Probabilistic cloning with supplementary information contained in the quantum states of two auxiliary systems

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Abstract
In probabilistic cloning with two auxiliary systems, we consider and compare three different protocols for the success probabilities of cloning. We show that, in certain circumstances, it may increase the success probability to add an auxiliary system to the probabilistic cloning machine having one auxiliary system, but we always can find another cloning machine with one auxiliary system having the same success probability as that with two auxiliary systems.

PACS: 03.67.-a, 03.65.Ud
Keywords: Probabilistic cloning; Quantum cloning; Quantum communication

1. Introduction

In quantum information processing, the unitarity and linearity of quantum physics lead to some impossibilities—the no-cloning theorem [1,2,3] and the no-deleting principle [4]. The linearity of quantum theory makes an unknown quantum state unable to be perfectly copied [1,2] and deleted [4], and two nonorthogonal states are not allowed to be precisely cloned and deleted, as a result of the unitarity [3,5,6], that is, for nonorthogonal pure states \(|\psi_1\rangle\) and \(|\psi_2\rangle\), no physical operation in quantum mechanic can exactly achieve the transformation \(|\psi_i\rangle \rightarrow |\psi_i\rangle|\psi_i\rangle \ (i = 1,2)\). This has been generalized to mixed states and entangled states [7,8]. Remarkably, these restrictions provide a valuable resource in quantum cryptography [9], because they forbid an eavesdropper to gain information on the distributed secret key without producing errors.

*This research is supported by the National Natural Science Foundation (Nos. 90303024, 60573006), the Research Foundation for the Doctoral Program of Higher School of Ministry of Education (No. 20050558015), and the Natural Science Foundation of Guangdong Province (No. 031541) of China.
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For detailed review of quantum cloning, we refer to [10]. We briefly recall some preliminaries regarding quantum cloning. In general, there are two kinds of cloners. One is the universal quantum copying machine firstly introduced by Bužek and Hillery [11], and this kind of machines is deterministic and does not need any information about the states to be cloned, so it is state-independent. The other kind of cloners is state-dependent, since it needs some information from the states to be cloned. Furthermore, this kind of cloning machines may be divided into three fashions of cloning: first is probabilistic cloning proposed firstly by Duan and Guo [12,13], and then by Chefles and Barnett [14] and Pati [15], that can clone linearly independent states with nonzero probabilities; second is deterministic clone first investigated by Bruß et al. [16] and then by Chefles and Barnett [17]; third is hybrid cloners studied by Chefles and Barnett [14], that combine deterministic cloners with probabilistic cloning.

Recently Jozsa [18] and Horodecki et al. [19] further clarified the no-cloning theorem and the no-deleting principle from the viewpoint of conservation of quantum information, and in light of this point of view two copies of any quantum state contain more information than one copy; in contrast, two classical states have only the same information as any one of the two states. Specifically, Jozsa [18] verified that if supplementary information, say a mixed state \( \rho_i \) is supplemented, then there is a physical operation

\[
|\psi_i\rangle \otimes \rho_i \rightarrow |\psi_i\rangle|\psi_i\rangle
\]

if and only if there exists physical operation

\[
\rho_i \rightarrow |\psi_i\rangle
\]

where by physical operation we mean a completely positive trace-preserving map, and \( |\psi_i\rangle \) is any given finite set of pure states containing no orthogonal pair of states. This result was called stronger no-cloning theorem and implies that the supplementary information must be provided as the copy \( |\psi_i\rangle \) itself, since the second copy can always be generated from the supplementary information, independently of the original copy. Therefore, this result may show the permanence of quantum information; that is, to get a copy of quantum state, the state must already exist somewhere. It is worth stressing that, if \( |\psi_i\rangle \) contain orthogonal pairs, then the stronger no-cloning theorem verified by Jozsa does not hold again. Indeed, recently, Azuma et al. [20] proved that, for any pair-wise nonorthogonal set of original states \( \{|\psi_i\rangle\}_{i=1,2,\ldots,n} \), if \( \{|\psi_i\rangle\}_{i=1,2,\ldots,n} \) is irreducible, i.e., it can not be divided into two nonempty sets \( S_1 \) and \( S_2 \) such that any state in \( S_1 \) orthogonal to any state in \( S_2 \), then there exists a set of supplementary states \( \{|\phi_i\rangle\}_{i=1,2,\ldots,n} \) such that the following transformation can not be achieved over local operation and classical communication (LOCC):

\[
|\psi_i\rangle|\phi_i\rangle \xrightarrow{\text{LOCC}} |\psi_i\rangle|\psi_i\rangle \quad (i = 1, 2, \cdots, n).
\]
As we stated above, cloning quantum states with a limited degree of success has been proved always possibly. A natural issue is that if the supplementary information is added in Duan and Guo’s probabilistic cloning [12,13] and Pati’s novel cloning machine (NCM) [15], then whether the optimal efficiency of the machine may be increased. This problem was positively addressed by Azuma et al. [21] and Qiu [22]. Azuma et al. [21] discussed probabilistic cloning with supplementary information contained in the quantum states of one auxiliary system. It turns out that when the set of input states contains only two states, the best efficiency of producing \( m \) copies is always achieved by a two-step protocol, in which the helping party first attempts to produce \( m - 1 \) copies from the supplementary state, and if it fails, then the original state is used to produce \( m \) copies. When the set of input states contains more than two states, such a property does not hold any longer. Qiu [22] dealt with the NCM with supplementary information, and presented an equivalent characterization of such a quantum cloning device in terms of a two-step cloning protocol in which the original and the supplementary parties are only allowed to communicate with classical channel.

In this Letter, we investigate probabilistic cloning with supplementary information contained in the quantum states of two auxiliary systems (for brevity, we sometimes call it two auxiliary systems) via three scenarios. This remainder of the paper is organized as follows. In Section 2, we provide related basic results and then introduce three protocols used in later sections. We describe the three protocols in terms of different communication channels between the original party and the two supplementary parties: first, the original party and two supplementary parties are in quantum communication; second, the original party and the first supplementary party are in quantum communication, but the first supplementary party and the second one are in classical communication; third, the original party and the first supplementary party are in classical communication, but the two supplementary parties are in quantum communication.

In Section 3, we prove our main result expressed by Corollary 4, Theorem 5 and Corollary 6; in particular, we show that, when the two states have the same \textit{a priori} probability chosen, the best efficiency of producing \( m \) copies is achieved by the first protocol and the third protocol. Furthermore, we also show that, in certain circumstances, by adding an auxiliary system, we may increase the maximum success probability of the probabilistic cloning with supplementary information contained in the quantum states of only \textit{one} auxiliary system. However, we always can find another probabilistic cloning with \textit{one} auxiliary system having the same success probability as that with \textit{two} auxiliary systems. Finally, in Section 4, we summarize our results obtained, mention some potential of applications, and address a number of related issues for further consideration.

2. Preliminaries

In this section, we serve to recall Jozsa’s stronger no-cloning theorem [18] and the prob-
probabilistic cloning with supplementary information contained in the quantum states of one auxiliary system dealt with by Azuma et al. [21]. Then we present three cloning protocols of probabilistic cloning with supplementary information contained in the quantum states of two auxiliary systems, that will be mainly discussed in Section 3. First we recollect Duan and Guo’s probabilistic cloning machine. We denote by

$$U_B(|\phi_i\rangle|\Sigma|P_B^{i}) = \sqrt{r_i^B} |\psi_i\rangle^{\otimes m}|\chi_b\rangle|P_B^{i}) + \sqrt{1-r_i^B}|\Psi_{bp}\rangle \quad (i = 1, 2, \ldots, n)$$  \hspace{1cm} (4)$$

a machine having the following properties: (i) it receives a quantum state from a given set \{|$\phi_i$\rangle\} as an input and return quantum states as an output \{|$\phi_i$\rangle^m\}, together with a normalized state |$\chi_b$\rangle and one bit of classical output \{|$P_B^{i}$\rangle\} indicating whether the transformation has been successful or not; (ii) when the input quantum state is \{|$\phi_i$\rangle\}, the transformation succeeds with probability \(r_i^B\), and the successful output states are \(m\) copies of \{|$\psi_i$\rangle\}. A necessary and sufficient condition for the existence of Duan and Guo’s probabilistic cloning machine is given by the following Theorem 1.

**Theorem 1 ([21]).** There exists a machine

$$U_B(|\phi_i\rangle|\Sigma|P_B^{i}) = \sqrt{r_i^B} |\psi_i\rangle^{\otimes m}|\chi_b\rangle|P_B^{i}) + \sqrt{1-r_i^B}|\Psi_{bp}\rangle \quad (i = 1, 2, \ldots, n),$$  \hspace{1cm} (5)$$

if and only if there are normalized states \{|$P_B^{0}$\rangle\} and \{|$P_B^{i}$\rangle\} \((i = 1, 2, \ldots, n)\) such that the matrix \(X - \sqrt{\Gamma}Y\sqrt{\Gamma}\) is positive semidefinite, where \(U_B\) is a unitary operator, \(|\Sigma\rangle\) is a blank state, \(|\chi_b\rangle\) is a normalized state, \(|\Psi_{bp}\rangle\) are normalized states of the composite system \(BP\), and \(|P_B^{i}\rangle|\Psi_{bp}\rangle = 0 \quad (i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n)\), \(X = [\langle \phi_i | \phi_j \rangle]\), \(Y = [\langle \psi_i | \psi_j \rangle^m (P_B^{i}|P_B^{j})]\) and \(\Gamma = \text{diag}(r_1^B, r_2^B, \ldots, r_n^B)\) are \(n \times n\) matrices.

If the number of the possible original states is two, then the necessary and sufficient condition is as follows:

**Corollary 2 ([21]).** Denote \(\eta_{\text{in}} = \langle \phi_1 | \phi_2 \rangle\) and \(\eta_{\text{out}} = \langle \psi_1 | \psi_2 \rangle\). There exists a machine

$$U_B(|\phi_i\rangle|P_B^{i}) = \sqrt{r_i^B} |\psi_i\rangle|P_B^{i}) + \sqrt{1-r_i^B}|\Psi_{bp}\rangle \quad (i = 1, 2),$$  \hspace{1cm} (6)$$

if and only if \(r_1^B \geq 0, r_2^B \geq 0, \) and \(\sqrt{(1-r_1^B)(1-r_2^B)} - \eta_{\text{in}} + \eta_{\text{out}} \sqrt{r_1^B} \sqrt{r_2^B} \geq 0\), where \(U_B\) is a unitary operator, \(|\Psi_{bp}\rangle\) are normalized states of the composite system \(BP\), \(|P_B^{i}\rangle\) \((i = 0, 1, 2)\) are probe states and normalized, and \(|P_B^{i}|\Psi_{bp}\rangle = 0 \quad (i = 1, 2; \quad j = 1, 2)\).

Jozsa [18] considered how much or what kind of supplementary information \(|\phi_i\rangle\) is required to make two copies \(|\psi_i\rangle|\psi_i\rangle\) from the original information \(|\psi_i\rangle\). He showed that for any mutually nonorthogonal set of original states \(|\psi_i\rangle\), whenever two copies \(|\psi_i\rangle|\psi_i\rangle\) are generated with the help of the supplementary information \(|\phi_i\rangle\), the state \(|\psi_i\rangle\) can be generated from the supplementary information \(|\phi_i\rangle\) alone, independently of the original state. This is described by the following theorem:
Theorem 3 (Stronger no-cloning theorem [18]). Let $|\psi_i\rangle$ ($i = 1, 2, ..., n$) be any finite set of pure states containing no orthogonal pairs of states. Let $|\phi_i\rangle$ be any other set of states indexed by the same labels. Then there is a physical operation

$$U_{AB}(|\psi_i\rangle|\phi_i\rangle|P^0_{AB}) = \sqrt{r_i^{AB}}|\psi_i\rangle|P^i_{AB}) + \sqrt{1 - r_i^{AB}}|\psi_i\rangle|P^0_{AB})$$

if and only if there is a physical operation

$$U_B(|\phi_i\rangle|P^0_B) = \sqrt{r_i^B}|\phi_i\rangle|P^i_B) + \sqrt{1 - r_i^B}|\phi_i\rangle|P^0_B)$$

where $U_{AB}$ and $U_B$ are unitary operators, $|P^i_{AB}\rangle$ ($i = 0, 1, \cdots, n$) and $|P^i_B\rangle$ ($i = 0, 1, \cdots, n$) are probe states and normalized, $|\Psi^i_{abp}\rangle$ are normalized states of the composite system ABP and $|\Psi^i_{bvp}\rangle$ are normalized states of the composite system BP and $|\Psi^i_{abvp}\rangle = 0$ ($i = 1, 2, \cdots, n; j = 1, 2, \cdots, n$). Here, the supplementary states $|\phi_i\rangle$ ($i = 1, 2, ..., n$) are thought of as some pure states.

Motivated by the stronger no-cloning theorem, Azuma et al. [21] discussed probabilistic cloning with supplementary information contained in the quantum states of one auxiliary system. They showed that when the set of input states contains only two states, the best efficiency of producing $m$ copies is always achieved by a two-step protocol, in which the helping party first attempts to produce $m - 1$ copies from the supplementary state, and if it fails, then the original state is used to produce $m$ copies. When the set of input states contains more than two states, such a property does not hold any longer. Now we consider probabilistic cloning with supplementary information contained in the quantum states of two auxiliary systems. We may have the following three protocols:

**Scenario I.** Alice, Bob, and Victor can use two one-way quantum channels to communicate each other. One is between Victor and Bob, and the other is between Bob and Alice. In this case, a single party having both the original and the supplementary information runs a machine described by

$$U(|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle|\Sigma\rangle|P^0\rangle) = \sqrt{r_i^I}|\psi_i\rangle|\Sigma\rangle|P^i\rangle + \sqrt{1 - r_i^I}|\Psi^i_{abvp}\rangle,$$

where $r_i^I$ is the success probability of cloning $|\psi_i\rangle$ with the input states $|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle$, $|\Sigma\rangle$ is a blank state, $|P^i\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi^i_{abvp}\rangle$ are normalized states of the composite system ABP, $\langle P^i|\Psi^i_{abvp}\rangle = 0$ ($i = 1, 2; j = 1, 2$), $U$ is a unitary operator, and, here and in the follow-up, $|\chi\rangle$ is a normalized state, playing as a memory.

Before presenting Scenario II, we should understand the fact that in the same condition, the success probability of cloning $m - 1$ copies is not less than that of cloning $m$ copies ($m \geq 2$).

According to Corollary 2, the condition required by cloning $m$ copies is that

$$\sqrt{(1 - r_1^B)(1 - r_2^B)} - \eta m + |\alpha|^m \sqrt{r_1^B r_2^B} > 0, \ i = 1, 2,$$
and the condition required by cloning \( m - 1 \) copies is that
\[
\sqrt{(1 - r^B_i)(1 - r^B_j)} - \eta_{in} + |\alpha|^{m-1}\sqrt{r^B_i r^B_j} \geq 0, \quad i, j = 1, 2. \tag{11}
\]
It is clear that if \( r^B_i \) satisfy Eq. (10), they also satisfy Eq. (11). That is to say, if we succeed in cloning \( m \) copies, we can clone \( m - 1 \) copies in the same success probability. So, the success probability of cloning \( m - 1 \) copies is not less than that of cloning \( m \) copies.

**Scenario II.** If the original information is held by Alice, and the supplementary information by Bob and Victor, respectively. In this case, Victor and Bob can use a one-way classical channel. Bob and Alice can use a one-way quantum channel. In this Scenario, there are two ways to clone \( m \) copies of input state \(|\psi_i\rangle\).

The first one is that: Victor, who possesses the supplementary state \(|\phi^{(2)}_i\rangle\), first runs the machine
\[
U_V(|\phi^{(2)}_i\rangle|\Sigma_i|P^0_V) = \sqrt{r^V_i}|\psi_i\rangle^{\otimes m}|\chi_v\rangle|P^i_V\rangle + \sqrt{1 - r^V_i}|\Psi^{i}_{vp}\rangle, \quad i = 1, 2, \tag{12}
\]
where \( r^V_i \) is the success probability of cloning \(|\psi_i\rangle\) with the input state \(|\phi^{(2)}_i\rangle, |\Sigma\rangle\) is a blank state, and \(|P^i_V\rangle (i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^{i}_{vp}\rangle\) are normalized states of the composite system \(VP\), \(\langle P^i_V|\Psi^{i}_{vp}\rangle = 0 \) \((i = 1, 2; \; j = 1, 2)\), \(U_V\) is a unitary operator, \(|\chi_v\rangle\) is a normalized state, then Victor tells Alice and Bob whether the trial was successful or not. In the successful case, Alice and Bob just leave their states \(|\psi_i\rangle|\phi^{(1)}_i\rangle\) as they are, and hence they obtain \(m\) copies in total. If Victor’s attempt has failed, Alice and Bob run the machine
\[
U_{AB}(|\psi_i\rangle|\phi^{(1)}_i\rangle|\Sigma_i|P^0_{AB}) = \sqrt{r^A_i |\psi_i\rangle^{\otimes m}|\chi_{ab}\rangle|P^i_{AB}\rangle + \sqrt{1 - r^A_i}|\Psi^{i}_{abp}\rangle, \quad i = 1, 2, \tag{13}
\]
where \( r^A_i \) is the success probability of cloning \(|\psi_i\rangle\) with the input states \(|\psi_i\rangle|\phi^{(1)}_i\rangle, |\Sigma\rangle\) is a blank state, and \(|P^i_{AB}\rangle (i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^{i}_{abp}\rangle\) are normalized states of the composite system \(ABP\), \(\langle P^i_{AB}|\Psi^{i}_{abp}\rangle = 0 \) \((i = 1, 2; \; j = 1, 2)\), \(U_{AB}\) is a unitary operator, \(|\chi_{ab}\rangle\) is a normalized state. So, the total success probability of cloning \(|\psi_i\rangle\) for input states \(|\psi_i\rangle|\phi^{(1)}_i\rangle|\phi^{(2)}_i\rangle\) in this protocol is given by \(r^I_i = r^V_i + (1 - r^V_i) r^A_i\). If we change the form of the equality, we get that \(r^I_i = r^A_i + (1 - r^A_i) r^V_i\).

The second one is as follows: Alice and Bob first run the machine (13), then Bob tells Victor whether the trial was successful or not. In the successful case, Victor just leaves his state \(|\phi^{(2)}_i\rangle\) as it is, and hence they obtain \(m\) copies in total. If Alice and Bob’s attempt has failed, Victor runs the machine
\[
U'_V(|\phi^{(2)}_i\rangle|\Sigma_i|P^{0}_{V'}) = \sqrt{r^V_i'|\psi_i\rangle^{\otimes m}|\chi_v\rangle'|P^{i'}_V\rangle + \sqrt{1 - r^V_i'|\Psi^{i'}_{vp}\rangle, \quad i = 1, 2, \tag{14}
\]
where \( r^V_i' \) is the success probability of cloning \(|\psi_i\rangle\) with the input state \(|\phi^{(2)}_i\rangle, |\Sigma\rangle\) is a blank state, and \(|P^{i'}_V\rangle (i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^{i'}_{vp}\rangle\) are normalized states of the composite system \(VP\), \(\langle P^{i'}_V|\Psi^{i'}_{vp}\rangle = 0 \) \((i = 1, 2; \; j = 1, 2)\), \(U'_V\) is a unitary operator,
$|\chi_i^r\rangle$ is a normalized state. So, the total success probability of cloning $|\psi_i\rangle$ for input states $|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle$ in this protocol is given by $r_i^{\text{III}} = r_i^{AB} + (1 - r_i^{AB})r_i^V$.

According to the fact that the success probability of cloning $m - 1$ copies is not less than that of cloning $m$ copies, we have $r_i^V \geq r_i^{\text{III}}$. So, $r_i^{\text{III}} \geq r_i^{\text{III}}$. That is to say, in Scenario II, the optimal performance is always achieved by the first way.

**Scenario III.** If the original information is held by Alice, and the supplementary information by Bob and Victor, respectively. In this case, Victor and Bob can use a one-way quantum channel. Bob and Alice can use a one-way classical channel. Similar to Scenario II, in this case, we can prove that the optimal performance is always achieved as follows: Victor and Bob, who possess the supplementary states $|\phi_i^{(1)}\rangle$ and $|\phi_i^{(2)}\rangle$, respectively, first run the machine

$$U_{BV}(|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle|\Sigma\rangle|P_{BV}^0\rangle) = \sqrt{r_i^{BV}}|\psi_i\rangle^\otimes m - 1|\chi_{bv}\rangle|P_i^{ BV}\rangle + \sqrt{1 - r_i^{ BV}}|\Psi_{bvp}\rangle, \ i = 1, 2,$$

where $r_i^{BV}$ is the success probability of cloning $|\psi_i\rangle$ with the input states $|\phi_i^{(1)}\rangle$ and $|\phi_i^{(2)}\rangle$, $|\Sigma\rangle$ is a blank state and $|P_{BV}^i\rangle (i = 0, 1, 2)$ are normalized states of the probe $P$, $|\Psi_{bvp}\rangle$ are normalized states of the composite system $BP$, $|P_{BV}^i\rangle|\Psi_{bvp}\rangle = 0 (i = 1, 2; j = 1, 2)$, $U_{BV}$ is a unitary operator, $|\chi_{bv}\rangle$ is a normalized state. Then Victor and Bob tell Alice whether the trial was successful or not. In the successful case, Alice just leaves her state $|\psi_i\rangle$ as it is, and hence they obtain $m$ copies in total. If the attempt of Victor and Bob has failed, Alice runs the machine

$$U_A(|\psi_i\rangle|\Sigma\rangle|P_{A}^0\rangle) = \sqrt{r_i^{A}}|\psi_i\rangle^\otimes m|\chi_a\rangle|P_i^{ A}\rangle + \sqrt{1 - r_i^{ A}}|\Psi_{ap}\rangle, \ i = 1, 2,$$

where $r_i^{A}$ is the success probability of cloning $|\psi_i\rangle$ with the input state $|\psi_i\rangle$, $|\Sigma\rangle$ is a blank state, and $|P_{A}^i\rangle (i = 0, 1, 2)$ are normalized states of the probe $P$, $|\Psi_{ap}\rangle$ are normalized states of the composite system $AP$, $|P_{A}^i\rangle|\Psi_{ap}\rangle = 0 (i = 1, 2; j = 1, 2)$, $U_{A}$ is a unitary operator, $|\chi_a\rangle$ is a normalized state. So, the total success probability of cloning $|\psi_i\rangle$ for input states $|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle$ in this protocol is given by $r_i^{\text{III}} = r_i^{BV} + (1 - r_i^{BV})r_i^A$.

Now, based on the three protocols above, we are ready to calculate their maximum success probabilities and investigate their relationships in next section.

3. Probabilistic cloning with supplementary information contained in the quantum states of two auxiliary systems

Throughout this paper, we consider $\{|\psi_1\rangle, |\psi_2\rangle\}$ as the original states. $\{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle\}$ and $\{|\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle\}$ are supplementary information contained in the quantum states of two auxiliary systems. For convenience, we denote $\langle\psi_1|\psi_2\rangle = \alpha$, $\langle\phi_1^{(1)}|\phi_2^{(1)}\rangle = \beta$, $\langle\phi_1^{(2)}|\phi_2^{(2)}\rangle = \gamma$. Let $P_i$ be the prior probability of $|\psi_i\rangle$, and, let $r_i^t$, $r_i^{\text{II}}$, $r_i^{\text{III}}$ be the total success probabilities in Scenario I, II, III, respectively. Let $r_i^t$, $r_i^{\text{II}}$, $r_i^{\text{III}}$ denote the success probabilities of
cloning \(|\psi_i\rangle\) in Scenario I, II, III, respectively. Clearly, we have the following relationships: 
\[ r^I = P_1 r^I_1 + P_2 r^I_2, \quad r^{II} = P_1 r^{II}_1 + P_2 r^{II}_2, \quad r^{III} = P_1 r^{III}_1 + P_2 r^{III}_2. \]
Moreover, by \(r^I_{max}, r^{II}_{max}, r^{III}_{max}\) we mean the maximum values of \(r^I, r^{II}, r^{III}\), respectively.

Before presenting the main result, we verify a corollary as follows, which will be used in the proof of Theorem 5.

**Corollary 4.** Denote \(\eta_{in} = |\langle \phi_1 | \phi_2 \rangle|\) and \(\eta_{out} = |\langle \psi_1 | \psi_2 \rangle|\). Suppose that \(\eta_{in} > \eta_{out}\). If there exists a machine
\[
U_B(\langle \phi_i | P^B_i \rangle) = \sqrt{r^B_i |\psi_i\rangle P^i_B} + \sqrt{1 - r^B_i |\Psi_{bp}\rangle}, \quad i = 1, 2, \tag{17}
\]
where \(U_B\) is a unitary operator, \(|\Psi^i_{bp}\rangle\) are normalized states of the composite system \(BP\), \(|P^i_B\rangle\) (\(i = 0, 1, 2\)) are probe states and normalized, \(|P^i_B|\Psi^j_{bp}\rangle = 0\) (\(i = 1, 2; j = 1, 2\)), \(r^B_i\) (\(i = 1, 2\)) are the success probabilities of cloning \(|\psi_i\rangle\), then we can conclude that \(r^B_1\) and \(r^B_2\) satisfy
\[
\frac{r^B_1 + r^B_2}{2} \leq \frac{1 - \eta_{in}}{1 - \eta_{out}}. \tag{18}
\]

**Proof.** According to Corollary 2, if there exists a machine
\[
U_B(\langle \phi_i | P^B_i \rangle) = \sqrt{r^B_i |\psi_i\rangle P^i_B} + \sqrt{1 - r^B_i |\Psi_{bp}\rangle}, \quad i = 1, 2, \tag{17}
\]
where \(U_B\) is a unitary operator, \(|\Psi^i_{bp}\rangle\) are normalized states of the composite system \(BP\), \(|P^i_B\rangle\) (\(i = 0, 1, 2\)) are probe states and normalized, \(|P^i_B|\Psi^j_{bp}\rangle = 0\) (\(i = 1, 2; j = 1, 2\)), then we have
\[
\sqrt{(1 - r^B_1)(1 - r^B_2)} - \eta_{in} + \eta_{out} \sqrt{r^B_1 r^B_2} \geq 0. \tag{19}
\]
And it is known that
\[
\sqrt{(1 - r^B_1)(1 - r^B_2)} - \eta_{in} + \eta_{out} \sqrt{r^B_1 r^B_2} \leq \frac{(1 - r^B_1) + (1 - r^B_2)}{2} - \eta_{in} + \eta_{out} \frac{r^B_1 + r^B_2}{2} \tag{20}
\]
where the equality holds if and only if \(r^B_1 = r^B_2\). Using Eqs. (19) and (20), we get
\[
\frac{(1 - r^B_1) + (1 - r^B_2)}{2} - \eta_{in} + \eta_{out} \frac{r^B_1 + r^B_2}{2} \geq 0. \tag{21}
\]
That is to say,
\[
1 - \frac{r^B_1 + r^B_2}{2} - \eta_{in} + \eta_{out} \frac{r^B_1 + r^B_2}{2} \geq 0. \tag{22}
\]
Thus, we conclude that
\[
\frac{r^B_1 + r^B_2}{2} \leq \frac{1 - \eta_{in}}{1 - \eta_{out}}. \tag{23}
\]
where the equality holds if and only if \( r_1^B = r_2^B \).

Now we can present the main result.

**Theorem 5.** The relationships of the three Scenarios above are as follows:

1. When \( |\beta| \leq |\alpha|^{m-1} \) or \( |\gamma| \leq |\alpha|^{m-1} \), the maximum success probabilities of the three Scenarios are equal to 1.

2. When \( |\beta| > |\alpha|^{m-1} \) and \( |\gamma| > |\alpha|^{m-1} \), there are two cases:
   1. If \( |\alpha|^{2m-2} < |\beta| |\alpha|^{m-1} \), we have \( r_{\text{max}}^I = r_{\text{max}}^{III} > r_{\text{max}}^{II} \).
   2. If \( |\alpha|^{m-1} < |\beta| |\alpha|^{m-1} \) and \( P_1 = P_2 \), we conclude that \( r_{\text{max}}^I = r_{\text{max}}^{III} \geq r_{\text{max}}^{II} \), and only if \( |\gamma| = 1 \) or \( |\beta| = 1 \), we get the equality \( r_{\text{max}}^I = r_{\text{max}}^{III} = r_{\text{max}}^{II} \).

**Proof.** 1. When \( |\gamma| \leq |\alpha|^{m-1} \) or \( |\beta| \leq |\alpha|^{m-1} \), from Corollary 2, there exists a machine described by

\[
U_V(|\phi_1^{(2)}\rangle |\Sigma\rangle |P_V^i\rangle) = \sqrt{r_V^i} |\psi_i\rangle^{\otimes m-1} |\chi_i\rangle |P_V^i\rangle + \sqrt{1 - r_V^i} |\Psi_{vp}\rangle, \quad i = 1, 2,
\]

satisfying \( r_V^i = r_V^2 = 1 \), where \( r_V^i \) is the success probability with the input state \(|\phi_1^{(2)}\rangle, |\Sigma\rangle \) is a blank state, and \(|P_V^i\rangle \) \((i = 0, 1, 2)\) are normalized states of the probe \( P \), \(|\Psi_{vp}\rangle \) are normalized states of the composite system \( VP \), \(|P_V^i \rangle |\Psi_{vp}\rangle = 0 \) \((i = 1, 2; \ j = 1, 2)\), \( U_V \) is a unitary operator, \(|\chi_i\rangle \) is a normalized state. So, we have the equality \( r_{\text{max}}^{Ii} = r_V^i + (1 - r_V^i) r_{\text{max}}^{AB} = 1 \).

Or, there exists a machine

\[
U_{AB}(|\psi_i\rangle |\phi^{(1)}_i\rangle |\Sigma\rangle |P_{AB}^0\rangle) = \sqrt{r_{\text{max}}^{AB}} |\psi_i\rangle^{\otimes m} |\chi_{ab}\rangle |P_{AB}^i\rangle + \sqrt{1 - r_{\text{max}}^{AB}} |\Psi_{abp}\rangle, \quad i = 1, 2,
\]

satisfying \( r_{\text{max}}^{AB} = r_{\text{max}}^{AB} = 1 \), where \( r_{\text{max}}^{AB} \) is the success probability with the input states \(|\psi_i\rangle, |\phi^{(1)}_i\rangle, |\Sigma\rangle \) is a blank state, and \(|P_{AB}^i\rangle \) \((i = 0, 1, 2)\) are normalized states of the probe \( P \), \(|\Psi_{abp}\rangle \) are normalized states of the composite system \( ABP \), \(|P_{AB}^i \rangle |\Psi_{abp}\rangle = 0 \) \((i = 1, 2; \ j = 1, 2)\), \( U_{AB} \) is a unitary operator, \(|\chi_{ab}\rangle \) is a normalized state. We also have \( r_{\text{max}}^{II} = r_V^i + (1 - r_V^i) r_{\text{max}}^{AB} = 1 \).

Because \( |\gamma| \leq |\alpha|^{m-1} \) or \( |\beta| \leq |\alpha|^{m-1} \), we have \(|\alpha \beta \gamma| \leq |\alpha|^{m-1} \), and thus there exists a machine

\[
U(|\psi_i\rangle |\phi^{(1)}_i\rangle |\phi^{(2)}_i\rangle |\Sigma\rangle |P^0\rangle) = \sqrt{r_i^I} |\psi_i\rangle^{\otimes m} |\chi_i\rangle |P^i\rangle + \sqrt{1 - r_i^I} |\Psi_{abp}\rangle, \quad i = 1, 2,
\]

satisfying \( r_i^I = r_i^2 = 1 \), where \( r_i^I \) is the success probability with the input states \(|\psi_i\rangle, |\phi^{(1)}_i\rangle, |\phi^{(2)}_i\rangle, |\Sigma\rangle \) is a blank state, and \(|P^i\rangle \) \((i = 0, 1, 2)\) are normalized states of the probe \( P \), \(|\Psi_{abp}\rangle \) are normalized states of the composite system \( ABVP \), \(|P^i \rangle |\Psi_{abp}\rangle = 0 \) \((i = 1, 2; \ j = 1, 2)\), \( U \) is a unitary operator, \(|\chi_i\rangle \) is a normalized state.
Using Eqs. (29,31,32), we conclude that this case, together with \(|\beta| \leq |\alpha|^{m-1}\), we have \(|\beta\gamma| \leq |\alpha|^m\), and thus there exists a machine

\[
U_{BV}(\phi^{(1)}_i|\phi^{(2)}_i|\Sigma|P^0_{BV})) = \sqrt{r^\text{BV}_i}|\psi_i\rangle^{\otimes m-1}|\chi_{bv}\rangle|P^i_{BV}) + \sqrt{1 - r^\text{BV}_i}|\Psi^i_{abvp}\rangle, \quad i = 1, 2, \tag{27}
\]

satisfying \(r^\text{BV}_1 = r^\text{BV}_2 = 1\), where \(r^\text{BV}_i\) is the success probability with the input states \(\phi^{(1)}_i\) and \(\phi^{(2)}_i\), \(|\Sigma\rangle\) is a blank state and \(|P^i_{BV}\rangle\) \((i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^i_{abvp}\rangle\) are normalized states of the composite system \(BV_{P}\), \(\langle P^i_{BV}|\Psi^i_{abvp}\rangle = 0\) \((i = 1, 2; \ j = 1, 2)\), \(U_{BV}\) is a unitary operator, \(|\chi_{bv}\rangle\) is a normalized state. So, we have \(r^{III}_i = r^{BV}_i + (1 - r^{BV}_i)r^A_i = 1\).

As a result, we conclude that \(r^{III}_i = r^{II}_i = r^I_i = 1\). So, in this case, \(r^{I\text{max}}_i = r^{III\text{max}}_i = r^{III\text{max}}_i = 1\).

2. When \(|\beta| > |\alpha|^{m-1}\) and \(|\gamma| > |\alpha|^{m-1}\), we have the two cases as follows:

1) If \(|\alpha|^{2m-2} < |\beta\gamma| \leq |\alpha|^{m-1}\), from Corollary 2, there exists a machine

\[
U(|\psi_i\rangle|\phi^{(1)}_i|\phi^{(2)}_i|\Sigma\rangle|P^0\rangle) = \sqrt{r^I_i}|\psi_i\rangle^{\otimes m}|\chi\rangle|P^i\rangle + \sqrt{1 - r^I_i}|\Psi^i_{abvp}\rangle, \quad i = 1, 2, \tag{28}
\]

with \(r^I_i = r^I_j = 1\), where \(r^I_i\) is the success probability with the input states \(|\psi_i\rangle|\phi^{(1)}_i|\phi^{(2)}_i\rangle\), \(|\Sigma\rangle\) is a blank state, and \(|P^i\rangle\) \((i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^i_{abvp}\rangle\) are normalized states of the composite system \(AB_{P}\), \(\langle P^i|\Psi^i_{abvp}\rangle = 0\) \((i = 1, 2; \ j = 1, 2)\), \(U\) is a unitary operator, \(|\chi\rangle\) is a normalized state. So, we conclude

\[r^{I\text{max}}_i = 1. \tag{29}\]

And there also exists a machine

\[
U_{BV}(\phi^{(1)}_i|\phi^{(2)}_i|\Sigma|P^0_{BV}) = \sqrt{r^\text{BV}_i}|\psi_i\rangle^{\otimes m-1}|\chi_{bv}\rangle|P^i_{BV}) + \sqrt{1 - r^\text{BV}_i}|\Psi^i_{abvp}\rangle, \quad i = 1, 2, \tag{30}
\]

with \(r^\text{BV}_1 = r^\text{BV}_2 = 1\), where \(r^\text{BV}_i\) is the success probability with the input states \(\phi^{(1)}_i\) and \(\phi^{(2)}_i\), \(|\Sigma\rangle\) is a blank state and \(|P^i_{BV}\rangle\) \((i = 0, 1, 2)\) are normalized states of the probe \(P\), \(|\Psi^i_{abvp}\rangle\) are normalized states of the composite system \(BV_{P}\), \(\langle P^i_{BV}|\Psi^i_{abvp}\rangle = 0\) \((i = 1, 2; \ j = 1, 2)\), \(U_{BV}\) is a unitary operator, \(|\chi_{bv}\rangle\) is a normalized state. So, we get \(r^{III}_i = r^{BV}_i + (1 - r^{BV}_i)r^A_i = 1\), and \(r^{III} = P_1r^{III}_1 + P_2r^{III}_2 = 1\). As a consequence, we have

\[r^{III\text{max}}_i = 1. \tag{31}\]

Since \(r^{II}_i \leq 1\) holds in any case and the success probability of \(Scenario II\) can not be 1 in this case, together with \(r^{II} = P_1r^{II}_1 + P_2r^{II}_2\), we clearly have \(r^{II} < 1\). So, we conclude

\[r^{III\text{max}}_i < 1. \tag{32}\]

Using Eqs. (29,31,32), we conclude that \(r^{I\text{max}}_i = r^{III\text{max}}_i = 1 > r^{III\text{max}}_i\).
2) If $|\alpha|^{m-1} < |\beta\gamma| \leq 1$ and $P_1 = P_2 = \frac{1}{2}$, we investigate the relationships between the three Scenarios as follows:

In Scenario I, there exists a machine

$$U(|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle|\Sigma\rangle|P^0\rangle) = \sqrt{r_i^V}(|\psi_i\rangle)^{\otimes m}|\chi\rangle|P^i_{\text{ABP}}\rangle + \sqrt{1 - r_i^V}|\Psi^{i}_{\text{abp}}\rangle, \quad i = 1, 2,$$

(33)

where $r_i^V$ is the success probability with the input states $|\psi_i\rangle|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle, |\Sigma\rangle$ is a blank state, and $|P^i\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi^{i}_{\text{abp}}\rangle$ are normalized states of the composite system $ABVP$, $\langle P^i|\Psi^{i}_{\text{abp}}\rangle = 0$ ($i = 1, 2; \ j = 1, 2$), $U$ is a unitary operator, $|\chi\rangle$ is a normalized state. In this machine, $\eta_{\text{in}} = |\alpha\beta\gamma|$ and $\eta_{\text{out}} = |\alpha|^m$. According to Corollary 4, we conclude that

$$r_i^V = \frac{r_i^V + r_i^P}{2} \leq \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m}.$$  

(34)

The equality (34) holds if and only if $r_1^V = r_2^V$. Therefore, we have

$$r_i^V_{\text{max}} = \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m}.  

(35)

In Scenario II, there also exists a machine

$$U_V(|\phi_i^{(2)}\rangle|\Sigma\rangle|P^0_{\text{V}}\rangle) = \sqrt{r_i^V}(|\psi_i\rangle)^{\otimes m-1}|\chi_{\text{V}}\rangle|P^i_{\text{V}}\rangle + \sqrt{1 - r_i^V}|\Psi^{i}_{\text{vp}}\rangle, \quad i = 1, 2,$$

(36)

where $r_i^V$ is the success probability with the input state $|\phi_i^{(2)}\rangle, |\Sigma\rangle$ is a blank state, and $|P^i_{\text{V}}\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi^{i}_{\text{vp}}\rangle$ are normalized states of the composite system $V_P$, $\langle P^i_{\text{V}}|\Psi^{i}_{\text{vp}}\rangle = 0$ ($i = 1, 2; \ j = 1, 2$), $U_V$ is a unitary operator, $|\chi_{\text{V}}\rangle$ is a normalized state. In this case, $\eta_{\text{in}} = |\gamma|$ and $\eta_{\text{out}} = |\alpha|^m$. According to Corollary 4, we conclude that

$$r_i^V + r_i^P \leq \frac{1 - |\gamma|}{1 - |\alpha|^m}.  

(37)

And there exists another machine

$$U_{\text{AB}}(|\psi_i\rangle|\phi_i^{(1)}\rangle|\Sigma\rangle|P^0_{\text{AB}}\rangle) = \sqrt{r_i^{AB}}(|\psi_i\rangle)^{\otimes m}|\chi_{\text{ab}}\rangle|P^i_{\text{AB}}\rangle + \sqrt{1 - r_i^{AB}}|\Psi^{i}_{\text{abp}}\rangle, \quad i = 1, 2,$$

(38)

where $r_i^{AB}$ is the success probability with the input states $|\psi_i\rangle|\phi_i^{(1)}\rangle, |\Sigma\rangle$ is a blank state, and $|P^i_{\text{AB}}\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi^{i}_{\text{abp}}\rangle$ are normalized states of the composite system $ABP$, $\langle P^i_{\text{AB}}|\Psi^{i}_{\text{abp}}\rangle = 0$ ($i = 1, 2; \ j = 1, 2$), $U_{\text{AB}}$ is a unitary operator, $|\chi_{\text{ab}}\rangle$ is a normalized state. In this machine, $\eta_{\text{in}} = |\alpha\beta|$ and $\eta_{\text{out}} = |\alpha|^m$. According to Corollary 4, we conclude that

$$r_i^{AB} + r_i^{AB} \leq \frac{1 - |\alpha\beta|}{1 - |\alpha|^m}.  

(39)$$
So, we have
\[
 r^{II} = \frac{r_Y^1 + (1 - r_Y^1)r^{AB}_1}{2} + \frac{r_Y^2 + (1 - r_Y^2)r^{AB}_2}{2} 
\]
(40)
\[
 = \frac{r_Y^1 + r_Y^2}{2} + \frac{r^{AB}_1 + r^{AB}_2}{2} - \frac{r_Y^1 r^{AB}_1 + r_Y^2 r^{AB}_2}{2} 
\]
(41)
\[
 \leq \frac{1 - |\gamma|}{1 - |\alpha|^m - 1} + \frac{1 - |\alpha\beta|}{1 - |\alpha|^m - 1} - \frac{1 - |\alpha\beta|}{1 - |\alpha|^m - 1} 
\]
(42)
\[
 = \frac{(1 - |\gamma|)(|\alpha\beta| - |\alpha|^m) + (1 - |\alpha\beta|)(1 - |\alpha|^m - 1)}{(1 - |\alpha|^m)(1 - |\alpha|^m - 1)} . 
\]
(43)

Also, the above equality holds if and only if \( r^{AB}_1 = r^{AB}_2 \) and \( r_Y^1 = r_Y^2 \). Consequently, we have
\[
 r^{II}_{\text{max}} = \frac{(1 - |\gamma|)(|\alpha\beta| - |\alpha|^m) + (1 - |\alpha\beta|)(1 - |\alpha|^m - 1)}{(1 - |\alpha|^m)(1 - |\alpha|^m - 1)} . 
\]
(44)

As well, we can compare the success probability of Scenario I with that of Scenario II. Because
\[
 r^{II}_{\text{max}} - r^{I}_{\text{max}} = \frac{(1 - |\gamma|)(|\alpha\beta| - |\alpha|^m) + (1 - |\alpha\beta|)(1 - |\alpha|^m - 1)}{(1 - |\alpha|^m)(1 - |\alpha|^m - 1)} - \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m} 
\]
(45)
\[
 = \frac{(1 - |\gamma|)|\alpha|^m(|\beta| - 1)}{(1 - |\alpha|^m)(1 - |\alpha|^m - 1)} 
\]
(46)
\[
 \leq 0 , 
\]
(47)
we get
\[
 r^{II}_{\text{max}} \leq r^{I}_{\text{max}} . 
\]
(48)

In scenario III, there exists a machine
\[
 U_{BV}(\phi^{(1)}_i|\phi^{(2)}_i|\Sigma|P_{BV}^0) = \sqrt{r^{BV}_1}\psi_i^{\otimes m-1}|\chi_{bv}\rangle|P_{BV}^i\rangle + \sqrt{1 - r^{BV}_1}\Psi_{bvp}^i , \ i = 1, 2, 
\]
(49)
where \( r_i^{BV} \) is the success probability with the input states \( \phi^{(1)}_i \) and \( \phi^{(2)}_i \), \( |\Sigma\rangle \) is a blank state and \( |P_{BV}^i\rangle \ (i = 0, 1, 2) \) are normalized states of the probe \( P, |\Psi_{bvp}^i\rangle \) are normalized states of the composite system \( BVP, \langle P_{BV}^i|\Psi_{bvp}^j\rangle = 0 \ (i = 1, 2; j = 1, 2) \), \( U_{BV} \) is a unitary operator, \( |\chi_{bv}\rangle \) is a normalized state. In this machine, \( \eta_{in} = |\beta\gamma| \) and \( \eta_{out} = |\alpha|^m - 1 \). According to Corollary 4, we conclude that
\[
 \frac{r_1^{BV} + r_2^{BV}}{2} \leq \frac{1 - |\beta\gamma|}{1 - |\alpha|^m - 1} . 
\]
(50)
And there exists a machine
\[
 U_{A}(\psi_i|\Sigma|P_{A}^0) = \sqrt{r^{A}_i}\psi_i^{\otimes m}|\chi_{A}\rangle|P_{A}^i\rangle + \sqrt{1 - r^{A}_i}|\Psi_{ap}^i\rangle , \ i = 1, 2, 
\]
(51)
where $r_A^i$ is the success probability with the input state $|\psi_i\rangle$, $|\Sigma\rangle$ is a blank state, and $|P_A^i\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi_{ap}^i\rangle$ are normalized states of the composite system $AP$, $\langle P_i A | \Psi_{ap}^j \rangle = 0$ ($i = 1, 2$; $j = 1, 2$), $U_A$ is a unitary operator, $|\chi_a\rangle$ is a normalized state. In this case, $\eta_{in} = |\alpha\rangle$ and $\eta_{out} = |\alpha|^m$. According to Corollary 4, we conclude that

$$r_A^1 + r_A^2 \leq \frac{1 - |\alpha|}{1 - |\alpha|^m}.$$  \hfill (52)

So, we have

$$r_{III} = r_{I}^B + (1 - r_{I}^B)r_A^1 + \frac{r_{II}^B + (1 - r_{II}^B)r_A^2}{2}.$$  \hfill (53)

$$= \frac{r_{I}^B + r_{II}^B}{2} + \frac{r_A^1 + r_A^2}{2} - \frac{r_{I}^B r_A^1 + r_{II}^B r_A^2}{2}.$$  \hfill (54)

$$\leq \frac{1 - |\beta\gamma|}{1 - |\alpha|^m - 1 - |\alpha|^m - 1 - |\alpha|^m} + \frac{1 - |\alpha|}{1 - |\alpha|^m - 1 - |\alpha|^m - 1 - |\alpha|^m}.$$  \hfill (55)

$$= \frac{(1 - |\beta\gamma|)(1 - |\alpha|^m) + (1 - |\alpha|)(1 - |\alpha|^m - 1) - (1 - |\beta\gamma|)(1 - |\alpha|)}{(1 - |\alpha|^m)(1 - |\alpha|^m - 1)}.$$  \hfill (56)

$$= \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m}.$$  \hfill (57)

Also, the above equality holds if and only if $r_{AB}^1 = r_{AB}^2$ and $r_V^1 = r_V^2$. As a result,

$$r_{III} = \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m}.$$  \hfill (58)

As well, we can compare the success probability of Scenario I with that of Scenario III. Because

$$r_{III}^\max - r_{I}^\max = \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m} - \frac{1 - |\alpha\beta\gamma|}{1 - |\alpha|^m} = 0,$$  \hfill (59)

we get

$$r_{III}^\max = r_{I}^\max.$$  \hfill (60)

By using Eqs. (45-48) and Eq. (60), we conclude that $r_{I}^\max = r_{III}^\max \geq r_{II}^\max$. If $|\gamma| \neq 1$ and $|\beta| \neq 1$, we have $r_{I}^\max > r_{II}^\max$, and only if $|\gamma| = 1$ or $|\beta| = 1$, we get the equality $r_{I}^\max = r_{III}^\max = r_{II}^\max$.  \hfill \□

Based on the above proof, we find that, by adding one auxiliary system, it is possible to increase the success probability for probabilistic cloning machine with one auxiliary system. However, given a probabilistic cloning machine with two auxiliary systems, we always can establish an probabilistic cloning machine with one auxiliary system, whose success probability is the same as that with two auxiliary systems. We further describe this by the following corollary.
Corollary 6. When \( P_1 = P_2, |\beta| > |\alpha|^{m-1}, |\gamma| > |\alpha|^{m-1} \) and \( |\beta| \neq 1, |\gamma| \neq 1 \), by means of adding an auxiliary system described by states \( \{|\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle\} \), we may increase the maximum success probability of the cloning machine with only one auxiliary system described by states \( \{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle\} \). However, given a probabilistic cloning machine with two auxiliary systems described by states \( \{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle\} \) and \( \{|\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle\} \), respectively, we always can find another probabilistic cloning machine with only one auxiliary system described by states \( \{|\phi_1^{(3)}\rangle, |\phi_2^{(3)}\rangle\} \), satisfying the condition: \( \langle \phi_1^{(3)} | \phi_2^{(3)} \rangle = \langle \phi_1^{(1)} | \phi_2^{(1)} \rangle \langle \phi_1^{(2)} | \phi_2^{(2)} \rangle \), such that the two cloning machines have the same maximum success probability.

Proof. Clearly, we have

\[
r_i^{AB} + (1 - r_i^{AB}) r_i^{V} \geq r_i^{AB},
\]

and according to Scenario II, we have

\[
r_i^{II} = r_i^{AB} + (1 - r_i^{AB}) r_i^{V}
\]

where \( r_i^{AB} \) are the success probabilities of cloning state \( |\psi_i\rangle \) with supplementary information contained in quantum states of one auxiliary system described by \( \{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle\} \). Let \( r^{AB} \) be the total success probability with supplementary information contained in the quantum states of one auxiliary system, that is to say,

\[
r^{AB} = P_1 r_1^{AB} + P_2 r_2^{AB}.
\]

Because \( r^{II} = P_1 r_1^{II} + P_2 r_2^{II} \), we conclude that

\[
r^{II} \geq r^{AB}.
\]

So, we have

\[
r_{max}^{II} \geq r_{max}^{AB},
\]

where \( r_{max}^{AB} \) denotes the maximum value of \( r^{AB} \).

According to Theorem 5 and Eq. (65), if \( P_1 = P_2, |\beta| > |\alpha|^{m-1}, |\gamma| > |\alpha|^{m-1} \) and \( |\beta| \neq 1, |\gamma| \neq 1 \), we have

\[
r_{max}^{I} > r_{max}^{AB}.
\]

That is to say, we may increase the maximum success probability of cloning through adding an auxiliary system described by states \( \{|\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle\} \).

When auxiliary system described by states \( \{|\phi_1^{(3)}\rangle, |\phi_2^{(3)}\rangle\} \) satisfies that

\[
\langle \phi_1^{(3)} | \phi_2^{(3)} \rangle = \langle \phi_1^{(1)} | \phi_2^{(1)} \rangle \langle \phi_1^{(2)} | \phi_2^{(2)} \rangle,
\]

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we can consider in Scenno III the product state $|\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle$ as a state $|\phi_i^{(3)}\rangle$ of one auxiliary system. So, by changing Scenario III, we can describe it as follows:

If the original information is held by Alice, and the supplementary information $\{|\phi_1^{(3)}\rangle, |\phi_2^{(3)}\rangle\}$ by John. John and Alice can use a one-way classical channel. In this case, the optimal performance is always achieved as follows: John first runs the machine

$$U_J(|\phi_i^{(3)}\rangle|\Sigma\rangle|P_{j0}^0\rangle) = \sqrt{r_j^I|\psi_i\rangle^{\otimes m-1}|\chi_j\rangle|P_{j}^i\rangle} + \sqrt{1-r_j^I}|\Psi_j^{i}\rangle, \quad i = 1, 2, \quad (68)$$

where $r_j^I$ is the success probability of cloning $|\psi_i\rangle$ with the input state $|\phi_i^{(3)}\rangle$, $|\Sigma\rangle$ is a blank state and $|P_{j}^i\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi_j^{i}\rangle$ are normalized states of the composite system $JP$, $\langle P_{j}^i|\Psi_j^{j}\rangle = 0$ ($i = 1, 2; j = 1, 2$), $U_J$ is a unitary operator, $|\chi_j\rangle$ is a normalized state, and afterwards John tells Alice whether the trial was successful or not. In the successful case, Alice just leaves her state $|\psi_i\rangle$ as it is, and hence they obtain $m$ copies in total. If the attempt of Victor and Bob has failed, Alice runs the machine

$$U_A(|\psi_i\rangle|\Sigma\rangle|P_{A0}^0\rangle) = \sqrt{r_i^A|\psi_i\rangle^{\otimes m}|\chi_A\rangle|P_{A}^i\rangle} + \sqrt{1-r_i^A}|\Psi_{ap}^i\rangle, \quad i = 1, 2, \quad (69)$$

where $r_i^A$ is the success probability of cloning $|\psi_i\rangle$ with the input state $|\psi_i\rangle$, $|\Sigma\rangle$ is a blank state, and $|P_{A}^i\rangle$ ($i = 0, 1, 2$) are normalized states of the probe $P$, $|\Psi_{ap}^i\rangle$ are normalized states of the composite system $AP$, $\langle P_{A}^i|\Psi_{ap}^j\rangle = 0$ ($i = 1, 2; j = 1, 2$), $U_A$ is a unitary operator, $|\chi_A\rangle$ is a normalized state. So, the total success probability of cloning $|\psi_i\rangle$ for input state $|\psi_i\rangle|\phi_i^{(3)}\rangle$ in this protocol is given by $r_{II} = r_i^I + (1-r_i^I)r_i^A$.

In this case, $\langle \phi_1^{(3)}|\phi_2^{(3)}\rangle = \langle \phi_1^{(1)}|\phi_2^{(1)}\rangle\langle \phi_1^{(2)}|\phi_2^{(2)}\rangle$. The process of calculating $r_{III_{\text{max}}}$ is similar to the proof of Theorem 5. As a result, we can consider $r_{III_{\text{max}}}$ as the maximum success probability with supplementary information contained in the quantum states of one auxiliary system $\{|\phi_1^{(3)}\rangle, |\phi_2^{(3)}\rangle\}$.

Furthermore, according to Theorem 5, when $P_1 = P_2$, the equality $r_{\text{max}} = r_{\text{III_{max}}}$ always holds. In other words, there always exists a machine with one auxiliary system whose maximum success probability of cloning is equal to that with two auxiliary systems.

It seems as if we could permanently increase the probability of success. Starting from one scheme with one auxiliary system $\{|\phi_1^{(1)}\rangle, |\phi_2^{(1)}\rangle\}$, we can add second auxiliary system $\{|\phi_1^{(2)}\rangle, |\phi_2^{(2)}\rangle\}$ and thus increase the probability of success. And we can always find another probabilistic cloning with one auxiliary system $\{|\phi_1^{(3)}\rangle, |\phi_2^{(3)}\rangle\}$ having the same probability of success as the previous one with two auxiliary systems. As we have now a probabilistic cloning with one auxiliary system, we can add second auxiliary system increasing the probability of success ones more. Continuing this “cyclic” argument, we can permanently increase the probability of success.

Nevertheless, in fact, it is not like this. According to Corollary 6, the probability increase should saturate at 1. Firstly, starting from one scheme with one auxiliary system
\{ |φ^{(1)}_1 \rangle, |φ^{(1)}_2 \rangle \}\), we can add second auxiliary system \{ |φ^{(2)}_1 \rangle, |φ^{(2)}_2 \rangle \} satisfying the condition 

\[ |β| > |α|^{m-1}, |γ| > |α|^{m-1}, |β| \neq 1, |γ| \neq 1, \] 

and thus increase the probability of success. Secondly, we can always find another probabilistic cloning with one auxiliary system described by states \{ |φ^{(3)}_1 \rangle, |φ^{(3)}_2 \rangle \}, satisfying the condition that 

\[ \langle φ^{(3)}_1 | φ^{(3)}_2 \rangle = \langle φ^{(1)}_1 | φ^{(1)}_2 \rangle \langle φ^{(2)}_1 | φ^{(2)}_2 \rangle, \] 

having the same probability of success as the previous one with two auxiliary systems. (However, we should note that the inner product of states \{ |φ^{(3)}_1 \rangle, |φ^{(3)}_2 \rangle \} is less than \{ |β| \}.) If we continue this “cyclic” argument, we can increase the probability of success. However, notably, when the inner product of states of auxiliary system is not more than \{ |α|^{m-1}, \} it does not satisfy the condition of Corollary 6. So, we can not permanently continue this “cyclic” argument. According to Theorem 5, when the inner product of states of auxiliary system is not more than \{ |α|^{m-1}, \}, the maximum success probability is 1.

4. Concluding remarks

In this Letter, we considered three probabilistic cloning protocols in terms of different communication channels between the original party and the two supplementary parties: first, the original party and two supplementary parties are in quantum communication; second, the original party and the first supplementary party are in quantum communication, but the first supplementary party and the second one are in classical communication; third, the original party and the first supplementary party are in classical communication, but the two supplementary parties are in quantum communication. In particular, we show that, when the two states have the same \textit{a priori} probability chosen, the best efficiency of producing \( m \) copies is achieved by the first and the third protocols. Furthermore, we also show that, in certain circumstances, by adding an auxiliary system, we may increase the maximum success probability of the probabilistic cloning with supplementary information contained in the quantum states of only one auxiliary system. However, we always can find another probabilistic cloning with one auxiliary system having the same success probability as that with two auxiliary systems.

Probabilistic cloning may get precise copies with a certain probability, so, improving the success ratio is of importance. We hope that our results would provide some useful ideas in preserving important quantum information, parallel storage of quantum information in a quantum computer, and quantum cryptography.

An interesting problem is what is the case for the two states to be cloned having different prior probabilities chosen. As well, novel cloning machine by Pati [15] with supplementary information contained in the quantum states of two auxiliary systems still merits consideration. Moreover, if the supplementary information is given as a mixed state or we have more than two auxiliary systems, then the probabilistic cloning devices are still worth considering. We would like to explore these questions in future.
Acknowledgement

The authors would like to thank the anonymous referee and Professor P.R. Holland, Editor for their invaluable comments and suggestions that greatly help to improve the quality of this paper.

References

