x^2} (e^\alpha - e^{- \alpha})$$

(17)

which, using equation (1), can be transformed into

$$Q' - \frac{1}{4\Delta x^2} = e^{- \alpha} (Q - \frac{1}{4\Delta x^2})$$

(18)

This equation possesses an infinity of solutions. However, its form indicates the existence of a relation between $Q$ and $\Delta x^2$ that is scale independant

$$Q = \frac{1}{4\Delta x^2}$$

(19)

This particular solution is the only one for which the relation between $\Delta p_q^2$ and $\Delta x^2$ is independant of the scale exponent $\alpha$. Furthermore, $\Delta p_q^2$ being proportional to the Hamiltonian functional, the generator of dynamics, one would expect that the latter keeps the same form in term of $\Delta x^2$, independently of $\alpha$. In other words, an observer should not be able to infer the value of $\alpha$ by doing only internal measurements of motion. This argument justifies the choice of solution (19) on physical ground.

We are, thus, led to the conclusion (19) that the supplementary term necessary to obtain a definition of $\Delta p_q^2$ that is compatible with the transformations (1), (2) is inversely proportional to $\Delta x^2$. As the latter quantity only depends on the probability density $\rho(x)$, it is obvious that $Q$ must be a functional of the form (4) that does not depend on the action $s(x)$ or any of its spatial derivatives.

One should keep in mind, at this level, that the precise definition of the quadratic position uncertainty, $\Delta x^2$, that appears in transformations (1) and (2) is still unknown at this level. This undeterminacy is now lifted by considering
In order to explain the transition from classical to quantum mechanics they assume that the classical momentum \( p \) and energy of a particle and only depends on the variables of that particle. Their fundamental statement is the following.

The causality condition means that the equations of motions generated by \( \mathcal{H}_q \) should be causal, i.e. the existence and unicity of their solutions should require only the specification of \( \rho(x) \) and \( s(x) \) on an initial surface. This condition, combined with equations (14) and (19), implies that \( \mathcal{Q} \) should only depend on \( \rho(x) \) and its first order space derivatives.

The second principle required in the Hall-Reginatto theory is the so-called independance condition, that is, the Hamiltonian of N non-interacting particles must be the sum of N terms. Each of these terms represents the kinetic energy of a particle and only depends on the variables of that particle.

Using these principles, Hall and Reginatto are able to prove that the unique functional form for \( \mathcal{Q} \) is

\[
\mathcal{Q} = \beta \int d^3x |\nabla \rho(x)^{1/2}|^2
\]

where \( \beta \) is a constant which must be put equal to one in order to find the quantum Hamiltonian functional which in the variables \( \rho(x) \) and \( s(x) \) reads

\[
\mathcal{H}_q = \int d^3x \left[ \frac{\rho(x)|\nabla s(x)|^2}{2m} + \frac{\hbar^2}{2m} |\nabla \rho(x)^{1/2}|^2 \right]
\]

Simultaneously, we obtain the complete determination of \( \Delta x \) that appears in equations (1) and (2) by using the relation (19). Interestingly, what is obtained is not the usual definition corresponding to the trace of the second order centered statistical moment of the position. The definition obtained here is, up to a constant factor, proportional to the classical Fisher length \( \frac{1}{\Delta \rho^{1/2}} \) associated to the position probability density \( \rho(x) \).

The functional \( \mathcal{H}_q \) generates the quantum time evolution of any functional \( \mathcal{A} \) of the algebra \( \mathcal{G} \) via equation (5) where \( \mathcal{H}_{cl} \) is to be replaced by \( \mathcal{H}_q \). When \( \mathcal{A} \) is specialized to \( s(x) \) this leads to the apparition of the quantum potential \( Q \) in the Hamilton-Jacobi equation (9)

\[
\partial_t s = -\frac{|\nabla s|^2}{2m} + \frac{\hbar^2}{2m} \rho^{1/2}
\]

while the continuity equation for \( \rho(x) \), equation (8), is preserved.

Finally, the Schrödinger equation is readily obtained from equation (23) and the continuity equation by performing the transformation from the variables \( \rho(x) \) and \( s(x) \) to the wave function variables \( \psi \) and \( \psi^* \)

\[
\psi = \rho^{1/2}e^{is/h}
\]
In the algebra defined by the Poisson bracket (7), this is a canonical transformation.

Let us summarize. We have derived the quantum evolution law for a free non-relativistic particle in 3-D flat space from the requirement that the quadratic uncertainties on position and momentum should satisfy the transformations rules (1) and (2) together with the causality and independence principles. The form in which we obtain quantum mechanics is that of the canonical field theory which has been introduced and studied from different points of view by various authors [7], [8], [9], [10]. None of these authors, however, derives quantum mechanics from an invariance principle as we do here.

Let us now consider the variance of the Schrödinger equation under the dilatations (10). By adding and subtracting adequate terms, the transformation of the Hamiltonian functional (22) under these dilatations can be cast in the explicit form

\[ H'_{q}[\rho, s, \nabla \rho, \nabla s] = \cosh \alpha \int d^3x \left[ \frac{\rho(x)|\nabla s(x)|^2}{2m} + \frac{\hbar^2}{2m} \left| \nabla \rho(x)^{1/2} \right|^2 \right] - \sinh \alpha \int d^3x \left[ \frac{\rho(x)|\nabla s(x)|^2}{2m} - \frac{\hbar^2}{2m} \left| \nabla \rho(x)^{1/2} \right|^2 \right] \]

(25)

where \( H'_{q} \), as a functional of \( \rho(x), s(x) \) and their respective spatial derivatives, is obtained from

\[ H'_{q}[\rho, s, \nabla \rho, \nabla s] \equiv H_{q}[\rho', s', \nabla \rho', \nabla s'] \]

(26)

in which \( \rho', s', \nabla \rho', \nabla s' \) are derived from equation (10).

The first term in the right hand side of equation (25) is proportional to \( H_{q}[\rho, s, \nabla \rho, \nabla s] \) while the second term contains a factor that is similar to \( H_{q}[\rho, s, \nabla \rho, \nabla s] \) up to a sign in the integral. Let us call \( K_{q} \) this factor

\[ K_{q}[\rho, s, \nabla \rho, \nabla s] \equiv \int d^3x \left[ \frac{\rho(x)|\nabla s(x)|^2}{2m} - \frac{\hbar^2}{2m} \left| \nabla \rho(x)^{1/2} \right|^2 \right] \]

(27)

The physical dimension of \( K_{q} \) is clearly the same as that of \( H_{q} \), i.e. it is an energy. As any functional belonging to the algebra \( G \), \( K_{q} \) is the generator of a one parameter continuous group. Let us denote by \( \tau \) the parameter of that group. Since \( K_{q} \) has the dimension of an energy, the dimension of \( \tau \) is that of a time. In terms of this new functional, the transformation (25) can be rewritten in a more compact notation as

\[ H'_{q} = \cosh \alpha \ H_{q} - \sinh \alpha \ K_{q} \]

(28)

while \( K_{q} \) can easily be shown to transform as

\[ K'_{q} = -\sinh \alpha \ H_{q} + \cosh \alpha \ K_{q} \]

(29)

Notice that these transformations are strictly equivalent to the equations (1), (2).

Hence, under the dilatations (10) the couple \( (H_{q}, K_{q}) \) transforms as a 2-D Minkowski vector under a Lorentz-like transformation. One easily shows that this induces the following transformations on the group parameters \( t \) and \( \tau \) respectively associated to \( H_{q} \) and \( K_{q} \)

\[ t' = \cosh \alpha \ t + \sinh \alpha \ \tau \]

(30)

and

\[ \tau' = \sinh \alpha \ t + \cosh \alpha \ \tau \]

(31)

Now, any functional \( A \) of \( G \) can be considered as a function of both \( t \) and \( \tau \), and its evolution in both times variables is given by

\[ \partial_{t}A = \{ A, H_{q} \} \]

(32)

and

\[ \partial_{\tau}A = \{ A, K_{q} \} \]

(33)
Under a dilatation transformation of parameter \( \alpha \) given by equation (10), the above equations (32) and (33) will, obviously, transform into

\[ \partial_t' A' = \{ A', H_q' \} \]  \hspace{1cm} (34)

and

\[ \partial_T' A' = \{ A', K_q' \} \]  \hspace{1cm} (35)

In other words, these equations are covariant under transformations (10).

The Schrödinger equation is a particular case of equation (32) for

\[ A = \psi = \rho^{1/2} e^{i s / \hbar} \]  \hspace{1cm} (36)

and has the usual form

\[ i \hbar \partial_t \psi = - \frac{\hbar^2}{2m} \nabla^2 \psi \]  \hspace{1cm} (37)

The wave function, \( \psi \), can also be considered as a function of \( \tau \). Its evolution equation in this parameter derives from equation (33) and writes

\[ i \hbar \partial_\tau \psi = - \frac{\hbar^2}{2m} \nabla^2 \psi' + \frac{\hbar^2}{m} \psi' \frac{\nabla^2 |\psi|}{|\psi|} \]  \hspace{1cm} (38)

The physical meaning of this equation is still an open question and is discussed in the conclusion of the present article. As a result of the above derivation, the system of equations (37) and (38) is covariant under the space dilatations and its transform reads

\[ i \hbar \partial_t' \psi' = - \frac{\hbar^2}{2m} \nabla^2 \psi' \]  \hspace{1cm} (39)

\[ i \hbar \partial_\tau' \psi' = - \frac{\hbar^2}{2m} \nabla^2 \psi' + \frac{\hbar^2}{m} \psi' \frac{\nabla^2 |\psi'|}{|\psi'|} \]  \hspace{1cm} (40)

where the transformation of the wave function

\[ \psi'(x) = e^{\frac{3\alpha}{4} [\psi(e^{\alpha} x)]^\frac{1+\alpha}{2}} [\psi(e^{-\alpha} x)]^\frac{1-\alpha}{2} \]  \hspace{1cm} (41)

directly derives from the dilatation mapping (10). The nonlinearity of transformation (41) is remarkable and contrasts with the linear transformation rules that generally are assumed in the studies of invariance groups of the Schrödinger equation \[14, 15, 16\]. The reason for that difference clearly appears when considering among others the article by P. Havas \[14\]. In this work, the transformation rules of both the classical Hamilton-Jacobi and the Schrödinger equations under spatial dilatations are studied. When considering the transformation of the Hamilton-Jacobi equation, the classical action \( s \) is supposed to transform as prescribed in our equation (10). However, when the transformation of the Schrödinger equation under dilatations is considered, the \( \psi \) function is assumed to transform as the square root of a density, i.e. as \( \rho^{1/2} \). This hypothesis does not take into account the fact that \( \psi \), as given by equation (36), is a function of both \( \rho^{1/2} \) and \( s \). As \( s \) in the quantum case obeys a modified Hamilton-Jacobi equation (23), there is no reason to assume that this quantity does not transform under dilatations. The reason to discard the transformation of \( s \) in the wave function in the above mentioned studies is unclear but it is perhaps related to the fact that this quantity appears in \( \psi \) via a complex phase factor of modulus one. However, there is no fundamental argument that can support this hypothesis. The roles of both \( s \) and \( \rho^{1/2} \) in the transformation of the Schrödinger equation is better appreciated when its decomposition in terms of the continuity equation (8) and the modified Hamilton-Jacobi equation (23) is considered.

Before ending this article, another approach to the transformations (1-2) or (28-29) should be mentioned. This was in fact the first we considered chronologically. These transformation rules can, indeed, be generated by the following element of the algebra \( G \) whose definition is

\[ S = \int d^3 x \rho(x)s(x) \]  \hspace{1cm} (42)
It represents the average on the position ensemble of the classical action or, up to a factor \( \hbar \), the ensemble average of the quantum phase.

An easy calculation using the definition (7) of the functional Poisson bracket gives

\[
\{ S, \mathcal{H}_q \} = \int d^3x \left[ \frac{\rho |\nabla \rho|^2}{2m} - \frac{\hbar^2}{2m} |\nabla \rho^{1/2}|^2 \right] = \mathcal{K}_q
\]

(43)

and

\[
\{ S, \mathcal{K}_q \} = \int d^3x \left[ \frac{\rho |\nabla \rho|^2}{2m} + \frac{\hbar^2}{2m} |\nabla \rho^{1/2}|^2 \right] = \mathcal{H}_q
\]

(44)

The infinitesimal transformation for the parameter \( \delta \alpha \) generated by \( S \) of any element \( A \) of the algebra \( \mathbb{G} \) is defined as

\[
A' = A + \delta \alpha \{ A, S \}
\]

(45)

Let us apply (45) respectively to both \( \mathcal{H}_q \) and \( \mathcal{K}_q \). It is easily shown that after exponentiating these infinitesimal transformations in order to generate the transformation for finite values of \( \alpha \) one recovers equations (28) and (29). As a consequence, transformations (1) and (2) are also recovered.

Note also that both generators \( \mathcal{H}_q \) and \( \mathcal{K}_q \) tend to \( \mathcal{H}_{\text{cl}} \) for \( \hbar \to 0 \), i.e. the times evolution in \( t \) and \( \tau \) become identical in the classical limit. This seems to indicate that the finiteness of \( \hbar \) is lifting a degeneracy that is intrinsical to classical mechanics, and splits the two time variables.

Another remarkable property that can be derived from the above relations is the fact that \( \mathcal{H}_q + i \mathcal{K}_q \) is a holomorphic function of \( t + i \tau \).

The nonlinear Schrödinger equation (40) in the variable \( \tau \) obtained here as a companion to the usual linear Schrödinger equation in the time \( t \) is not a newcomer in physics. It has been postulated, though in the time \( t \) variable and in different contexts, by several authors [13], [17], [18]. It belongs to the class of Weinberg’s nonlinear Schrödinger equations [14]. This equation admits a nonlinear superposition principle [20]. It has been studied as a member of the general class of nonlinear Schrödinger equations generated by the so-called nonlinear gauge transformations introduced by Doebner and Goldin [21]. The evolution generated by this equation in the \( \tau \) variable is nonunitary as \( \mathcal{K}_q \) can not be reduced to the quantum average of a Hermitian operator. In addition, one easily shows that together with the functionals generating translations, rotations and Galilean boosts, \( \mathcal{K}_q \) constitutes a functional canonical representation of the Galilei algebra. Another important property is that equation (40) also implies the continuity equation for the probability density function \( \rho \). Hence, though non-unitary, it has nice physical properties such as Galilean invariance and conservation of the probability norm.

In conclusion, our postulate of transformations (1) and (2) of the position and momentum uncertainties under space dilatations is shown to force the addition of a new contribution to the classical average energy of a free particle. This term happens to be exactly the so-called quantum potential [6] and the new form of the energy obtained corresponds to the standard quantum average of the energy of a free particle. Hence, the emergence of quantum mechanics from classical dynamics seems to be related to an invariance principle preserving the Heisenberg inequality. Other natural conditions are also necessary. These are causality and the additivity of the energy for non-interacting particles.

The consequences of this result are multiple and mostly unknown up to now. Many questions remain open and among them, the following are coming to the mind.

What is the physical meaning of equation (40) and of the temporal parameter \( \tau \)? In relation with this question, it is intriguing to notice that while for a free particle one has \( \partial_t \Delta x^2 > 0 \) and \( \partial_t \Delta p^2 = 0 \) in the physical time \( t \), it can be shown that in the \( \tau \) evolution the product \( (\partial_\tau \Delta x^2)(\partial_\tau \Delta p^2) \) is always negative. This is reminiscent of the process of state vector reduction in position measurement in which \( \Delta x^2 \to 0 \) while \( \Delta p^2 \to +\infty \), or conversely if one is measuring momentum. Would this \( \tau \) evolution be related in some way to the nonunitary process that physicists like R.Penrose [22] are trying to identify for the description of the wave function collapse? The difference with these approaches lies, fundamentally, in the fact that they always consider the reduction process in the usual time.

Another interesting question emerging from the above framework concerns the consequences of requiring local invariance under the dilatations (10), i.e dilatations with space dependent parameter \( \alpha(x) \). Would this requirement result in the existence of a new fundamental interaction field? Preliminary work seems to indicate that this is the case and the obtained gauge field is not the electromagnetic one.

Finally, the most exciting question is about the picture of space-time that would emerge from the combination of the special or general relativistic invariance with the quantum invariance described here.
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