GRAVITATIONAL WAVE SCINTILLATION BY A STELLAR CLUSTER

G. Congedo, F. De Paolis, P. Longo, A. A. Nucita, D. Vetrugno
Department of Physics and INFN, University of Lecce, CP 193, I-73100 Lecce, Italy

Asghar Qadir
Centre for Advanced Mathematics and Physics at the National University of Science and Technology, Rawalpindi, Pakistan
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

Received (received date)
Revised (revised date)

The diffraction effects on gravitational waves propagating through a stellar cluster are analyzed in the relevant approximation of Fresnel diffraction limit. We find that a gravitational wave scintillation effect - similar to the radio source scintillation effect - comes out naturally, implying that the gravitational wave intensity changes in a characteristic way as the observer moves.

Keywords: gravitational waves: general

1. Introduction

General Relativity predicts the existence of gravitational waves that propagate in the vacuum with the speed of light. Analogously to electromagnetism, it is expected that gravitational waves might also be scattered by intervening particles \(^1\), \(^2\), \(^3\) and micro-lensed by compact objects along the line of sight \(^4\), \(^5\), \(^6\). Moreover, gravitational waves will be diffracted as they pass through a distribution of objects, each of which acts as an obstacle (or slit). Of course, these effects are expected to become apparent only if the linear sizes of the intervening particles are comparable to the gravitational radiation wavelength in the range \(10^7 - 10^{15}\) cm. We consider a stellar cluster as the diffracting object and concentrate in particular on that possibly hosted in the center of the Milky Way and surrounding the central black hole Sgr A*.

There are two types of diffraction phenomena discussed: the Fraunhoffer \(^7\) and Fresnel approximation. In our case only the second is relevant since the first approximation involves infinite source-cluster and cluster-observer distances. We determine the conditions under which the diffraction effects may be measurable.

It turns out that a gravitational wave scintillation effect, analogous to the well
known scintillation of radio sources, comes out naturally implying that the gravitational wave intensity changes in a characteristic way with the motion of the observer.

The paper is structured as follows: in Section 2 we describe the parameters of the cluster mass density profile used to study the Fresnel diffraction. In Section 3 is given the mathematical tools of the Fresnel theory, in the particular geometry described there. In Section 4 we present the main result of this paper: namely, the scintillation pattern of the gravitational waves as they pass through the stellar cluster at the galactic center. Finally, in Section 5 we present some conclusions.

2. The Mass Distribution Model

The ESO and Keck teams have continuously monitored the region of the galactic center for several years so that a cluster of stars (at a distance < 1”) has been identified. In particular, Ghez et al. have reported on the observations of the main sequence S2 star (with mass $M_{S2} \approx 15 M_\odot$) orbiting the central black hole with a Keplerian period of $\approx 15$ yrs. This has allowed the mass contained within $R \approx 4.87 \times 10^{-3}$ pc to be constrained to $M(R) \approx 3.67 \times 10^6 M_\odot$. The most plausible model is that this mass is in the form of a super-massive black hole.

However, there is a possibility that a small fraction of the inferred central mass is in the form of a diffuse stellar cluster which surrounds the central black hole. In this case, according to the dynamical observations towards the galactic center, we require that the total mass satisfies the condition $M(R) = M_{BH} + M_{CL}(R)$, $M_{BH}$ and $M_{CL}(R)$ being the black hole and the cluster mass within $R$, respectively.

Of course, the cluster mass and density distribution are still unknown so that, as a toy model, we can assume that the stellar component follows a Plummer density profile given by

$$\rho_{CL}(r) = \rho_0 f(r) , \quad \text{with} \quad f(r) = \left[ 1 + \left( \frac{r}{r_c} \right)^2 \right]^{-5/2} ,$$

where the cluster central density $\rho_0$ is given by

$$\rho_0 = \frac{M_{CL}}{\int_0^{R_{CL}} \frac{M_{CL}}{4\pi r^2 f(r)} \, dr} ,$$

$R_{CL}$ being the cluster radius. Useful information is provided by the black hole mass fraction, $\lambda_{BH} = M_{BH}/M$ and its complement, $\lambda_{CL} = 1 - \lambda_{BH}$. As one can see, the requirement given in eq. (2) implies that $M(r) \to M_{BH}$ for $r \to 0$. Hence, the total mass density profile $\rho(r)$ is given by

$$\rho(r) = \lambda_{BH} M \delta(3) (\vec{r}) + \rho_0 f(r)$$

and the mass contained within $r$ is

$$M(r) = \lambda_{BH} M + \int_0^r 4\pi r'^2 \rho_0 f(r') \, dr' .$$
Considerations on both the dynamics and evolution of the stellar cluster surrounding the central black hole allow us to constrain the possible range of values for $\lambda_{BH}$ to be above about 0.9. In the following for definiteness we will assume that $\lambda_{BH} \simeq 0.99$.

Assuming that all the cluster components have the same mass ($\simeq 1 \, M_\odot$), we can estimate the average distance between two neighboring stars as $\langle d(r) \rangle \simeq [M_\odot/\rho(r)]^{1/3}$.

3. The Fresnel Diffraction Approximation

Consider a distant source emitting gravitational radiation. Assume that there is a distribution of objects between the source and the observer, each of which acts as an obstacle (or slit). Let $(X_0, Y_0)$, $(X_1, Y_1)$ and $(X_2, Y_2)$ be three reference frames (with mutually parallel axes) in the planes of the observer, diffraction and source, respectively chosen such that the origins are collinear.

Using the Fresnel approximation for interaction of electromagnetic waves with dust particles distributed along the line of sight gives the diffraction amplitude $A_0$ on the plane of the observer (at distance $z_0$ from the diffraction plane) is

$$A_0(x_0, y_0) = \frac{e^{ikz_0}}{i\lambda z_0} \int \int_{-\infty}^{\infty} A_1'(x_1, y_1) e^{i\kappa [d^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2]} dx_1 dy_1,$$

where

$$A_1'(x_1, y_1) = \frac{A_2 e^{i\kappa r_{12}}}{z_1} e^{i\kappa \delta(x_1, y_1)}$$

is the amplitude just past the diffraction plane. In eq. (5) $A_2$ is the source amplitude, $r_{12} \simeq z_1$ is the distance of the source from a generic point on the diffraction plane, and $z_1$ is the distance of the source from the center of the diffraction plane. It is worth noting that when the wave moves through the diffraction plane, a change of phase does occur due to the optical path difference $\delta(x_1, y_1)$ which, in turn, depends on the coordinates of each obstacle. For incoming plane waves generated by a point-like source, the factor $A = A_2 e^{i\kappa r_{12}}/z_1$ is a real constant so that the previous integral becomes

$$A_0(x_0, y_0) = \frac{A e^{i\kappa z_0}}{2i\pi R_F^2} \int \int_{-\infty}^{\infty} e^{i\kappa \delta(x_1, y_1)} e^{i\frac{(x_0-x_1)^2 + (y_0-y_1)^2}{2R_F^2}} dx_1 dy_1,$$

where the Fresnel radius is defined as

$$R_F = \sqrt{\frac{\lambda z_0}{2\pi}}.$$

It is assumed that the screen is a step of optical path $\delta$ parallel to the $Y_1$ axis, described by a Heaviside distribution in the $(X_1, Y_1)$ plane $\delta(x_1, y_1) = \delta \times H(x_1)$. This case is realistic since, at the Fresnel scale, the edge of a particle
distribution can be considered as a straight line that divides the plane into two regions. Under these assumption, the integral in eq. (7) turns out to be

\[
A_0(x_0, y_0) = A e^{ikz_0} \left[ 1 + e^{i\kappa \delta} - \frac{1}{2} \left( 1 + S(X_0) + C(X_0) - i(S(X_0) - C(X_0)) \right) \right], \tag{9}
\]

where \( S(X) \) and \( C(X) \) are Fresnel integral and \( X_0 = x_0 / (\sqrt{\pi} R_F) \) is the reduced variable corresponding to \( x_0 \). Thus, the intensity on the plane of the observer is

\[
I_0(x_0, y_0) = A_0(x_0, y_0) A_0^*(x_0, y_0) = A^2 \iota_0(X_0), \tag{10}
\]

\( \iota_0(X_0) \) being the relative intensity given by

\[
\iota_0(X_0) = 1 - (S(X_0) - C(X_0)) \sin(\kappa \delta) + \left[ S(X_0)^2 + C(X_0)^2 - \frac{1}{2} \right] \left[ 1 - \cos(\kappa \delta) \right]. \tag{11}
\]

In the next Section, we apply the previous formalism to the case of monochromatic gravitational radiation (with wavelength \( \lambda_{GW} \)) passing through the stellar cluster possibly surrounding the galactic center black hole. Taking the optical path \( \delta \) to be of the same order of magnitude as the average distance \( \langle d(r) \rangle \) between the stars in the cluster, we give the condition under which Fresnel diffraction occurs. Moreover, we shall find that a characteristic scintillation pattern appears in the plane of the observer.

4. Gravitational Wave Scintillation

The Fresnel diffraction limit entails that \( D > \sim R_F \), where \( D \) is the circular hole diameter and \( R_F \) is the Fresnel radius. In order for there to be diffraction by the stellar cluster, the optical path \( \delta \) should be of the same order of \( D \) that is \( \sim \langle d \rangle \), i.e. the average distance between neighboring stars. Therefore, we get

\[
\langle d \rangle > \sim \sqrt{\frac{\lambda z_0}{2\pi}}, \tag{12}
\]

implying

\[
\lambda_{GW} \lesssim \frac{2\pi \langle d \rangle^2}{z_0}. \tag{13}
\]

Thus, in order to have diffractive effects by the star cluster in the Fresnel limit, we need gravitational radiation with wavelength less than the mean spacing squared divided by the distance of the observer from the plane of diffraction that we take to be \( z_0 \sim 8 \text{kpc} \).

Next we consider a gravitational wave source behind the stellar cluster at the galactic center. The source might be, for example, an extragalactic system of inspiral binary black holes or a single spinning pulsar or a binary system of neutron stars at the outer edge of the Galaxy, which are assumed to be detectable by a gravitational wave telescope (like VIRGO, LIGO, LISA or ASTROD) with an integration time \( T \) (for details see e.g. Thorne [16]). In the following, we shall mainly concentrate on diffraction by the star cluster at the galactic center and, due to
the expected average distance between neighboring stars, the most relevant gravitational radiation sources are pulsars lying behind that cluster. Therefore, we will consider in the following analysis the VIRGO interferometer.

In Fig. 1, we present the scintillation pattern obtained from eq. (11) in the plane of the observer for an incoming monochromatic gravitational wave whose wavelength is $\lambda_{GW} \simeq 1.5 \times 10^7$ cm. The diffracting stellar cluster has parameters $\lambda_{BH} = 0.99$, $\rho_0 = 1.15 \times 10^{-11}$ g cm$^{-3}$ and $r_c = 5.8$ mpc, corresponding to an average stellar distance of $\langle d \rangle \simeq 10^{15}$ cm evaluated at a distance of $r \simeq r_c$ from the cluster center. The position $X_0 = 0$ corresponds to the case of perfect alignment among the source, the cluster center and the observer. Of course, as one can see from Fig. 1 as the observer moves along the direction $X_0$, it passes through a series of maxima and minima of the gravitational wave intensity.

![Fig. 1](image-url)

**Fig. 1.** The gravitational wave pattern in the plane of the observed is shown. Here, the diffracting stellar cluster has parameters $\lambda_{BH} = 0.99$, $\rho_0 = 1.15 \times 10^{-11}$ g cm$^{-3}$ and $r_c = 5.8$ mpc, corresponding to an average stellar distance of $\langle d \rangle \simeq 10^{15}$ cm evaluated at a distance of $r \simeq r_c$ from the cluster center.

Analogously to Fig. 1, the gravitational wave pattern obtained with $\langle d \rangle \simeq 10^{14}$ cm, evaluated at the distance $r \simeq 0.1r_c$, is reported in Fig. 2.

As one can see by comparing Fig. 2 with Fig. 1, even though the frequency of the curves remains the same, the height of the curves decreases and the principal maximum shifts from $i_0 \simeq 1.8$ to $i_0 \simeq 1.4$. If one considers a different gravitational radiation wavelength, which is to say a different wave source, we obtain analogous scintillation patterns to those presented in Fig. 1 (for $\langle d \rangle \simeq 10^{14}$ cm) or Fig. 2 (for $\langle d \rangle \simeq 10^{15}$ cm).

Obviously, the positions of the maxima do not change in the dimensionless variable $X_0$. However, different gravitational radiation diffraction patterns (for different
incoming wavelengths) are obtained in the variable $x_0$. As is clear, each figure has been normalized to the incoming gravitational radiation intensity ($I_0 = 1$). As such, as the observer moves, the scintillation effect implies a change in the wave intensity that, in principle, might be used to infer information about the source and the cluster parameters (in particular on $\langle d \rangle$).

As an example, consider a gravitational wave (with wavelength $\lambda_{GW} \simeq 1.5 \times 10^7$ cm) interacting with the stellar cluster at the center of the Galaxy. By inspecting Fig. 1, it is clear that the distance between the principal maximum and the next minimum is $\Delta X_0 \simeq 1.5$. Hence, if the observer moves around the center of the Milky Way with the typical orbital speed $v_\odot = 220$ km s$^{-1}$, the time-scale $\Delta T \simeq 1$ year is obtained. For such a cluster we do not consider binary sources because of the Fresnel diffraction limits [13]. Obviously, if we take into account other types of clusters (with different physical parameters) it is also possible to get diffractive effects with different gravitational wave sources, like black hole binaries, etc.

In order for the diffraction effect to be measurable, it is clear that the integration time $T$, required by a gravitational wave interferometer to detect a signal over the typical instrument noise, has to be less than the traveling time, $T_\odot$, for Earth to move from a maximum to the next minimum of the scintillation pattern. Here, an optimal detection sensitivity or, roughly speaking a signal-to-noise ratio $\simeq 1$, is assumed. Consequently, the integration time becomes [13]

$$T = \left[ \frac{h(f)}{h_0} \right]^2$$

(14)
where \( h(f) \) is the experimental sensitivity curve (in unit of \( \sqrt{Hz} \)) and \( h_0 \) is the expected (dimensionless) gravitational wave amplitude at the characteristic frequency \( f \).

In Tab. 1, we give the relevant parameters for three pulsars and the resulting values for \( T_\odot \) and \( T \) to be evaluated by taking into account the VIRGO sensitivity threshold. As one can see, for the Crab pulsar the scintillation effect is clearly detectable, for PSR 0021-72C the effect is marginally detectable while for PSR 1604-00 it is far from detectability, at least for the VIRGO interferometer.

**Table 1.** Here the physical parameters for some known pulsars together with the characteristic frequency \( f \) at which the objects mostly emit are reported. The expected gravitational wave amplitude \( h_0 \) and the experimental sensitivity curve \( h(f) \) (taken from VIRGO sensitivity threshold) at that frequency are also reported. The integration time \( T \) and the traveling time \( T_\odot \) are given with the minimum average distance \( \langle d \rangle \) between two neighboring stars (see eq. 13).

<table>
<thead>
<tr>
<th>Pulsar</th>
<th>Crab</th>
<th>PSR0021-72C (47TUC)</th>
<th>PSR1604-00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ) (ms)</td>
<td>33.2</td>
<td>5.757</td>
<td>421.8</td>
</tr>
<tr>
<td>( P \times 10^{-15} ) (s/s)</td>
<td>422.6</td>
<td>0.04</td>
<td>0.30</td>
</tr>
<tr>
<td>( f ) (Hz)</td>
<td>60.423</td>
<td>347.403</td>
<td>4.742</td>
</tr>
<tr>
<td>( h_0 )</td>
<td>( 3.59 \times 10^{-25} )</td>
<td>( 8.36 \times 10^{-27} )</td>
<td>( 2.68 \times 10^{-27} )</td>
</tr>
<tr>
<td>( h(f) )</td>
<td>( 7 \times 10^{-25} )</td>
<td>( 5 \times 10^{-23} )</td>
<td>( 2 \times 10^{-20} )</td>
</tr>
<tr>
<td>( T_\odot )</td>
<td>( \simeq 3.75 \text{ y} )</td>
<td>( \simeq 1.563 \text{ y} )</td>
<td>( \simeq 13.8 \text{ y} )</td>
</tr>
<tr>
<td>( T )</td>
<td>( \simeq 10.6 \text{ h} )</td>
<td>( \simeq 1.1 \text{ y} )</td>
<td>( \simeq 2 \times 10^6 \text{ y} )</td>
</tr>
<tr>
<td>( \langle d \rangle ) (cm)</td>
<td>( 1.4 \times 10^{15} )</td>
<td>( 5.68 \times 10^{15} )</td>
<td>( 5.98 \times 10^{14} )</td>
</tr>
</tbody>
</table>

In Tab. 2, we give the same quantities as in Tab. 1 but for three different kinds of pulsar parameters: a typical radio pulsar, a typical new-born pulsar and a millisecond pulsar. As one can see, only for new-born pulsars, even with rather small \( P \) values, the scintillation effect would be detectable by the VIRGO detector.

**Table 2.** This table is analogous to Tab. 1 but for three different classes of pulsar parameters.

<table>
<thead>
<tr>
<th>( P ) (s)</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ) (s/s)</td>
<td>( 10^{-15} )</td>
<td>( 10^{-15} )</td>
<td>( 10^{-19} )</td>
</tr>
<tr>
<td>( f ) (Hz)</td>
<td>20</td>
<td>200</td>
<td>( \simeq 2000 )</td>
</tr>
<tr>
<td>( h_0 )</td>
<td>( \simeq 10^{-26} )</td>
<td>( \simeq 3 \times 10^{-26} )</td>
<td>( \simeq 10^{-27} )</td>
</tr>
<tr>
<td>( h(f) )</td>
<td>( 5 \times 10^{-22} )</td>
<td>( 4 \times 10^{-21} )</td>
<td>( 10^{-22} )</td>
</tr>
<tr>
<td>( T_\odot )</td>
<td>( \simeq 6.5 \text{ y} )</td>
<td>( \simeq 2 \text{ y} )</td>
<td>( \simeq 0.92 \text{ y} )</td>
</tr>
<tr>
<td>( T )</td>
<td>( \simeq 79 \text{ y} )</td>
<td>( \simeq 18.5 \text{ d} )</td>
<td>( \simeq 300 \text{ y} )</td>
</tr>
<tr>
<td>( \langle d \rangle )</td>
<td>( 2.42 \times 10^{15} )</td>
<td>( 7.66 \times 10^{14} )</td>
<td>( 2.42 \times 10^{15} )</td>
</tr>
</tbody>
</table>

*See VIR-NOT-PER-1390-51 available online at [http://www.virgo.infn.it/senscurve](http://www.virgo.infn.it/senscurve)*
5. Concluding Remarks

The General Theory of Relativity predicts the existence of gravitational waves. In accordance with what is currently observed for electromagnetic waves interacting with obstacles (or slits), we expect that also gravitational radiation may undergo diffraction. As usual, it is possible to have diffraction effects only if the incident radiation wavelength and the intervening obstacle size are of the same order of magnitude. Hence, due to the typical large wavelength of the considered radiation (i.e. $10^7 - 10^{15}$ cm for spinning neutron stars), only systems of objects on galactic scale may act as diffractive objects. We have considered the diffraction of gravitational waves by a stellar cluster possibly hosted at the center of our Galaxy and found, under the conditions for which the Fresnel diffraction occurs, that a scintillation of the gravitational wave signal comes out naturally.

As we have shown, this effect, analogous to the well known scintillation of radio waves due to a medium along the line of sight of a distant source, produces a characteristic pattern on the plane of the observer. We have also shown that the typical crossing time $T_\odot$ (i.e. the time required by the observer to move across a minimum to the next maximum of the gravitational wave scintillation pattern) is of the order of a few years. For a particularly strong gravitational wave signal and in particular regions of the frequency band, the typical integration time of instruments like VIRGO and/or LIGO (which have a rather similar sensitivity threshold curve) may be as short as few hours thus allowing a good sampling of the scintillation pattern.

We emphasize that, as observed in Section 4, new-born pulsars, i.e. pulsars that were born within the last $\sim 10^4$ years are the best candidates to look for the gravitational wave scintillation effect by the star cluster lying at the galactic center with the VIRGO instrument.

It is worth noting that the next generation of gravitational wave interferometers (such as Advanced VIRGO and Advanced LIGO) will have a sensitivity greater than that allowed by the present instrument configurations. For example, Advanced LIGO $^b$ will have more than an order of magnitude greater sensitivity than initial LIGO implying a reduction by at least two orders of magnitude of the integration time. If this is the case, there will be a real possibility of using the gravitational radiation signals to probe deep into our Universe and search for effects like the scintillation of gravitational waves. Detecting this effect will give an independent way of estimating the main parameters of the intervening stellar cluster like the average stellar distance $\langle d \rangle$ and the cluster central mass density. The microlensing effect on the gravitational waves due to the black hole at the galactic center has been investigated elsewhere.$^5$.

---

$^b$See the web page at [http://www.ligo.caltech.edu/advLIGO/](http://www.ligo.caltech.edu/advLIGO/)
References