Hidden Symmetry of Higher Dimensional Kerr–NUT–AdS Spacetimes

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It is well known that 4-dimensional Kerr–NUT–AdS spacetime possesses the hidden symmetry associated with the Killing–Yano tensor. This tensor is ‘universal’ in the sense that there exist coordinates where it does not depend on any of the free parameters of the metric. Recently the general higher dimensional Kerr–NUT–AdS solutions of the Einstein equations were obtained. We demonstrate that all these metrics with arbitrary rotation and NUT parameters admit a universal Killing–Yano tensor. We give an explicit presentation of the Killing–Yano tensor and associated second rank Killing tensor and briefly discuss their properties.

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There are several reasons why higher dimensional black hole solutions attracted a lot of attention recently. The string theory is consistent only when the number of spacetime dimensions is either 10 or 26. Black holes in the string theory were widely discussed in connection with the problem of microscopical explanation of the black hole entropy. Also in the recent models with large extra dimensions it is assumed that one or more additional spatial dimensions are present. In such models one expects mini black hole production in the high energy collisions of particles. Mini black holes can serve as a probe of the extra dimensions. At the same time their interaction with the brane, representing our physical world, can give the information about the brane properties.

Higher dimensional non–rotating black hole solutions were found long time ago by Tangherlini [1]. The solutions for rotating black holes, which are higher dimensional generalization of the Kerr metric, were obtained by Myers and Perry (MP metrics) [2]. More recently the MP solutions were generalized to include the cosmological constant [3, 4, 5]. These solutions are of special interest in connection with their possible applications for the study of AdS/CFT correspondence. Further generalization of the higher dimensional Kerr–AdS solutions which includes also the NUT parameters was found in [3, 7].

The higher dimensional Kerr–NUT–AdS metrics are stationary and axisymmetric; they possess the Killing vectors which generate the time translation and rotations in the independent 2D planes of rotation. In this paper we show that besides these evident symmetries all higher dimensional Kerr–NUT–AdS metrics, describing the rotating black holes with arbitrary rotation and NUT parameters in an asymptotically AdS spacetime, have a new hidden symmetry.

Hidden symmetries of 4D black hole solutions of the Einstein equations are well known. The study of them has begun when Carter [8] discovered that the Hamilton–Jacobi equation in the (charged) Kerr metric allows the separation of variables. Walker and Penrose [9] demonstrated that this separability is a consequence of the existence of an irreducible second rank Killing tensor

\[ K_{\mu\nu} = K_{(\mu\nu)}, \quad K_{(\mu\nu;\lambda)} = 0. \]  

Penrose and Floyd [10] found that this Killing tensor is a ‘square’ of a more fundamental antisymmetric Killing–Yano (KY) tensor [11]

\[ f_{\mu\nu} = f_{[\mu\nu]}, \quad f_{\mu(\nu;\lambda)} = 0. \]  

The second rank Killing tensor can be expressed in terms of \( f_{\mu\nu} \) as follows

\[ K_{\mu\nu} = f_{\mu\nu} f_\nu^\alpha. \]  

Carter [12] showed that the Killing–Yano tensor itself is derivable from the existence of a ‘Killing–Maxwell’ form (an analogue of what we call below a potential \( b \)).

The existence of the Killing–Yano tensor for the Kerr metric is a consequence of the fact that this metric belongs to the Petrov type D. Collinson [13] proved that if a vacuum solution of 4D Einstein equations admits a KY tensor it belongs to the type D. All the vacuum type D solutions were obtained by Kinnersley [14]. Demiański and Francaviglia [15] showed that in the absence of acceleration these solutions admit the KY tensor.

The type D solutions of the Einstein–Maxwell equations with the cosmological constant allow a convenient representation in the form of the Plebański–Demiański metric [10] (for a recent review and reinterpretation of parameters in this solution see [17]). A subclass of solutions without acceleration studied by Plebański [18] possesses a Killing–Yano tensor. The Plebański metric reads

\[ ds^2_4 = Q_p (d\tau - r^2 d\sigma)^2 - Q_r (d\tau + p^2 d\sigma)^2 + \frac{dp^2}{Q_r} + \frac{d\sigma^2}{Q_p}, \]  

where, in the absence of electric and magnetic charges,

\[ Q_p = \frac{\gamma + 2lp - \epsilon p^2 - \lambda p^4}{r^2 + p^2}, \quad Q_r = \frac{\gamma - 2mr + \epsilon r^2 - \lambda r^4}{r^2 + p^2}. \]
This metric obeys the equation $R_{\mu\nu} = 3\lambda g_{\mu\nu}$. Its form is invariant under the rescaling of the coordinates $p \to \alpha p$, $r \to \alpha r$, $\tau \to \alpha^{-1} \tau$, $\sigma \to \alpha^{-3} \sigma$. Under this transformation the cosmological constant parameter $\lambda$ is invariant, while the other parameters change. One can always use this transformation to fix the magnitude of one of the parameters, say $\epsilon$. Afterwards the parameters $(m, \gamma, l)$ are related to the mass, angular momentum and NUT charge (see, e.g., [6]). The Killing–Yano tensor (which under the scaling transformation is multiplied by a constant) reads

$$f^{(4)} = r dp \wedge (d\tau - r^{-2} d\sigma) + pd\tau \wedge (d\tau + p^2 d\sigma).$$

It is interesting that in the chosen coordinates its form does not depend on the parameters $(\lambda, \gamma, m, l)$. Moreover, one can easily check, using GRTensor, that $f^{(4)}$ remains the KY tensor for the solutions of the cosmological Einstein–Maxwell equations (11) when the electric and magnetic charges are included in (5). Let us emphasize that this universality property is valid only for a specially chosen coordinate system, but the very existence of such coordinates is rather non-trivial.

The hidden symmetries of higher dimensional rotating black holes were first discovered for 5D MP metrics in [19, 20]. Namely, it was demonstrated that both, the Hamilton–Jacobi and massless scalar field equations, allow the separation of variables; the corresponding Killing tensor was obtained. This result was generalized for the MP metrics in an arbitrary number of dimensions, provided that their rotation parameters can be divided into two classes, and within each of the classes the rotation parameters are equal one to another [21]. A similar result is valid in the presence of the cosmological constant [22, 23] and NUT parameters [6, 24, 25]. Recently we explicitly demonstrated that the KY and Killing tensors exist for any MP metric with arbitrary rotation parameters [26]. We generalize now this result to the Kerr–NUT–AdS spacetimes.

Our starting point is the expression for the general higher dimensional Kerr–NUT–AdS metrics obtained recently [7]. For notation convenience, we deal with an analytical continuation of these metrics. Let $D$ denotes the total number of spacetime dimensions. We define $n = [D/2]$ and for brevity $\varepsilon = D - 2n$, $m = n - 1 + \varepsilon$. The metrics read

$$ds^2 = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + Q_\mu \sum_{k=0}^{m-1} A^{(k)}(d\psi_k)^2 - \frac{\varepsilon c}{A^{(n)}} \sum_{k=0}^{m} A^{(k)}(d\psi_k)^2,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^{n} (x_\nu^2 - x_\mu^2), \quad c = \prod_{k=1}^{m} a_k^2,$$

$$X_\mu = (-1)^{2\mu} \frac{\varepsilon c}{x_\mu^2} \prod_{k=1}^{m} (a_k^2 - x_\mu^2) + 2M_\mu (-x_\mu)^{1-\varepsilon},$$

$$A^{(k)} = \prod_{\nu_1 < \cdots < \nu_k} x_{\nu_1}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1}^2 \cdots x_{\nu_k}^2.$$  \hspace{1cm} (8)

The primes on the sum and product symbols indicate that the index $\nu = \mu$ is omitted. In odd dimensions the apparent singularity for $c = 0$ is removed when $c$ is ‘absorbed’ in the definition of $\psi_\mu$.

The physical metrics with proper signature are recovered when a standard radial coordinate $r = -ix_n$ is introduced. The metrics possess $(n + \varepsilon)$ Killing vectors $\delta_{\psi_k}$. $\Psi_0$ plays the role of the time coordinate. The meaning of the parameters is the following: $a_k$ denote $m$ ‘rotation’ parameters, $M_\alpha$ for $\alpha = 1 \ldots (n-1)$ denote the ‘NUT’ parameters, $M = -i^{1+\varepsilon} M_\mu$ is the mass and $\lambda = -g^2$ is proportional to the cosmological constant [27]. In odd dimensions one of the ‘NUT’ parameters may be eliminated due to the scaling symmetry (see [7] for details). Therefore these metrics constitute $(D - 1 - \varepsilon)$-parametric solutions of the cosmological Einstein equations $R_{\mu\nu} = (D - 1)\lambda g_{\mu\nu}$. Let us finally remark that the formulas (7) and (8) are applicable also in $D = 3$ where one recovers 2-parametric BTZ black hole [28].

The connection with the Kerr–AdS metrics [4, 5] is established through the ‘Jacobi’ transformation of the (constrained) ‘latitude’ Boyer–Lindquist coordinates $(i = 1 \ldots n, a_{n+1} = 0)$

$$\mu_\alpha^2 = \prod_{\alpha=1}^{n-1} (a_\alpha^2 - x_\alpha^2) / \prod_{k=1}^{n} (a_\alpha^2 - a_k^2),$$

and the due rescaling of $\psi_k$ coordinates.

The inverse metrics read

$$\langle \partial_\alpha \rangle^2 = \sum_{\mu=1}^{n} Q_\mu U_\mu \left[ \sum_{k=0}^{m} (-1)^k x_\mu^{2(m-k)} \partial_\psi_k \right]^2 + \sum_{\mu=1}^{n} Q_\mu \partial_\psi_k^2 - \frac{\varepsilon}{cA^{(n)}} \langle \partial_\psi_k \rangle^2.$$  \hspace{1cm} (10)

Our claim is that the metrics (7) admit the $(D - 2)$–rank Killing–Yano tensor and the second rank Killing tensor in a general case the KY tensor is defined as a $p$–form $f$ which obeys the equation

$$\nabla_{\epsilon_1 f_{\epsilon_2} \epsilon_3 \cdots \epsilon_{p+1}} = 0.$$  \hspace{1cm} (11)

The associated second rank Killing tensor is

$$K_{\mu\nu} = \frac{1}{(p-1)!} f_{\mu \epsilon_1 \cdots \epsilon_{p-1} \nu} \epsilon_1^\alpha \cdots \epsilon_{p-1}^\alpha.$$  \hspace{1cm} (12)

We shall follow the procedure established in [20]. Instead of dealing with the KY tensor of the rank $(D-2)$, it is technically easier to consider its dual tensor $k_{\mu\nu}$ related to $f$ as

$$f_{\alpha_1 \cdots \alpha_{D-2}} = (\ast k)_{\alpha_1 \cdots \alpha_{D-2}} = \frac{1}{2} \epsilon_{\alpha_1 \cdots \alpha_{D-2} \nu \kappa \delta} k^{\nu \kappa \delta}.$$  \hspace{1cm} (13)

Here $\epsilon_{\alpha_1 \cdots \alpha_{D}}$ is the totally antisymmetric tensor

$$\epsilon_{\alpha_1 \cdots \alpha_{D}} = \sqrt{-g} \epsilon_{\alpha_1 \cdots \alpha_{D}}.$$  \hspace{1cm} (14)
It is possible to show that $k$ is a conformal Killing–Yano (CYK) tensor \cite{29,30}, obeying the equation

$$k_{\alpha\beta;\gamma} + k_{\gamma\beta;\alpha} = \frac{2}{D-1}(g_{\alpha\gamma} k^\beta_{\beta;\sigma} + g_{\beta}(\alpha k^\gamma_{\gamma;\sigma})),$$ \hspace{1cm} (15)

If the second rank CYK tensor $k$ is closed, $dk = 0$, the form $f$ defined by \cite{13} is the KY tensor \cite{31}. In the latter case the form $k$ can be written (at least locally) as $k = db$, where $b$ is a one–form (potential).

We propose the following ansatz for potential $b$

$$2b = \sum_{k=0}^{n-1} A^{(k+1)} d\psi_k.$$ \hspace{1cm} (16)

Its exterior derivative

$$2k = 2db = \sum_{k=0}^{n-1} da^{(k+1)} \wedge d\psi_k$$ \hspace{1cm} (17)

is the 2–rank CKY tensor. We have explicitly checked, using GR Tensor, that the equations \cite{13} are satisfied for $D \leq 9$. We strongly believe that they are valid in any number of dimensions.

The quantities $A^{(k)}$ and $A_{(k)}$ contain no dependence on the parameters specifying the solution \cite{7}. The special structure of $k_{\alpha\beta}$ together with the form of inverse metrics \cite{13} imply that $k^{\alpha\beta}$ has the same property. The determinant of \cite{7} is

$$g = (-cA^{(n)})^\tau [\det A^{(k)}]^2.$$ \hspace{1cm} (18)

In this expression $A^{(k)}$ is understood as the $n \times n$ matrix. The relation \cite{13} implies that the KY tensor $f_{\mu_1...\mu_{D-2}}$ has no dependence on the parameters (up to a not essential common constant factor $\sqrt{c}$ in odd dimensions). In other words, $f$ is universal.

Using \cite{10, 13, 17} and \cite{18} one can easily obtain KY tensor $f$ in an explicit form for an arbitrary number of dimensions. Since $D = 3$ case is trivial ($f$ is a Killing vector) we shall give the expressions in $D = 4, 5$ and 6.

In $D = 4$ we denote $x_1 = p$, $x_2 = i\tau$, $\psi_0 = \tau$, $\psi_1 = \sigma$.

Then we recover the KY tensor \cite{6}.

In $D = 5$ we find

$$f^{(5)} = x_1 dx_1 \wedge h(x_2) + x_2 dx_2 \wedge h(x_1),$$ \hspace{1cm} (19)

where we have defined

$$h(x) = d\psi_0 \wedge d\psi_1 + x^2 d\psi_0 \wedge d\psi_2 + x^4 d\psi_1 \wedge d\psi_2.$$ \hspace{1cm} (20)

Finally in $D = 6$ let us denote $x_1 = x$, $x_2 = y$, $x_3 = z$.

Then we have

$$f^{(6)} = z(x^2 - y^2) dx \wedge dy \wedge h(z) + y(x^2 - z^2) dx \wedge dz \wedge h(y) + x(y^2 - z^2) dy \wedge dz \wedge h(x).$$ \hspace{1cm} (21)

The existence of a KY tensor immediately implies the existence of a Killing tensor. With the help of CKY one can rewrite the formula \cite{12} as

$$K^\alpha_{\mu\nu} = Q_{\mu\nu} - \frac{1}{2} g_{\mu\nu} Q^\alpha_{\alpha},$$ \hspace{1cm} (22)

where $Q_{\mu\nu} = k_{\mu\alpha} k^\alpha_{\nu}$ is the conformal Killing tensor. The components of (conformal) Killing tensor are most easily written in ‘mixed’ indices. We find

$$Q^\alpha_{\beta} = \sum_{\mu=1}^{n} \delta^\alpha_{\mu} \delta^\beta_{\mu} a^2 + \sum_{\mu=1}^{n-1} \delta^\alpha_{\mu} \delta^\beta_{\mu} A^{(k+1)} - \sum_{k=0}^{m-1} \delta^\alpha_{\mu} \delta^\beta_{\mu}.$$ \hspace{1cm} (23)

Evidently, $Q^\alpha_{\alpha} = 2A^{(1)}$, and so we have

$$K^\alpha_{\beta} = Q^\alpha_{\beta} - A^{(1)} \delta^\alpha_{\beta}.$$ \hspace{1cm} (24)

In this ‘mixed’ form the (conformal) Killing tensor is universal. With both indices down, $K_{\mu\nu}$ possesses the same structure of diagonal and nondiagonal sectors as the metric.

Similar to the Myers–Perry case the constructed KY and Killing tensors have direct connection with the isometries of the spacetime. We find

$$\xi^\mu = \frac{1}{D-1} k^{\sigma\mu},$$ \hspace{1cm} (25)

and also

$$\eta^\mu = -K^\mu_{\alpha} \xi^\alpha = (\partial \psi_0)^\mu.$$ \hspace{1cm} (26)

To summarize, we claim that all the general Kerr–NUT–AdS metrics \cite{7} in any number of dimensions and with arbitrary parameters possess both the Killing–Yano and Killing tensors. The existence of an additional integral of motion is implied. However, for $D \geq 6$ the total number of integrals of motion, connected with the Killing vectors and one additional Killing tensor, is not sufficient for the separation of variables in the Hamilton–Jacobi, Dirac, and Klein–Gordon equations. An interesting open question is whether there exist other non–trivial Killing–Yano and Killing tensors in the spacetime \cite{7}. It would be also interesting to understand whether in higher dimensions there exists a deep relation between the hidden symmetries and the algebraic type of the metric, similar to what has been observed in the four dimensional case.

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[25] Davis P 2006 Class. Quant. Grav. 23 3607
[27] We use notations ‘rotation’ and ‘NUT’ parameters to stress that the meaning of these parameters depends on the choice of the scaling fixing condition.
[31] Cariglia M 2004 Class. Quant. Grav. 21 1051