Baxter $Q$–operator for graded $SL(2|1)$ spin chain

A.V. Belitsky$^a$, S.É. Derkachov$^{b,a}$, G.P. Korchemsky$^c$, A.N. Manashov$^{d,e}$

$^a$Department of Physics, Arizona State University
Tempe, AZ 85287-1504, USA

$^b$St. Petersburg Department of Steklov Mathematical Institute,
Russian Academy of Sciences, 191023 St.-Petersburg, Russia

$^c$Laboratoire de Physique Théorique$^1$, Université de Paris XI
91405 Orsay Cédex, France

$^d$Department of Theoretical Physics, St.-Petersburg State University
199034, St.-Petersburg, Russia

$^e$Institut für Theoretische Physik, Universität Regensburg
D-93040 Regensburg, Germany

Abstract

We study an integrable noncompact superspin chain model that emerged in recent studies of the dilatation operator in the $\mathcal{N} = 1$ super-Yang-Mills theory. It was found that the latter can be mapped into a homogeneous Heisenberg magnet with the quantum space in all sites corresponding to infinite-dimensional representations of the $SL(2|1)$ group. We extend the method of the Baxter $Q$–operator to spin chains with supergroup symmetry and apply it to determine the eigenspectrum of the model. Our analysis relies on a factorization property of the $\mathcal{R}$–operators acting on the tensor product of two generic infinite-dimensional $SL(2|1)$ representations. It allows us to factorize an arbitrary transfer matrix into a product of three ‘elementary’ transfer matrices which we identify as Baxter $Q$–operators. We establish functional relations between transfer matrices and use them to derive the TQ-relations for the $Q$–operators. The proposed construction can be generalized to integrable models based on supergroups of higher rank and, in distinction to the Bethe Ansatz, it is not sensitive to the existence of the pseudovacuum state in the quantum space of the model.

$^1$Unité Mixte de Recherche du CNRS (UMR 8627).
# Contents

1. Introduction 2

2. Noncompact $SL(2|1)$ spin chain 6

   2.1. Representation of the $SL(2|1)$ superalgebra 7
   2.2. Reducible $SL(2|1)$ representations
      2.2.1. (Anti)chiral infinite-dimensional representations 9
      2.2.2. Finite dimensional typical representations 10
      2.2.3. Finite-dimensional atypical representations 12
      2.2.4. Traces over reducible representations 12
   2.3. Invariant scalar product
      2.3.1. General case 15
      2.3.2. Reduction to the chiral representation 17

3. Baxter $Q$–operators as transfer matrices 18

   3.1. Factorized $R$–matrix 18
   3.2. Definition of $Q$–operators 21
   3.3. $Q$–operator for $\mathcal{N} = 1$ super-Yang-Mills theory 24
   3.4. Analytical properties of the $Q$–operators 27

4. Factorized transfer matrices 28

   4.1. Finite-dimensional transfer matrices 29
   4.2. Infinite-dimensional (anti)chiral transfer matrices 30
   4.3. Finite-dimensional (anti)chiral transfer matrices 31
   4.4. Factorization of (anti)chiral transfer matrices 32

5. Baxter equations 34

   5.1. TQ-relations 34
   5.2. Nested TQ-relations 36

6. Nested Bethe Ansatz 37

   6.1. Polynomial transfer matrices 38
   6.2. Polynomial $Q$–operators
      6.2.1. TQ-relations 40
      6.2.2. TQ-relations in the chiral limit 43
      6.2.3. Eigenspectrum in the chiral limit 44
   6.3. Matching quantum numbers
      6.3.1. Chiral limit 46

7. Conclusion 49

A Reducible representations of the $SL(2|1)$ 51

B Calculation of the normalization factors 54

C Matrix representation of the $R$–operators 56

D Integral representation of $R^{(3)}$–operator 57

E Factorization of the transfer matrix 59
1. Introduction

Integrable lattice spin chain models with supergroup symmetries play an important rôle in various areas of theoretical physics ranging from condensed matter to supersymmetric gauge theories. In particular, these models arose in studies of strongly correlated electronic systems in relation with high $T_c$ superconductivity, in the quantum Hall effect and recently made their appearance on both sides of the gauge/string correspondence.

In condensed matter physics, the interest in the one-dimensional supersymmetric $t - J$ model has been renewed by Anderson’s suggestion that two-dimensional systems may share common features with one-dimensional systems \[1\]. This model describes electrons on a one-dimensional lattice with a Hamiltonian that includes nearest-neighbor hopping ($t$) and nearest-neighbor spin exchange and charge interactions ($J$). The Hilbert space of the model is constrained to exclude double occupancy so that at a given lattice site there are only three possible electronic states: the Fock vacuum $|0\rangle$, spin-up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ states. For special values of the couplings, $J = 2t$, the model can be mapped into an integrable Heisenberg magnet with the spin operators in each site being generators of the three-dimensional atypical representation of the $SL(2|1)$ group \[2, 3\]. Its exact eigenspectrum can be found within the nested Bethe ansatz approach.

In supersymmetric Yang-Mills (SYM) theories, integrable lattice spin chain models appeared in studies of the scale dependence of composite light-cone single-trace operators

\begin{equation}
\mathcal{O}(z_1, \theta_1, \ldots, z_N, \theta_N) = \text{tr} [\Phi(z_1 n, \theta_1) \ldots \Phi(z_N n, \theta_N)] ,
\end{equation}

built from chiral superfields $\Phi(zn_\mu, \theta)$ “living” on the light ray defined by a light-like vector, $n_\mu^2 = 0$. The expansion of $\Phi(zn_\mu, \theta)$ in powers of $\theta$ produces bosonic, $\phi$, and fermionic, $\chi$, fields\[1\] which are assumed to be holomorphic functions of $z$ at the origin

\begin{equation}
\Phi_{N=1}(zn_\mu, \theta) = \chi(zn_\mu) + \theta \phi(zn_\mu) = \sum_{k \geq 0} z^k \cdot \chi_k + \theta z^k \cdot \phi_k ,
\end{equation}

with $\phi_k = (n \cdot \partial)^k \phi(0)$ and $\chi_k = (n \cdot \partial)^k \chi(0)$. In gauge theories with $\mathcal{N} > 1$ supercharges, the superfield depends on $\mathcal{N}$ Grassmann variables $\theta^A$ (with $A = 1, \ldots, N$) and its expansion involves more terms. The scale dependence of the operators \[1.1\] is driven by the dilatation operator which can be calculated in gauge theory as a series in the coupling constant. To one-loop order and in the multi-color limit, the dilatation operator in super-Yang-Mills theories with $\mathcal{N}$–supercharges takes the form

\begin{equation}
\mathbb{H}_N = H_{12} + \ldots + H_{N-1,N} + H_{N1} .
\end{equation}

The two-particle Hamiltonian, say $H_{12}$, acts only on the 1st and 2nd superfields inside the trace \[1.1\] and is given by the following integral operator \[4\]

\begin{equation}
H_{12} \Phi(Z_1) \Phi(Z_2) = \int_0^1 \frac{d\alpha}{\alpha} \left\{ 2\Phi(Z_1) \Phi(Z_2) \right. \\
\left. - (1 - \alpha)^j \left[ \Phi((1 - \alpha)Z_1 + \alpha Z_2) \Phi(Z_2) + \Phi(Z_1) \Phi((1 - \alpha)Z_2 + \alpha Z_1) \right] \right\} .
\end{equation}

\[1\] In $\mathcal{N} = 1$ SYM theory, the fields $\chi(zn_\mu)$ and $\phi(zn_\mu)$ can be identified as a helicity $(-1/2)$ component of the gaugino field and a helicity $(-1)$ component of the gauge field strength.
Here $j = (3 - N)$ is twice the superconformal spin of the superfields $\Phi(Z_k) \equiv \Phi(z_k n, \theta_k)$ (with $k = 1, 2$) and a short-hand notation is used for the sum of vectors in the $(N + 1)\text{-dimensional}$ superspace $\beta Z_1 + \alpha Z_2 \equiv (\beta z_1 + \alpha z_2, \beta \theta_1^j + \alpha \theta_2^j)$. The dilatation operator $H_N$ defined in this way can be mapped into a Hamiltonian of a (graded) integrable Heisenberg $SL(2|N)$ magnet. The length of the spin chain equals the number of superfields entering $|1\rangle$ and its eigenspectrum determines the spectrum of anomalous dimensions of the operators $|1\rangle$ to one-loop accuracy.

The appearance of the global $SL(2|N)$ symmetry in gauge theory is not of course accidental and has a clear physical origin. The Lagrangian of SYM theory with $N$ supercharges is invariant under the $SU(2, 2|N)$ group of superconformal transformations. The $SL(2|N)$ symmetry of the one-loop dilatation operator arises as a reduction of this symmetry for the light-like operators $|1\rangle$. The superfield $\Phi(z_k n, \theta_k)$ belongs to an irreducible chiral representation of the $SL(2|N)$ group labeled by its superconformal spin $j$. The corresponding graded vector space $V_j$ defines the quantum space in $k\text{-th}$ site of the lattice model so that the Hilbert space for the Hamiltonian is given by the tensor product of $N$ copies of this space, $V_j \otimes^N$. According to (1.2), for $N = 1$ the linear vector space $V_j$ is spanned by the monomials $V_j = \text{span}\{z^k, \theta^k | k \in \mathbb{N}\}$ and, therefore, it is necessarily infinite-dimensional. This should be compared to the supersymmetric $t - J$ model in which case the corresponding $SL(2|1)$ representation is three-dimensional. Still, one can associate with each site of the $t - J$ model a superfield given by a linear combination of three states

$$\Phi_{tJ}(z, \theta) = 1 \cdot |\uparrow\rangle + z \cdot |\downarrow\rangle + \theta \cdot |0\rangle,$$

where $\{1, \theta, z\}$ define the basis of the graded linear space $v_1$ corresponding to the atypical fundamental representation of the $SL(2|1)$. The Hamiltonian of the $t - J$ model can be realized as an operator acting on the product of superfields $\Phi_{tJ}(z_1, \theta_1) \ldots \Phi_{tJ}(z_N, \theta_N) \in v_1 \otimes^N$. One of the advantages of dealing with superfields is that $\Phi_{tJ}(z_1, \theta_1)$ can be realized as an invariant component of reducible (but indecomposable) $SL(2|1)$ representation $V_{j = -1}$. This allows one to treat in a unifying manner both compact and noncompact graded spin chain models. In both cases, the Hilbert space of the model contains a pseudovacuum state and this opens up a possibility to construct the nested Bethe ansatz solution. For noncompact super-spin chains, the number of eigenstates is infinite for a finite length of the spin chain $N$ and completeness of the Bethe ansatz proves to be an extremely nontrivial issue. This calls up for an alternative approach which does not rely on the existence of the pseudovacuum state in the Hilbert space of the model and which is particularly suitable for solving the eigenproblem for noncompact graded spin chains. We shall demonstrate in the present paper that such an approach is offered by the method of the Baxter $Q-$operator.

Another motivation for developing the Baxter $Q-$operator method for spin chains with supergroup symmetries comes from two seemingly unrelated areas: the description of the transition between plateaux in the integer quantum Hall effect and studies of the AdS/CFT correspondence between supersymmetric Yang-Mills theories and strings on a nontrivial curved background. In both cases one has to deal with quantization of sigma models on noncompact supergroup target spaces – the problem that turns out to be extremely difficult to solve. As a way out, one can try to ‘discretize’ the sigma model and construct a lattice spin chain of length $N$ that would flow into the former in the continuum limit $N \to \infty$. It has been proposed to look for such models among integrable graded spin chains with the Hilbert space of the type $(V \otimes \bar{V}) \otimes^{N/2}$ with $V$ and $\bar{V}$ being conjugated infinite-dimensional representations of supergroups. However, the tensor product $V \otimes \bar{V}$ contains in general irreducible components which have neither the highest, nor the lowest weights vectors and, as a consequence, the nested Bethe ansatz is not
applicable. The situation here is quite similar to that for the \( SL(2) \) magnet with the spin operators being generators of the principal series of the \( SL(2; \mathbb{C}) \) \[12\] or lattice sinh-Gordon model \[13\]. In these cases, the Bethe ansatz can not be applied by the same token as before whereas the method of the Baxter \( Q \)–operator allows one to determine the exact eigenspectrum of the model.

The Baxter \( Q \)–operator is one of the corner-stones of quantum integrable systems \[14\] and it has been discussed in a variety of contexts varying from conformal field theories \[15, 16\] to classical Bäcklund transformations \[17\]. Originally developed for the six-vertex model \[14\], the method provides a general framework to solve the eigenproblem for transfer matrices in a variety of integrable lattice models. Defined as a trace of the monodromy operator over some auxiliary space \( V \), the transfer matrix \( T_V(u) \) depends on the spectral parameter \( u \) and belongs to a commutative family of operators acting on the quantum space of the model, \[ T_V(u), T_{V'}(u') = 0. \]

\[1.6\]

The number of independent \( Q \)–operators depends on the rank of the symmetry group. A distinguished feature of these operators is that all transfer matrices of the model and, as a consequence, the Hamiltonian of the model can be expressed in terms of \( Q_a(u) \). At present, there exists no regular procedure for constructing \( Q \)–operators in a generic lattice integrable model and the number of models for which the Baxter method has been developed is rather limited. The latter include homogeneous Heisenberg magnets based on classical \( SL(2) \) and \( SL(3) \) symmetry and their spin-offs.

In the \( SL(2) \) invariant homogeneous spin chain one can explicitly construct two Baxter operators \( Q_{\pm}(u) \) \[18, 19\]. They satisfy an operatorial \textit{second-order} finite difference equation, the so-called TQ-relation

\[ \tau_N^{(1/2)}(u) Q_{\pm}(u) = (u + s)^N Q_{\pm}(u + 1) + (u - s)^N Q_{\pm}(u - 1), \]

\[1.7\]

and verify the Wronskian condition

\[ Q_+(u)Q_-(u + 1) - Q_+(u + 1)Q_-(u) = \left[ \frac{\Gamma(-u - s)}{\Gamma(-u + s)} \right]^N. \]

\[1.8\]

Here, the half-integer spin \( s \) labels irreducible representations of the \( SL(2) \) group and \( \tau_N^{(1/2)}(u) \) is the transfer matrix \( T_V(u) \) with the auxiliary space \( V \) being two-dimensional spin–1/2 representation of the \( SL(2) \). The eigenvalues of the operator \( Q_+(u) \) are polynomials in \( u \) that we shall denote as \( P_m^{(s)}(u) \) (with nonnegative integer \( m \) defining the total \( SL(2) \) spin of the model).

At the same time, the eigenvalues of the second operator are \textit{meromorphic functions} of \( u \) which can be represented as \( Q_-(u) = \left[ \Gamma(1 - u - s) \right]^N \times (\text{analytical function}) \) as far as the order and position of poles is concerned. Another remarkable feature of the Baxter operators is that the Hamiltonian of the \( SL(2) \) spin chain can be expressed in terms of the ‘polynomial’ \( Q \)–operator as

\[ H_{\text{SL}(2)} = (\ln Q_+(s))' - (\ln Q_+(s))', \]

\[1.9\]
where the prime denotes a derivative with respect to the spectral parameter.

For the $SL(3)$ invariant homogeneous spin chain, one already encounters three $Q-$operators \[20\,\,21\,\,22\]. Similarly to \[17\], three $Q-$operators satisfy the same TQ-relation. Among them only one $Q-$operator is polynomial in $u$ and the eigenvalues of remaining two operators are meromorphic functions. The Baxter equation now takes the form of a finite difference equation of the third order and involves two transfer matrices with the auxiliary space corresponding to two fundamental three-dimensional representations of the $SL(3)$. The Wronskian relation involves all three $Q-$operators simultaneously and it can be cast in a determinant form.

In the present paper we extend the method of the Baxter $Q-$operator to integrable spin chain models with supergroup symmetry. More precisely, we present an explicit construction of the $Q-$operators for the homogeneous Heisenberg magnet with the quantum space in all sites corresponding to the infinite-dimensional $SL(2|1)$ representations $[j_q, \bar{j}_q]$ labeled by a pair of spins $j_q$ and $\bar{j}_q$. We shall argue that the model has three different Baxter operators $Q_a(u)$ (with $a = 1, 2, 3$). These operators have a number of unusual properties as compared with models based on classical Lie symmetry. Namely, two operators, $Q_1(u)$ and $Q_3(u)$, verify the same TQ-relation which takes the form of a second-order finite difference equation analogous to \[17\]. For instance, in the chiral limit $\bar{j}_q = 0$ and $j_q \neq 0$, relevant for the $\mathcal{N} = 1$ SYM theory, the TQ-relation reads (for $a = 1, 3$)

\[
\begin{align*}
\left[\tau_N(u)\bar{\tau}_N(u + j_q) - (u(u + j_q))^N\right] Q_a(u) &= u^N \left[\tau_N(u + j_q) - (u + j_q)^N\right] Q_a(u - 1) + (u + j_q)^N \left[\tau_N(u) - u^N\right] Q_a(u + 1),
\end{align*}
\]

where $\tau_N(u)$ and $\bar{\tau}_N(u)$ are two transfer matrices with the auxiliary space corresponding to three-dimensional atypical representations of the $SL(2|1)$. An important difference with \[17\] is that the dressing factors themselves now depend on the $SL(2|1)$ transfer matrices. As a consequence, there exists no Wronskian relation for the operators $Q_1(u)$ and $Q_3(u)$. The TQ-relation for the remaining operator, $Q_2(u)$, is a finite-difference equation of the first order. In the chiral limit, it reads

\[
\begin{align*}
\left[\bar{\tau}_N(u + j_q - 1) - (u + j_q - 1)^N\right] Q_2(u) &= \left[\tau_N(u) - u^N\right] Q_2(u - 1).
\end{align*}
\]

Among the three $SL(2|1)$ Baxter operators only $Q_1(u)$ is not polynomial in $u$. Under appropriate normalization, its eigenvalues are meromorphic functions of $u$ and their pole structure is similar to that for eigenvalues of the $SL(2)$ operator $Q_-(u)$. Finally, we will demonstrate that in analogy with the $SL(2)$ relation \[13\], the dilatation operator in the $\mathcal{N} = 1$ SYM theory, Eqs. \[13\] and \[14\], is given by a logarithmic derivative of the polynomial operator $Q_3(u)$ in the chiral limit $\bar{j}_q = 0$ and $j_q = 2$

\[
\begin{align*}
\mathbb{H}_{SL(2|1)} &= (\ln Q_3(0))' - (\ln Q_3(-j_q))'.
\end{align*}
\]

Being combined with the TQ-relations \[10\], this leads to an exact solution to the eigenproblem for the $SL(2|1)$ spin chain Hamiltonian \[13\].

The present construction of $Q-$operators for the graded $SL(2|1)$ spin chain makes use of the approach developed in Ref. \[22\] in application to the $SL(3)$ spin chain. The two main ingredients of our analysis are (i) the factorization property of the $\mathcal{R}-$operators \[23\] acting on the tensor product of two generic, infinite-dimensional $SL(2|1)$ representations and (ii) property of the transfer matrices with the auxiliary space corresponding to a reducible \[15\,\,16\,\,22\] (but in general indecomposable) $SL(2|1)$ representation. These properties allow us to factorize an arbitrary transfer matrix into a product of three ‘elementary’ transfer matrices which we identify
as Baxter $Q$–operators. In addition, they lead to functional relations between the $SL(2|1)$ transfer matrices including those with the auxiliary space corresponding to finite-dimensional representations of the $SL(2|1)$. Such representations naturally arise as invariant components of a bigger infinite-dimensional reducible representation. As a result, a generic finite-dimensional transfer matrix can be expressed as a difference of two infinite-dimensional transfer matrices each given by a product of three $Q$–operators. Remarkably, this relation can be cast into the form of the Baxter TQ-relations. It also leads to functional relations between finite-dimensional transfer matrices which are in agreement with similar relations obtained in Ref. [24]. The above construction can be straightforwardly generalized to integrable models based on supergroups of higher rank and, in distinction to the Bethe Ansatz, it is not sensitive to the existence of the pseudovacuum state in the quantum space of the model.

The outline of the paper is as follows. In Section 2, we introduce the $SL(2|1)$ superalgebra and present its realization most suitable for the analysis of the model (1.3). Then, we define generic infinite-dimensional representations of the $SL(2|1)$ group and describe in detail the structure of reducible indecomposable $SL(2|1)$ representations which play a pivotal rôle throughout our analysis. In Section 3, we review the formalism of factorized $R$–operators and demonstrate that a generic infinite-dimensional transfer matrix can be factorized into the product of three mutually commuting operators that we identify as $Q$–operators. In Section 4, we combine together properties of reducible $SL(2|1)$ representations and factorized expressions for the corresponding transfer matrices to obtain a representation for various finite- and infinite-dimensional transfer matrices in terms of the Baxter operators. In Section 5, we argue that the obtained relations yield a hierarchy between the transfer matrices and identify one of the relations as the TQ-equation for the Baxter operators. In Section 6, we present an exact solution of the eigenproblem for the model (1.3) based on the TQ-relations and establish the correspondence with the nested Bethe ansatz solution. Section 7 contains concluding remarks. Several appendices give detailed derivation of some results used in the body of the paper.

2. Noncompact $SL(2|1)$ spin chain

As was already mentioned, noncompact $SL(2|1)$ spin chains naturally appear in supersymmetric $\mathcal{N} = 1$ Yang-Mills theory. The $SL(2|1)$ symmetry arises as a reduction of the full superconformal symmetry group $SU(2,2|1)$ of the four-dimensional gauge theory on the light-cone. Gauge theory leads to a particular realization of the $SL(2|1)$ algebra on the space of functions in the superspace $\mathcal{Z} = (z, \theta, \bar{\theta})$ that we shall employ throughout this paper. As we will argue, this representation is advantageous as far as the construction of the Baxter operators is concerned.

A general superfield $\Phi(z, \theta, \bar{\theta})$ is defined as a function of ‘even’ $z$ and ‘odd’ $\theta$ and $\bar{\theta}$ variables verifying the standard anti-commutation relations $\theta^2 = \bar{\theta}^2 = 0$ and $\{\theta, \bar{\theta}\} = \theta\bar{\theta} + \bar{\theta}\theta = 0$. A single superfield comprises four independent functions

$$\Phi(z, \theta, \bar{\theta}) = \chi(z) + \theta\phi(z) + \bar{\theta}\bar{\phi}(z) + \theta\bar{\theta}\psi(z), \quad (2.1)$$

which are assumed to be holomorphic functions of $z$ around the origin. In the chiral limit, the superfield does not depend on $\bar{\theta}$ or, equivalently, $\bar{\phi}(z) = \psi(z) = 0$. The superfield parameterizes the quantum space in each site of the spin chain. If its expansion in powers of $z$ is not truncated, this space is infinite-dimensional and, therefore, the corresponding spin chain is called noncompact. Otherwise, the superfield is a polynomial in $z$ of a finite degree and the corresponding spin chain is compact.
2.1. **Representation of the SL(2|1) superalgebra**

The superfield (2.1) forms a representation of the $SL(2|1)$ algebra labeled by two spins $j$ and $\bar{j}$. Its variation under the $SL(2|1)$ transformations is given by

$$\delta_c \Phi(z, \theta, \bar{\theta}) = G_{j\bar{j}} \cdot \Phi(z, \theta, \bar{\theta})$$

(2.2)

with the operator $G_{j\bar{j}}$ being a linear combination of four even $L^+, L^0, B$ and four odd $V^\pm, \bar{V}^\pm$ generators. Using the technique of induced representations, they can be realized as first order differential operators acting on the super-coordinates of the superfields $\Phi(z, \theta, \bar{\theta})$.

- The operators $L^-, V^-$ and $\bar{V}^-$ decrease the power in $z$, $\theta$ and $\bar{\theta}$,

$$L^- = -\partial_z, \quad V^- = \partial_\theta + \frac{1}{2} \bar{\theta} \partial_{\bar{\theta}}, \quad \bar{V}^- = \partial_{\bar{\theta}} + \frac{1}{2} \theta \partial_\theta .$$

(2.3)

- The operators $L^+, V^+$ and $\bar{V}^+$ increase the power in $z$, $\theta$ and $\bar{\theta}$,

$$V^+ = z \partial_\theta + \frac{1}{2} \bar{\theta} z \partial_z + \frac{1}{2} \bar{\theta} \theta \partial_\theta + j \bar{\theta},$$

$$\bar{V}^+ = z \partial_{\bar{\theta}} + \frac{1}{2} \theta z \partial_z + \frac{1}{2} \theta \bar{\theta} \partial_{\bar{\theta}} + j \theta,$$

$$L^+ = z^2 \partial_z + z \theta \partial_\theta + z \bar{\theta} \partial_{\bar{\theta}} + (j + \bar{j}) z + \frac{1}{2} (j - \bar{j}) \bar{\theta} \theta .$$

(2.4)

- The operators $J$ and $\bar{J}$ preserve the power in $z$, $\theta$ and $\bar{\theta}$,

$$J = L^0 + B = z \partial_z + \bar{\theta} \partial_{\bar{\theta}} + j,$$

$$\bar{J} = L^0 - B = z \partial_z + \theta \partial_\theta + \bar{j} .$$

(2.5)

Equations (2.3) – (2.5) define the infinitesimal $SL(2|1)$ transformations of the superfields carrying the superconformal spins $j$ and $\bar{j}$. In a global form the transformations generated by the Cartan generators (2.5) look like

$$e^{\lambda J} \cdot \Phi(z, \theta, \bar{\theta}) = e^{\lambda J} \Phi(\lambda z, \theta, \lambda \bar{\theta}) ,$$

$$e^{\lambda \bar{J}} \cdot \Phi(z, \theta, \bar{\theta}) = e^{\lambda \bar{J}} \Phi(\lambda z, \lambda \theta, \bar{\theta}) ,$$

(2.6)

while for other generators similar relations can be found in Ref. [25]. For our purposes it is convenient to introduce the operators $E^{AB}$ (with $A, B = 1, 2, 3$)

$$E^{11} = J \quad \quad E^{12} = -V^+ \quad \quad E^{13} = L^+$$

$$E^{21} = -\bar{V}^- \quad \quad E^{22} = \bar{J} - J \quad \quad E^{23} = -\bar{V}^+$$

$$E^{31} = L^- \quad \quad E^{32} = V^- \quad \quad E^{33} = -\bar{J} .$$

(2.7)

Then, the generators of the superconformal transformations (2.3) – (2.5) satisfy the graded $SL(2|1)$ commutation relations [26]

$$[E^{AB}, E^{CD}] \equiv E^{AB} E^{CD} - (-1)^{(A+B)(C+D)} E^{CD} E^{AB}$$

$$= \delta_{CB} E^{AD} - (-1)^{(A+B)(C+D)} \delta_{AD} E^{CB} ,$$

(2.8)
where the indices run over $A, B, C, D = 1, 2, 3$ and the grading is chosen as $\bar{1} = 3 = 0$ and $\bar{2} = 1$. The $SL(2|1)$ algebra has an obvious automorphism

$$J \mapsto J, \quad V^\pm \mapsto V^\pm, \quad L^\pm \mapsto L^\pm,$$

(2.9)

which amounts to substituting $\theta \mapsto \bar{\theta}$ and $j \mapsto \bar{j}$ in (2.3) – (2.5).

Following [27, 28, 29, 26], one can construct the $SL(2|1)$ Casimir operators$^2$ $C_p (p = 1, 2, 3, \ldots)$

$$C_1 = \sum_A E^{AA} = 0,$$

$$C_2 = \frac{1}{2} \sum_{A,B} (-1)^B E^{AB} E^{BA} = J \bar{J} + L^+ L^- - V^+ V^- - V^+ V^-,$$

(2.10)

$$C_3 = \frac{1}{2} \sum_{A,B,C} (-1)^{B+C} E^{AB} E^{BC} E^{CA} = \frac{1}{2} (J - \bar{J} + \frac{1}{3}) J \bar{J} + \ldots .$$

Here ellipses denote terms involving the lowering operators $V^-$, $\bar{V}^-$ and $L^-$ to the right and, therefore, vanishing when applied to the lowest weight (see Eq. (2.13) below). One can verify using (2.3) – (2.5) that the Casimir operators are diagonal for the superfield $\Phi(z, \theta, \bar{\theta})$

$$C_2 \cdot \Phi(z, \theta, \bar{\theta}) = j \bar{j} \Phi(z, \theta, \bar{\theta}),$$

$$C_3 \cdot \Phi(z, \theta, \bar{\theta}) = \frac{1}{2} (j - \bar{j} + \frac{1}{3}) j \bar{j} \Phi(z, \theta, \bar{\theta}).$$

(2.11)

Both Casimirs vanish in the (anti)chiral limit $j = 0$ (or $\bar{j} = 0$).

Let us denote by $[j, \bar{j}]$ the $SL(2|1)$ representation the field $\Phi(z, \theta, \bar{\theta})$ belongs to. For generic values of the spins $j$ and $\bar{j}$, this representation is infinite-dimensional and irreducible. However the representation $[j, \bar{j}]$ becomes reducible (but, in general, indecomposable) for some special values of the spins. This property plays a crucial rôle in our analysis and we shall describe it in details in the next subsection.

The even generators of the $SL(2|1)$ superalgebra form the $SL(2) \otimes U(1)$ subalgebra. The operators $L^\pm$ and $L^0 = \frac{1}{2} (J + \bar{J})$ belong to the $SL(2)$ subalgebra while $B = \frac{1}{2} (J - \bar{J})$ defines the $U(1)$ charge. Substituting the superfield in (2.2) by its expression (2.1) one finds that four functions of $z$ entering (2.1) form four different representations under the action of the $SL(2) \otimes U(1)$ generators. This corresponds to the decomposition of a typical $SL(2|1)$ representation $[j, \bar{j}]$ over the $SL(2) \otimes U(1)$ multiplets [27, 28, 29, 26]

$$[j, \bar{j}] = D_\ell(b) \oplus D_{\ell+\frac{1}{2}}(b - \frac{1}{2}) \oplus D_{\ell+\frac{1}{2}}(b + \frac{1}{2}) \oplus D_{\ell+1}(b),$$

(2.12)

where $D_\ell(b)$ stands for the $SL(2) \otimes U(1)$ representation labeled by the conformal $SL(2)$ spin $\ell$ and the $U(1)$ charge $b$

$$\ell = \frac{1}{2} (j + \bar{j}), \quad b = \frac{1}{2} (j - \bar{j}).$$

(2.13)

### 2.2. Reducible $SL(2|1)$ representations

By construction, the representation $[j, \bar{j}]$ is spanned by the superfields (2.1) which are assumed to be analytical functions of $z$ around the origin. Let us denote the corresponding representation

$^2$The definition of the Casimirs in Ref. [27, 28, 29, 26] involves the $GL(2|1)$ generators $e^{AB}$. They are related to the $SL(2|1)$ generators as $E^{AB} = e^{AB} - \text{str} e^{AB}$ so that $\text{str} E^{AB} = 0$. 

8
space as $\mathcal{V}_{jj}$. The basis in the infinite-dimensional linear graded space $\mathcal{V}_{jj}$ can be chosen as

$$\mathcal{V}_{jj} = \text{span}\left\{z^k, z^k\theta, z^k\bar{\theta}, z^k\theta\bar{\theta} \mid k \in \mathbb{N}\right\}. \quad (2.14)$$

One identifies among these states the lowest weight $\Omega = 1$. It is annihilated by all lowering $SL(2|1)$ generators$^3$ and diagonalizes the Cartan generators

$$L^-\Omega = V^-\Omega = \bar{V}^-\Omega = 0, \quad J\Omega = j\Omega, \quad \bar{J}\Omega = \bar{j}\Omega. \quad (2.15)$$

Applying the raising operators (2.4) to the lowest weight $\Omega = 1$ one can construct the invariant graded linear space

$$\mathcal{V}_\Omega = \text{span}\left\{(L^+)^k\Omega, \ (L^+)^k\bar{V}^+\Omega, \ (L^+)^kV^+\Omega, \ (L^+)^k\bar{V}^+V^+\Omega \mid k \in \mathbb{N}\right\}. \quad (2.16)$$

These states are given by a linear combination of the basis vectors (2.14) of the same Grassmann parity. Their explicit form can be found in Ref. [25].

For generic $j$ and $\bar{j}$, the two spaces are isomorphic, $\mathcal{V}_{jj} = \mathcal{V}_\Omega$. There are however special values of the spins $j$ and $\bar{j}$ for which some basis vectors in (2.16) vanish identically. In that case, nonvanishing states in (2.16) still form the $SL(2|1)$ invariant space but it is now a subspace of $\mathcal{V}_{jj}$. In other words, the representation $[j, \bar{j}]$ becomes reducible and $\mathcal{V}_\Omega$ defines its invariant component. The corresponding values of spins are:

- $\bar{j} = 0$ for $j \neq 0$ (chiral limit);
- $j = 0$ for $\bar{j} \neq 0$ (antichiral limit);
- $j = \bar{j} = 0$;
- $j + \bar{j} = -n$ with $n$ is positive integer.

Let us examine the four cases one after another and decompose $\mathcal{V}_{j\bar{j}}$ over irreducible components.

2.2.1. (Anti)chiral infinite-dimensional representations

For $\bar{j} = 0$, one finds from (2.4) that $V^+ \cdot 1 = 0$ and, therefore, half of the basis vectors in (2.16) vanish identically indicating that the corresponding $SL(2|1)$ representation $[j, 0]$ becomes reducible. The nonvanishing vectors in (2.16) define an infinite-dimensional $SL(2|1)$ invariant subspace that we shall denote as $\mathcal{V}_j$. It is convenient to introduce two supercovariant derivatives

$$D = -\partial_\theta + \frac{1}{2}\theta\partial_z, \quad \bar{D} = -\bar{\partial}_\theta + \frac{1}{2}\bar{\theta}\partial_z \quad (2.17)$$

satisfying $D^2 = \bar{D}^2 = 0$ and $\{D, \bar{D}\} = -\partial_z$. Then, one can verify that for $\bar{j} = 0$ the basis vectors in (2.16) are annihilated by the operator $D$ and, therefore, the space $\mathcal{V}_j$ coincides with its kernel

$$\mathcal{V}_j = \text{ker} \ D = \text{span}\left\{1, \theta z^k, (z + \frac{1}{2}\bar{\theta}\theta)^{k+1} \mid k \in \mathbb{N}\right\}. \quad (2.18)$$

Notice that the $\bar{\theta}$—dependence of states in $\mathcal{V}_j$ can be removed by a shift $z \mapsto z - \frac{1}{2}\bar{\theta}\theta$. The basis in the quotient space $\mathcal{V}_{j,0}/\mathcal{V}_j$ can be constructed by imposing the antichirality conditions $\bar{D}\Phi_\pm = 0$ and $D\Phi_\pm \neq 0$ for the basis vectors,

$$\mathcal{V}_{j,0}/\mathcal{V}_j = \text{span}\left\{\theta, \theta z^{k+1}, (z - \frac{1}{2}\bar{\theta}\theta)^{k+1} \mid k \in \mathbb{N}\right\}. \quad (2.19)$$

$^3$In virtue of $\{V^-, \bar{V}^-\} = -L^-$, the relation $L^-\Omega = 0$ follows from the remaining two.
Under the $SL(2|1)$ transformations, the states $\Phi_+ \in \mathcal{V}_{j,0}/\mathcal{V}_j$ mix with the states from the invariant subspace $\Phi_+ \in \mathcal{V}_j$ while the opposite is prohibited. This implies that for $\tilde{j} = 0$ the $SL(2|1)$ generators $G_{j0}$, Eqs. (2.13) – (2.15), take a block-triangular form in the basis $(\Phi_+, \Phi_-)$

\[
G_{j0} \cdot \Phi_i^+ = \Phi_i^+ [G^+]^k_i \\
G_{j0} \cdot \Phi_i^- = \Phi_i^- [G^-]^\beta_i + \Phi_i^+ [G^+]^\alpha_i
\]  

(2.20)

where $G_{\pm \pm}$ are (infinite-dimensional) graded matrices. This property can be depicted graphically as demonstrated in Fig. 1. The upper diagonal block $G_{++}$ defines a (infinite-dimensional) representation of the $SL(2|1)$ superalgebra to which we shall refer as the chiral $SL(2|1)$ representation of spin $j$ and denote as $[j]_+$. The corresponding representation space is defined in (2.18). The lower diagonal block $G_{--}$ defines yet another $SL(2|1)$ representation which is isomorphic to the chiral $SL(2|1)$ representation of spin $j + 1$ (see Appendix A for details).

We conclude that for $\tilde{j} = 0$ the $SL(2|1)$ representation $[j,0]$ is reducible but indecomposable. It is given by a semidirect sum of two chiral $SL(2|1)$ representations of spins $j$ and $j + 1$

\[
[j,0] = [j]_+ \mathfrak{D} [j + 1]_+.
\]  

(2.21)

Here ‘$\mathfrak{D}$’ stands for the semidirect sum with the first summand being an invariant subspace of the whole representation space. Making use of the automorphism of the $SL(2|1)$ superalgebra (2.9), one can obtain from (2.21) the decomposition of the $SL(2|1)$ representation $[0, \tilde{j}]$

\[
[0, \tilde{j}] = [\tilde{j}]_- \mathfrak{D} [\tilde{j} + 1]_-,
\]  

(2.22)

where $[\tilde{j}]_-$ denotes the antichiral $SL(2|1)$ representation. It is spanned by the states $\bar{\Phi}(z, \theta, \bar{\theta}) \in \bar{\mathcal{V}}_j$ which verify the condition $\bar{D} \Phi = 0$. The vector space $\bar{\mathcal{V}}_j$ can be obtained from (2.18) by substituting $\theta = \bar{\theta}$.

\[
\bar{\mathcal{V}}_j = \ker \bar{D} = \text{span} \left\{ 1, \bar{\theta} z^k, (z - \frac{1}{2} \bar{\theta} \theta)^{k+1} \mid k \in \mathbb{N} \right\}.
\]  

(2.23)

So far we did not specify the value of the spin $j$ (or $\tilde{j}$) and did not discuss reducibility of the representation $[j]_+$ (or $[\tilde{j}]_-\)$. We will address this question in Section 2.2.3 and show that the (anti)chiral $SL(2|1)$ representations $[-n]_+$ and $[-n]_-$ are in turn reducible for $n \in \mathbb{N}$ in which case they contain a finite-dimensional invariant component.

2.2.2. Finite dimensional typical representations

For $j = \tilde{j} = 0$ or $j + \tilde{j} = -n$ (with $n \in \mathbb{Z}_+$) the representation space $\mathcal{V}_{j,j}$, Eq. (2.11), has a finite-dimensional invariant subspace that we shall denote as $v_{n/2b}$ (with $b = \frac{1}{2}(j - \tilde{j})$).

Indeed, for these values of the spins, the state

\[
\hat{\Omega} = (z + \frac{1}{2} \theta \bar{\theta})^{-j} (z - \frac{1}{2} \theta \bar{\theta})^{-\tilde{j}} = z^{-(j+\tilde{j})} - \frac{1}{2} (j - \tilde{j}) z^{-(j+\tilde{j}) - 1} \theta \bar{\theta}
\]  

(2.24)

belongs to (2.16) and takes the form $\hat{\Omega} \sim (L^+)^{-j-\tilde{j}} \Omega$ with $\Omega = 1$. It is annihilated by all raising operators $V^+ \hat{\Omega} = \hat{V}^+ \hat{\Omega} = L^+ \hat{\Omega} = 0$ and, therefore, it defines the highest weight in $\mathcal{V}_\Omega$. As a consequence, the $SL(2|1)$ invariant space (2.16) becomes finite-dimensional. For $j = \tilde{j} = 0$, it
contains only one state $v_{00} = \{1\}$, while for $j + \bar{j} = -n$, it has the dimension $\dim v_{n/2b} = 4n$ and is spanned by the states

$$v_{n/2b} = \text{span}\left\{1, \theta, \bar{\theta}, z^k, \bar{z}^k, \theta z^k, \bar{\theta}z^k, \bar{\Omega} | 1 \leq k \leq n-1\right\},$$

(2.25)

where $z_{\pm} = z \pm \frac{1}{2}\theta\bar{\theta}$ and $\bar{\Omega}$ is given by (2.24). The underlying $SL(2|1)$ representation is known in the literature [27, 28, 29, 26] as the typical representation and we shall denote it as $(b, n/2)$.

Subsequent analysis goes along the same lines as in Section 2.2.1. For $j + \bar{j} = -n$, the $SL(2|1)$ generators $G_{j\bar{j}}$, Eqs. (2.3) – (2.5), take a block-triangular form in the basis $\Phi_+ \in v_{n/2b}$ and $\Phi_- \in V_{j\bar{j}}/v_{n/2b}$ similarly to (2.20). The upper diagonal block $G_{++}$ represents the $SL(2|1)$ generators of the typical representation $(b, n/2)$ while the lower diagonal block $G_{--}$ can be mapped into generators of the infinite-dimensional representation $[-\bar{j}, -j]$ described in Section 2.1 (see Appendix A for details). We conclude that for $j + \bar{j} = -n$ (with $n$ being positive integer) the $SL(2|1)$ representation $[j, \bar{j}]$ is reducible and it admits the following decomposition

$$[j, \bar{j}] = (b, n/2) \triangleleft [-\bar{j}, -j],$$

(2.26)

where $b = (j - \bar{j})/2$ and $(b, n/2)$ is a finite-dimensional typical $SL(2|1)$ representation (2.23).

For $j = \bar{j} = 0$ the situation is more subtle. The invariant subspace $v_{00}$ contains only one state – the lowest weight 1, while the quotient space $V_{00}/v_{00}$ is given by a direct sum

$$V_{00}/v_{00} = \mathcal{V}_+ \oplus \mathcal{V}_-, \quad \mathcal{V}_\pm = \text{span}\{\theta_\pm, \theta_\pm z^{k+1}, \bar{z}_\pm^{k+1} | k \in \mathbb{N}\},$$

(2.27)

where $\theta_+ = \theta$, $\theta_- = \bar{\theta}$ and $z_\pm = z \pm \frac{1}{2}\theta\bar{\theta}$. The action of the $SL(2|1)$ generators (2.3) – (2.5) on $\mathcal{V}_\pm$ defines the infinite-dimensional chiral and antichiral representations, $[1]_+$ and $[1]_-$, respectively (see Appendix A for details). As a result, for $j = \bar{j} = 0$ the infinite-dimensional $SL(2|1)$ representation $[0, 0]$ admits the following decomposition (see Fig. 1)

$$[0, 0] = (0, 0) \triangleleft ([1]_+ \oplus [1]_-)$$

(2.28)

where $(0, 0)$ stands for a trivial one-dimensional $SL(2|1)$ representation.
2.2.3. Finite-dimensional atypical representations

For \( j = 0 \) and \( j = -n \) one encounters the situation when reducibility conditions discussed in two previous subsections are satisfied simultaneously. According to (2.21), the representation \([-n,0]\) decomposes into a semi-direct sum of two infinite-dimensional chiral representations \([-n,+]_+\) and \([1-n]_+\). The latter representations are also reducible for positive integer \( n \).

For \( j = -n \) the chiral representation \([-n,+]_+\) is spanned by the states (2.18). A unique feature of \( \mathcal{V}_{-n} \) is that it contains the highest weight vector \( \hat{\Omega} = (z + \frac{1}{2} \bar{\theta} \theta)^n \) which is annihilated by all raising \( SL(2|1) \) generators (2.4). As a consequence, the space \( \mathcal{V}_{-n} \) contains a finite-dimensional invariant subspace

\[
\mathcal{v}_n = \text{span}\{1, \theta, z_+, \theta z, z_+^2, \ldots, \theta z^{n-1}, z_+^n\},
\]

(2.29)

with \( z_+ = z + \frac{1}{2} \bar{\theta} \theta \). It has the dimension \( \dim \mathcal{v}_n = 2n + 1 \) and all states in \( \mathcal{v}_n \) are annihilated by the supercovariant derivative \( D \), Eq. (2.17). We shall denote the corresponding \( SL(2|1) \) representation as \((n)_+\). It is known in the literature [27, 28, 29, 26] as the atypical \( SL(2|1) \) representation. As before, the \( SL(2|1) \) generators take a block-triangular form on \( \mathcal{v}_n \oplus \mathcal{V}_{-n}/\mathcal{v}_n \).

Acting on the quotient space \( \mathcal{V}_{-n}/\mathcal{v}_n \), they define the \( SL(2|1) \) representation \([n+1]_-\), i.e., the anti-chiral infinite-dimensional representation of spin \( n + 1 \) (see Appendix A for details).

We conclude that the chiral representation \([-n]_+\) (with positive integer \( n \)) decomposes into a semidirect sum of atypical and antichiral representations,

\[
[-n]_+ = (n)_+ \circ [n+1]_-.
\]

(2.30)

Making use of the automorphism of the \( SL(2|1) \) superalgebra, Eq. (2.9), this relation can be extended to reducible antichiral representations,

\[
[-n]_- = (n)_- \circ [n+1]_+.
\]

(2.31)

Here \((n)_-\) is yet another atypical \( SL(2|1) \) representation spanned by the states

\[
\bar{\mathcal{v}}_n = \text{span}\{1, \bar{\theta}, z_-, \bar{\theta} z, z_-^2, \ldots, \bar{\theta} z^{n-1}, z_-^n\},
\]

(2.32)

with \( z_- = z - \frac{1}{2} \bar{\theta} \theta \).

It is well known [27 28 29 26] that the atypical representations \((n)_+\) and \((n)_-\) also appear as invariant components of the typical representation \((b,n/2)\) for \( b = \pm n/2 \)

\[
(-n/2,n/2) = (n)_+ \circ (n-1)_+,
\]

\[
(n/2,n/2) = (n)_- \circ (n-1)_-.
\]

(2.33)

In our analysis these relations appear as consistency conditions for the relations (2.26) and (2.21) with \( j = -n, \bar{j} = 0 \) and for the relations (2.26) and (2.22) with \( j = 0, \bar{j} = -n \), respectively.

2.2.4. Traces over reducible representations

One of the fundamental objects in lattice integrable models is the transfer matrix. It is defined as a (super)trace of the so-called monodromy operator over a particularly chosen \( SL(2|1) \) representation space \( \mathcal{V}_{j\bar{j}} \) (see Eq. (3.18) below). As we will argue in Section 3, a crucial role in constructing the Baxter operator is played by transfer matrices with \( \mathcal{V}_{j\bar{j}} \) being a reducible \( SL(2|1) \) representation.
Applying the results of this section, one can decompose a supertrace of an arbitrary linear operator on $V_{j,\bar{j}}$ into a sum of supertraces evaluated over irreducible components of $V_{j,\bar{j}}$. Let

$$\vec{e} = \{1, \theta z^k, \bar{\theta} z^k, z_+^{k+1}, z_-^{k+1} \mid k \in \mathbb{N}\} \quad (2.34)$$

be a basis of $V_{j,\bar{j}}$ and let us assign the grading $(-1)^{\bar{e}_k} = 1$ and $(-1)^{\bar{e}_k} = -1$ to correspondingly ‘even’ and ‘odd’ vectors in this basis. Then, an arbitrary linear operator, say $O$, can be represented as a (infinite-dimensional) graded matrix

$$O \cdot e_i = \sum_k e_k O_{ki}, \quad (2.35)$$

with $O_{ki}$ possessing the grading $(-1)^{\bar{O}_{ki}} = (-1)^{\bar{e}_i + \bar{e}_k}$. The supertrace is defined then as

$$\text{str}_{V_{j,\bar{j}}} O = \sum_i (-1)^{\bar{e}_i} O_{ii}. \quad (2.36)$$

As an example relevant for further analysis, let us choose $O$ to be the $SL(2|1)$ generators (2.7) and realize them as finite dimensional matrices on the spaces $v_1$ and $\bar{v}_1$, Eqs. (2.29) and (2.32), corresponding to the atypical representations $(1)_+$ and $(1)_-$, respectively. It is convenient to choose the basis on $v_1$ as $e_1 = -z_+, e_2 = \theta, e_3 = 1$ with the grading $\bar{1} = 3 = 0$ and $\bar{2} = -1$. The $SL(2|1)$ generators are given by the differential operators (2.3) – (2.5) with $j = -1$ and $\bar{j} = 0$. Then, one applies (2.35) and finds after some algebra

$$(E^{AB})_{kl} = (e^{AB})_{kl} - \delta_{kl} \text{str} e^{AB} = \delta_k^A \delta_l^B - (-1)^{A} \delta^{AB} \delta_{kl}, \quad (2.37)$$

where $(e^{AB})_{kl} = \delta_k^A \delta_l^B$ are the $GL(2|1)$ generators of the fundamental representation. Similar expressions for the generators of the atypical representation $(1)_-$ can be obtained from (2.37) with a help of the automorphism (2.9).

We have demonstrated in this section, that for reducible indecomposable $SL(2|1)$ representations $[j,\bar{j}]$ the generators $G_{j,\bar{j}}$ take a block-diagonal form in an appropriately chosen basis on $V_{j,\bar{j}}$. Obviously, the same property holds true for an arbitrary linear operator $O$ depending on $G_{j,\bar{j}}$. This allows one to rewrite its supertrace over a ‘big’ space, $\text{str}_{V_{j,\bar{j}}} O$, as a sum of supertraces over diagonal blocks corresponding to various irreducible components of $[j,\bar{j}]$. In this way, one finds the following relations:

- From (2.26), for $j = -\frac{1}{2}n + b$ and $\bar{j} = -\frac{1}{2}n - b$,

$$\text{str}_{V_{j,\bar{j}}} O = \text{str}_{V_{-j,-\bar{j}}} O + \text{str}_{v_{n/2}, b} O. \quad (2.38)$$

- From (2.21) and (2.22), for $j \neq 0$ and $\bar{j} \neq 0$, respectively,

$$\text{str}_{V_{j,\bar{j}}} O = \text{str}_{V_{j,\bar{j}}} O - \text{str}_{V_{j+1,\bar{j}}} O, \quad \text{str}_{V_{0,\bar{j}}} O = \text{str}_{V_{\bar{j}}} O - \text{str}_{V_{\bar{j}+1}} O. \quad (2.39)$$

- From (2.30) and (2.31),

$$\text{str}_{V_{-n}} O = \text{str}_{v_{n}} O - \text{str}_{v_{n+1}} O, \quad \text{str}_{V_{-n}} O = \text{str}_{v_{n}} O - \text{str}_{v_{n+1}} O. \quad (2.40)$$
From (2.33),
\[
\text{str}_{v_{n/2,n/2}} O = \text{str}_{v_n} O - \text{str}_{v_{n-1}} O, \\
\text{str}_{\bar{v}_{n/2,n/2}} O = \text{str}_{\bar{v}_n} O - \text{str}_{\bar{v}_{n-1}} O.
\]
(2.41)

Notice the minus sign in the right-hand side of (2.39). It comes about due to the fact that the lowest weights in the space \(V_j\), Eq. (2.18), and the quotient \(V_{j,0}/V_j\), Eq. (2.19), are given by 1 and \(\bar{\theta}\), respectively, and have different Grassmann parity. The minus sign in the right-hand side of (2.40) and (2.41) has the same origin.

Later in the paper we shall heavily use the relations (2.38) – (2.41) with the operator \(O\) coinciding with the monodromy operator for the \(SL(2|1)\) spin chain. In that case, the supertrace of \(O\) over the \(SL(2|1)\) invariant space defines the transfer matrix of the model. Depending on the choice of this (auxiliary) space, one can distinguish six different transfer matrices summarized in Table 1. There, the third column sets up the notation for the transfer matrix and the fourth column specifies the dimension of the corresponding auxiliary space. In what follows, we shall refer to the transfer matrices with a (in)finite-dimensional auxiliary space as (in)finite-dimensional ones.

<table>
<thead>
<tr>
<th>Representation</th>
<th>Vector space</th>
<th>Transfer matrix</th>
<th>Dimension</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>([j, \bar{j}])</td>
<td>(V_{j,\bar{j}})</td>
<td>(T_{j,\bar{j}}(u))</td>
<td>(\infty)</td>
<td>(2.14)</td>
</tr>
<tr>
<td>([j]_+)</td>
<td>(V_j)</td>
<td>(T_j(u))</td>
<td>(\infty)</td>
<td>(2.18)</td>
</tr>
<tr>
<td>([\bar{j}]_-)</td>
<td>(\bar{V}_j)</td>
<td>(\bar{T}_j(u))</td>
<td>(\infty)</td>
<td>(2.23)</td>
</tr>
<tr>
<td>((b,n/2))</td>
<td>(v_{n/2,b})</td>
<td>(t_{n/2,b}(u))</td>
<td>(4n)</td>
<td>(2.25)</td>
</tr>
<tr>
<td>((n)_+)</td>
<td>(v_n)</td>
<td>(t_n(u))</td>
<td>(2n + 1)</td>
<td>(2.29)</td>
</tr>
<tr>
<td>((n)_-)</td>
<td>(\bar{v}_n)</td>
<td>(\bar{t}_n(u))</td>
<td>(2n + 1)</td>
<td>(2.32)</td>
</tr>
</tbody>
</table>

Table 1: Notations for the \(SL(2|1)\) representations and the corresponding transfer matrices used throughout the paper.

Equations (2.38) – (2.41) allow us to establish relations between the transfer matrices listed in Table 1. A remarkable feature of these relations, that we shall explore in Section 4, is that \textit{finite-dimensional} transfer matrices can be expressed as a difference of \textit{infinite-dimensional} ones. This suggests that infinite-dimensional transfer matrices should serve as building blocks in the construction of the Baxter \(Q\)–operator. Indeed, we will show in Section 3 that the \(Q\)–operators can be identified as the \(SL(2|1)\) transfer matrices \(T_{j,\bar{j}}(u)\) for special values of the spins \(j\) and \(\bar{j}\) (see Eqs. (3.24) below).

2.3. Invariant scalar product

Instead of dealing with infinite-dimensional matrices (2.36), it is more advantageous to realize \(O\) as an integral operator on the space of functions \(\Phi(z, \theta, \bar{\theta}) \in V_{j,\bar{j}}\) endowed with an \(SL(2|1)\) invariant scalar product. In what follows we shall assume that the spins \(j\) and \(\bar{j}\) take real values only.
2.3.1. General case

For two arbitrary states belonging to an infinite-dimensional vector space \( \mathcal{V}_{j,j} \) the scalar product is defined as

\[
\langle \Phi_2 | \Phi_1 \rangle_{jj} = \int [DZ]_{jj} (\Phi_2 (z, \theta, \bar{\theta}))^* \Phi_1 (z, \theta, \bar{\theta}) ,
\]

where \( Z = (z, \theta, \bar{\theta}) \) parameterizes the superspace and the integration is performed over complex \( z \) and four “odd” variables

\[
\int [DZ]_{jj} = \frac{j + \bar{j}}{J J} \int_{|z| \leq 1} d^2 z \int d\theta d\theta^* \int d\bar{\theta} d\bar{\theta}^* \mu_{jj} (Z, Z^*).
\]

Here, '*' denotes the complex conjugation which acts on even and odd coordinates according to

\[ Z^* = (z^*, \theta^*, \bar{\theta}^*) , \quad (\theta \bar{\theta})^* = \bar{\theta}^* \theta^* , \]

with \( \theta, \bar{\theta}, \theta^* \) and \( \bar{\theta}^* \) being mutually independent Grassmann variables. The integration measure in (2.43) is given by

\[
\mu_{jj} (Z, Z^*) = \frac{1}{\pi} (1 - z_+ z_+^* - \theta \theta^*)^j (1 - z_- z_-^* - \bar{\theta} \bar{\theta}^*)^j ,
\]

with

\[ z_\pm = z \pm \frac{1}{2} \theta \bar{\theta} , \quad z_\pm^* = z^* \pm \frac{1}{2} \theta^* \bar{\theta}^* . \]

In Eq. (2.43), the integration goes over a unit disk in the complex \( z \)-plane, \( d^2 z = dz dz^* \) and the integration over the Grassmann variables is performed according to

\[
\int d\theta d\theta^* (c_0 + c_1 \theta + c_2 \theta^* + c_3 \theta \theta^*) = c_3
\]

with arbitrary constants \( c_i \). A similar relation holds upon the substitution \( (\theta, \theta^*) \mapsto (\bar{\theta}, \bar{\theta}^*) \).

The \( SL(2|1) \) scalar product (2.42) represents a natural generalization of the \( SL(2) \) scalar product for holomorphic functions \( \phi(z) \)

\[
\langle \phi_2 | \phi_1 \rangle_s = \frac{2s - 1}{\pi} \int_{|z| \leq 1} d^2 z (1 - zz^*)^{2s-2} (\phi_2 (z))^* \phi_1 (z) .
\]

Here the (half)integer positive \( s \) defines the spin of the \( SL(2) \) representation to which the states \( \phi_{1,2}(z) \) belong. We recall that the states \( \Phi(z, \theta, \bar{\theta}) \) can be decomposed over the \( SL(2) \) multiplets carrying the spins \( \ell, \ell + \frac{1}{2} \) and \( \ell + 1 \), Eq. (2.12). Indeed, substituting \( \Phi_{1,2}(Z) \) in (2.42) with its expansion (2.12) in powers of \( \theta \) and \( \bar{\theta} \) and performing the integration over Grassmann variables, one can express the \( SL(2|1) \) scalar product \( \langle \Phi_2 | \Phi_1 \rangle \) as a sum of the \( SL(2) \) scalar products (2.47) between the functions \( \phi(z), \chi(z), \bar{\chi}(z) \) and \( \varphi(z) \). To save space we do not present the explicit expression.

The hermitian conjugation of the \( SL(2|1) \) generators \( G_{jj} \) with respect to the scalar product (2.42) is defined conventionally as

\[
\langle \Phi_2 | G_{jj} | \Phi_1 \rangle = \langle G_{jj}^\dagger | \Phi_2 | \Phi_1 \rangle .
\]
Replacing the generators by their explicit expressions \((2.3) - (2.5)\), one integrates by parts in both sides of \((2.48)\) and finds after some algebra

\[
(L^\pm)^\dagger = -L^\mp, \quad (\bar{V}^\pm)^\dagger = V^\mp, \quad J^\dagger = J, \quad \bar{J}^\dagger = \bar{J}.
\]  

(2.49)

Using these relations, one verifies that the Casimirs \((2.10)\) are hermitian operators, \(C_j^p = C_p\), and the scalar product \((2.42)\) is invariant under (complexified) \(SL(2|1)\) transformations \((2.22)\), \(\delta_G \langle \Phi_2 | \Phi_1 \rangle = 0\).

Using the scalar product \((2.42)\) one can realize an arbitrary \(SL(2|1)\) invariant operator \(O\) as an integral operator on \(\mathcal{V}_{j,\bar{j}}\)

\[
O \cdot \Phi(W) = \int [DZ]_{jj} O(W, Z^*) \Phi(Z)
\]

(2.50)

where \(\Phi(W)\) is an arbitrary test function and the kernel of the operator, \(O(W, Z^*)\), depends on two sets of variables \(W = (w, \vartheta, \bar{\vartheta})\) and \(Z^* = (z^*, \theta^*, \bar{\theta}^*)\). In particular, the unity operator in \(\mathcal{V}_{j,\bar{j}}\) has the following integral representation

\[
1 \cdot \Phi(W) = \int [DZ]_{jj} K_{j\bar{j}}(W, Z^*) \Phi(Z) = \Phi(W),
\]

(2.51)

with the reproducing kernel \(K_{j\bar{j}}(W, Z^*)\) being

\[
K_{j\bar{j}}(W, Z^*) = (1 - w_- z^*_+ - \bar{\vartheta} \bar{\theta}^*)^{-\bar{j}} (1 - w_+ z^*_+ - \vartheta \theta^*)^{-j},
\]

(2.52)

where \(w_\pm = w \pm \frac{1}{2} \bar{\vartheta} \vartheta\) and \(z^*_\pm = z^* \pm \frac{1}{2} \theta^* \bar{\theta}\). To verify \((2.51)\) it suffices to substitute \(\Phi(Z)\) with one of the basis vectors \((2.34)\) and perform the integration.

Let us demonstrate that the scalar product \((2.42)\) is positively definite for \(j, \bar{j} > 0\). An arbitrary state \(\Phi(Z) \in \mathcal{V}_{j\bar{j}}\) can be decomposed over the graded basis \((2.33)\) as

\[
\Phi(z, \theta, \bar{\theta}) = \sum_{n \geq 0} \chi_1(n) \cdot z^+_n + \chi_2(n) \cdot z^-_n + \phi(n) \cdot \theta z^n + \bar{\phi}(n) \cdot \bar{\theta} z^n,
\]

(2.53)

with \(z^\pm = z \pm \frac{1}{2} \bar{\theta} \vartheta\). Calculating the scalar product of the basis vectors, one can express the norm of \(\Phi(z, \theta, \bar{\theta})\) in terms of the expansion coefficients \(\chi_{1,2}, \phi\) and \(\bar{\phi}\). The basis vectors \(z^+_n, \theta z^n\) and \(\bar{\theta} z^n\) diagonalize simultaneously the \(U(1)\) charge \(B = \frac{1}{2} (J - \bar{J})\), Eq. \((2.5)\), and the \(SL(2)\) Cartan operator \(L^0 = \frac{1}{2} (J + \bar{J})\), Eq. \((2.5)\). By virtue of \((2.49)\), the two operators are hermitian with respect to the scalar product \((2.42)\) and, therefore, the basis vectors with different values of the \(U(1)\) charge and the \(SL(2)\) spin are orthogonal to each other. As a result, the norm of the state \((2.53)\) is given by

\[
\langle \Phi | \Phi \rangle_{j\bar{j}} = \sum_{n \geq 0} \sigma(n) \left( \sum_{i, k = 1,2} \chi_i(n) g^{ik}(n) \chi_k(n) \phi(n) \bar{\phi}(n) / j + \bar{\phi}^*(n) \phi(n) / \bar{j} \right),
\]

(2.54)

where the notation was introduced for \(\sigma(n) = n! \Gamma(j + \bar{j} + 1) / \Gamma(n + 1 + j + \bar{j})\) and

\[
g^{ik} = \begin{bmatrix} (j + n) / j & 1 \\ 1 & (\bar{j} + n) / \bar{j} \end{bmatrix}.
\]

(2.55)

This matrix is positively definite for \(j, \bar{j} > 0\).
2.3.2. Reduction to the chiral representation

We demonstrated in Section 2.2.1 that for $\tilde{j} = 0$ the representation space $V_{j,0}$ contains an invariant subspace $V_j$. It is spanned by the states (2.18), which admit the expansion (2.53) with $\chi_2(n) = \tilde{\phi}(n) = 0$. According to (2.54), the states $\Phi_+ \in V_j$ have a finite norm with respect to the scalar product (2.42) as $\tilde{j} \to 0$

Let us apply (2.42) to determine the scalar product on $V_j$. The states $\Phi(z, \theta, \bar{\theta}) \in V_j$ verify the chirality condition $D \bar{\Phi}(z, \theta, \bar{\theta}) = 0$ and, as a consequence, their dependence on $\bar{\theta}$ can be eliminated by a shift in $z$

$$\Phi(z, \theta, \bar{\theta}) = e^{\frac{1}{2} \bar{\theta} \theta} \partial_z \bar{\Phi}(z, \theta),$$  \hspace{1cm} (2.56)

with $\bar{\Phi}(z, \theta) = \Phi(z, \theta, 0)$. Let substitute this relation into (2.42) and examine the integral in the right-hand side of (2.42) in the limit $\tilde{j} \to 0$. One shifts the integration variables as $z \to z - \frac{1}{2} \bar{\theta} \theta$, $z^* \to z^* - \frac{1}{2} \theta^* \bar{\theta}^*$, performs integration over $\bar{\theta}$ and $\theta^*$ and, finally, obtains the scalar product on the space of functions $\bar{\Phi}(z, \theta) \in V_j$ as

$$\langle \bar{\Phi}_2 | \bar{\Phi}_1 \rangle_j = \int [DZ]_j \left( \bar{\Phi}_2(z, \theta) \right)^* \bar{\Phi}_1(z, \theta).$$  \hspace{1cm} (2.57)

Here the integration goes over the unit disk in the complex $z-$plane and two Grassmann variables

$$\int [DZ]_j = \int_{|z| \leq 1} d^2 z \int d\theta d\theta^* \mu_j(Z, Z^*)$$  \hspace{1cm} (2.58)

with $Z = (z, \theta)$, $Z^* = (z^*, \theta^*)$ and the integration measure given by

$$\mu_j(Z, Z^*) = \frac{1}{\pi} (1 - zz^* - \theta \theta^*)^{-j-1}.$$  \hspace{1cm} (2.59)

By construction, the scalar product (2.57) is invariant under the $SL(2|1)$ transformations

$$\delta_G \bar{\Phi}(z, \theta) = \hat{G}_j \cdot \bar{\Phi}(z, \theta),$$  \hspace{1cm} (2.60)

with the operators $\hat{G}_j$ related to the $SL(2|1)$ generators (2.3) - (2.5) as $\hat{G}_j = e^{-\bar{\theta} \theta \partial_z/2} G_{j,0} e^{\bar{\theta} \theta \partial_z/2}$. They are given by differential operators acting on $z$ and $\theta$ variables only.

Making use of (2.57), one defines invariant operators on $V_j$

$$O \cdot \bar{\Phi}(W) = \int [DZ]_j O(W, Z^*) \bar{\Phi}(Z),$$  \hspace{1cm} (2.61)

where the kernel $O(W, Z^*)$ depends on $W = (w, \vartheta)$ and $Z^* = (z^*, \theta^*)$. The unity operator takes the form

$$1 \cdot \bar{\Phi}(W) = \int [DZ]_j K_j(W, Z^*) \bar{\Phi}(Z) = \bar{\Phi}(W),$$  \hspace{1cm} (2.62)

with the reproducing kernel expressed by

$$K_j(W, Z^*) = (1 - wz^* - \vartheta \theta^*)^{-j}.$$  \hspace{1cm} (2.63)

To verify the relation (2.62) one substitutes a test function by its general expression $\bar{\Phi}(W) = \sum_{n \geq 0} c_1 w^n + c_2 \vartheta w^n$ and performs the integration.
The relations \((2.50)\) and \((2.61)\) allow one to manipulate operators acting on infinite-dimensional representation spaces, \(\mathcal{V}_{j,j}\) and \(\mathcal{V}_j\), respectively. For instance, the product of operators on \(\mathcal{V}_{j,j}\) corresponds to the convolution of their integral kernels

\[
O_1 O_2 \cdot \Phi(W) = \int [D\mathcal{Z}_1]_{ij} \int [D\mathcal{Z}_2]_{ij} O_1(W, Z_1^*) O_2(Z_1, Z_2^*) \Phi(Z_2).
\]  

(2.64)

The supertrace \((2.36)\) over \(\mathcal{V}_{j,j}\) then reads

\[
\text{str}_{\mathcal{V}_{j,j}} O = \int [D\mathcal{Z}]_{ij} O(\mathcal{Z}, \mathcal{Z}^*). \]  

(2.65)

Finally, the operator acting on the tensor product \(\mathcal{V}_{j_1,j_1} \otimes \ldots \otimes \mathcal{V}_{j_N,j_N}\) is represented by an integral kernel depending on \(N\) pairs of coordinates \(\{\mathcal{W}, \mathcal{Z}^*\} \equiv \{\mathcal{W}_k, \mathcal{Z}_k^*| 1 \leq k \leq N\}\)

\[
O \cdot \Phi(\mathcal{W}_1, \ldots, \mathcal{W}_N) = \int [D\mathcal{Z}_1]_{j_1,j_1} \ldots \int [D\mathcal{Z}_N]_{j_N,j_N} O(\{\mathcal{W}, \mathcal{Z}^*\}) \Phi(\mathcal{Z}_1, \ldots, \mathcal{Z}_N). \]  

(2.66)

Similar relations also hold for the operators in \(\mathcal{V}_j\). In the next Section, we will apply them to define the Baxter \(Q\)-operator as an integral operator on the quantum space of the model.

3. Baxter \(Q\)-operators as transfer matrices

The construction of noncompact integrable \(SL(2|1)\) spin chains relies on the \(\mathcal{R}\)-operator which depends on a spectral parameter and acts on the tensor product of two infinite-dimensional \(SL(2|1)\) representations as

\[
\mathcal{R}(u) : \mathcal{V}_{j_1,j_1} \otimes \mathcal{V}_{j_2,j_2} \mapsto \mathcal{V}_{j_1,j_1} \otimes \mathcal{V}_{j_2,j_2}.
\]  

(3.1)

In addition, it obeys the Yang-Baxter equations

\[
\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v),
\]

\[
\mathcal{R}_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}_{12}(u-v),
\]  

(3.2)

where in the first relation both sides are defined on \(\mathcal{V}_{j_1,j_1} \otimes \mathcal{V}_{j_2,j_2} \otimes \mathcal{V}_{j_3,j_3}\) and each \(\mathcal{R}_{nm}\)-operator acts on \(n^{th}\) and \(m^{th}\) spaces only, for instance, \(\mathcal{R}_{12}(u) = \mathcal{R}(u) \otimes \mathbb{1}\). In the second relation, \(L_k(u)\) is the \(SL(2|1)\) Lax operator [30] acting on the tensor product \(\mathcal{V}_{j_k,j_k} \otimes \mathcal{V}_1\) with \(\mathcal{V}_1\) being the (fundamental) three-dimensional atypical \(SL(2|1)\) representation, Eq. \((2.29)\).

For compact (typical and atypical) \(SL(2|1)\) representations, solutions to the Yang-Baxter equation are well known [30] [31]. In the context of noncompact spins one has to deal however with solutions to \((3.2)\) for the infinite-dimensional representations \((3.1)\). The latter have been studied in Ref. [25].

3.1. Factorized \(R\)-matrix

The Lax operator \(L_k(u)\) is given by a \(3 \times 3\) graded matrix whose entries are differential operators representing the \(SL(2|1)\) generators on the ‘quantum space’ \(\mathcal{V}_{j_k,j_k}\) (see Eq. \((3.1)\) below). As such, \(L_k(u)\) depends on three parameters – two spins, \(j_k\) and \(\bar{j}_k\), and the spectral parameter \(u\).
Solving the second relation in (3.2), it is convenient to view the Lax operators $L_1(u)$ and $L_2(v)$ as functions of the following combinations of the above parameters

$$
\begin{align*}
    u_1 &= u + j_1, \quad u_2 = u + j_1 - \bar{j}_1, \quad u_3 = u - \bar{j}_1, \\
    v_1 &= v + j_2, \quad v_2 = v + j_2 - \bar{j}_2, \quad v_3 = v - \bar{j}_2,
\end{align*}
$$

so that $L_1 \equiv L_1(u_1, u_2, u_3)$ and $L_2 \equiv L_2(v_1, v_2, v_3)$. Then, the second relation in (3.2) can be rewritten as

$$
\hat{R}_{12}(u - v)L_1(u_1, u_2, u_3)L_2(v_1, v_2, v_3) = L_1(v_1, v_2, v_3)L_2(u_1, u_2, u_3)\hat{R}_{12}(u - v),
$$

(3.4)

where $\hat{R}_{12}(u) = \Pi_{12}\hat{R}_{12}(u)$ and the notation was introduced for the (graded) permutation operator $\Pi_{12}$. For an arbitrary state in the tensor product $V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2}$ it permutes the $Z = (z, \theta, \bar{\theta})$—coordinates in the two spaces according to

$$
\Pi_{12} \cdot \Phi(Z_1, Z_2) = \Phi(Z_2, Z_1).
$$

(3.5)

One notices that in (3.4) the $\hat{R}_{12}$—operator interchanges the arguments of two Lax operators, $(u_1, u_2, u_3) \leftrightarrow (v_1, v_2, v_3)$. This transformation can be split into three steps, first, exchanging $u_3 \leftrightarrow v_3$, then $u_2 \leftrightarrow v_2$ and, finally, $u_1 \leftrightarrow v_1$. Each step is governed by a certain $\mathcal{R}^{(a)}$—operator (with $a = 1, 2, 3$) leading to the following factorized expression for the $\mathcal{R}$—matrix [23].

$$
\mathcal{R}(u - v) = \Pi \mathcal{R}^{(1)}(u_1 - v_1)\mathcal{R}^{(2)}(u_2 - v_2)\mathcal{R}^{(3)}(u_3 - v_3).
$$

(3.6)

The order in which the spectral parameters $u_j$ and $v_j$ are interchanged in (3.4) is not important. This allows one to write down six different expressions for the $\mathcal{R}$—operator containing the product of operators $\mathcal{R}^{(a)}(u_a - u_a)$ but in different order. One can show that these expressions coincide up to an overall normalization factor.

There is the following important difference between the operators $\mathcal{R}(u)$ and $\mathcal{R}^{(a)}(u)$. In distinction with the former, the $\mathcal{R}^{(a)}$—operators map $V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2}$ into the tensor product of two yet another $SL(2|1)$ representations:

$$
\begin{align*}
    \mathcal{R}^{(1)}(u) : & \quad V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2} \mapsto V_{j_1, \bar{j}_1 - u} \otimes V_{j_2, \bar{j}_2 + u} \\
    \mathcal{R}^{(2)}(u) : & \quad V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2} \mapsto V_{j_1 - u, j_1 + u} \otimes V_{j_2 + u, j_2 - u} \\
    \mathcal{R}^{(3)}(u) : & \quad V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2} \mapsto V_{j_1 + u, \bar{j}_1} \otimes V_{j_2 - u, \bar{j}_2}.
\end{align*}
$$

(3.7)

For $u \neq 0$ the operators $\mathcal{R}^{(1)}(u)$ and $\mathcal{R}^{(3)}(u)$ only modify the spins in the antichiral and chiral sectors, respectively, while the operator $\mathcal{R}^{(1)}(u)$ changes the spins in both sectors simultaneously

$$
\mathcal{R}^{(a)}(u) \left( G_{j_1, \bar{j}_1} + G_{j_2, \bar{j}_2} \right) = \left( G_{j_1, \bar{j}_1} + G_{j_2, \bar{j}_2} \right) \mathcal{R}^{(a)}(u),
$$

(3.8)

and the sum of chiral and antichiral spins is separately preserved, $j_1 + j_2 = j'_1 + j'_2$ and $\bar{j}_1 + \bar{j}_2 = \bar{j}'_1 + \bar{j}'_2$. Here $G_{j, \bar{j}}$ denote the $SL(2|1)$ generators, Eqs. (2.3) – (2.5), and the spins $j'_1, j'_2$ and $\bar{j}'_1, \bar{j}'_2$ can be read from the right-hand side of (3.7). Examining the right-hand side of (3.6) on the tensor product $V_{j_1, \bar{j}_1} \otimes V_{j_2, \bar{j}_2}$, one finds from (3.4) that the $\mathcal{R}^{(1)}$— and $\mathcal{R}^{(2)}$—operators act on different tensor products $V_{j'_1, \bar{j}'_1} \otimes V_{j'_2, \bar{j}'_2}$ with spins $j'_1, j'_2$ and $\bar{j}'_1, \bar{j}'_2$ depending on the $u$— and $v$—parameters. One can check using (3.5) that the $\mathcal{R}$—operator (3.6) satisfies (3.1) provided that these parameters verify the relations $(u_2 - u_3) - (v_2 - v_3) = j_1 - j_2$ and $(u_1 - u_2) - (v_1 - v_2) = j_1 - \bar{j}_2$, in agreement with (3.3). The operators $\mathcal{R}^{(a)}(u)$ (with $a = 1, 2, 3$) possess a number of remarkable properties:
Figure 2: Diagrammatical representation of the integral operators $R^{(1)}(u)$ and $R^{(3)}(u)$, Eq. (3.13). The arrow line with the index $(\alpha, \bar{\alpha})$ and the end-points $\mathcal{W}$ and $Z^*$ represents for the kernel $\mathcal{K}_{\alpha, \bar{\alpha}}(\mathcal{W}, Z^*)$.

- For $u = 0$, the $R^{(a)}$–operator does not affect the arguments of the Lax operators and, as a consequence, $R^{(a)}(u = 0)$ is proportional to the identity operator. We choose the normalization as
  \[ R^{(a)}(u = 0) = 1 \quad (a = 1, 2, 3). \]  
  (3.9)

- In virtue of (3.9), for special values of the spins $j_2$ and $\bar{j}_2$, the operator $R(u)$ reduces to a single $R^{(a)}$–operator
  \[
  R(u) = \begin{cases}
  \Pi R^{(3)}(u), & \text{for } (j_2 = j_1 + u, \bar{j}_2 = \bar{j}_1) \\
  \Pi R^{(2)}(-u), & \text{for } (j_2 = j_1 + u, \bar{j}_2 = \bar{j}_1 - u) \\
  \Pi R^{(1)}(u), & \text{for } (j_2 = j_1, \bar{j}_2 = \bar{j}_1 - u)
  \end{cases}
  \]  
  (3.10)

We recall that the $R$–operator acts on the tensor product $\mathcal{V}_{j_1, \bar{j}_1} \otimes \mathcal{V}_{j_2, \bar{j}_2}$, Eq. (3.1), and, in all three cases in (3.10), the space $\mathcal{V}_{j_2, \bar{j}_2}$ depends explicitly on the spectral parameter $u$.

- For $a > b$, the operators $R^{(a)}_{12}(v)$ and $R^{(b)}_{23}(v)$ commute with each other
  \[ R^{(a)}_{12}(u)R^{(b)}_{23}(v) = R^{(b)}_{23}(v)R^{(a)}_{12}(u), \]  
  (3.11)

where the definition of the $R^{(nm)}_{nm}$–operators on $\mathcal{V}_{j_1, \bar{j}_1} \otimes \mathcal{V}_{j_2, \bar{j}_2} \otimes \mathcal{V}_{j_3, \bar{j}_3}$ is analogous to that of the $R^{(nm)}_{nm}$–operators in (3.12).

It is straightforward to verify that the operators entering both sides of (3.11) interchange the argument of three Lax operators $L_1 L_2 L_3$ in the same way and, therefore, they are proportional to each other. Explicit calculations show that the corresponding proportionality factor equals 1 for $a > b$ and it is different from 1 for $a < b$.

Similarly to (2.50), the operators $R^{(a)}(u)$ can be realized on the tensor product of graded linear spaces $\mathcal{V}_{j_1, \bar{j}_1} \otimes \mathcal{V}_{j_2, \bar{j}_2}$ as integral operators
  \[ R^{(a)}(u) \Phi(\mathcal{W}_1, \mathcal{W}_2) = \int [DZ]_{j_1 \bar{j}_1} [DZ]_{j_2 \bar{j}_2} R^{(a)}(u) (\mathcal{W}_1, \mathcal{W}_2; Z^*_1, Z^*_2) \Phi(Z_1, Z_2), \]  
  (3.12)

where the integration measure $\int [DZ]_{j \bar{j}}$ is defined in (2.43) and $\Phi(\mathcal{W}_1, \mathcal{W}_2)$ is a test function on $\mathcal{V}_{j_1, \bar{j}_1} \otimes \mathcal{V}_{j_2, \bar{j}_2}$. As in (2.61), the product of $R^{(a)}$–operators is represented by a convolution of the
corresponding integral kernels. The integral kernels of the operators $R^{(1)}(u)$ and $R^{(3)}(u)$ can be expressed in terms of the $SL(2|1)$ reproducing kernels \[2.52\]

$$
R^{(1)}_u(W_1, W_2; Z_1^*, Z_2^*) = \left. r^{(1)}_u \, \mathcal{K}_{j_1 \bar{j}_1}(W_1, Z_1^*) \mathcal{K}_{0, -u}(W_1, Z_2^*) \mathcal{K}_{j_2 \bar{j}_2 + u}(W_2, Z_2^*) \right.
$$

$$
R^{(3)}_u(W_1, W_2; Z_1^*, Z_2^*) = \left. r^{(3)}_u \, \mathcal{K}_{j_1 + u, \bar{j}_1}(W_1, Z_1^*) \mathcal{K}_{-u, 0}(W_2, Z_2^*) \mathcal{K}_{j_2 \bar{j}_2}(W_2, Z_2^*) \right. \tag{3.13}
$$

The factors $r^{(1)}_u$ and $r^{(3)}_u$ fix the normalization of the operators. Let us choose them as \(^4\)

$$
r^{(1)}_u = e^{-i\pi u/2} \frac{j_2}{j_2 + u} \frac{\Gamma(u + j_2 + j_2 + 1)}{\Gamma(j_2 + j_2 + 1)},
$$

$$
r^{(3)}_u = e^{i\pi u/2} \frac{\Gamma(u + j_1 + \bar{j}_1 + 1)}{\Gamma(j_1 + j_1 + 1)}. \tag{3.14}
$$

It is convenient to introduce a diagrammatic representation of the kernels \[3.13\] in terms of Feynman diagrams. Let us represent the reproducing kernel $K_{j_1 \bar{j}_1}(W, Z^*)$ as an arrow line connecting the points $W$ and $Z^*$ and carrying the pair of indices $j_1, \bar{j}_1$. Then, the kernels of the $R^{(1)}$- and $R^{(3)}$-operators, Eq. \[3.13\], can be depicted as two 'zig-zag' diagrams shown in Fig. 2(a) and (b), respectively.

The remaining operator $R^{(2)}(u)$ can be realized as a differential operator acting on $\mathcal{V}_{j_1 \bar{j}_1} \otimes \mathcal{V}_{j_2 \bar{j}_2}$

$$
R^{(2)}(u) = \frac{j_2}{j_2 + u} \left[ \left( 1 + u \frac{\theta_{12} \bar{D}_2}{j_2} \right) \left( 1 - u \frac{\bar{\theta}_{12} D_1}{j_1} \right) + u \left( z_{1+} - z_{2+} + \theta_1 \bar{\theta}_2 \right) \frac{D_1 \bar{D}_2}{j_1 j_2} \right], \tag{3.15}
$$

where the notation was introduced for $z_{k\pm} = z_k \pm \frac{1}{2} \theta_1 \bar{\theta}_k, \, \theta_{12} = \theta_1 - \theta_2$ and $\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2$. Also, $D_1 = -\partial_{\theta_1} + \frac{1}{2} \theta_1 \partial_{z_1}$ and $\bar{D}_2 = -\partial_{\bar{\theta}_2} + \frac{1}{2} \bar{\theta}_2 \partial_{z_2}$ denote the supercovariant derivatives \[2.17\] acting on the first and second arguments of $\Phi(Z_1, Z_2) \in \mathcal{V}_{j_1 \bar{j}_1} \otimes \mathcal{V}_{j_2 \bar{j}_2}$, respectively. The same operator can be realized as an integral operator \[3.12\] with the kernel

$$
R^{(2)}_u(W_1, W_2; Z_1^*, Z_2^*) = R^{(2)}(u) \cdot \mathcal{K}_{j_1 \bar{j}_1}(W_1, Z_1^*) \mathcal{K}_{j_2 \bar{j}_2}(W_2, Z_2^*), \tag{3.16}
$$

where supercovariant derivatives $\bar{D}_1$ and $D_2$ entering \[3.15\] act on $W_1$ and $W_2$, respectively. Notice that the operator $(u + j_2)R^{(2)}(u)$ is a quadratic function of the spectral parameter $u$ with operator-valued coefficients.

### 3.2. Definition of $Q$–operators

Having the solutions to the Yang-Baxter equation \[3.2\] at our disposal, we can construct the transfer matrix for the $SL(2|1)$ spin chain of length $N$. The Hilbert space in each site is identified with the $SL(2|1)$ representation space $\mathcal{V}_{j_3 \bar{j}_3}$. The quantum space of the model is given by the direct product of the Hilbert spaces over the entire lattice with spins taking the same values in all sites

$$
\mathcal{H}_N = \mathcal{V}_{j_{3\bar{3}}} \otimes \ldots \otimes \mathcal{V}_{j_{3\bar{3}}} . \tag{3.17}
$$

Let $\mathcal{V}_{j_3 \bar{j}_3}$ be some reference $SL(2|1)$ representation space (see Table 1) and let us denote by $R_{n0}(u)$ the $R$–operator acting on the tensor product of a quantum space in the $n^{th}$ site and the auxiliary

\(^4\)We introduced the phases $e^{\pm i\pi u/2}$ to avoid factors $(-1)^u$ in expressions for the transfer matrices (see Eq. \[4.6\] below).
space $V_{jj}$. By definition, the transfer matrix $T_{jj}(u)$ is defined as a supertrace of their product over all sites

$$T_{jj}(u) = \text{str}_{jj} \left[ R_{N0}(u) \ldots R_{10}(u) \right]. \quad (3.18)$$

It follows from the Yang-Baxter equation that the transfer matrices with different values of spins in the auxiliary space form a commutative family of operators

$$[T_{jj}(u), T_{j', j''}(v)] = 0, \quad (3.19)$$

and, therefore, they serve as generating functionals of the Hamiltonian and of (an infinite number of) integrals of motion.

Making use of, we obtain the following identities

$$T_{j_q + u, j_q}(u) = \text{str}_{j_q + u, j_q} \left[ \Pi_{N0} R_{N0}^{(3)}(u) \ldots \Pi_{10} R_{10}^{(3)}(u) \right],$$

$$T_{j_q + u, j_q - u}(u) = \text{str}_{j_q + u, j_q - u} \left[ \Pi_{N0} R_{N0}^{(2)}(-u) \ldots \Pi_{10} R_{10}^{(2)}(-u) \right], \quad (3.20)$$

$$T_{j_q, j_q - u}(u) = \text{str}_{j_q, j_q - u} \left[ \Pi_{N0} R_{N0}^{(1)}(u) \ldots \Pi_{10} R_{10}^{(1)}(u) \right],$$

where $j_q$ and $\bar{j}_q$ are the $SL(2|1)$ spins of the quantum space (3.17) and $\Pi_{k0}$ is the graded permutation operator, Eq. (3.5). For $u = 0$ one applies (3.9) to get

$$T_{j_q, \bar{j}_q}(0) = \mathbb{P}, \quad (3.21)$$

where the notation was introduced for the operator of cyclic permutations on (3.17)

$$\mathbb{P} \Phi(\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_N) = \Phi(\mathbb{Z}_2, \mathbb{Z}_3, \ldots, \mathbb{Z}_1). \quad (3.22)$$

Analogously to the $R$-operator, Eq. (3.6), a general infinite-dimensional transfer matrix (3.18) can be factorized into the product of the three operators (3.21). Calculation goes along the same lines as for the $SL(3)$ spin chain and details can be found in Appendix E. The resulting factorized expression for the transfer matrix reads

$$T_{jj}(w) = \mathbb{P}^{-2} T_{j_q, \bar{j}_q - w_1}(w_1) T_{j_q - w_2, j_q + w_2}(-w_2) T_{j_q + w_3, \bar{j}_q}(w_3), \quad (3.23)$$

with the spectral parameters $w_1 = w - j + j_q$, $w_2 = w - j + \bar{j} + j_q - \bar{j}_q$ and $w_3 = w + \bar{j} - j_q$. In Eq. (3.23), the dependence of the right-hand side on the spins of the auxiliary space resides in the $w-$parameters. This suggests to introduce the following operators

$$Q_3(u) = T_{j_q - \bar{j}_q + u, j_q}(u - \bar{j}_q),$$

$$Q_2(u) = T_{j_q - u, j_q + u}(\bar{j}_q - j_q - u),$$

$$Q_1(u) = T_{j_q, j_q - u}(u + j_q), \quad (3.24)$$

and rewrite the infinite-dimensional transfer matrix (3.23) as

$$T_{jj}(u) = \mathbb{P}^{-2} Q_1(u - j) Q_2(u - j + \bar{j}) Q_3(u + \bar{j}). \quad (3.25)$$

As follows from (3.19), the $Q$-operators defined in this way form a commutative family of operators in the quantum space of the model (3.17). Combining together (3.24) and (3.25) one
Figure 3: Diagrammatical representation of the operator $Q_3(u + \tilde{q})$. The indices specify the values of spins $\alpha_3 = (-u, 0)$ and $\beta_3 = (j_q + u, \tilde{j}_q)$.

finds that the $Q$–operators coincide with the cyclic permutation operator for special values of the spectral parameter

$$Q_1(-j_q) = Q_2(\tilde{j}_q - j_q) = Q_3(\tilde{j}_q) = \mathbb{P}. \quad (3.26)$$

To identify the operators (3.24) as Baxter operators for the $SL(2|1)$ spin chain we have to establish the corresponding TQ-relations. This will be done in Section 5.

According to (3.24), the $Q$–operators are defined as infinite-dimensional transfer matrices (3.20) built from $R^{(a)}$–operators. To determine their explicit form, one has to evaluate super-traces over infinite-dimensional graded spaces in the right-hand side of (3.20). This can be done using the integral representation for the $R^{(a)}$–operators, Eq. (3.12). In this way, the Baxter $Q$–operators can be realized as integral operators (2.66) on the quantum space of the model (3.17)

$$Q_a(u) \cdot \Phi(W_1, \ldots, W_N) = \int [DZ_1]_{j_q \bar{j}_q} \cdots \int [DZ_N]_{j_q \bar{j}_q} Q^{(a)}(\{W, Z^*\}) \Phi(Z_1, \ldots, Z_N), \quad (3.27)$$

with $\{W, Z^*\} \equiv \{W_k, Z^*_k \mid 1 \leq k \leq N\}$. Let us start with the operator $Q_3(u)$ and examine the operator $\Pi_{k0} R^{(3)}_{k0}(u)$ entering the first relation in (3.20)

$$\Pi_{k0} R^{(3)}_{k0}(u) \Phi(W_k, W_0) = \int [DZ_k]_{j_q \bar{j}_q} \int [DZ_0]_{j_0 \bar{j}_0} R^{(3)}(W_0, W_k; Z^*_k, Z^*_0) \Phi(Z_k, Z_0) = r_u^{(3)} \int [DZ_k]_{j_q \bar{j}_q} K_{j_q + u, \bar{j}_q}(W_0, Z^*_k) K_{-u, 0}(W_0, Z^*_k) \Phi(Z_k, W_k), \quad (3.28)$$

where the subscripts ‘0’ and ‘k’ refer to the auxiliary space and to the quantum space in $k^{th}$ site, respectively. Here in the second relation we applied (3.13) and performed $Z_0$–integration with a help of (3.21). Substituting (3.28) into the first relation in (3.20) one obtains the integral kernel of the operator $Q_3(u)

$$Q_3(u + \tilde{q}) := \rho_3(u) \prod_{k=1}^N K_{-u, 0}(W_k, Z^*_k) K_{j_q + u, \bar{j}_q}(W_{k+1}, Z^*_k) \quad (3.29)$$

where $\rho_3(u) = [\text{e}^{i\pi u/2} \Gamma(u + j_q + \tilde{j}_q + 1)/\Gamma(j_q + \tilde{j}_q + 1)]^N$ and periodic boundary conditions are imposed, $W_{N+1} = W_1$. Using the diagrammatical technique introduced earlier, the integral kernel (3.29) can be represented as a zig-zag diagram shown in Fig. 3.
Table 1) and the dilatation operator (1.3) acts on the Hilbert space Eq. (2.13). As a consequence, they form an irreducible chiral (1.2) carry the $V$ with the one-loop dilatation operator of the latter, Eqs. (1.3) and (1.4). The quantum space of the model (3.17) for arbitrary spins $j$ in the context of the $\mathcal{N}=0$, Eq. (2.21). Let us apply the map $\mathcal{V}_{(j_0,0)} \mapsto \mathcal{V}_{(j_q)}$ to construct the $Q$–operators (3.24) in the quantum space $(\mathcal{V}_{(j_q)})^{\otimes N}$. The states $\Phi \in (\mathcal{V}_{(j_0,0)})^{\otimes N}$ are projected onto $\tilde{\Phi} \in (\mathcal{V}_{(j_q)})^{\otimes N}$ as

$$\Phi(Z_1, \ldots, Z_N) = \prod_{k=1}^{N} e^{\frac{i}{2} \bar{\theta}_k \theta_k \partial_{\bar{z}_k}} \tilde{\Phi}(Z_1, \ldots, Z_N),$$

with $\mathcal{Q}_1(u - j_q) := \rho_1(u) \int \prod_{k=1}^{N} [\mathcal{D}Y_k]_{j_q,j_q-u} \mathcal{K}_j \mathcal{K}_{j_q} (\mathcal{W}_{k+1}, Y_k^*) \mathcal{K}_{j_q,j_q} (\mathcal{Y}_k, Z_k^*) \mathcal{K}_{0,-u} (\mathcal{Y}_{k+1}, Y_k^*), \quad (3.30)$

with $\mathcal{Y}_{N+1} = \mathcal{Y}_1$ and $\rho_1(u) = [e^{-i\pi u/2} (\bar{j}_q - u) \Gamma(j_q + \bar{j}_q + 1)/\Gamma(j_q + \bar{j}_q + 1 - u)]^N$. We do not display the explicit expression for the integral operators $\mathcal{Q}_2(u)$ since it will not be important for our purposes.

### 3.3. $Q$–operator for $\mathcal{N}=1$ super-Yang-Mills theory

As was explained in the Introduction, infinite-dimensional $SL(2|1)$ spin chains naturally appear in the context of the $\mathcal{N}=1$ super-Yang-Mills theory – the Hamiltonian of the former coincides with the one-loop dilatation operator of the latter, Eqs. (1.3) and (1.4). The $\mathcal{N}=1$ superfields (1.2) carry the $SL(2)$ spin $\ell = 1$ and the $U(1)$ charge $b = 1$, or equivalently $j = 2$ and $\bar{j} = 0$, Eq. (2.13). As a consequence, they form an irreducible chiral $SL(2|1)$ representation $[2]_+$ (see Table I) and the dilatation operator (1.3) acts on the Hilbert space

$$\mathcal{H}^{(\mathcal{N}=1)} = \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_2.$$
or equivalently \( \hat{\Phi}(Z_1, \ldots, Z_N) = \Phi(Z_1, \ldots, Z_N)|_{\theta_1 = \cdots = \theta_N = 0} \). Here, \( Z_k = (z_k, \theta_k, \bar{\theta}_k) \) and \( Z_k = (z_k, \theta_k) \) (with \( k = 1, \ldots, N \)). Going over to the Baxter \( \hat{Q} \)-operators one finds

\[
Q_a(u)\hat{\Phi}(Z_1, \ldots, Z_N) = Q_a(u)\Phi(Z_1, \ldots, Z_N)|_{\theta_1 = \cdots = \theta_N = 0},
\]

(3.33)

where \( Q_a(u) \) is given by (3.24) and \( Q_a(u) \) denotes the Baxter operator on \((\mathbb{V}_{j_q})^\otimes N\). In the right-hand side of this relation, one applies (3.32), replaces \( Q_a(u) \) by its integral representation (3.27), performs the integration over \( \int \tilde{\theta}_k d\tilde{\theta}_k^* \) (with \( k = 1, \ldots, N \)) and obtains after some algebra

\[
Q_a(u)\Phi(W_1, \ldots, W_N) = \int [DZ_1]_{j_q} \ldots \int [DZ_N]_{j_q} \hat{Q}_u^{(3)}(\{W, Z^*\}) \Phi(Z_1, \ldots, Z_N).
\]

(3.34)

Here, the integration measure \( \int [DZ]_{j_q} \) was defined in Eqs. (2.58) and (2.59). The kernel \( \hat{Q}_u^{(3)}(\{W, Z^*\}) \) depends on the variables \( W_k = (w_k, \bar{\theta}_k) \) and \( Z_k^* = (z_k^*, \theta_k^*) \) (with \( k = 1, \ldots, N \)).

Equation (3.34) defines the \( Q \)-operators acting on the quantum space \( \mathbb{V}_{j_q} \otimes \cdots \otimes \mathbb{V}_{j_q} \). Let us apply it to construct the operator \( Q_3(u) \). One substitutes (3.29) into the right-hand side of (3.33) and notices that for \( j_q \to 0 \) the normalization factor \( \hat{Q}_u^{(3)} \) in (3.33) scales as \( 1/j_q \). Carefully examining the right-hand side of (3.33) one finds that \( \hat{Q}_u^{(3)}(\{W, Z^*\}) \) approaches a finite value as \( j_q \to 0 \)

\[
\frac{\hat{Q}_u^{(3)}(\{W, Z^*\})}{\rho(u)} = \rho(u) \frac{N}{\prod_{k=1}^N (1 - w_k z_k^* - \partial_k \theta_k^*)^u (1 - w_{k+1} z_k^* - \partial_{k+1} \theta_k^*)^{-j_q - u}}
\]

(3.35)

with \( \rho(u) = \left[ e^{i\pi u / 2} \Gamma(u + j_q + 1)/\Gamma(j_q + 1) \right]^N \). Here, \( W_k = (w_k, \bar{\theta}_k) \) and \( Z_k^* = (z_k^*, \theta_k^*) \) and periodic boundary conditions \( W_{N+1} = W_1 \) are imposed. It convenient to remove the factor \( \rho(u) \) from the right-hand side of (3.33) by changing the normalization of the operator \( Q_3(u) \to Q_3(u)/\rho(u) \). The resulting expression for the integral kernel can be then rewritten in terms of the reproducing kernels (2.63) as

\[
Q_3(u)|_{(\mathbb{V}_{j_q})^\otimes N} := \prod_{k=1}^N K_{-u}(W_k, Z_k^*) K_{u+j_q}(W_{k+1}, Z_k^*).
\]

(3.36)

Analogously to (3.28), this expression admits a diagrammatic representation as a zig-zag diagram shown in Fig. 3.

The operator \( Q_3(u) \) admits yet another integral representation which is extremely useful for comparison with the \( N = 1 \) dilatation operator (1.3) and (1.4). It is based on the following identity

\[
\frac{1}{A^a B^b} = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \int_0^1 d\alpha \frac{\alpha^{a-1}(1 - \alpha)^{b-1}}{[\alpha A + (1 - \alpha) B]^{a + b}}.
\]

(3.37)

Applying this transformation to (3.35), one can combine two \( Z_k \)-dependent ‘propagators’ in the right-hand side of (3.35) into a single factor \( (1 - w_k' z_k^* - \partial_k' \theta_k^*)^{-j_q} \) (with \( w_k' = w_k \alpha + w_{k+1} (1 - \alpha) \) and \( \theta_k' = \theta_k \alpha + \theta_{k+1} (1 - \alpha) \)) which coincides in its turn with the reproducing kernel \( K_{j_q}(\alpha W_k + (1 - \alpha) W_{k+1}, Z_k^*) \), Eq. (2.63). Then, the subsequent \( Z_k \)-integration is trivial making use of the property of the reproducing kernel (2.62). Finally, we obtain a multiple contour integral representation for the Baxter operator

\[
Q_3(u)\Phi(W_1, \ldots, W_N) = \left[ \frac{\Gamma(j_q)}{\Gamma(-u) \Gamma(u + j_q)} \right]^N \int_0^1 \prod_{k=1}^N d\alpha_k \alpha_k^{-u-1}(1 - \alpha_k)^{j_q + u-1} \times \Phi(\alpha_1 W_1 + (1 - \alpha_1) W_2, \ldots, \alpha_N W_N + (1 - \alpha_N) W_1),
\]

(3.38)
where \( W_k = (w_k, \vartheta_k) \) and the notation was introduced for a linear combination of (super-)coordinates \( \alpha W + \beta W' = (\alpha w + \beta w', \alpha \vartheta + \beta \vartheta') \).

Let us substitute \( \vartheta_1 = \ldots = \vartheta_N = 0 \) in both sides of (3.38). In this limit, the superfield \( \Phi(W) \), Eq. (1.12), reduces to its lowest component \( \chi(w_k) \) which, in its turn, belongs to the \( SL(2) \otimes U(1) \) multiplet \( D_{\ell}(b) \) with \( \ell = b = j_q/2, \) Eq. (2.12). Then, the \( Q \)-operator in (3.38) acts on the tensor product \( (D_{\ell}(b))^{\otimes N} \) and coincides with the known expression for the \( SL(2) \) Baxter operator \( Q_+(u) \) [IS 32]

\[
[Q_3(u) \Phi(W_1, \ldots, W_N)] \big|_{\vartheta_k = 0} = Q_+^{SL(2)}(u + \ell) [\Phi(W_1, \ldots, W_N)] \big|_{\vartheta_k = 0}. \tag{3.39}
\]

In other words, the operator (3.38) can be considered as a lift of the \( SL(2) \) Baxter \( Q \)-operator from the light-cone into the superspace \( W = (w, \vartheta) \). The relation (3.39) also suggests that for eigenstates of the \( SL(2|1) \) spin chain independent of ‘odd’ \( \vartheta \)-variables, the corresponding eigenvalues of the \( Q \)-operator coincide with eigenvalues of the \( SL(2) \) Baxter \( Q \)-operator. We will return to this issue in Section 6.3.

Let us establish the relation between the \( Q \)-operator (3.38) and the dilatation operator (1.3) and (1.4). As a first step, one examines (3.38) for \( u \to 0 \). In this limit, the leading contribution to the right-hand side of (3.38) comes from the integration in the vicinity of \( \alpha_k = 0 \). Expanding the integrand in powers of \( \alpha_k \) and performing the integration one gets

\[
Q_3(u) \Phi(W_1, \ldots, W_N) = \mathbb{P} \left[ 1 + u \mathbb{H}_N^+ + \mathcal{O}(u^2) \right] \Phi(W_1, \ldots, W_N), \tag{3.40}
\]

where \( Q_3(u = 0) = \mathbb{P} \) is the operator of cyclic permutations [3222], in agreement with the normalization condition (3.26) (for \( j_q = 0 \)). The operator \( \mathbb{H}_N^+ \) has a structure of a nearest-neighbor Hamiltonian, \( \mathbb{H}_N^+ = H_{12}^+ + \ldots + H_{N1}^+ \), with the two-particle kernel \( H_{12}^+ \) acting locally in \( k \)th and \((k + 1)\)th sites as

\[
H_{k,k+1}^+ \Phi(\ldots, Z_k, Z_{k+1}, \ldots) = \int_0^1 \frac{d\alpha}{\alpha} (1 - \alpha)^{j_q - 1} \\
\times \left\{ \Phi(\ldots, Z_k, Z_{k+1}, \ldots) - \Phi(\ldots, Z_k, \alpha Z_k + (1 - \alpha)Z_{k+1}, \ldots) \right\}. \tag{3.41}
\]

Next, we examine the expansion of \( Q_3(u) \) around \( u = -j_q \), or equivalently \( Q_3(-u - j_q) \) for \( u \to 0 \). In this case, the leading contribution to the right-hand side of (3.38) comes from the integration in the vicinity of \( \alpha_k = 1 \) and one gets

\[
Q_3(u - j_q) \Phi(W_1, \ldots, W_N) = \left[ 1 - u \mathbb{H}_N^- + \mathcal{O}(u^2) \right] \Phi(W_1, \ldots, W_N), \tag{3.42}
\]

where \( \mathbb{H}_N^- = H_{12}^- + \ldots + H_{N1}^- \) and

\[
H_{k,k+1}^- \Phi(\ldots, Z_k, Z_{k+1}, \ldots) = \int_0^1 \frac{d\alpha}{\alpha} (1 - \alpha)^{j_q - 1} \\
\times \left\{ \Phi(\ldots, Z_k, Z_{k+1}, \ldots) - \Phi(\ldots, (1 - \alpha)Z_k + \alpha Z_{k+1}, Z_{k+1}, \ldots) \right\}. \tag{3.43}
\]

We observe that the Hamiltonians \( H_{k,k+1}^+ \) and \( H_{k,k+1}^- \) are not invariant under the permutations of \( k \)th and \((k + 1)\)th particles while their sum is.
The Hamiltonian of the $SL(2|1)$ spin chain is defined as

$$H_{(V_{jq})^\otimes N} = H_N^+ + H_N^- = H_{12} + \ldots + H_{N1},$$  \hspace{1cm} (3.44)

with $H_{k,k+1} = H_{k,k+1}^+ + H_{k,k+1}^-$. In (3.44), the subscript in the left-hand side indicates that the quantum space of the model. Making use of (3.40) and (3.42), the Hamiltonian and the operator of cyclic permutations can be expressed as

$$H = (\ln Q_3(0))' - (\ln Q_3(-j_q))',$$

$$P = Q_3(0)/Q_3(-j_q),$$  \hspace{1cm} (3.45)

where prime in the first relation denotes a derivative with respect to the spectral parameter. Written in this form, the two operators do not depend on the normalization of the $Q-$operators.

The first relation in (3.45) has a striking similarity with a similar relation for the $SL(2)$ spin chain, Eq. (1.9). Moreover, it follows from (3.39) that the $SL(2|1)$ Hamiltonian coincides with the $SL(2)$ Hamiltonian (1.9) when projected onto eigenstates independent of ‘odd’ $\vartheta-$variables.

Comparing the $SL(2|1)$ Hamiltonian, Eqs. (3.44), (3.41) and (3.43), with the $N=1$ dilatation operator, Eqs. (1.3) and (1.4), one concludes that the two operators coincide for $j_q = 3 - N = 2$ up to an additive c-number correction. There is however the following difference between the two models. The dilatation operator acts on the single-trace operators (1.1) which are invariant under cyclic permutation of the superfields. This leads to the additional constraint $P = 1$ in the $N = 1$ theory.

### 3.4. Analytical properties of the $Q-$operators

The $Q-$operators satisfy finite-difference TQ-equations whose explicit form will be established in Section 5. To determine uniquely their solutions, one has to specify analytical properties of the $Q-$operators as functions of the spectral parameter $u$. According (3.24), the $u-$dependence enters through the arguments of the transfer matrices and spins of the auxiliary space. In this subsection, we will use (3.24) to determine analytical properties of the operators $Q_a(u)$.

Let us start with the operator $Q_2(u)$. It follows from (3.24) and (3.20) that it is given by

$$Q_2(u - j_q + \bar{j}_q) = \text{str}_{V_{jq-u,jq+u}} \left[ \Pi_{N0} R_{00}^{(2)}(u) \ldots \Pi_{10} R_{10}^{(2)}(u) \right].$$  \hspace{1cm} (3.46)

Here the $R^{(2)}-$operators in all sites act on the tensor product $V_{jq} \otimes V_{jq-u,jq+u}$ and are given by the differential operators (3.15) with $j_1 = j_q$, $\bar{j}_1 = \bar{j}_q$, $j_2 = j_q - u$ and $\bar{j}_2 = \bar{j}_q + u$. It is easy to see that for these values of spins the operator $R^{(2)}(u)$ is a quadratic function of $u$ with operator-valued coefficients. In analogy with (2.35), the operator $\Pi_{10} R_{k0}^{(2)}(u)$ entering (3.46) can be represented in the linear auxiliary space $V_{jq-u,jq+u}$ by (finite-dimensional) matrices whose entries are at most quadratic in $u$. Their explicit form can be found in Appendix C. Multiplying these matrices and taking their supertrace afterwards one obtains from (3.46) that, in general, the operator $Q_2(u - j_q + \bar{j}_q)$ is a polynomial in $u$ of degree $2N$.

---

\footnote{We will demonstrate in Section 6.2.3, that this Hamiltonian naturally arises as the first term in the expansion of the chiral transfer matrix $\mathbb{T}_{j_q}(u)$ (see Table 1) around $u = 0$.}
The analysis of the operators $Q_1(u)$ and $Q_3(u)$ goes along the same lines. From (3.24) and (3.26) one gets

$$Q_3(u + j_q) = \text{str}_{V_{j_q+j_q}} \left[ \Pi_{N_0} R^{(3)}_{N_0}(u) \ldots \Pi_{10} R^{(3)}_{10}(u) \right],$$

$$Q_1(u - j_q) = \text{str}_{V_{j_q-j_q-u}} \left[ \Pi_{N_0} R^{(1)}_{N_0}(u) \ldots \Pi_{10} R^{(1)}_{10}(u) \right],$$

where the $R^{(3)}$- and $R^{(1)}$- operators are given by (3.12) and (3.13) with $j_1 = j_q$, $\tilde{j}_1 = \tilde{j}_q$ and spins in the auxiliary space equal, correspondingly, to $j_2 = j_q + u$, $\tilde{j}_2 = \tilde{j}_q$ and $j_2 = \tilde{j}_q$, $\tilde{j}_2 = j_q - u$. As before, one examines analytical properties of matrices representing the $R^{(1)}$- and $R^{(3)}$- operators on the tensor product $V_{j_q,j_q} \otimes V_{j_2,j_2}$ with the spins $j_2$ and $\tilde{j}_2$ specified above. The explicit form of these matrices can be found in Appendix C. One finds that matrix elements of the operator $R^{(1)}(u)$ are entire functions of $u$. For the operator $R^{(3)}(u)$ its matrix elements are meromorphic functions of $u$ which admit the following representation

$$[R^{(3)}(u)]_{ik} = e^{i\pi u/2} \Gamma(u + j_q + \tilde{j}_q + 1) p_{ik}(u),$$

with $p_{ik}(u)$ being polynomial in $u$. This suggests that the operator $Q_1(u)$ (or, more precisely, its eigenvalues) should be entire functions of $u$ while the operator $Q_3(u)/[e^{i\pi u/2} \Gamma(u + j_q + 1)]^N$ should be polynomial in $u$. A delicate point however is that the supertrace in the right-hand side of (3.47) is given by an infinite sum over matrix elements of the $R^{(a)}$-operators and it is not obvious that analytical properties of the two are the same. This can be checked by applying the integral operators, $Q_1(u)$ and $Q_3(u)$, Eqs. (3.34), (3.29) and (3.30), to an arbitrary test function and examining analytical properties of the resulting Feynman integrals.

4. Factorized transfer matrices

We demonstrated in the previous section that the transfer matrix evaluated over the auxiliary space $V_{j\bar{j}}$ is factorized into a product of three mutually commuting $Q-$operators

$$T_{j\bar{j}}(u) = \text{str}_{V_{j\bar{j}}} [R_{N_0}(u) \ldots R_{10}(u)] = \mathbb{P}^{-2} Q_1(u - j) Q_2(u - j + \tilde{j}) Q_3(u + \tilde{j}).$$

Here the operators $T_{j\bar{j}}(u)$ and $Q_a(u)$ act on the quantum space of the model (3.17) while the operator $R_{k0}(u)$ acts on the tensor product of the quantum space in $k$th site and the auxiliary space. Notice that the dependence of the transfer matrix $T_{j\bar{j}}(u)$ on the spins of the auxiliary space, $j$ and $\tilde{j}$, resides in the arguments of the $Q-$operators only.

The relation (4.1) holds true for arbitrary values of the spins $j$ and $\tilde{j}$, that is, for generic infinite-dimensional $SL(2|1)$ representation $[j, \tilde{j}]$ (see Table 1). We have seen in Section 2.2, that for certain values of the spin, the representation $[j, \tilde{j}]$ becomes reducible and it can be decomposed into a (semidirect) sum of irreducible components. Whenever the representation $[j, \tilde{j}]$ becomes reducible, the corresponding representation space can be decomposed as $V_{j\bar{j}} = V^+ \oplus V^-$. Then, the $R-$operator acting on the tensor product $V_{jq\bar{j}} \otimes V_{j\bar{j}}$ has a block triangular form according to the pattern of decomposition of space $V_{j\tilde{j}}$ shown schematically in Fig. 1

$$R_{V_{jq\bar{j}} \otimes V_{j\bar{j}}}(u) = \begin{pmatrix} R^+(u) & * \\ 0 & R^-(u) \end{pmatrix},$$

(4.2)
where ‘*’ denotes an off-diagonal operator whose explicit form is not relevant for our purposes. Here $\mathbb{R}^+(u)$ defines the $\mathcal{R}$-operator on the invariant subspace $\mathcal{V}_{j_0 j_q} \otimes \mathbb{V}^+$, while $\mathbb{R}^-(u)$ represents the same operator on the quotient space $\mathcal{V}_{j_0 j_q} \otimes \mathbb{V}^-$.

By definition, the $\mathcal{R}$-operator satisfies the Yang-Baxter equation (3.2). Substituting $\mathcal{R}_{ik}(u)$ in (3.2) with (4.1) one finds that the operators $\mathbb{R}^+(u)$ and $\mathbb{R}^-(u)$ also satisfy the same equation. Moreover, evaluating the transfer matrix (4.1) with the $\mathcal{R}$-operators given by (4.2), one concludes that it is given by a sum of two (mutually commuting) transfer matrices $T^\pm(u) = \text{str}_{\mathbb{V}^\pm} [\mathbb{R}^\pm_{00}(u) \ldots \mathbb{R}^\pm_{n0}(u)]$ evaluated over ‘smaller’ auxiliary spaces $\mathbb{V}^\pm$.

To make this relation more precise, let us consider the values of the spins $j$ and $\bar{j}$ for which the representation $[j, \bar{j}]$ becomes reducible.

### 4.1. Finite-dimensional transfer matrices

We have shown in Section 2.2.2 that for $j = -n/2 + b$ and $\bar{j} = -n/2 - b$ (with $n$ positive integer) the representation $[j, \bar{j}]$ decomposes as (2.26) so that the space $\mathcal{V}_{j \bar{j}}$ has a finite-dimensional invariant subspace $v_{n/2,b}$, Eq. (2.25). Then, the $\mathcal{R}$-matrix has the form (4.2) with $\mathbb{R}^+(u)$ defined on the tensor product of the quantum space $\mathcal{V}_{j_0 j_q}$ and typical $SL(2|1)$ representation space $v_{n/2,b}$. The corresponding transfer matrix $T^+(u)$ is just the transfer matrix over the finite-dimensional auxiliary space $v_{n/2,b}$

$$t_{n/2,b}(u) = \text{str}_{v_{n/2,b}} \left[ \mathbb{R}_{00}, \ldots, \mathbb{R}_{10}(u) \right], \quad \mathbb{R}^+(u) = \mathcal{R}_{j_0 \bar{j} 0} \otimes v_{n/2,b}(u).$$  \hspace{1cm} (4.4)

The operator $\mathbb{R}^-(u)$ defines the $\mathcal{R}$-operator on the tensor product $\mathcal{V}_{j_0 \bar{j} q} \otimes \mathcal{V}_{-j, -\bar{j}}$. More precisely, it verifies the same Yang-Baxter equation (3.2) as the operator $\mathcal{R}_{j_0 \bar{j} \bar{j}} \otimes \mathcal{V}_{-j, -\bar{j}}(u)$ and, therefore, the two operators coincide modulo an overall normalization factor

$$\mathbb{R}^-(u) = c(u) \mathcal{R}_{j_0 \bar{j} \bar{j}} \otimes \mathcal{V}_{-j, -\bar{j}}(u).$$  \hspace{1cm} (4.5)

Notice that under our definition of the $\mathcal{R}$-operators, Eqs. (3.6) and (3.9), the normalization of operators entering this relation is uniquely fixed. To determine $c(u)$ it is sufficient to apply both sides of (4.5) to some reference state belonging to the quotient $\mathcal{V}_{j \bar{j}} / v_{n/2,b}$. Explicit calculations show that $c(u) = 1$ (see Appendix B for details). By definition (4.3), the operator $T^-(u)$ is the transfer matrix built from the operators (4.5). According to (4.1), it equals $T_{-j, -\bar{j}}(u)$ and one finds from (4.3)

$$T_{j\bar{j}}(u) = t_{n/2,b}(u) + T_{-j, -\bar{j}}(u).$$  \hspace{1cm} (4.6)

Then, one replaces the $\mathcal{T}$-operators by their expression (4.1) in terms of the $Q$-operators and arrives at the following relation

$$t_{n/2,b}(u + b) = P^{-2} Q_2(u - b) [Q_1(u + n/2)Q_3(u - n/2) - Q_1(u - n/2)Q_3(u + n/2)].$$  \hspace{1cm} (4.7)

We notice that the dependence of the transfer matrix on $b$ resides in the first factor only \(^7\) and, therefore,

$$\frac{t_{n/2,b}(u + b)}{t_{n/2,b}(u + b')} = \frac{Q_2(u - b)}{Q_2(u - b')}$$  \hspace{1cm} (4.8)

\(^6\)Additional simplifications of the $\mathcal{R}$-operator occur when the $SL(2|1)$ representation in the quantum space $[j_q, \bar{j}_q]$ is reducible.

\(^7\)We recall that the $Q$-operators only depend on spins in the quantum space, $j_q$ and $\bar{j}_q$, and the spectral parameter.
for arbitrary $b$ and $b'$. Choosing $b' = u + j_q - \bar{j}_q$ in this relation and taking into account the relation \(3.26\), one finds
\[
t_{n/2b}(u + b) = \mathbb{P}^{-1} \mathcal{Q}_2(u - b)t_{n/2u + j_q - \bar{j}_q}(2u + j_q - \bar{j}_q).
\]
(4.9)

It follows from this relation that the operator $\mathcal{Q}_2(u)$ is given by a ratio of two finite-dimensional transfer matrices with the auxiliary spaces carrying the same $SL(2)$ spin $n/2$ and different values of the $U(1)$ charge. By construction, the operator $\mathcal{Q}_2(u)$ is a polynomial in $u$ with the operator valued coefficients (see Section 3.4). We will demonstrate in Section 6 that up to an overall $c$-valued normalization factor the finite-dimensional transfer matrices $t_{n/2b}(u)$ enjoy the same property for arbitrary $b$. Going over to eigenvalues in both sides of \((4.9)\), one observes that $\mathcal{Q}_2(u)$ divides $t_{n/2b}(u + 2b)$, that is, all roots of the former are also roots of the latter.

For $b = \pm n/2$, the expression for the transfer matrix $t_{n/2,b}$ can be simplified further. The reason for this is that the typical representation $(b, n/2)$ becomes reducible and it decomposes into a semidirect sum of two atypical representations \((2.33)\). Together with \((2.41)\) this suggests that for $b = \pm n/2$ the finite-dimensional transfer matrix $t_{n/2, \pm n/2}(u)$ is given by a sum of two atypical transfer matrices. The explicit expressions will be given below (see Eqs. \((4.18)\) and \((4.19)\)).

### 4.2. Infinite-dimensional (anti)chiral transfer matrices

For $\bar{j} = 0$ the representation $[j, 0]$ decomposes into a semidirect sum of two (infinite-dimensional) chiral representations of $[j]_+$ and $[j + 1]_+$, Eq. \((2.21)\). As before, the $\mathcal{R}$–operator on the tensor product $\mathcal{V}_{j_q} \otimes \mathcal{V}_{j_0}$ has a triangular form \((4.2)\). The operators $\mathcal{R}^+(u)$ and $\mathcal{R}^-(u)$ are related to the $\mathcal{R}$–operator on the tensor products $\mathcal{V}_{j_q} \otimes \mathcal{V}_j$ and $\mathcal{V}_{j_q} \otimes \mathcal{V}_{j+1}$, respectively, as
\[
\mathcal{R}^+(u) = \mathcal{R}_{j_q \otimes j}(u), \quad \mathcal{R}^-(u) = \alpha(u - j)\mathcal{R}_{j_q \otimes j+1}(u).
\]
(4.10)

The calculation of the normalization factor $\alpha(u)$ goes along the same lines as in \((4.7)\). One applies the second relation to the same reference state belonging to the quotient $\mathcal{V}_{j_0}/\mathcal{V}_j$ and matches its both sides. In this way one obtains (see Appendix B for details)
\[
\alpha(u) = i\frac{(u + j_q)(u - \bar{j}_q)}{u + j_q - \bar{j}_q - 1}.
\]
(4.11)

Let us introduce a notation for the transfer matrix evaluated over the chiral representation space
\[
\mathcal{T}_j(u) = \text{str}_{\mathcal{V}_j}[\mathbb{R}_{0\mathcal{V}_j}(u) ... \mathbb{R}_{1\mathcal{V}_j}(u)], \quad \mathbb{R}(u) = \mathcal{R}_{j_q \otimes j}(u).
\]
(4.12)

Then, it follows from \((4.3)\) and \((4.1)\) that
\[
\mathcal{T}_{j,0}(u) = \mathcal{T}_j(u) - \alpha(u - j)\mathcal{T}_{j+1}(u) = \mathbb{P}^{-2} \mathcal{Q}_1(u - j)\mathcal{Q}_2(u - j)\mathcal{Q}_3(u), \quad \text{where the additional factor } "(-1)" \text{ in front of the second term has the same origin as in } (2.39).
\]
(4.13)

It is straightforward to generalize this consideration to the anti-chiral transfer matrices $\bar{\mathcal{T}}_j(u)$. In this case, for $j = 0$ and $\bar{j} \neq 0$, the representation $[0, \bar{j}]$ decomposes as in \((2.22)\) and the relation between the transfer matrices $\mathcal{T}_{0j}(u)$ and $\bar{\mathcal{T}}_j(u)$ reads
\[
\mathcal{T}_{0j}(u) = \bar{\mathcal{T}}_j(u) - \alpha(u + \bar{j} + 1)\mathcal{T}_{j+1}(u) = \mathbb{P}^{-2} \mathcal{Q}_1(u)\mathcal{Q}_2(u + \bar{j})\mathcal{Q}_3(u + \bar{j}), \quad \text{with the normalization factor } \alpha(u) \text{ defined in } (4.11).
\]
(4.14)
4.3. Finite-dimensional (anti)chiral transfer matrices

For $j = -n$ (with $n$ positive integer), the chiral representation $[j]_+$ decomposes into a semidirect sum of the atypical representation $(n)_+$ and the antichiral representation $(n+1)_-$. As in (4.2), the $R$–operator on the tensor product $V_{j\bar{j}q} \otimes V_{-n}$ has a block-triangular form with the upper diagonal block given by $R^{V_{j\bar{j}q} \otimes V_{n}(u)}$ and the lower diagonal block proportional to $R^{V_{j\bar{j}q} \otimes \bar{V}_{n+1}(u)}$. The $R$–operator on $V_{j\bar{j}q} \otimes \bar{V}_{-n}$ admits a similar representation. As a result, for the chiral and anti-chiral transfer matrices one finds (see Appendix B for details)

$$T_{-n}(u) = t_n(u) - [\alpha(u + n + 1)]^{-N} T_{n+1}(u),$$

$$\bar{T}_{-n}(u) = \bar{t}_n(u) - [\alpha(u - n)]^N \bar{T}_{n+1}(u).$$

(4.15)

Here, the notation was introduced for the finite-dimensional atypical transfer matrix (see Table II)

$$t_n(u) = \text{str}_{\bar{v}_n} \left[ R_{N0}(u) \ldots R_{10}(u) \right], \quad \bar{R}(u) = R^V_{j\bar{j}q} \otimes V_n(u),$$

and the transfer matrix $\bar{t}_n(u)$ is defined similarly, with $v_n$ replaced by $\bar{v}_n$.

Let us now establish a relation between finite-dimensional atypical transfer matrices, $t_n(u)$ and $\bar{t}_n(u)$, and the $Q$–operators. For $b = n/2$, or equivalently $j = 0$ and $\bar{j} = -n$, one obtains from (4.16)

$$t_{n/2,n/2}(u) = T_{0,-n}(u) - T_{n,0}(u).$$

(4.17)

One applies (4.13) and (4.14), takes into account (4.15) and obtains two different representations for $t_{n/2,n/2}(u)$ (with $n \geq 1$)

$$t_{n/2,n/2}(u) = \bar{t}_n(u) - [\alpha(u - n + 1)]^{-N} \bar{t}_{n-1}(u)$$

$$= \mathbb{P}^{-2} Q_2(u - n) [Q_1(u) Q_3(u - n) - Q_1(u - n) Q_3(u)].$$

(4.18)

In a similar manner, for $b = -n/2$, or equivalently $j = -n$ and $\bar{j} = 0$ one gets

$$t_{n/2,-n/2}(u) = t_n(u) - [\alpha(u + n)]^N t_{n-1}(u)$$

$$= \mathbb{P}^{-2} Q_2(u + n) [Q_1(u + n) Q_3(u) - Q_1(u) Q_3(u + n)].$$

(4.19)

To obtain $t_n(u)$ and $\bar{t}_n(u)$ from these relations they have to be supplemented by the expression for $t_0(u)$ and $\bar{t}_0(u)$. For $n = 0$ the (anti)chiral auxiliary spaces $v_0$ (and $\bar{v}_0$) in (4.16) contain only one basis vector $\{1\}$ and, as a consequence, the transfer matrices $t_0(u) = \bar{t}_0(u)$ reduce to a c-number. Its value can be found as (see Appendix B for details)

$$t_0(u) = \bar{t}_0(u) = \left[ -\xi \frac{\Gamma(u + 1 + j_q)(u + j_q - \bar{j}_q)}{\Gamma(-u + \bar{j}_q)} \right]^N,$$

(4.20)

with $\xi = e^{-i\pi(j_q + \bar{j}_q)/2}/(j_q \bar{j}_q)$. Combining together the relations (4.18) – (4.20), one can express the transfer matrices $t_n(u)$ and $\bar{t}_n(u)$ (with $n \geq 1$) in terms of the $Q$–operators.
4.4. Factorization of (anti)chiral transfer matrices

Let us demonstrate that the (anti)chiral transfer matrices $\mathbb{T}_j(u)$ and $\overline{\mathbb{T}}_j(u)$ can also be expressed in terms of $\mathcal{Q}$–operators. We return to the first relation in (4.13) and observe that the difference of the two $\mathbb{T}$–operators carrying the spins $j$ and $j+1$ is proportional to the operator $\mathcal{Q}_3(u)$ which in its turn is $j$–independent. In the same manner, the difference of the $\overline{\mathbb{T}}$–operators in (4.14) is proportional to $\mathcal{Q}_1(u)$. This suggests to write down the transfer matrices in a factorized form

\[ \mathbb{T}_j(u) = \mathbb{P}^{-2} \mathcal{Q}_3(u) \mathcal{Q}_{12}(u + 1 - j)/(-\beta(u - j)), \]
\[ \overline{\mathbb{T}}_j(u) = \mathbb{P}^{-2} \mathcal{Q}_1(u) \mathcal{Q}_{23}(u + j)/\beta(u + j), \]

where $\mathcal{Q}_{12}(u)$ and $\mathcal{Q}_{23}(u)$ are some operators commuting with the $\mathcal{Q}$–operators. Here, for the later convenience, we introduced the normalization factor satisfying the condition

\[ \frac{\beta(u - 1)}{\beta(u)} = [\alpha(u)]^N. \]

Combining together (4.13), (4.14) and (4.21) one obtains the relations

\[ \beta(u) \mathcal{Q}_1(u) \mathcal{Q}_2(u) = \mathcal{Q}_{12}(u) - \mathcal{Q}_{12}(u + 1), \]
\[ \beta(u) \mathcal{Q}_3(u) \mathcal{Q}_2(u) = \mathcal{Q}_{23}(u) - \mathcal{Q}_{23}(u + 1), \]

which establish the correspondence between the two sets of the $\mathcal{Q}$–operators. Here the rationale behind the subscript ‘12’ is that the $\mathcal{Q}_{12}$–operator is related to the product of $\mathcal{Q}$–operators with the subscripts ‘1’ and ‘2’.

To elucidate the origin of (4.21) and the meaning of the new operators, we observe that the transfer matrix $\mathbb{T}_j(u)$ is built from the $\mathcal{R}$–operators (4.12) acting on the tensor product $\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_j$. These operators appear as an upper diagonal block in the expression for the ‘big’ operator $\mathcal{R}_{\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j,0}}$, Eq. (4.12) for $\bar{j} = 0$. The latter operator has the factorized form (3.6) and a question arises whether $\mathcal{R}_{\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j}}$ admits a similar factorized representation. To start with, let us examine the action of the $\mathcal{R}^{(a)}$–operators entering (3.6) on the tensor product $\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j,0}$. Applying (3.7) for $j_1 = j_2, \bar{j}_1 = \bar{j}_2, \bar{j} = j$, and $\bar{j}_2 = 0$, we notice that $\mathcal{R}^{(3)}(u)$ does not modify the zero value of the antichiral spin in the auxiliary space, while the $\mathcal{R}^{(1)}$– and $\mathcal{R}^{(2)}$–operators move it away from zero. In other words, the operator $\mathcal{R}^{(3)}(u)$ maps a reducible auxiliary space $\mathcal{V}_{j,0}$ into a reducible one $\mathcal{V}_{j,u,0}$ and, as a consequence, it has a block-triangular form (4.22). The same property does not hold however for the $\mathcal{R}^{(1)}$– and $\mathcal{R}^{(2)}$–operators separately but it is restored in their product $\mathcal{R}^{(12)}(u) \equiv \mathcal{R}^{(1)}(u)\mathcal{R}^{(2)}(u - \bar{j}_q)$ so that the $\mathcal{R}$–operator in (3.6) acquires a block-triangular form (4.22). This suggests to use the upper diagonal block of $\mathcal{R}^{(12)}(u)$ to construct a new transfer matrix analogous to (4.20). Indeed, let us denote by $\mathcal{P}_0$ the operator that projects $\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j,0}$ onto its invariant component $\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j}$. Then, the chiral transfer matrix (4.12) can be expressed as

\[ \mathbb{T}_j(u) = \text{str}_{\mathcal{V}_{j,0}} [\mathcal{P}_0 \mathbb{R}_{00}(u) \ldots \mathbb{R}_{10}(u)], \quad \mathbb{R}(u) = \mathcal{R}_{\mathcal{V}_{j\bar{j}q} \otimes \mathcal{V}_{j,0}}(u). \]

One substitutes the $\mathcal{R}$–operator with (3.6) and repeats the same steps that led to the factorized expression for the transfer matrix (E.3) to obtain the first relation in (4.21) with

\[ \mathcal{Q}_{12}(u + 1 - j) \sim \text{str}_{\mathcal{V}_{j,0}} \{ \mathcal{P}_0 \mathcal{\Pi}_{00} \mathcal{R}_{00}^{(12)}(u) \ldots \mathcal{\Pi}_{10} \mathcal{R}_{10}^{(12)}(u) \}, \]
and \( R^{(12)}(u) = R^{(1)}(u)R^{(2)}(u - j_q) \). The analysis of the antichiral transfer matrix \( \hat{T}_j(u) \) goes along the same lines with the only difference that it is now the operator \( R^{(1)}(u) \) that preserves zero value of the chiral spin, \( J = 0 \), in the auxiliary space and the \( \mathcal{Q}_{23} \)--operator is built from \( R^{(2)}(u) \equiv R^{(2)}(u + j_q)R^{(3)}(u) \).

Substituting (4.21) into (4.26) one can express the finite-dimensional atypical transfer matrices \( t_n(u) \) and \( \bar{t}_n(u) \) (with \( n \geq 0 \)) in terms of the \( \mathcal{Q} \)--operators

\[
t_n(u) = \frac{P^{-2} [Q_1(u) Q_{23}(u + n + 1) - Q_3(u) Q_{12}(u + n + 1)]}{\beta(u + n)},
\]

\[
\bar{t}_n(u) = \frac{P^{-2} [Q_1(u) Q_{23}(u - n) - Q_3(u) Q_{12}(u - n)]}{\beta(u - n)}.
\]

for an arbitrary integer \( n \). For \( n = 0 \) the two relations in (4.26) coincide in virtue of (4.23) leading to \( t_0(u) = \bar{t}_0(u) \), in agreement with (4.20). They can be further simplified by excluding \( Q_1 \) and \( Q_3 \) with a help of (4.23) leading to

\[
t_0(u) Q_2(u) = \frac{P^{-2} [Q_{12}(u) Q_{23}(u + 1) - Q_{23}(u) Q_{12}(u + 1)]}{[\beta(u)]^2},
\]

with \( \beta(u) \) given by (4.22).

Being combined together, Eqs. (4.23) and (4.28) allow us to express three different \( \mathcal{Q} \)--operators in terms of two operators \( Q_{12}(u) \) and \( Q_{23}(u) \) only. Obviously, the same holds true for the transfer matrices defined in (4.11), (4.21) and (4.26). To save space we do not present here their explicit expressions. One of the consequences of this remarkable property is that a generic infinite-dimensional transfer matrix \( \mathcal{T}_{j,j}(u) \) can be expressed in terms of (anti)chiral transfer matrices\(^8\), \( \mathcal{T}_j(u) \) and \( \hat{T}_j(u) \),

\[
\mathcal{T}_{j,j}(u) = \left[ T_j(u + \bar{j}) \hat{T}_j(u - j) - \gamma(u - 2b) T_{j+1}(u + \bar{j}) \hat{T}_{j+1}(u - j) \right] / t_0(u - 2b).
\]

where \( b = (j - \bar{j})/2 \) and the notation was introduced for

\[
\gamma(u) = [\alpha(u)/\alpha(u + 1)]^N
\]

with \( \alpha(u) \) given by (4.11). A similar relation also holds between finite-dimensional transfer matrices (4.17) and (4.20)

\[
t_{\ell,\bar{n}}(u) = \left[ t_{\ell,\bar{n}}(u - \bar{n}) \bar{t}_{\bar{n}}(u + n) - \gamma(u - 2b) t_{\ell-1,\bar{n}}(u - \bar{n}) \bar{t}_{\bar{n}-1}(u + n) \right] / t_0(u),
\]

for \( \ell > 0 \). For nonnegative integer \( n \), \( t_n(u) \) coincides with the atypical transfer matrix (4.16), while for negative integer \( n \) it is related to the transfer matrix \( \bar{t}_{-n-1}(u) \) through (4.27).

The entire hierarchy of the transfer matrices is summarized in Fig. 5. We observe that all transfer matrices can be expressed in terms of two operators only, \( \mathcal{T}_j(u) \) and \( \hat{T}_j(u) \). Together with (4.21), this allows one to express the \( SL(2|1) \) transfer matrices listed in Table 1 in terms of Baxter \( \mathcal{Q} \)--operators. So far, we introduced two families of mutually commuting operators, \( \mathcal{Q}_a(u) \) (with \( a = 1, 2, 3 \)) and \( \mathcal{Q}_{12}(u), \mathcal{Q}_{23}(u), \) and one more operator \( \mathcal{Q}_{13}(u) \) will be defined later in Eq. (5.12). Similar to the transfer matrices, \( \mathcal{Q} \)--operators also form a hierarchy shown in Fig. 6.

\(^8\)Another way to get this relation is to start with (4.28), shift the spectral paramater as \( u \rightarrow u - j + \bar{j} \), multiply both sides of the relation by \( \mathcal{Q}_1(u - j) \mathcal{Q}_3(u + j) \) and, then, apply (4.11) and (4.21).
5. Baxter equations

Let us now establish the TQ-relations between the $Q$-operators and the atypical transfer matrices $t_1(u)$ and $\bar{t}_1(u)$ defined over the three-dimensional auxiliary spaces $\nu_1$, Eq. (2.29) and $\bar{\nu}_1$, Eq. (2.32), respectively. It is well known [30] that these transfer matrices are generating functions for local integrals of motion of the $SL(2|1)$ spin chain (see Eqs. (6.10) and (6.3) below).

5.1. TQ-relations

Let us derive TQ-relations for each Baxter operator.

Operator $Q_2(u)$

We apply the relations (4.18) and (4.19) for $n = 1$ to get

$$\frac{Q_2(u)}{Q_2(u - 1)} = \frac{t_{1/2,-1/2}(u - 1)}{t_{1/2,1/2}(u)}.$$  \hspace{1cm} (5.1)

From (4.18) and (4.19), one can express the transfer matrices entering this relation in terms of $t_1(u)$ and $\bar{t}_1(u)$ and c-valued functions $t_0(u) = \bar{t}_0(u)$, Eq. (4.20), as

$$t_{1/2,1/2}(u) = \bar{t}_1(u) - \Delta_{-}(u),$$

$$t_{1/2,-1/2}(u) = t_1(u) - \Delta_{+}(u + 1),$$  \hspace{1cm} (5.2)

where the notation was introduced for

$$\Delta_{-}(u) = t_0(u)/\alpha^N(u) = \left[(u + j_q - \bar{j}_q - 1)\Delta(u - 1)\right]^N,$$

$$\Delta_{+}(u) = t_0(u - 1)\alpha^N(u) = \left[(u + j_q)(u - \bar{j}_q) \left/ \left(u + j_q - \bar{j}_q\right)\Delta(u - 1)\right.\right]^N$$  \hspace{1cm} (5.3)

with the normalization factor

$$\Delta(u) = -i\xi \frac{\Gamma(u + j_q + 1)}{\Gamma(-u + j_q)}(u + j_q - \bar{j}_q + 1),$$  \hspace{1cm} (5.4)
and \( \xi = e^{-i\pi(j_q+\bar{j}_q)/2}/(j_q\bar{j}_q) \). We recall that the operator \( Q_2(u) \) is polynomial in \( u \) and, therefore, the right-hand side of (5.1) is a rational (operator-valued) function of \( u \). This property is not obvious in the right-hand side of (5.1) since the (anti)chiral transfer matrices \( t_1(u) \) and \( \bar{t}_1(u) \) involve terms proportional to the ratio of \( \Gamma \)-functions (see Eqs. (6.10) and (5.4) below). We will demonstrate in Section 6.1, that such terms cancel against each other.

**Operators \( Q_3(u) \) and \( Q_1(u) \)**

For \( n = \bar{n} = 1 \), or equivalently \( \ell = 1 \) and \( b = 0 \), the transfer matrix \( t_{\ell,b} \) admits two equivalent representations, Eqs. (4.30) and (4.37),

\[
t_{1,0}(u) = \frac{\bar{t}_1(u+1)t_1(u-1) - \gamma(u) \bar{t}_0(u+1)t_0(u-1)}{t_0(u)} = \mathcal{P}^{-2} Q_2(u)\left[Q_1(u+1)Q_3(u-1) - Q_1(u-1)Q_3(u+1)\right].
\]

Supplementing the second relation with a similar expression for the transfer matrices \( t_{1/2, \pm 1/2}(u) \), Eq. (4.7),

\[
t_{1/2, \pm 1/2}(u \pm 1) = \pm \mathcal{P}^{-2} Q_2(u)\left[Q_1(u \pm 1)Q_3(u) - Q_1(u)Q_3(u \pm 1)\right]
\]

one finds that \( Q_3(u) \) satisfies the functional relation

\[
t_{1,0}(u)Q_3(u) = t_{1/2,1/2}(u + 1)Q_3(u - 1) + t_{1/2,-1/2}(u - 1)Q_3(u + 1).
\]

Replacing the transfer matrices by their expressions (5.5) and (5.2) in terms of atypical transfer matrices, one obtains

\[
[\bar{t}_1(u+1)t_1(u-1) - \Delta_+(u)\Delta_-(u+1)]Q_3(u) = t_0(u)[\bar{t}_1(u+1) - \Delta_-(u+1)]Q_3(u-1) + t_0(u)[t_1(u-1) - \Delta_+(u)]Q_3(u+1).
\]

The operator \( Q_1(u) \) satisfies the same equation due to the symmetry of (5.5) and (5.6) under exchange of the \( Q_1 - \) and \( Q_3 - \)operators.

**Operators \( Q_{23}(u) \) and \( Q_{12}(u) \)**

Let us start with the following identity

\[
t_1(u-1)\bar{t}_1(u) - t_0(u-1)\bar{t}_0(u) = \mathcal{P}^{-4} [Q_1(u)Q_3(u-1) - Q_1(u-1)Q_3(u)]
\]

\[
\times [Q_{12}(u-1)Q_{23}(u+1) - Q_{12}(u+1)Q_{23}(u-1)]/(\beta(u)\beta(u-1)).
\]

It can be verified by replacing the transfer matrices in the left-hand side by their expressions (4.28) in terms of \( Q \)-operators. As follows from (4.18) and (4.19), the first factor in the right-hand side of (5.9) is given by \( \mathcal{P}^{-2}t_{1/2,1/2}(u)/Q_2(u-1) = \mathcal{P}^{-2}t_{1/2,-1/2}(u-1)/Q_2(u) \). Then, one multiplies both sides of (5.9) by \( Q_2(u) \), excludes the \( Q_{12} \)-operator with a help of (4.28) and gets the following relation for the operator \( Q_{23} \)

\[
[t_1(u-1)\bar{t}_1(u) - t_0(u)\bar{t}_0(u-1)]Q_{23}(u) = \Delta_-(u)t_{1/2,-1/2}(u-1)Q_{23}(u-1) + \Delta_+(u)t_{1/2,1/2}(u)Q_{23}(u+1),
\]
where the ‘dressing factors’ $\Delta_\pm(u)$ were defined in (5.3). Finally, one applies (5.2) and obtains

$$
[t_1(u - 1)\bar{t}_1(u) - \Delta_+(u)\Delta_-(u)] Q_{23}(u) = \Delta_-(u) [t_1(u - 1) - \Delta_+(u)] Q_{23}(u + 1) + \Delta_+(u) [\bar{t}_1(u) - \Delta_-(u)] Q_{23}(u + 1)
$$

(5.11)

The operator $Q_{12}(u)$ satisfies the same relation.

**Operator $Q_{13}(u)$**

As we will see in Section 5.2, it proves convenient to introduce the following operator

$$
Q_{13}(u) = \mathbb{P}^{-2} [Q_1(u + 1)Q_3(u) - Q_1(u)Q_3(u + 1)].
$$

(5.12)

The chain of ancestor relations of $Q$-operators is shown in Fig. 6. Then, the transfer matrices (5.6) can be factorized into a product of two $Q$-operators as

$$
t_{1/2, -1/2}(u - 1) = Q_2(u)Q_{13}(u - 1),
$$

$$
t_{1/2, 1/2}(u + 1) = Q_2(u)Q_{13}(u).
$$

(5.13)

This leads to the following relation for the operator $Q_{13}(u)$

$$
\frac{Q_{13}(u - 1)}{Q_{13}(u)} = \frac{t_{1/2, -1/2}(u - 1)}{t_{1/2, 1/2}(u + 1)} = \frac{t_1(u - 1) - \Delta_+(u)}{t_1(u + 1) - \Delta_-(u + 1)},
$$

(5.14)

which should be compared with the analogous relation (5.1) for the operator $Q_2(u)$.

Notice that the TQ-relations (5.8) and (5.11) for the operators $Q_3(u)$ and $Q_{23}(u)$ (as well as for $Q_1(u)$ and $Q_{12}(u)$) are finite difference equations of the second-order, while the TQ-relations (5.1) and (5.14) for the operators $Q_2(u)$ and $Q_{13}(u)$ are of the first order only.

### 5.2. Nested TQ-relations

The TQ-relations (5.1), (5.8) and (5.11) involve one $Q$-operator and two transfer matrices, $t_1(u)$ and $\bar{t}_1(u)$. To make a comparison with the nested Bethe Ansatz it is convenient to exclude the antichiral transfer matrix $\bar{t}_1(u)$ from the TQ-relation by trading it for another $Q$-operator.

Let us start with (5.11) and notice that the combination of transfer matrices in front of $Q_{23}(u)$ in the left-hand side can be rewritten with a help of (5.2) as $t_1(u - 1)t_{1/2, 1/2}(u) + \Delta_-(u)t_{1/2, -1/2}(u - 1)$. Then, one divides both sides of (5.11) by $t_{1/2, 1/2}(u)Q_{23}(u)$ and applies (5.1) to get

$$
t_1(u - 1) = \Delta_+(u)\frac{Q_{23}(u + 1)}{Q_{23}(u)} + \Delta_-(u) \left[ \frac{Q_{23}(u - 1)}{Q_{23}(u)} - 1 \right] \frac{Q_2(u)}{Q_2(u - 1)}
$$

(5.15)

This relation also holds true with $Q_{23}(u)$ being replaced by $Q_{12}(u)$.

The analysis of (5.8) or, equivalently, (5.7) goes along the same lines. One uses (5.2) to verify that

$$
t_{1,0}(u) = (t_{1/2, 1/2}(u + 1)t_1(u - 1) + [\alpha(u + 1)]^{-N}t_{1/2, -1/2}(u - 1)t_0(u + 1)) / t_0(u).
$$

(5.16)
Dividing both sides of (5.17) by \( t_{1/2,1/2}(u + 1)Q_3(u) \), one obtains

\[
t_1(u - 1) = t_0(u) \frac{Q_3(u - 1)}{Q_3(u)} + t_0(u) \left[ \frac{Q_3(u + 1)}{Q_3(u)} - \frac{t_0(u + 1)}{t_0(u)\alpha^N(u + 1)} \right] \frac{t_{1/2,-1/2}(u - 1)}{t_{1/2,1/2}(u + 1)}. \tag{5.17}
\]

Here, in distinction with (5.15) and (5.1), the ratio of the transfer matrices cannot be expressed in terms of the \( Q_3 \) operator. Instead, making use of (5.14), it can be simplified leading to yet another expression for the transfer matrix in terms of the operators \( Q_3(u) \) and \( Q_{13}(u) \)

\[
t_1(u - 1) = t_0(u) \frac{Q_3(u - 1)}{Q_3(u)} + \left[ t_0(u) \frac{Q_3(u + 1)}{Q_3(u)} - \Delta_-(u + 1) \right] \frac{Q_{13}(u - 1)}{Q_{13}(u)}. \tag{5.18}
\]

Equations (5.15) and (5.18) provide two different representations of the transfer matrix \( t_1(u - 1) \) in terms of two pairs of the \( Q \)-operators: \( (Q_2, Q_{23}) \) and \( (Q_3, Q_{13}) \). One might expect that there exist two more representations for \( t_1(u - 1) \) which involve two pairs of the operators \( (Q_2, Q_{13}) \) and \( (Q_3, Q_{23}) \). Indeed, the former representation follows immediately from (5.13) and (5.12)

\[
t_1(u - 1) - \Delta_+(u) = Q_2(u)Q_{13}(u - 1) \tag{5.19}
\]

Lastly, one applies (5.12) and (4.26) to verify that

\[
Q_{13}(u)Q_{23}(u + 1) = t_0(u + 1)\beta(u + 1)Q_3(u) - t_0(u)\beta(u)Q_3(u + 1)
= \beta(u) [\Delta_-(u + 1)Q_3(u) - t_0(u)Q_3(u + 1)]. \tag{5.20}
\]

Using this relation one excludes the operator \( Q_{13}(u) \) from (5.18) and obtains the following representation for the transfer matrix

\[
t_1(u - 1) - \Delta_+(u) = \left[ 1 - \frac{Q_{23}(u + 1)}{Q_{23}(u)} \right] \left[ t_0(u) \frac{Q_3(u - 1)}{Q_3(u)} - \Delta_+(u) \right]. \tag{5.21}
\]

Eqs. (5.15), (5.18), (5.19) and (5.21) provide four different expressions for the transfer matrix \( t_1(u) \) in terms of various Baxter \( Q \)-operators.

We remind that the TQ-relations stay invariant under the substitution of the operators \( Q_3(u) \) and \( Q_{23}(u) \) with the operators \( Q_1(u) \) and \( Q_{12}(u) \), respectively. The reason why we wrote the TQ-relations in terms of the former operators only is that, as we will show in the next section, their eigenvalues are given (up to an overall normalization factor) by polynomials in \( u \).

### 6. Nested Bethe Ansatz

The transfer matrix \( t_1(u) \) and the \( Q \)-operators commute with each other and, therefore, can be diagonalized simultaneously. In this section, we will apply the obtained TQ-relations to find the eigenspectrum of the transfer matrices. The nested Bethe Ansatz provides an alternative approach to solving the same eigenproblem. It relies on the existence of a pseudovacuum state in the quantum space of the model and leads to expressions for the eigenvalues of the transfer matrices in terms of two sets of Bethe roots. The \( SL(2|1) \) spin chain has in fact three different
pseudo-vacuum states and, as a consequence, one can construct three different nested Bethe Ansatz solutions \[33, 34, 35, 36].

In this section, we will establish the correspondence between the TQ-relations for Baxter $Q-$operators and nested Bethe Ansatz relations. In particular, we will identify the Bethe roots as zeros of polynomial eigenvalues of certain $Q-$operators and demonstrate that three TQ-relations, Eqs. \(5.15\), \(5.18\) and \(5.21\), are in the one-to-one correspondence with three different nested Bethe Ansatz solutions of the $SL(2|1)$ spin chain.

In the nested Bethe Ansatz, eigenvalues of the transfer matrices are given by expressions similar to \(5.15\) with $Q-$operators replaced by polynomials parameterized by the Bethe roots. The latter are uniquely fixed from the condition that the transfer matrix should be polynomial in $u$. It is important to stress that our definition of the transfer matrix $t_1(u)$, Eq. \(4.16\), differs from the conventional one $\tau_N(u)$ (see Eq. \(6.3\) below) used in the nested Bethe Ansatz. The main difference is that the former operator is built from the $R-$matrices while the latter is constructed from the Lax operators. We will show that the two operators differ by an overall normalization factor and, therefore, polynomiality of $t_1(u)$ can be restored by taking this factor out.

### 6.1. Polynomial transfer matrices

Local integrals of motion of the $SL(2|1)$ spin chain are generated by the auxiliary transfer matrices built from the chiral and antichiral Lax operators, $L(u)$ and $\bar{L}(u)$, respectively \[30\]. By definition,

$$L(u) = u + \sum_{A,B=1,2,3} (-)^B e^{AB} E^{BA} = \begin{pmatrix} u + J & -\bar{V}^- & L^- \\ -\bar{V}^+ & u + J - \bar{J} & V^- \\ L^+ & -\bar{V}^+ & u - \bar{J} \end{pmatrix}$$

(6.1)

and

$$\bar{L}(u) = u + \sum_{A,B=1,2,3} (-)^\bar{B} \bar{e}^{AB} \bar{E}^{BA} = \begin{pmatrix} u + J & -V^- & L^- \\ -V^+ & u + J - J & V^- \\ L^+ & -V^+ & u - J \end{pmatrix},$$

(6.2)

where $E^{BA}$ are the $SL(2|1)$ generators \[27\] and the $3 \times 3$ graded matrices $e^{AB}$ and $\bar{e}^{AB}$ represent the generators of the three-dimensional atypical representations $v_1$ and $\bar{v}_1$, Eqs. \(2.29\) and \(2.32\), respectively. The corresponding auxiliary transfer matrices are defined as

$$\tau_N(u) = \text{str}_{v_1} [L_N(u) \ldots L_1(u)] = u^N + i^2 q_2 u^{N-2} + \ldots + i^N q_N$$

$$\bar{\tau}_N(u) = \text{str}_{\bar{v}_1} [\bar{L}_N(u) \ldots \bar{L}_1(u)] = u^N + i^2 \bar{q}_2 u^{N-2} + \ldots + i^N \bar{q}_N,$$

(6.3)

where the supertrace is taken over three-dimensional auxiliary space. Defined in this way, the transfer matrices $\tau_N(u)$ and $\bar{\tau}_N(u)$ are polynomials in $u$ of degree $N$. The operator valued $q-$ and $\bar{q}-$coefficients are local integrals of motion of the model. It follows from \(6.3\) that the operators $q_2$ and $\bar{q}_2$ are proportional to the quadratic Casimir operator \(2.10\)

$$q_2 = \bar{q}_2 = -C_2 + N j_q \bar{j}_q,$$

(6.4)

with $C_2 = \sum_{A,B=1,2,3} (-)^B E^{AB} E^{BA}$ and $E^{AB} = \sum_{n=1}^N E_n^{AB}$ is given by the sum of the $SL(2|1)$ generators over all sites. The remaining charges $q_k$ and $\bar{q}_k$ (with $k \geq 3$) are given by homogeneous polynomials of degree $k$ in the $SL(2|1)$ generators. In distinction with $q_2$, the operators $q_k \neq \bar{q}_k$
are not self-adjoint with respect to the scalar product \( (2.12) \). Instead one finds using \( (2.49) \) that
\[
q_k^+ = \bar{q}_k \quad \mapsto \quad (\tau_N(u))^+ = (-1)^N \tau_N(-u^*). \quad (6.5)
\]
To match the auxiliary transfer matrix \( (6.3) \) into \( (4.16) \), we have to identify the Lax operator as a special case of the \( R \)-operator.

By definition \( (6.1) \), the Lax operator \( L(u) \) acts on the tensor product of quantum space \( \mathcal{V}_{j_q \bar{j}_q} \) and three-dimensional chiral representation \( v_1 \) and satisfies the Yang-Baxter equation. As such it can be identified (modulo an overall normalization factor and a shift of the spectral parameter) with one of the \( R \)-operators, \( L(u) \sim R_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u + c) \) or \( L(u) \sim \mathcal{R}_{v_1 \otimes \mathcal{V}_{j_q \bar{j}_q}}(u) \). Indeed, an explicit calculation shows (see Appendix B) that
\[
\mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u) = \Delta(u) L(u + 1), \quad (6.6)
\]
where the normalization factor \( \Delta(u) \) is given by \( (5.4) \). According to \( (6.2) \), the antichiral Lax operator \( \bar{L}(u) \) can be obtained from the chiral operator \( L(u) \), Eq. \( (6.1) \), by replacing the auxiliary space with \( \bar{v}_1 \). Together with \( (6.5) \) this leads to \( \mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u) \sim \bar{L}(u + 1) \). The transfer matrix \( \bar{t}_1(u) \), Eq. \( (4.16) \), is built from the operators \( \mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u) \) rather than \( \mathcal{R}_{v_1 \otimes \mathcal{V}_{j_q \bar{j}_q}}(u) \). It is straightforward to verify that \( \bar{L}(u + 1) \) and \( (\mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(-u))^{-1} \) obey the same Yang-Baxter equation \( (3.2) \) and, therefore, the two operators coincide up to an overall normalization factor. The latter can be calculated as before, by examining the action of both operators on the same reference state leading to
\[
\mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u + 1) = \Delta(u)(j_q + u)(\bar{j}_q - u)\bar{L}^{-1}(-u). \quad (6.7)
\]
This relation can be further simplified if one of the spins in the quantum space vanishes, \( j_q \bar{j}_q = 0 \). In particular, for \( j_q = 0 \) one has \( \bar{L}(-u)\bar{L}(u + j_q) = -u(u + j_q) \) yielding \( (6.8) \)
\[
\mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u + 1) = \Delta(u)\bar{L}(u + j_q). \quad (6.8)
\]
In a generic case, for \( j_q \bar{j}_q \neq 0 \), it is convenient to introduce the operator \( \tilde{L}(u) \) related to the inverse Lax operator
\[
\tilde{L}(u) = -(u + \bar{j}_q)(u - j_q)(u + 1)\tilde{L}^{-1}(-u + j_q - \bar{j}_q) = (u + 1)(u + \bar{L}(0)) + (\tilde{L}(0) + j_q)(\tilde{L}(0) - \bar{j}_q). \quad (6.9)
\]
The Lax operators \( L(u) \) and \( \tilde{L}(u) \) are linear and quadratic functions in \( u \), respectively. Then, one deduces from \( (6.6) \) and \( (6.7) \) that, up to an overall normalization, the operators \( \mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u) \) and \( \mathcal{R}_{\mathcal{V}_{j_q \bar{j}_q} \otimes v_1}(u) \) have the same property. This allows us to reveal analytical properties of the transfer matrices \( t_1(u) \) and \( \bar{t}_1(u) \), Eq. \( (4.10) \).

Substituting \( (6.6) \) and \( (6.7) \) into \( (4.16) \) we obtain
\[
t_1(u) = \frac{\tau_N(u + 1)}{[\Delta(u)]^N},
\]
\[
\bar{t}_1(u) = \frac{\tau_2(u + j_q - \bar{j}_q - 1)}{[\Delta(u - 1)/(u + j_q - \bar{j}_q)]^N}. \quad (6.10)
\]
\footnote{The expression for \( \Delta(u) \), Eq. \( (5.4) \), involves a singular factor \( \xi = e^{-i\pi(j_q + j_\bar{q})/2}/(j_q\bar{j}_q) \). In what follows it is tacitly assumed that this factor is removed in the (anti)chiral limit \( j_q \bar{j}_q \to 0 \) by changing the normalization of the \( \mathcal{R} \)-operators.}
The TQ-relations (5.8) and (5.11) are finite-difference equations and they stay invariant under the
relation (5.8) as eigenvalues of the spectral parameter. It is convenient to parameterize them as
out in Section 3.4. As we argued there, the operators TQ-relations with additional conditions on their solutions. These conditions were worked
out in (6.11) can be expressed in terms of the integrals of motion \( q_k \) and Casimir operators, e.g.,

\[ i^2 \bar{q}_2 = C_2 \]
\[ i^3 \bar{q}_3 = i^3 q_3 - 2C_3 + \frac{1}{3} C_2 + (N - 1)C_2 + 2(j_q - \bar{j}_q)C_2 - N(j_q - \bar{j}_q)j_q \bar{j}_q, \]

where \( C_2 \) and \( C_3 \) are given by (2.10) with the \( SL(2|1) \) generators acting on the quantum space (3.17).

In the chiral limit \( \bar{j}_q = 0 \), one has \( \bar{L}(0)(\bar{L}(0) + j_q) = 0 \), or equivalently \( \bar{L}(u) = (u + 1)\bar{L}(u) \), so that (6.11) is expressed in terms of the antichiral transfer matrix

\[ \bar{\tau}_2(u)|_{j_q=0} = (u + 1)^N \bar{\tau}(u). \]

As a consequence, the second relation in (6.10) simplifies to (see footnote 9)

\[ t_1(u)|_{j_q=0} = \tau_N(u + 1) [\Delta(u)]^N, \]
\[ \bar{t}_1(u)|_{j_q=0} = \bar{\tau}_N(u + j_q - 1) [\Delta(u - 1)]^N, \]

where \( \bar{\tau}_N(u) \) is a polynomial in \( u \), Eqs. (6.3) and (6.5).

### 6.2. Polynomial \( Q \)-operators

The TQ-relations (5.8) and (5.11) are finite-difference equations and they stay invariant under the multiplication of the \( Q \)-operators by an arbitrary periodic function with period 1. In order to fix this ambiguity and uniquely determine eigenvalues of the \( Q \)-operators, one has to supplement the TQ-relations with additional conditions on their solutions. These conditions were worked out in Section 3.4. As we argued there, the operators \( Q_2(u) \) and \( Q_3(u)/[e^{i\pi u/2} \Gamma(1 + u + j_q)]^N \) are polynomial in \( u \) and, therefore,

\[ Q_2(u)|\Psi_q\rangle = Q_2(u)|\Psi_q\rangle, \]
\[ Q_3(u)|\Psi_q\rangle = [e^{i\pi u/2} \Gamma(1 + u + j_q)]^N Q_3(u)|\Psi_q\rangle, \]

where \( |\Psi_q\rangle \) stands for the eigenstate of the \( SL(2|1) \) spin chain Hamiltonian and the corresponding eigenvalues \( Q_2(u) \) and \( Q_3(u) \) are polynomials in \( u \). The operator \( Q_1(u) \) verifies the same TQ-relation (5.8) as \( Q_3(u) \), but in distinction with the latter its eigenvalues are entire functions of the spectral parameter. It is convenient to parameterize them as

\[ Q_1(u)|\Psi_q\rangle = [\xi e^{-i\pi u/2} / \Gamma(-u - j_q)]^N Q_1(u)|\Psi_q\rangle, \]

where \( \xi = e^{-i(j_q + \bar{j}_q)/2} / (j_q \bar{j}_q) \) and \( Q_1(u) \) is a meromorphic function with poles located at \( u + j_q \in \mathbb{N} \) of maximal order \( N \).
Let us now examine the operator $Q_{13}(u)$. This operator enters into the factorized expression for the transfer matrix $t_{1/2,-1/2}(u - 1)$, Eq. (5.13). Taking into account the relations (5.2) – (5.4) one finds from (5.13)

$$Q_2(u)Q_{13}(u - 1) = [\Delta(u - 1)]^N \left[ \tau_N(u) - \left( (u + j_q)(u - \bar{j}_q)/(u + j_q - \bar{j}_q) \right)^N \right].$$

Since $Q_2(u)$ divides the transfer matrix $t_{1/2,-1/2}(u - 1)$ (see (4.9)), and $\tau_N(u)$ is a polynomial in $u$, the operator $Q_{13}(u - 1)/[\Delta(u - 1)/(u + j_q - \bar{j}_q)]^N$ is also polynomial so that

$$Q_{13}(u)|\Psi_q\rangle = \left[ \frac{\Delta(u)}{u + j_q - \bar{j}_q + 1} \right]^N Q_{13}(u)|\Psi_q\rangle,$$

where $\Delta(u)$ is determined in (6.4) and $Q_{13}(u)$ is a polynomial in $u$. We notice that the second factor in the right-hand side of (6.17) simplifies in the chiral limit $\bar{j}_q = 0$. As a result, $Q_{13}(u - 1)/\Delta(u - 1)$ ought to be polynomial yielding

$$Q_{13}(u)|_{\bar{j}_q = 0} = (u + j_q + 1)^N Q_{13}^{(0)}(u),$$

with $Q_{13}^{(0)}(u)$ being yet another polynomial.

It is instructive to compare properties of the functions $Q_1(u)$ and $Q_3(u)$ with those of the eigenvalues of the Baxter operators for the $SL(2)$ spin chain. In that case, one encounters two operators $Q_{\pm}(u)$ analogous to $Q_1$ and $Q_2$. The eigenvalues of the operators $Q_{+}(u)$ and $Q_{-}(u)$ are correspondingly polynomials and meromorphic functions of $u$. They satisfy the second-order finite difference TQ-relation (1.7) and verify the Wronskian relation (1.8). Equation (5.12) is an analog of the Wronskian relation for the $SL(2|1)$ spin chain. Going over to the eigenvalues in both sides of (5.12), one gets

$$Q_1(u + 1)Q_3(u) - Q_1(u)Q_3(u + 1) = Q_{13}(u)e^{2i\theta_q}\left[ \frac{\Gamma(-u - 1 - j_q)}{\Gamma(-u + j_q)} \right]^N,$$

where $\theta_q$ is the quasimomentum, $\mathbb{P}|\Psi_q\rangle = e^{i\theta_q}|\Psi_q\rangle$. In distinction with (1.3), the right-hand side of this relation involves the eigenvalues of the operator $Q_{13}(u)$.

Let us parameterize the eigenvalues of the operators $Q_{12}(u)$ and $Q_{23}(u)$ as

$$Q_{12}(u + 1)|\Psi_q\rangle = \beta(u)\left[ \xi e^{-i\pi u/2}(u + j_q - \bar{j}_q)/\Gamma(-u - j_q) \right]^N Q_{12}(u + 1)|\Psi_q\rangle,$n

$$Q_{23}(u + 1)|\Psi_q\rangle = \beta(u)\left[ e^{i\pi u/2}(u + j_q - \bar{j}_q)\Gamma(u + j_q + 1) \right]^N Q_{23}(u + 1)|\Psi_q\rangle,$$

with the same normalization factor $\beta(u)$ as in (1.21) and (1.22). Substituting the second relation into (5.20) and (1.24), one finds

$$Q_{23}(u)Q_{13}(u - 1) = (u + j_q - \bar{j}_q)^N Q_3(u - 1) - (u + j_q)^N Q_3(u),$$

$$Q_2(u)Q_3(u) = (u - \bar{j}_q)^N Q_{23}(u) - (u + j_q - \bar{j}_q)^N Q_{23}(u + 1).$$

Due to invariance of the TQ-relations under the exchange of the $Q$–operators, the same relation holds true upon the substitution $Q_3(u)$ and $Q_{23}(u)$ with $Q_1(u)$ and $Q_{12}(u)$, respectively. As before, $Q_{13}(u - 1)$ divides the right-hand side of the first relation in (6.22) and, therefore, the
functions $Q_{23}(u)$ and $Q_{12}(u)$ have the same analytical properties as $Q_3(u)$ and $Q_1(u)$, respectively. So, $Q_{23}(u)$ is polynomial and $Q_{12}(u)$ is a meromorphic function of $u$. One finds from (4.28) the two functions satisfy a Wronskian-like condition,

$$Q_{12}(u)Q_{23}(u+1) - Q_{12}(u+1)Q_{23}(u) = Q_2(u)e^{2\theta u}[\frac{\Gamma(-u-j_q)}{\Gamma(1-u+j_q)}]^N,$$

which should be compared with (6.20) and (6.8).

### 6.2.1. TQ-relations

The eigenvalues of the operators $Q_2(u)$, $Q_3(u)$, $Q_{13}(u)$ and $Q_{23}(u)$ are parameterized by four polynomials, Eqs. (6.15), (6.18) and (6.21), respectively. To determine these polynomials one has to examine the TQ-relations and replace the transfer matrices by their eigenvalues (6.10) and (6.3). In this way, one obtains three different expressions for the transfer matrix (6.3).

From (5.15),

$$\tau_N(u) = -(u+j_q-j_q-1)^N\frac{Q_2(u)}{Q_2(u-1)} + (u-j_q-1)^N\frac{Q_{23}(u-1)}{Q_{23}(u)}\frac{Q_2(u)}{Q_2(u-1)} + (u+j_q)^N\frac{Q_{23}(u+1)}{Q_{23}(u)}.$$

From (5.18),

$$\tau_N(u) = (u-j_q)^N\frac{Q_3(u-1)}{Q_3(u)} - (u+j_q+1)^N\frac{Q_{23}(u-1)}{Q_{23}(u)}\frac{Q_3(u)}{Q_3(u)} - (u-j_q-1)^N\frac{Q_{13}(u-1)}{Q_{13}(u)}.$$

From (5.21),

$$\tau_N(u) = (u-j_q)^N\frac{Q_3(u-1)}{Q_3(u)} - (u+j_q-j_q+1)^N\frac{Q_{23}(u-1)}{Q_{23}(u)}\frac{Q_3(u)}{Q_3(u)} + (u+j_q)^N\frac{Q_{23}(u+1)}{Q_{23}(u)}.$$

By construction, the $Q$-functions entering these relations are polynomials in $u$ of a finite degree. As such, they can be parameterized by their roots

$$Q_2(u) = c_2\prod_{k=1}^{n_2}(u-\lambda_k^{(2)}), \quad Q_3(u) = c_3\prod_{k=1}^{n_3}(u-\lambda_k^{(3)}),$$

$$Q_{13}(u) = c_{13}\prod_{k=1}^{n_{13}}(u-\lambda_k^{(13)}), \quad Q_{23}(u) = c_{23}\prod_{k=1}^{n_{23}}(u-\lambda_k^{(23)}),$$

where $n$’s are nonnegative integers and $c$’s are normalization constants. Equations (6.24) involve four different sets of roots. The ratios of $Q$-functions in the right-hand side of (6.25) and (6.26) are meromorphic functions in $u$ whereas the left-hand side involves the polynomial transfer matrix (6.3). Matching analytical properties of both sides of the relations (6.25) and (6.26), one equates to zero residues at all poles and arrives at a system of coupled nested Bethe equations for the roots of $Q$-polynomials.

The polynomials (6.27) satisfy the additional relations (6.28) that can be considered as consistency conditions for the system (6.25) and (6.26). In addition, one finds from (5.19) that

$$Q_2(u)Q_{13}(u-1) = (u+j_q-j_q)^N\tau_N(u) - [(u+j_q)(u-j_q)]^N = (Nj_q\bar{j}_q - q_2)u^{2N-2} + O(u^{2N-3}),$$

(6.28)
with $\tau_N(u)$ defined in Eq. (6.3). Substituting (6.27) into (6.28) and examining the asymptotic behavior of both sides for $u \to \infty$, one finds $n_{13} = 2(N - 1) - n_2$ and $n_{23} = n_2 + n_3 - N + 1$. Thus, the four polynomials in (6.27) depend on two integers only, $n_2$ and $n_3$. Since $n_{13} \geq 0$ and $n_{23} \geq 0$, they have to satisfy the relations

$$n_2 \leq 2(N - 1), \quad n_2 + n_3 \geq N - 1.$$  \hspace{1cm} (6.29)

The nested TQ-relations (6.24) – (6.28) involve one polynomial transfer matrix $\tau_N(u)$ and two $Q$-polynomials. There exists another set of the TQ-relations, Eqs. (5.1), (5.8), (5.11) and (6.14), which involve only one $Q$–operator and two transfer matrices. Going over to the eigenvalues in both sides of these relations, one can obtain the TQ-relations between the corresponding polynomials $Q_2(u)$, $Q_3(u)$, $Q_{23}(u)$, $Q_{13}(u)$ and polynomial transfer matrices $\tau_N(u)$ and $\tilde{\tau}_{2N}(u)$, Eq. (6.10). In the next subsection, we will present these relations in the chiral limit (see Eqs. (6.34) and (6.35)).

### 6.2.2. TQ-relations in the chiral limit

In the chiral limit, $j_q \neq 0$ and $\bar{j}_q = 0$, one finds from (6.28) and from the first relation in (6.22) that $(u + j_q)^N$ divides the polynomial $Q_{13}(u - 1)$, in agreement with (6.19). One deduces from (6.27) and (6.48) that $Q_{13}^{(0)}(u)$ is a polynomial of degree $n_{13} - N$.

The TQ-relations (6.24) – (6.28) take the following form for $\bar{j}_q = 0$

$$\tau_N(u) = \left[ (u - 1)^N \frac{Q_{23}(u - 1)}{Q_{23}(u)} - (u + j_q - 1)^N \right] \frac{Q_2(u)}{Q_2(u - 1)} + (u + j_q)^N \frac{Q_{23}(u + 1)}{Q_{23}(u)}$$

$$= u^N \frac{Q_3(u - 1)}{Q_3(u)} + (u + j_q)^N \left[ \frac{Q_3(u + 1)}{Q_3(u)} - 1 \right] \frac{Q_{13}^{(0)}(u - 1)}{Q_{13}^{(0)}(u)}$$

$$= u^N + Q_2(u)Q_{13}^{(0)}(u - 1).$$  \hspace{1cm} (6.30)

In the first three relations, the condition for the right-hand side to be polynomial in $u$ leads to three systems of nested Bethe ansatz equations for the roots of the $Q$–polynomials. They coincide with three different nested Bethe ansatz solutions of the $SL(2|1)$ spin chain obtained in Refs. [33, 34, 35, 36]. For $\bar{j}_q = 0$ one finds from (6.22)

$$Q_{23}(u)Q_{13}^{(0)}(u - 1) = Q_3(u - 1) - Q_3(u),$$

$$Q_2(u)Q_3(u) = u^N Q_{23}(u) = (u + j_q)^N Q_{23}(u + 1).$$  \hspace{1cm} (6.31)

Finally, one takes into account the relations (6.13) and (6.14) and obtains from (5.1) and (6.14) in the chiral limit

$$\frac{Q_2(u)}{Q_2(u - 1)} = \frac{\tau_N(u) - u^N}{\tau_N(u + j_q - 1) - (u + j_q - 1)^N},$$  \hspace{1cm} (6.32)

$$\frac{Q_{13}^{(0)}(u)}{Q_{13}^{(0)}(u - 1)} = \frac{\bar{\tau}_N(u + j_q) - (u + j_q)^N}{\bar{\tau}_N(u) - u^N}. $$  \hspace{1cm} (6.33)
From (5.8), on finds
\[
\left[ \tau_N(u) \bar{\tau}_N(u + j_q) - (u(u + j_q))^N \right] Q_3(u) = u^N \left[ \bar{\tau}_N(u + j_q) - (u + j_q)^N \right] Q_3(u - 1) + (u + j_q)^N \left[ \tau_N(u) - u^N \right] Q_3(u + 1),
\] (6.34)
while Eq. (5.11) yields
\[
\left[ \tau_N(u) \bar{\tau}_N(u + j_q - 1) - (u(u + j_q - 1))^N \right] Q_{23}(u) = (u - 1)^N \left[ \tau_N(u) - u^N \right] Q_{23}(u + 1) + (u + j_q)^N \left[ \bar{\tau}_N(u + j_q - 1) - (u + j_q - 1)^N \right] Q_{23}(u + 1)
\] (6.35)
The functions \(Q_1(u)\) and \(Q_{12}(u)\) satisfy the same relations as \(Q_3(u)\) and \(Q_{23}(u)\), respectively. The corresponding Wronskian relations are given by (6.20) and (6.28). The relations (6.34) and (6.35) look similar to the TQ-relations for the \(SL(2)\) magnet, Eq. (1.7). The only difference is that the dressing factors now depend on the transfer matrices which in their turn depend on the integrals of motion. This explains why the Wronskian relations (6.20) and (6.28) involve additional \(Q\)-functions in the right-hand side.

### 6.2.3. Eigenspectrum in the chiral limit

In the chiral limit, one applies (3.45) and obtains the eigenvalues of the Hamiltonian and the cyclic permutation operator in terms of the \(Q_3\)-polynomial as
\[
E = (\ln Q_3(0))' - (\ln Q_3(-j_q))',
\]
\[
e^{i\theta_q} = Q_3(0)/Q_3(-j_q).\] (6.36)
Here the quasimomentum \(\theta_q\) satisfies the relation \(e^{iN\theta_q} = 1\) in virtue of \(\mathbb{P}^N = 1\).

It is well known [36] that in lattice integrable models the Hamiltonian appears as a first term in the expansion of the so-called fundamental transfer matrix in powers of the spectral parameter. The auxiliary space for this transfer matrix coincides with the quantum space in each site. For the \(SL(2|1)\) spin chain in the chiral limit, it can be identified as \(\mathbb{V}_{j_q}\) and the corresponding fundamental transfer matrix is \(\mathbb{T}_{j_q}(u)\) (see Table 1). Let us demonstrate that \(\mathbb{T}_{j_q}(u)\) has the following expansion for \(u \to 0\)
\[
\mathbb{T}_{j_q}(u) \sim \mathbb{P} \exp \left( u \mathbb{H} + \mathcal{O}(u^2) \right),\] (6.37)
where \(\mathbb{P}\) is the cyclic permutation operator and \(\mathbb{H}\) is the Hamiltonian of the \(SL(2|1)\) spin chain. One finds from the first relation in (4.21) that
\[
\mathbb{T}_{j_q}(u) = -\mathbb{P}^2 \left( \xi e^{i\pi j_q/2} \right)^N \left[ u\Gamma(1 + u + j_q)/\Gamma(-u) \right]^N Q_3(u)Q_{12}(u + 1 - j_q)
\] (6.38)
where we replaced the operators \(Q_3(u)\) and \(Q_{12}(u)\) by their eigenvalues, Eqs. (6.15) and (6.24), respectively. We recall that the pre-factor \(\left( \xi e^{i\pi j_q/2} \right)^N\) is singular for \(j_q \to 0\) (see footnote 3) and it can be removed by changing a normalization of the transfer matrix. To reproduce (6.37) one has to exclude \(Q_{12}(u + 1 - j_q)\) from (6.38). To this end, one applies the first relation in (4.26) for \(n = 0\) and obtains after some algebra in the chiral limit
\[
Q_1(u - j_q)Q_{23}(u + 1 - j_q) - Q_3(u - j_q)Q_{12}(u + 1 - j_q) = \mathbb{P}^2 \left[ -\frac{\Gamma(-u)}{\Gamma(j_q - u)} \right]^N.
\] (6.39)
We observe that the right-hand side of this relation has a pole of order \(N\) at \(u = 0\). Since \(Q_{23}(u)\) and \(Q_{3}(u)\) are polynomials, it can only be generated by \(Q_{1}(u)\) and \(Q_{12}(u)\) which are meromorphic functions indeed. One deduces from (6.20) that \(Q_{1}(u - j_{q}) \sim Q_{13}(u - 1 - j_{q})\Gamma(-u) + \mathcal{O}(u^{0})\) but the residue at the pole \(u = 0\) vanishes in the chiral limit in virtue of (6.19). Thus, the first term in the left-hand side of (6.39) approaches a finite value for \(u \to 0\) whereas the second term scales as \(1/u^{N}\). Then, combining together (6.39) and (6.38) we finally obtain

\[
\mathcal{T}_{j_{q}}(u) = c(u) \left[ \frac{Q_{3}(u)}{Q_{3}(u - j_{q})} + \mathcal{O}(u^{N}) \right],
\]

(6.40)

with \(c(u) = [-u\Gamma(1 + u + j_{q})/\Gamma(-u + j_{q})]^{N}\). Expanding this relation at small \(u\) and taking into account (3.45) one arrives at (6.37).

6.3. Matching quantum numbers

Solutions to the nested TQ-relations (6.24) – (6.26) are parameterized by nonnegative integers \(n_{2}\) and \(n_{3}\). They determine the degrees of \(Q\)-polynomials in (6.27) and verify the condition (6.29). In this subsection we demonstrate that \(n_{2}\) and \(n_{3}\) have a simple physical meaning – they define the total \(SL(2|1)\) spins, \(J\) and \(\bar{J}\), carried by the eigenstates of the spin chain \(|\Psi_{q}\rangle\).

The eigenstates of the \(SL(2|1)\) spin chain belong to the quantum space (3.17) and they can be classified according to irreducible \(SL(2|1)\) representations entering the tensor product (3.17). Let us choose \(|\Psi_{q}\rangle\) to be the lowest weight vectors in these representations. The remaining eigenstates can be obtained from \(|\Psi_{q}\rangle\) by applying the raising operators. Being the lowest weights, the eigenstates \(|\Psi_{q}\rangle\) diagonalize the operators \(J\) and \(\bar{J}\), Eq. (2.5), and the Casimirs (2.10) acting in the quantum space of the model (3.17)

\[
\mathbb{C}_{2}|\Psi_{q}\rangle = J\bar{J}|\Psi_{q}\rangle, \quad \mathbb{C}_{3}|\Psi_{q}\rangle = \frac{1}{2} \left( J - \bar{J} + \frac{1}{3} \right) J\bar{J}|\Psi_{q}\rangle.
\]

(6.41)

The total chiral and antichiral spins take the form

\[
J = m + N j_{q}, \quad \bar{J} = \bar{m} + N \bar{j}_{q},
\]

(6.42)

with \(m\) and \(\bar{m}\) nonnegative integer. According to (2.10) these integers define the transformation properties of the eigenstates under dilatations and \(U(1)\) rotations

\[
\Psi_{q}(\lambda^{2}z, \lambda\theta, \lambda\bar{\theta}) = \lambda^{m+\bar{m}}\Psi_{q}(z, \theta, \bar{\theta}),
\]

\[
\Psi_{q}(z, \lambda^{-1}\theta, \lambda\bar{\theta}) = \lambda^{m-\bar{m}}\Psi_{q}(z, \theta, \bar{\theta}),
\]

(6.43)

where \((z, \theta, \bar{\theta}) \equiv \{z_{k}, \theta_{k}, \bar{\theta}_{k}|1 \leq k \leq L\}\) denotes the coordinates in the quantum space (3.17). In other words, \((m + \bar{m})/2\) defines the scaling dimension of the eigenstates while \(m - \bar{m}\) defines its \(U(1)\) charge.

The Casimir operators \(\mathbb{C}_{2}\) and \(\mathbb{C}_{3}\) enter into expressions for the conserved charges, Eqs. (6.4) and (6.12), and, therefore, determine the leading asymptotic behavior of the transfer matrices (6.3) and (6.11) at large \(u\). Let us consider the TQ-relation (5.1) and replace the transfer matrices by their explicit expressions, Eqs. (5.2) and (6.10). Then, one examines the asymptotic behavior of the right-hand side of (5.1) for large \(u\) and finds with a help of (6.11) and (6.12)

\[
\frac{Q_{2}(u)}{Q_{2}(u - 1)} = 1 + \left( \frac{2\mathbb{C}_{3}}{\mathbb{C}_{2}} - \frac{1}{3} + N(j - \bar{j}) + N - 1 \right) u^{-1} + \mathcal{O}(u^{-2})
\]

\[
= 1 + (N - 1 + m - \bar{m}) u^{-1} + \mathcal{O}(u^{-2}).
\]

(6.44)
Here we replaced the Casimir operators by their corresponding eigenvalues and took into account (6.15) and (6.41). Since \( Q_2(u) \) is a polynomial of degree \( n_2 \), Eq. (6.27), one deduces from (6.44) that \( n_2 = N - 1 + m - \tilde{m} \). In a similar manner, one examines asymptotic behavior of both sides of (6.26) for large \( u \), takes into account (6.3) and (6.4) and obtains after some algebra

\[-q_2 + N j_q \tilde{j}_q = JJ = (Nj_q + n_3)(Nj_q + n_{23}) .\]  

(6.45)

Together with \( n_{13} = 2(N - 1) - n_2 \) and \( n_{23} = n_2 + n_3 - N + 1 \) this fixes the values of integers as

\[
\begin{align*}
    n_2 &= N - 1 + m - \tilde{m} , \\
    n_3 &= \tilde{m} , \\
    n_{13} &= N - 1 - m + \tilde{m} , \\
    n_{23} &= m .
\end{align*}
\]

(6.46)

In distinction with \( n_3 \) and \( n_{23} \), the values of \( n_2 \) and \( n_{13} \) are bounded from above, \( 0 \leq n_2, n_{13} \leq 2(N - 1) \). Since \( n_2 \) and \( n_{13} \) take nonnegative values only, possible values of integers \( m \) and \( \tilde{m} \) are subject to the constraint \( |m - \tilde{m}| \leq N - 1 \). As before, simplifications occur in the chiral limit.

### 6.3.1. Chiral limit

For \( \tilde{j}_q = 0 \), the quantum space in each site is given by the chiral \( SL(2|1) \) representation \( \mathbb{V}_{j_q} \) and the Hilbert space of the model is \( (\mathbb{V}_{j_q})^{\otimes N} \). According to (2.18), the eigenstates \( \Psi_q \in (\mathbb{V}_{j_q})^{\otimes N} \) verify the chirality condition (with \( k = 1, \ldots, N \))

\[
D_k \Psi_q(z, \theta, \bar{\theta}) \equiv (-\partial_{\bar{\theta}} + \frac{1}{2} \theta \partial_z) \Psi_q(z, \theta, \bar{\theta}) = 0
\]

(6.47)

which fixes the dependence of the wave functions on \( \bar{\theta} \)-variables as \( \Psi_q = \Psi_q(z_+, \theta) \) with \( z_+ = z + \frac{1}{2} \theta \bar{\theta} \). Examining the transformation properties of \( \Psi_q(z_+, \theta) \) under (6.43), one finds that \( \bar{m} - m \geq 0 \) in the chiral limit. Moreover, for \( m - \bar{m} = 0 \) the wave function \( \Psi_q \) does not depend on \( \theta \)'s and it is a function of \( z_+ \) only. Since \( \Psi_q(z_+) \) is the lowest weight, it has also to be annihilated by the lowering operators (2.3). This leads (up to an overall normalization) to \( \Psi_q = 1 \) or, equivalently, \( m = \bar{m} = 0 \). Thus, in the chiral limit, possible values of the integers \( m \) and \( \bar{m} \) are \( m = \bar{m} = 0 \), or \( 1 \leq m - \bar{m} \leq N - 1 \).

Let us examine the TQ-relations (6.30) for different values of the integers \( m \) and \( \bar{m} \).

**m = m = 0**

In this case the eigenstate has the form \( \Psi_q = 1 \). It coincides with a pseudovacuum state in the nested Bethe Ansatz and, therefore, it does not have any Bethe roots associated with it. Indeed, one deduces from (6.46) that for \( m = \bar{m} = 0 \) the polynomials \( Q_3(u) \) and \( Q_{23}(u) \) are reduced to a c-number, \( Q_3(u) = Q_{23}(u) = 1 \), and obtains from the third relation in (6.30) the corresponding transfer matrix as \( \tau_N(u) = u^N \). Then, one applies (6.31) to obtain (up to an overall normalization factor)

\[
Q_2(u) = u^N - (u + j_q)^N , \quad Q_{13}^{(0)}(u) = 0 .
\]

(6.48)

One finds from (6.30) the corresponding energy and quasimomentum as

\[
E = 0 , \quad e^{i\theta} = 1 .
\]

(6.49)
\( \bar{m} - m = N - 1 \)

In this case, the eigenstate \( \Psi_q(z_+, \theta) \) has the \( U(1) \) charge \( N - 1 \) and, therefore, it is proportional to a homogeneous polynomial in \( \theta \)'s of degree \( (N - 1) \). Requiring that \( \Psi_q(z_+, \theta) \) should be annihilated by the lowering operators \( (2.3) \), one finds

\[
\Psi_q(z_+, \theta) = (\theta_1 - \theta_2)(\theta_2 - \theta_3) \ldots (\theta_{N-1} - \theta_N) \varphi_q(z_+) \sim V^- \theta_1 \ldots \theta_N \phi_q(z_+),
\]

(6.50)

with \( \phi_q(z_+) \) being a translation invariant function of \( z_{k,+} \) (with \( k = 1, \ldots, N \)) and \( V^- = \sum_k \partial \theta_k + \frac{1}{2} \theta_k \partial z_k \) being the lowering operator in \( (\Psi_{j_q})^\otimes N \). As we will see in a moment, the function \( \phi_q(z_+) \) coincides with the eigenstates of the \( SL(2) \) magnet of spin \( s = (1 + j_q)/2 \).

For \( \bar{m} - m = N - 1 \) one finds from (6.46) that \( n_2 = 0 \) and, therefore, \( Q_2(u) = 1 \). Its substitution into (6.30) yields

\[
\tau_N(u) + (u + j_q - 1)^N = (u - 1)^N \frac{Q_{23}(u - 1)}{Q_{23}(u)} + (u + j_q) \frac{Q_{23}(u + 1)}{Q_{23}(u)}. \tag{6.51}
\]

Shifting the spectral parameter as \( u \rightarrow u - \frac{1}{2}(j_q - 1) \) one identifies this relation as the Baxter equation (1.7) for the \( SL(2) \) magnet of length \( N \) and spin \( s = \frac{1}{2}(1 + j_q) = \frac{1}{2} + \ell \). Denoting its polynomial solution as \( P_m^{(s)}(u) \) one finds

\[
Q_{23}(u) = P_m^{(\ell + 1/2)}(u + \ell - \frac{1}{2}), \tag{6.52}
\]

with \( m \geq 0 \). Plugging this expression into (6.51) and expanding both sides in powers of \( u \) one can identify the conserved charges (6.3). Making use of (6.31) and (6.19) one determines the remaining polynomials as (up to an overall normalization)

\[
Q_3(u) = u^N Q_{23}(u) - (u + j_q)^N Q_{23}(u + 1),
\]

(6.53)

\[
Q_{13}^{(0)}(u) = \tau_N(u + 1) - (u + 1)^N.
\]

In agreement with (6.46), they have degree \( N - 1 + m \) and \( 2(N - 1) \), respectively. Inserting (6.32) and (6.52) into (6.36) one finds that the corresponding energy and quasimomentum are related to their counterparts in the \( SL(2) \) spin chain as

\[
E = \ln P_m^{(\ell + 1/2)}(\ell + \frac{1}{2}) - P_m^{(\ell + 1/2)}(-\ell - \frac{1}{2}) = E_m^{(\ell + 1/2)},
\]

(6.54)

\[
e^{i\theta} = (-1)^{N-1} P_m^{(\ell + 1/2)}(\ell + \frac{1}{2}) / P_m^{(\ell + 1/2)}(-\ell - \frac{1}{2}) = (-1)^{N-1} e^{i\theta_m^{(\ell + 1/2)}},
\]

where the superscript in the right-hand side refers to the spin of the \( SL(2) \) magnet.

\( \bar{m} - m = 1 \)

In this case the eigenstate \( \Psi_q(z_+, \theta) \) has a unit \( U(1) \) charge and, therefore, it is given by a linear combination of \( \theta \)'s with prefactors depending on \( z_+ \)-variables only. The latter are fixed from the requirement that \( \Psi_q(z_+, \theta) \) has to be annihilated by the lowering generators (2.3) leading to

\[
\Psi_q(z_+, \theta) = \left( \sum_{k=1}^N \theta_k \partial z_{k,+} \right) \chi_q(z_+) = \bar{V}^- \chi_q(z_+), \tag{6.55}
\]

47
with $\chi_q(z_+)$ being a translation invariant function of $z_{k,+}$ (with $k = 1, \ldots, N$) and $\tilde{\mathcal{V}}^- = \sum_k \theta_{q_k} + \frac{1}{2} \theta_k \partial_z \chi_k$ being the lowering operator in $(\mathcal{V}_q)^{\otimes N}$. Again, we will show at the end of this subsection that $\chi_q(z_+)$ coincides with eigenstates of the $\mathcal{S}\mathcal{L}(2)$ magnet of spin $s = j_q/2$.

It follows from (6.46) and (6.19) that $Q_{13}^{(0)}(u)$ a polynomial of degree $n_{13} = N = \bar{m} - m - 1$ and, therefore, for $\bar{m} - m = 1$ it reduces to a c-number $Q_{13}^{(0)}(u) = 1$. One applies the second relation in (6.30) to find

$$\tau_N(u) + (u + j_q)^N = u^N \frac{Q_3(u - 1)}{Q_3(u)} + (u + j_q)^N \frac{Q_3(u + 1)}{Q_3(u)}.$$  

(6.56)

Similarly to (6.51), one substitutes $u \to u - j_q/2$ and matches the resulting relation into the Baxter equation for the $\mathcal{S}\mathcal{L}(2)$ magnet of length $N$ and spin $s = j_q/2 = \ell$. As a result, the polynomial solution to (6.56) reads (up to a normalization factor)

$$Q_3(u) = P_{m+1}^{(\ell)}(u + \ell)$$  

(6.57)

with $m \geq 0$. Substituting this relation into (6.31) one obtains the remaining polynomials as

$$Q_2(u) = \tau_N(u) - u^N,$$

$$Q_{23}(u) = Q_3(u - 1) - Q_3(u).$$  

(6.58)

It is straightforward to verify that the obtained expressions for the $Q$–polynomials verify the TQ-relations (6.32) – (6.35). From (6.57) and (6.35) one finds the corresponding energy and quasimomentum as

$$E = \left( \ln P_{m+1}^{(\ell)}(\ell) \right)' - \left( P_{m+1}^{(\ell)}(-\ell) \right)' = E_{m+1}^{(\ell)},$$

$$e^{i\theta} = P_{m+1}^{(\ell)}(\ell)/P_{m+1}^{(\ell)}(-\ell) = e^{i\theta_{m+1}}.$$  

(6.59)

where in distinction with (6.54) the spin of the $\mathcal{S}\mathcal{L}(2)$ magnet equals $\ell$.

**2 $\leq \bar{m} - m \leq N - 2$**

This case is only realized for the spin chain of length $N \geq 4$. The eigenstate $\Psi_q(z_+, \theta)$ carries the $\mathcal{U}(1)$ charge equal to $\bar{m} - m$ and it is given by a homogeneous polynomial in $\theta$’s of degree $\bar{m} - m$ with the coefficient given by $z_+$–dependent functions. In distinction to (6.30) and (6.55), these functions are, in general, independent of each other. Going over to $Q$–polynomials, one notices that the eigenstates (6.50) and (6.55) corresponds to ‘extreme’ solutions to the TQ-relations (6.30) when one of the polynomials, $Q_2(u)$ or $Q_{13}^{(0)}(u)$, reduces to a c-number. For $2 \leq \bar{m} - m \leq N - 2$ the two polynomials have degrees $N - 1 - (\bar{m} - m)$ and $(\bar{m} - m) - 1$, respectively. Their explicit form can be found from the TQ-relations (6.30).

We have demonstrated in this subsection that for $\bar{m} - m = 1$ and $\bar{m} - m = N - 1$ solutions to the ‘chiral’ TQ-relations (6.30) are expressed in terms of $Q$–polynomials for the $\mathcal{S}\mathcal{L}(2)$ magnet. For $N = 2, 3$ there are no other solutions to the TQ-relations whereas for $N \geq 4$ there exist additional solutions with $2 \leq \bar{m} - m \leq N - 2$. This property can be understood as follows. We recall that in the chiral limit, $\bar{j}_q = 0$, the $\mathcal{S}\mathcal{L}(2|1)$ spin chain describes the dilatation operator in the $\mathcal{N} = 1$ SYM theory. The product of $N$ superfields, one for each site,
\(\Phi_N(Z) \equiv \text{tr} [\Phi(z_1, \theta_1) \cdots \Phi(z_N, \theta_N)]\) defines the quantum space of the \(SL(2|1)\) magnet. The chiral superfield \(\Phi(z, \theta) = \chi(z) + \phi(z)\) describes a ‘short’ \(SL(2) \otimes U(1)\) multiplet built from two fields \(\phi(z)\) and \(\chi(z)\) carrying the \(SL(2)\) spins \(\ell\) and \(\ell + \frac{1}{2}\) (with \(j_q = 2\ell\)) and the \(U(1)\) charges \(\ell\) and \(\ell - \frac{1}{2}\), respectively. Then, expansion of the single-trace operator in powers of ‘odd’ variables looks as

\[\Phi_N(Z) = \chi_N(z) + \ldots + \phi_N(z) \prod_{k=1}^N \theta_k ,\]  

(6.60)

where \(\chi_N(z) = \text{tr} [\chi(z_1) \ldots \chi(z_N)]\) and \(\phi_N(z) = \text{tr} [\phi(z_1) \ldots \phi(z_N)]\). The dilatation operator ‘mixes’ together different components of the sum carrying the same number of \(\theta\)’s. A distinguished feature of the two components, \(\chi_N(z)\) and \(\phi_N(z)\), is that the dilatation operator acts on them autonomously. For such states, corresponding to the so-called maximal helicity operators \([37]\), the dilatation operator can be mapped into a Hamiltonian of the \(SL(2)\) magnet of spin \(\ell\) and \(\ell + \frac{1}{2}\), respectively. The state \(\chi_N(z)\) is a descendant of the \(SL(2|1)\) lowest weight vector \([6.55]\) with \(\bar{m} - m = 1\) while \(\theta_1 \ldots \theta_N \phi_N(z)\) is a descendant of the lowest weight \([6.50]\) with \(\bar{m} - m = N - 1\).

In both cases, the \(SL(2|1)\) Hamiltonian effectively reduces to the Hamiltonian of the \(SL(2)\) spin chain of length \(N\). Thus, for \(\bar{m} - m = 1\) and \(\bar{m} - m = N - 1\) the solution to the \(SL(2|1)\) and \(SL(2)\) Baxter equations are related to each other and the same relation holds true between the energy spectrum of two models.

We remind that the integers \(m\) and \(\bar{m}\) are related to the total spin of the \(SL(2|1)\) spin chain \([6.12]\). A distinguished feature of this model as compared with conventional compact spin chains is that the total spin can now take arbitrarily large values and the energy spectrum of the model is not restricted from above for a finite length of the spin chain. In terms of the Baxter \(Q\)-operators, this property implies that the \(\text{TQ-relations} \ [6.31]\) admit an infinite number of polynomial solutions \(Q_3(u)\) parameterized by \(m\) and \(\bar{m}\). For small \(m\) and \(\bar{m}\) they can be worked out explicitly while for large \(m\) and/or \(\bar{m}\) one can construct asymptotic solutions to the \(\text{TQ-relations}\) by applying the semiclassical approach of Ref. \([38]\).

7. Conclusion

In this paper, we developed an approach for systematic construction of the Baxter \(Q\)-operators in integrable noncompact spin chain models with Lie supergroup symmetry. We exposed the formalism by applying it to the generalized homogeneous Heisenberg magnet with the \(SL(2|1)\) symmetry. Apart from being the simplest case which allows us to demonstrate all essential features of our approach, the model has a wide spectrum of applications, ranging from superconductivity to dynamics of four-dimensional gauge theories.

The central rôle in our analysis is played by noncompact transfer matrices defined over generic infinite-dimensional \(SL(2|1)\) representations in the auxiliary space. We have demonstrated that, in comparison with conventional compact transfer matrices, they have a number of remarkable properties. Firstly, a generic noncompact transfer matrix is factorized into the product of three ‘special’ noncompact transfer matrices. We argued that the latter can be identified as three Baxter operators \(Q_a(u)\) (with \(a = 1, 2, 3\)). Secondly, for certain values of spins, infinite-dimensional \(SL(2|1)\) representations become reducible indecomposable and their irreducible components define both infinite-dimensional (chiral and antichiral) and finite-dimensional (typical and atypical) representations of the \(SL(2|1)\). This property leads to the hierarchy between the corresponding transfer matrices and allows one to express all transfer matrices in terms of the \(Q\)-operators.
Combining the two properties together, we derived finite difference equations for the Baxter operators, the so-called TQ-relations. In distinction with higher rank classical Lie groups, the TQ-relations for the $SL(2|1)$ group are of the second order at most. Two out of the three Baxter operators, $Q_1(u)$ and $Q_3(u)$, satisfy the same second order finite-difference equation while the operator $Q_2(u)$ obeys a finite-difference equation of the first order. The former equation is quite similar in structure to the TQ-relation for the $SL(2)$ magnet. Important difference being however that its c-number dressing factors are now replaced by (operator valued) compact transfer matrices.

The TQ-relations are invariant under the multiplication of the $Q-$operators by an arbitrary periodic function with period 1 and, therefore, they have to be supplemented by additional conditions on their solutions. To deduce these conditions one needs an explicit expression for the $Q-$operators. We demonstrated that for the $SL(2|1)$ magnet under consideration the eigenvalues of the operators $Q_3(u)$ and $Q_2(u)$ are polynomials in $u$ and the eigenvalues of the operator $Q_1(u)$ are meromorphic functions of $u$. This property is intimately related to the fact that the quantum space of the model contains a pseudovacuum state which is annihilated by the lowering $SL(2|1)$ generators in all sites. It also allows one to demonstrate the equivalence of the Baxter $Q-$operator method and the nested Bethe ansatz approach. Parameterizing the polynomial eigenvalues of the operators $Q_3(u)$ and $Q_2(u)$ by their roots, we showed that the TQ-relations for the Baxter operators lead to a system of coupled equations for the roots which coincide with similar relations in the nested Bethe ansatz solution for the $SL(2|1)$ magnet.

The advantage of the Baxter $Q-$operator method, though, is that it does not rely on the existence of the pseudovacuum and, therefore, it can be applied to the models which do not possess such a state. Indeed, the derivation of the TQ-relations is based on the decomposition of infinite-dimensional $SL(2|1)$ representations over irreducible components and it is not sensitive to a detailed structure of the representation space. The latter information is encoded in analytical properties of the $Q-$operators. Polynomiality of the $Q-$operators is in one-to-one correspondence with the existence of the pseudovacuum state. If the quantum space of the model does not have it, the nested Bethe ansatz is not applicable and the eigenvalues of the $Q-$operators are, in general, meromorphic functions of $u$. To identify their analytical properties (position and order of poles, asymptotic behavior at infinity) one has to explicitly construct the corresponding $Q-$operators. Integrable $SL(2|1)$ spin chain with the quantum space of the form $(\mathbb{V} \otimes \bar{\mathbb{V}})^{\otimes N/2}$ mentioned in the Introduction provides an example of the model to which the nested Bethe ansatz is not applicable. It would be interesting to apply our approach to this model and work out its exact solution using the method of the Baxter $Q-$operator.

In this paper, we have only focused on the eigenvalue problem for the $SL(2|1)$ magnet, relating its energy spectrum to that of the Baxter $Q-$operators. Another advantage of these operators is that they can be also used to construct the eigenfunctions of the $SL(2|1)$ magnet in the representation of separated variables (SoV) \[39\]. In spite of the fact that the SoV method has been formulated awhile ago, the number of models for which it has been successfully implemented is limited. In the SoV representation, the eigenfunction factorizes into a product of functions depending on a single separated variable. In the case of the $SL(2)$ spin chain, going over through an explicit construction of the SoV representation, one can show that the latter functions coincide with polynomial eigenvalues of the Baxter $Q-$operator \[32\]. It is expected that this property is rather general and it should also hold for spin chains with Lie (super)symmetry of high rank including the $SL(2|1)$ group. This question deserves additional study.

The construction of $Q-$operators can be extended to spin chains with the $SL(2|N)$ sym-
These models have recently attracted attention in light of the gauge/string duality, most notably for the maximally supersymmetric $\mathcal{N} = 4$ gauge theory. We expect that a generic noncompact $SL(2|\mathcal{N})$ transfer matrices can be factorized into products of $(\mathcal{N} + 2)$ distinct $Q$–operators. Two of them, $Q_1(u)$ and $Q_{N+2}(u)$, are analogous to the $SL(2)$ Baxter $Q$–operators while the remaining operators $Q_a$ (with $a = 2, \ldots, \mathcal{N} + 1$) should reflect a nontrivial $SU(\mathcal{N})$ group structure of the model. We demonstrated in this paper that for $\mathcal{N} = 1$, the operators $Q_1(u)$ and $Q_{N+1}(u)$ for the $SL(2|1)$ magnet can be obtained from the $SL(2)$ operators by a lift from the light-cone to the $\mathcal{N} = 1$ superspace $z \mapsto Z = (z, \theta, \bar{\theta})$. Going over from $\mathcal{N} = 1$ to higher $\mathcal{N}$, one should simply enlarge the number of ‘odd’ dimensions in the superspace $Z = (z, \theta^A, \bar{\theta}^A)$ with $A = 1, \ldots, \mathcal{N}$. In other words, for arbitrary $\mathcal{N}$, the Baxter operators $Q_1(u)$ and $Q_{N+2}(u)$ can be represented by the same Feynman diagrams as shown in Figs. 3 and 4. The only difference with the $\mathcal{N} = 1$ expressions is that the reproducing kernel and the integration measure should be modified to take into account the contribution from extra $(\mathcal{N} - 1)$ ‘odd’ coordinates in the superspace. Moreover, in the chiral limit, the operator $Q_{N+2}(u)$ has the form identical to (3.38). The only change is that for arbitrary $\mathcal{N}$ the Baxter operator in (3.38) acts in the space of functions defined in the $(\mathcal{N} + 1)$–dimensional superspace $W = (w, \theta^A)$. Remarkably enough, substitution of the operator $Q_{N+2}(u)$ into (3.45) yields the Hamiltonian which coincides with the one-loop dilatation operator in the $\mathcal{N}$–extended SYM theory, Eqs. (1.3) and (1.4). A detailed study of the $SL(2|\mathcal{N})$ Baxter $Q$–operators will be presented elsewhere.

### Acknowledgements

We are grateful to D. Karakhanyan for collaboration at an early stage of the project. Three of us (A.B., A.M. and S.D.) would like to thank Laboratoire de Physique Théorique (Orsay) for hospitality. G.K. is most grateful to A. Tsvelik for discussions. The work was supported in part by the U.S. National Science Foundation under grant no. PHY-0456520 (A.B.), by the RFFI grant 05-01-00922 and the DFG grant 436 Rus 17/9/06 (S.D.), by the ANR under grant BLAN06-3_143793 (G.K.) and by the Helmholtz Association under contract VH-NG-004 (A.M.).

### A Reducible representations of the $SL(2|1)$

The $SL(2|1)$ representation $[j, \bar{j}]$ is reducible for values of spins $j$ and $\bar{j}$ specified in Section 2.2.

### $\bar{j} = 0$

In this case, the $SL(2|1)$ generators defined in Eqs. (2.3) – (2.5) depend on the spin $j$. Denoting them as $G_j$ one finds that the superconformal derivative $D$, (2.17), intertwines the $SL(2|1)$ generators of spin $j$ and $j + 1$

$$DG_j = (-1)^{\bar{G}}G_{j+1}D$$

(A.1)

with the grading $\bar{G} = 0$ and $G = 1$ for even and odd generators, respectively. Let us consider the state $\Phi_+(z, \theta, \bar{\theta}) \in \mathcal{V}_{j,0}$ which satisfies the chirality condition $D\Phi_+(z, \theta, \bar{\theta}) = 0$. Applying both sides of (A.1) to $\Phi_+(z, \theta, \bar{\theta})$ one finds that the state $G_j \Phi_+(z, \theta, \bar{\theta})$ is also annihilated by the supercovariant derivative, $DG_j \Phi_+(z, \theta, \bar{\theta}) = 0$. Therefore, the zero modes of the operator $D$

\footnote{The $j$–dependence resides in three generators $J, V^+$ and $L^+$ only.}
form the $SL(2|1)$ invariant subspace $V_j$, Eq. (2.18). From $D = -e^{\frac{i\theta}{2}} \partial_{\theta} e^{-\frac{i\theta}{2}}$ one finds that the states $\Phi_+(z, \theta, \bar{\theta}) \in V_j$ have the following form

$$D \Phi_+ = 0 \quad \Rightarrow \quad \Phi_+(z, \theta, \bar{\theta}) = \chi(z_+) + \theta \phi(z_+) \quad (A.2)$$

with $z_+ = z + \frac{1}{2} \bar{\theta} \theta$ and $\chi(z)$ and $\phi(z)$ being analytical inside the unit disk $|z| < 1$. Expanding the right-hand side of (A.2) in powers of $z$ one finds that the basis vectors in the graded linear space $V_j$ are given by homogeneous polynomials in $\theta$ and $z + \frac{1}{2} \bar{\theta} \theta$ only, Eq. (2.18). The states (A.2) form the chiral $SL(2|1)$ representation $[j]_+$. Similar to (2.12), it can be decomposed over the $SL(2) \otimes U(1)$ multiplets $[27, 28, 29, 26]$.

$$[j]_+ = D_\ell(\ell) \oplus D_{\ell+1/2}(\ell - \frac{1}{2}) \quad (A.3)$$

with $\chi(z_+) \in D_\ell(\ell)$ and $\phi(z_+) \in D_{\ell+1/2}(\ell - \frac{1}{2})$.

In Eq. (2.20), the diagonal blocks $G_{++}$ and $G_{--}$ define two different representations of the $SL(2|1)$ generators on the invariant subspace, $V_j$, and the quotient, $V_{j,0}/V_j$, respectively. One applies both sides of (A.1) to $\Phi_- \in V_{j,0}/V_j$ and makes use of (2.20) together with $D \Phi_+ = 0$ to obtain

$$G_{j+1} D \Phi_+ = (\pm 1)^{\ell} D G_j \Phi_+ = (\pm 1)^{\ell} D \Phi_+ [G_{--}]^{3a} \quad (A.4)$$

It follows from this relation that (infinite-dimensional) graded matrix $G_{--}$ represents the $SL(2|1)$ generators of spin $j + 1$ on the space spanned by the states $D \Phi_-$ with $\Phi_- \in V_{j,0}/V_j$. Since $D \cdot D \Phi_-= 0$, this space is isomorphic to the chiral $SL(2|1)$ irreps

$$D \Phi_- \in V_{j+1} = \text{span}\left\{1, \theta z^k, (z + \frac{1}{2} \bar{\theta} \theta)^{k+1} | k \in \mathbb{N} \right\}. \quad (A.5)$$

The $SL(2|1)$ generators on $V_{j+1}$ are given by the same expressions as before, Eqs. (2.3) - (2.5), with $j$ replaced by $j + 1$ and $j = 0$.

$$j + \bar{j} = n$$

Let us introduce the following operator

$$\mathcal{I} = j (D \bar{D})^n - \bar{j} (\bar{D} D)^n = (-\partial_z)^n \left[ j D \bar{D} - \bar{j} \bar{D} D \right] \quad (A.6)$$

where $D$ and $\bar{D}$ are supercovariant derivatives (217). One can verify that the $SL(2|1)$ generators $G_{j,\bar{j}}$ defined in (2.3) - (2.5) satisfy the relation

$$\mathcal{I} G_{j,\bar{j}} = G_{-j,\bar{-j}} \mathcal{I} \quad (A.7)$$

and, therefore, $\mathcal{I}$ intertwines the corresponding $SL(2|1)$ representations $[j, \bar{j}]$ and $[-\bar{j}, -j]$. This relation is analogous to (A.1) and, as before, it implies that zero modes of the operator $\mathcal{I}$ belong to the $SL(2|1)$ invariant subspace $v_{n/2, b} = \ker \mathcal{I}$, Eq. (2.25),

$$\mathcal{I} \Phi_+(z, \theta, \bar{\theta}) = 0 \quad \Rightarrow \quad \Phi_+(z, \theta, \bar{\theta}) \in v_{n/2, b} \quad (A.8)$$

The basis in the quotient space $V_{j,\bar{j}}/v_{n/2, b}$ can be constructed by applying the raising operators $V^+, \bar{V}^+$ and $L^+$ to a given reference state $\Omega'$ which does not belong to (2.25) and has a smallest possible degree (= $n$) in $z$. One possible choice could be

$$\Omega' = z^n - \frac{1}{2} \alpha z^{n-1} \theta \bar{\theta} \quad (A.9)$$
with \( \alpha \neq j - \tilde{j} \) so that \( \Omega' \) is different from the highest weight \( [2,24] \). Notice that this state is not the lowest weight in \( V_{j,j} \). However, the states \( V^−\Omega', V^−\Omega' \) and \( L^−\Omega' \) belong to \( v_{n/2,b} \) and, therefore, equal zero in the quotient \( V_{j,j}/v_{n/2,b} \). Similar to \( [A.4] \), one applies both sides of \( (A.7) \) to \( \Phi_− \in V_{j,j}/v_{n/2,b} \) and obtains that the states \( \mathcal{I}\Phi_− \) form the \( SL(2|1) \) representation \([−\tilde{j},−j] \)

\[
\mathcal{I} \Phi_−(z,\theta,\tilde{\theta}) \in \mathbb{V}_{−j,−j}
\]

with \( \mathcal{I}\Omega' \sim 1 \).

\( j = \tilde{j} = 0 \)

The invariant subspace \( v_{00} \) contains only one state \( 1 \) and the quotient space \( \mathbb{V}_{0,0}/v_{00} \) takes the form \( [2,27] \). The spaces \( \mathbb{V}_+ \) and \( \mathbb{V}_− \) contain the states \( \Omega_+ = \tilde{\theta} \) and \( \Omega_− = \theta \), respectively, such that the vectors \( V^−\Omega_±, V^−\Omega_± \) and \( L^−\Omega_± \) either vanish or have zero projection onto \( \mathbb{V}_{0,0}/v_{00} \). This allows one to realize \( \mathbb{V}_+ \) and \( \mathbb{V}_− \) as lowest weight representations built over these two states. The reference states \( \Omega_± \) verify the (anti)chirality conditions

\[
D \Omega_+ = \bar{D} \Omega_− = 0
\]

and the same condition is fulfilled for all states inside \( \mathbb{V}_± \). One applies both sides of \( (A.1) \) to \( \Phi_−(z,\theta,\tilde{\theta}) \in \mathbb{V}_− \) and finds that for \( j = 0 \) the states \( D \Phi_−(z,\theta,\tilde{\theta}) \) form the \( SL(2|1) \) invariant chiral representation space \([1]_+ \), Eq. \( (2.18) \). Notice that the value of the chiral spin is shifted from \( j = 0 \) to \( j = 1 \) as a consequence of \( (A.1) \). The same result can also be obtained by examining the eigenvalues of the Cartan operators \( J \) and \( \tilde{J} \) for the reference state \( \Omega_−, J\Omega_− = \Omega_− \) and \( \tilde{J}\Omega_− = 0 \). In a similar manner, the chiral representation \( \mathbb{V}_+ \) can be mapped into the antichiral representation \([1]_− \).

\( j = −n, \tilde{j} = 0 \)

According to \( [2,21] \) the representation \([−n,0] \) decomposes into a semidirect sum of two chiral representations \([−n]_+ \) and \([−n+1]_+ \) which are both indecomposable reducible for \( n \geq 1 \). Let us consider the representation \([−n]_+ \) and introduce the operator

\[
\mathcal{I} = −\bar{D}(D\bar{D})^n = −\bar{D}(−\partial_z)^n
\]

It annihilates the invariant subspace \( v_n \) defined in \( [2,23] \), \( \mathcal{I} \Phi_+ = 0 \) for \( \Phi_+ \in v_n \), and intertwines the \( SL(2|1) \) representations \([−n]_+ \) and \([n+1]_− \)

\[
\mathcal{I}G_{−n,0} = (−1)^{\tilde{\theta}}G_{0,1+n}I
\]

The quotient space \( \mathbb{V}_{−n}/v_n \) is spanned by the states

\[
\mathbb{V}_{−n}/v_n = \text{span} \{ θz^n, z_+^{n+1}, θz^{n+1}, z_+^{n+2}, \ldots \}
\]

As before, one applies both sides of \( (A.13) \) to \( \Phi_− \in \mathbb{V}_{−n}/v_n \) and finds that \( \mathcal{I} \Phi_− \) form the antichiral \( SL(2|1) \) representation \([n+1]_− \)

\[
\mathcal{I} \Phi_− \in \mathbb{V}_{n+1} = \text{span} \{ 1, \tilde{\theta}z^k, z_+^{k+1} \mid k \in \mathbb{N} \}
\]

with \( z_± = z ± \frac{1}{2}θ\tilde{\theta} \). One concludes that \([−n]_+ \) decomposes into a semidirect sum of \( v_n \) and \( \mathbb{V}_{n+1} \).
B Calculation of the normalization factors

In this Appendix we calculate the normalization factors entering the expressions for the $R-$matrix, Eqs. (4.3) and (4.10). In all cases, the calculation goes through the same main steps. Let $[j, \bar{j}]$ be a reducible indecomposable representation of the $SL(2|1)$ and let $V^+$ be invariant subspace of $V_{j,\bar{j}}$. The $R-$operator acting on the tensor product $V_{2q,\bar{q}} \otimes V_{j,\bar{j}}$ has a block-triangular form (4.2). The upper diagonal block $R^+(u)$ defines the $R-$operator on $V_{2q,\bar{q}} \otimes V^+$. It can be defined by applying both sides of (4.2) to the same test vector $\Phi^+ \in V^+$

$$R_{V_{2q,\bar{q}} \otimes V^+}(u)\Phi^+ = R_{V_{2q,\bar{q}} \otimes V_{j,\bar{j}}}(u)\Phi^+, \quad (B.1)$$

The expression in the right-hand side of this relation can be evaluated using explicit expression for the operator $R_{V_{2q,\bar{q}} \otimes V_{j,\bar{j}}}(u)$ from Ref. [25]. The invariant subspace can be defined as a kernel of a certain operator $I$, so that $I\Phi^+ = 0$ for $\Phi^+ \in V^+$. The same operator maps the quotient space $V_{2q,\bar{q}}/V^+$ into yet another representation, say $V_{j,\bar{j}}$, leading to

$$I R_{V_{2q,\bar{q}} \otimes V_{j,\bar{j}}}(u)\Phi^- = c(u)R_{V_{2q,\bar{q}} \otimes V_{j',\bar{j}'}}(u)I\Phi^-, \quad (B.2)$$

where $\Phi^-$ is an arbitrary test vector in $V_{j,\bar{j}}/V^+$. Using explicit expressions for the $R-$ and $I-$operators, one can evaluate both sides of this relation and, then, determine the normalization factor $c(u)$. The calculation can be simplified by choosing $\Phi_-$ to be the lowest weight in the tensor product $V_{2q,\bar{q}} \otimes V_{j,\bar{j}}$. Then, the vector $I\Phi^-$ is automatically the lowest weight in $V_{2q,\bar{q}} \otimes V_{j',\bar{j}'}$ and the two $R-$operators entering (B.2) are diagonalized simultaneously. As a result, $c(u)$ is given by the ratio of the corresponding eigenvalues.

A complete classification of the lowest weights in the tensor product of two generic $SL(2|1)$ representations $V_{2q,\bar{q}} \otimes V_{j,\bar{j}}$ can be found in Ref. [25]. One finds among them the states $1$ and $(\bar{\theta}_1 - \bar{\theta}_2)(z_1 - z_2)^k$ (with $k \in \mathbb{N}$) that we shall use as reference states $\Phi^-$ in (B.2). The action of the $R-$operator on these states looks like

$$R_{V_{2q,\bar{q}} \otimes V_{j,\bar{j}}}(u) \cdot 1 = r(j, \bar{j}), \quad R_{V_{2q,\bar{q}} \otimes V_{j,\bar{j}}}(u) \cdot \bar{\theta}_1 z_{12}^k = r_k(j, \bar{j}) \bar{\theta}_1 z_{12}^k, \quad (B.3)$$

where $\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2$, $z_{12} = z_1 - z_2$ and the notation was introduced for

$$r_k(j, \bar{j}) = \xi(-1)^k e^{i\pi(j+\bar{j})/2} (u + j_q - j + \bar{j}) (u - j_q - j + \bar{j}) \Gamma(k + u + j_q + \bar{j} + 1) \Gamma(k - u + j_q + \bar{j} + 1)$$

$$r(j, \bar{j}) = \frac{u + j_q - j - \bar{j}}{u + j_q - j + \bar{j}} r_0(j, \bar{j}), \quad (B.4)$$

with $\xi = e^{-i\pi(j_q+\bar{j}/2)}/(j_{q}\bar{j}_{q})$.

$$j + \bar{j} = -n$$

Let us choose a reference state in (B.2) as

$$\Phi^-(Z_1, Z_2) = (\bar{\theta}_1 - \bar{\theta}_2)(z_1 - z_2)^n, \quad I \Phi^-(Z_1, Z_2) = (\bar{\theta}_1 - \bar{\theta}_2), \quad (B.5)$$

where the operator $I$ is given by the differential operator (A.6) acting on $Z_2-$coordinates. In Eq. (B.2), this operator intertwines the $SL(2|1)$ representations $V_{j,\bar{j}}$ and $V_{-\bar{j},-j}$ leading to $c(u) = r_n(j, \bar{j})/r_0(-\bar{j}, -j) = 1$, in agreement with (4.3).
\[ j = 0 \]

According to (A.5) and (A.5), the operator \( I = D \) intertwines the \( SL(2|1) \) representations \( V_{j,0} \) and \( V_{j+1} \). One chooses the reference state in (B.2) as \( \Phi^+ = \bar{\theta}_1 - \bar{\theta}_2 \) and obtains

\[
D_2 \mathcal{R}_{V_{j,q}} \otimes V_{j,0}(u) \cdot \bar{\theta}_{12} = c(u) \mathcal{R}_{V_{j,q}} \otimes V_{j+1}(u) \cdot 1 = c(u) \mathcal{R}_{V_{j,q}} \otimes V_{j+1,0}(u) \cdot 1,
\]

where in the last relation we took into account that \( V_{j+1} \) is an invariant subspace of \( V_{j+1,0} \). One applies (B.3) and (B.4) to get \( c(u) = r_0(j,0)/r(j+1,0) = \alpha(u-j) \) in agreement with (4.11).

\[ j = \bar{j} = 0 \]

The space \( \mathcal{V}_{0,0} \) has invariant subspace \( v_0 = \{ 1 \} \). For \( j = \bar{j} = 0 \) one finds from (B.3) and (B.6)

\[
\mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{0,0}(u) \cdot 1 = \mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{0,0}(u) \cdot 1 = r(0,0)
\]

and, therefore, \( t_0(u) \cdot 1 = [r(0,0)]^N \), in agreement with (4.20).

\[ j = 0, \quad \bar{j} = -n \]

The operator \( I = (-D)(-\partial_{z_2})^n \) intertwines the representations \( \bar{\mathcal{V}}_{-n} \) and \( \mathcal{V}_{n+1} \). Choosing \( \Phi^- = \bar{\theta}_{12} \bar{z}_{12} \) in (B.2) one finds

\[
(-D_2)(-\partial_{z_2})^n \mathcal{R}_{V_{j,q}} \otimes \bar{\mathcal{V}}_{-n}(u) \bar{\theta}_{12} \bar{z}_{12}^n = c(u) n! \mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{n+1}(u) \cdot 1
\]

and, therefore, \( c(u) = r_n(0,-n)/r(n+1,0) = \alpha(u-n) \) leading to (4.15).

**Relation between Lax and \( \mathcal{R} \)-operators**

Let us examine the relation (6.7). The operators entering both sides of (6.7) act in the tensor product \( V_{j,q} \otimes \bar{\mathcal{V}}_1 \). The basis in the three-dimensional space \( \bar{\mathcal{V}}_1 \) can be chosen as \( \{ 1, \bar{\theta}_2, z_2 - \frac{1}{2} \bar{\theta}_2 \theta_2 \} \), Eq. (2.32), while the basis in \( V_{j,q} \) can be defined as in (2.1). One verifies that the states 1 and \( \bar{\theta}_{12} \) belong to \( V_{j,q} \otimes \bar{\mathcal{V}}_1 \) and define there the lowest weight vectors. One uses the explicit expression for the Lax operator (B.2) to obtain

\[
\bar{L}(u+1) \cdot 1 = (1 - u - j_q), \quad \bar{L}(u+1) \cdot \bar{\theta}_{12} = (-u + \bar{j}_q - j_q) \bar{\theta}_{12}.
\]

The operator \( \mathcal{R}_{V_{j,q}} \otimes \bar{\mathcal{V}}_1(u) \) appears as an upper diagonal block of the operator \( \mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{0,-1}(u) \) and, therefore, one gets from (B.3)

\[
\mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{0,-1}(u) \cdot 1 = r(0,-1), \quad \mathcal{R}_{V_{j,q}} \otimes \mathcal{V}_{0,-1}(u) \cdot \bar{\theta}_{12} = r_0(0,-1) \bar{\theta}_{12}.
\]

One uses (B.9) and (B.10) to verify with a help of (B.4) that

\[
\Delta(u-1) = \frac{r(0,-1)}{u - j_q - 1} = \frac{r_0(0,-1)(u + j_q - \bar{j}_q)}{(u + j_q - 1)(u - j_q - 1)},
\]

in agreement with (6.7) and (5.4). In the similar manner, one uses the relations \( \mathcal{R}_{V_{j,q}} \otimes \bar{\mathcal{V}}_1(u) \cdot 1 = r(-1,0) \) and \( \bar{L}(u+1) \cdot 1 = (u + 1 - \bar{j}_q) \) to verify (6.6) with \( \Delta(u) = r(-1,0)/(u - j_q + 1) \).
C Matrix representation of the $R$–operators

The operators $R^{(a)}(u)$ (with $a = 1, 2, 3$) are defined in Eqs. (3.12) – (3.16). According to (3.7), they map the tensor product of two infinite-dimensional $SL(2|1)$ representations $V_{j_1,j_1} \otimes V_{j_2,j_2}$ into tensor product of two yet another representations $V_{j'_1,j'_1} \otimes V_{j'_2,j'_2}$ with $j_1 + j_2 = j'_1 + j'_2$ and $\bar{j}_1 + \bar{j}_2 = \bar{j}'_1 + \bar{j}'_2$. Both tensor products can be decomposed over irreducible components as 29, 20, 25

$$[j_1,\bar{j}_1] \otimes [j_2,\bar{j}_2] = [j_{12},\bar{j}_{12}] + \sum_{n \geq 1} 2[j_{12} + n, \bar{j}_{12} + n] + [j_{12} + n - 1, \bar{j}_{12} + n] + [j_{12} + n, \bar{j}_{12} + n - 1],$$  \hspace{1cm} (C.1)

where $j_{12} = j_1 + j_2$, $\bar{j}_{12} = \bar{j}_1 + \bar{j}_2$ and the factor 2 inside the sum takes into account multiplicity. The $R^{(a)}$–operators map each irreducible component in the right-hand side of (C.1) into similar component in the tensor product $[j'_1,\bar{j}'_1] \otimes [j'_2,\bar{j}'_2]$ carrying the same spins. In particular, they transform the lowest weight vectors in $V_{j_1,j_1} \otimes V_{j_2,j_2}$ into those in $V_{j'_1,j'_1} \otimes V_{j'_2,j'_2}$. In the right-hand side of (C.1), the lowest weight of the $[j_{12},\bar{j}_{12}]$ representation is 1 and the $R^{(a)}$–operators act as

$$R^{(a)}(u) \cdot 1 = r^{(a)}_u.$$  \hspace{1cm} (C.2)

For $n \geq 1$ the lowest weights in four $SL(2|1)$ components in (C.1) are 25

$$e_i = \{(Z_{12} + \frac{1}{2} \theta_{12} \bar{\theta}_{12})^n, (Z_{12} - \frac{1}{2} \theta_{12} \bar{\theta}_{12})^n, \bar{\theta}_{12} Z_{12}^n, \theta_{12} Z_{12}^n\},$$  \hspace{1cm} (C.3)

where $Z_{12} = z_1 - z_2 + \frac{1}{2} \theta_1 \bar{\theta}_2 + \frac{1}{2} \bar{\theta}_1 \theta_2$ and $1 \leq i \leq 4$. Then, the $R^{(a)}$–operators can be represented by a graded $4 \times 4$ matrix

$$R^{(a)}(u) \cdot e_i = \sum_{k=1}^4 e_k [R^{(a)}(u)]_{ki}, \hspace{1cm} [R^{(a)}(u)]_{ki} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix},$$  \hspace{1cm} (C.4)

where ‘*’ denote entries that may take nonvanishing values.

The calculation of $r^{(a)}_u$ and the matrices $[R^{(a)}(u)]_{ki}$ is straightforward with a help of (3.12) – (3.16). The expressions for $r^{(1)}_u$ and $r^{(3)}_u$ are given in (3.14), while $r^{(2)}_u$ can be easily found from (3.15) as $r^{(2)}_u = j_2/(j_2 + u)$. In what follows, we present explicit expressions for the matrices $[R^{(a)}(u)]_{ki}$ evaluated for $n \geq 1$ and for the spins $j_1$, $\bar{j}_1$, $j_2$ and $\bar{j}_2$ taking the same values as in the definition of the $Q$–operators, Eqs. (3.20) and (3.24). Namely, $j_1 = j_q$, $\bar{j}_1 = \bar{j}_q$ and the remaining spins are fixed as follows:

$\bullet$ $j_2 = j_q$, $\bar{j}_2 = \bar{j}_q - j_q - u$ :

$$R^{(1)}(u + j_q) := e^{-i\pi(u + j_q)/2} \frac{\Gamma(n + 1 + j_q + \bar{j}_q)}{\Gamma(n + 1 - u + j_q)} \begin{pmatrix} -\frac{u+j_q-j_q}{j_q} & -\frac{(u+j_q)j_q}{(n+j_q+j_q)j_q} & 0 & 0 \\ 0 & -\frac{n+u-j_q}{n+j_q+j_q} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{u+j_q-j_q}{j_q} \end{pmatrix};$$  \hspace{1cm} (C.5)
\[ j_2 = \tilde{j}_q - u, \quad \tilde{j}_2 = j_q + u : \]
\[
\mathcal{R}^{(2)}(u + j_q - \tilde{j}_q) := \begin{pmatrix}
\frac{u + j_q}{j_q} & 0 & 0 & 0 \\
- \frac{(n + j_q)(u + j_q - \tilde{j}_q)}{j_q^2} & \frac{u - j_q}{j_q} & 0 & 0 \\
0 & 0 & - \frac{(u - j_q)(u + j_q)}{j_q^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad (C.6)
\]

\[ j_2 = j_q - \tilde{j}_q + u, \quad \tilde{j}_2 = \tilde{j}_q : \]
\[
\mathcal{R}^{(3)}(u - \tilde{j}_q) := e^{i\pi(\tilde{j}_q - u)/2} \frac{\Gamma(n + 1 + u + j_q)}{\Gamma(n + 1 + j_q + \tilde{j}_q)} \begin{pmatrix}
1 & \frac{(n + j_q)(u + j_q - \tilde{j}_q)}{j_q(n + u + j_q)} & 0 & 0 \\
0 & 0 & - \frac{(n + j_q)(u + j_q - \tilde{j}_q)}{j_q(n + u + j_q)} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{u + j_q - \tilde{j}_q}{j_q}
\end{pmatrix}. \quad (C.7)
\]

For \( j_2 = j_q \) and \( \tilde{j}_2 = \tilde{j}_q \) these matrices reduce to unity matrix in agreement with (3.9).

The relations (C.5) – (C.7) are in agreement with expressions for the corresponding matrix elements from the second paper in [23]. Notice that the operators \( \mathcal{R}_3(u_1, u_3, u_1|v_1), \mathcal{R}_3(u_1, u_2|v_2, v_3) \) and \( \mathcal{R}_1(u_1, v_1, v_2, v_3) \) introduced there coincide with the operators \( \mathcal{R}^{(a)}(u) \) (with \( a = 1, 2, 3 \)) up to the normalization factors
\[
\mathcal{R}^{(3)}(u_3 - v_3) = e^{i\pi(u_3 - v_3)/2} \frac{u_3 - v_3}{u_2 - u_3} \mathcal{R}_3(u_1, u_2, u_3|v_1),
\]
\[
\mathcal{R}^{(2)}(u_2 - v_2) = \frac{u_2 - v_2}{(u_1 - u_2)(u_2 - v_3)} \mathcal{R}_2(u_1, u_2|v_2, v_3),
\]
\[
\mathcal{R}^{(1)}(u_1 - v_1) = e^{-i\pi(u_1 - v_1)/2} \frac{u_1 - v_1}{u_1 - v_2} \mathcal{R}_1(u_1, v_1, v_2, v_3), \quad (C.8)
\]
with the \( v \)– and \( u \)–parameters given by (3.3).

**D  Integral representation of \( \mathcal{R}^{(3)} \)–operator**

In this Appendix, we demonstrate that the operator \( \mathcal{R}^{(3)}(u) \) entering the factorized expression for the \( \mathcal{R} \)–operator (3.6) can be realized in the space of functions \( \Phi(W_1, W_2) \in \mathcal{V}_{j_1, \tilde{j}_1} \otimes \mathcal{V}_{j_2, \tilde{j}_2} \) as an integral operator (3.13). The calculations for \( \mathcal{R}^{(1)} \)–operator are identical with minor modifications.

The operator \( \mathcal{R}^{(3)}(u) \) acts in the tensor product \( \mathcal{V}_{j_1, \tilde{j}_1} \otimes \mathcal{V}_{j_2, \tilde{j}_2} \) and changes the spins of the vector spaces as in (3.7). This translates into the following relation
\[
\mathcal{R}^{(3)}(u) \left( G_{j_1, \tilde{j}_1} + G_{j_2, \tilde{j}_2} \right) = \left( G_{j_1, u_1, j_1} + G_{j_2, u_2, j_2} \right) \mathcal{R}^{(3)}(u), \quad (D.1)
\]
where \( G_{j, \tilde{j}} \) denote the \( SL(2|1) \) generators in \( \mathcal{V}_{j, \tilde{j}} \), Eqs. (2.23) – (2.25). By definition, the operator \( \mathcal{R}^{(3)}(u_3 - v_3) \) interchanges the spectral parameters \( u_3 \leftrightarrow v_3 \) in the product of two Lax operators.
Finally, taking into account the relation (C.2),
\[ [\mathcal{R}^{(3)}(u), w_2] = [\mathcal{R}^{(3)}(u), \vartheta_2] = [\mathcal{R}^{(3)}(u), \bar{\vartheta}_2] = 0, \quad (D.2) \]
where \( W_2 = (w_2, \vartheta_2, \bar{\vartheta}_2) \) denote the coordinates in the graded spaces \( \mathcal{V}_{j_2 j_2} \) and \( \mathcal{V}_{j_2 - u j_2} \). Equation (D.2) implies that \( \mathcal{R}^{(3)}(u) \) acts nontrivially only on the \( W_1 \) — coordinates of a test function
\[ \mathcal{R}^{(3)}(u) \Phi(W_1, W_2) = \int [DZ]_{j_1 j_2} R^{(3)}_u(W_1; Z^*_1) \Phi(Z_1, W_2). \quad (D.3) \]
The same relation can be rewritten in terms of the reproducing kernel (2.52) as
\[ R^{(3)}_u(W_1; Z^*_1) = \mathcal{R}^{(3)}(u) \mathcal{K}_{j_1 j_2}(W_1; Z^*_1). \quad (D.4) \]
Indeed, substituting this relation into (D.3) one performs \( Z_1 \)— integration with a help of (2.51) and arrives at the identity. The reproducing kernel \( \mathcal{K}_{jj}(W; Z^*) \), Eq. (2.52), can be rewritten as a sequence of finite \( SL(2|1) \) transformations applied to the lowest weight vector in \( \mathcal{V}_{j_1 j_2} \), namely,
\[ \mathcal{K}_{jj}(W; Z^*) = e^{-\theta_j V_j^+} e^{-\theta_j^* V_j^+} e^{z_j^* L_j^+} \cdot 1, \quad (D.5) \]
where \( Z^* = (z^*, \theta^*, \bar{\theta}^*) \) plays the rôle of the transformation parameters and the raising generators \( V_j^+, V^+_j \) and \( L_j^+ \) are given by the differential operators (2.4) acting on functions depending on \( W = (w, \vartheta, \bar{\vartheta}) \). This fact combined together with the commutation relations (D.1) and (D.2) allows one to find the integral kernel \( R^{(3)}(W_1; Z^*_1) \) algebraically.

At the first step, one applies (D.5) to rewrite the product of two reproducing kernels as
\[ \mathcal{K}_{j_1 j_2}(W_1; Z^*_1) \mathcal{K}_{j_2 j_2}(W_2; Z^*_2) = e^{-\theta_{j_1}^* (V_{j_1 j_1}^+ + V^+_{j_2 j_2})} e^{-\theta_{j_2}^* (V_{j_2 j_1}^+ + V^+_{j_2 j_2})} e^{z_{j_1}^* \cdot (L_{j_1,j_1}^- + L_{j_2 j_2}^+)} \cdot 1. \quad (D.6) \]
Let us apply the operator \( \mathcal{R}^{(3)}(u) \) to both sides of this relation. In the left-hand side, it only acts on the \( W_1 \)— dependent kernel, while in the right-hand side it can be moved to the right across the exponent by virtue of (D.1) so that
\[ \mathcal{K}_{j_2 j_2}(W_2; Z^*_2) \left[ \mathcal{R}^{(3)}(u) \mathcal{K}_{j_1 j_2}(W_1; Z^*_1) \right] = e^{-\theta_{j_1}^* (V_{j_1,j_1}^+ + V^+_{j_2 j_2})} e^{-\theta_{j_2}^* (V_{j_1,j_2}^+ + V^+_{j_2 j_2})} e^{z_{j_2}^* \cdot (L_{j_1,j_1}^- + L_{j_2 j_2}^+)} \mathcal{R}^{(3)}(u) \cdot 1. \quad (D.7) \]
Finally, taking into account the relation (C.2), \( \mathcal{R}^{(3)}(u) \cdot 1 = r^{(3)}_u \), one finds from (D.7), (D.4) and (D.6)
\[ R^{(3)}_u(W_1; Z^*_1) = r^{(3)}_u \mathcal{K}_{j_1 + u j_1}(W_1; Z^*_1) \mathcal{K}_{j_2 - u j_2}(W_2; Z^*_2) / \mathcal{K}_{j_2 j_2}(W_2; Z^*_2) \]
\[ = r^{(3)}_u \mathcal{K}_{j_1 + u j_1}(W_1; Z^*_1) \mathcal{K}_{- u, 0}(W_2; Z^*_1). \quad (D.8) \]
Matching the relations (D.3) and (D.8) into (3.12) one finds the integral kernel of the operator \( \mathcal{R}^{(3)}(u) \)
\[ R^{(3)}_u(W_1, W_2; Z^*_1, Z^*_2) = R^{(3)}_u(W_1; Z^*_1) \mathcal{K}_{j_2 j_2}(W_2, Z^*_2), \quad (D.9) \]
which coincides with (3.13).
Derivation of the factorized expression for the transfer matrix (3.23) relies on the commutativity property of the \( R^{(a)} - \) operators, Eq. (3.11). To simplify manipulations with integral operators \( R^{(a)}_{k0}(u) \), it is convenient to introduce a diagrammatic representation of their integral kernel \( R^{(a)}(W_k, W_0; Z^*_k, Z^*_0) \) as shown in Fig. 7. The arguments of the kernel define the coordinates of four end-points of the diagram. The product of operators in the left-hand side of (3.11) is represented by the integral kernel

\[
R^{(a)}_{12}(u) R^{(b)}_{23} := \int [DZ'_2] j_2 \bar{j}_2 R^{(a)}(W_1, W_2; Z^*_1, Z'^*_2) R^{(b)}(Z'_2, W_3; Z^*_2, Z^*_3)
\]  

where the \( SL(2|1) \) representation \([j_2, \bar{j}_2] \) in the ‘intermediate’ space is \([j_2, -j_2 - v] \) for \( b = 1 \), \([j_2 - v, \bar{j}_2 + v] \) for \( b = 2 \) and \([j_2 + v, \bar{j}_2] \) for \( b = 3 \). Convolution of two integral kernels in (E.1) can be represented as the diagram shown in the left panel of Fig. 8. In the similar manner, the right panel of Fig. 8 corresponds to convolution of the kernels in the right-hand side of (3.11).

Let us start with a general expression for the transfer matrix (3.18) and substitute the \( R - \) operators by the factorized expression (3.6) (see left panel in Fig. 9).

\[
R_{k0}(w) = \Pi_{k0} R^{(12)}_{k0}(w_1, w_2) R^{(3)}_{k0}(w_3),
\]

where \( R^{(12)}_{k0}(w_1, w_2) = R^{(1)}_{k0}(w_1) R^{(2)}_{k0}(w_2) \), the permutation operator \( \Pi_{k0} \) was defined in (3.5) and the parameters \( w_a = u_a - v_a \) are given by (3.3) with \( w = u - v \). Next, one makes use of cyclicity...
Figure 9: Steps in factorization of the transfer matrix $T_{j\bar{j}}(w)$ for spin chain of length $N = 2$.

of the supertrace in (3.18) to move the right-most operator $\mathcal{R}^{(3)}_{10}(w_3)$ in front of $\mathcal{R}^{(12)}_{10}(w_1, w_2)$ (see central panel in Fig. 9) and, then, applies the relation (3.11) shown in Fig. 8 to interchange $\mathcal{R}^{(3)}$ and $\mathcal{R}^{(12)}$—operators. The result is displayed in Fig. 9, to the right and reads in the symbolic form

$$T_{j\bar{j}}(w) = \text{str} \left[ \Pi_{N_0} \mathcal{R}^{(12)}_{N_0}(w_1, w_2) \ldots \Pi_{10} \mathcal{R}^{(12)}_{10}(w_1, w_2) \right] \mathbb{P}^{-1} \text{str} \left[ \Pi_{N_0} \mathcal{R}^{(3)}_{N_0}(w_3) \ldots \Pi_{10} \mathcal{R}^{(3)}_{10}(w_3) \right],$$

(E.3)

where $\mathbb{P}$ is the operator of cyclic permutation (3.22). The last factor in this relation can be identified as the transfer matrix $T_{j_q + \bar{j}_q \bar{j}_q}(w_3)$, Eq. (3.20). Repeating the above consideration for the first factor in Eq. (E.3), we eventually arrive at

$$T_{j\bar{j}}(w) = \text{str} \left[ \Pi_{N_0} \mathcal{R}^{(1)}_{N_0}(w_1) \ldots \Pi_{10} \mathcal{R}^{(1)}_{10}(w_1) \right] \mathbb{P}^{-1}$$

$$\times \text{str} \left[ \Pi_{N_0} \mathcal{R}^{(2)}_{N_0}(w_2) \ldots \Pi_{10} \mathcal{R}^{(2)}_{10}(w_2) \right] \mathbb{P}^{-1} \text{str} \left[ \mathbb{P}_{10} \mathcal{R}^{(3)}_{10}(w_3) \ldots \mathbb{P}_{N_0} \mathcal{R}^{(3)}_{N_0}(w_3) \right].$$

This relation can be rewritten in terms of the transfer matrices (3.20) as follows

$$T_{j\bar{j}}(w) = T_{j_q \bar{j}_q - w_1}(w_1) \mathbb{P}^{-1} T_{j_q - w_2, \bar{j}_q + w_2}(-w_2) \mathbb{P}^{-1} T_{j_q + w_3, \bar{j}_q}(w_3)$$

$$= \mathbb{P}^{-2} T_{j_q \bar{j}_q - w_1}(w_1) T_{j_q - w_2, \bar{j}_q + w_2}(-w_2) T_{j_q + w_3, \bar{j}_q}(w_3),$$

(E.4)

with the spectral parameters $w_1 = w - j + j_q$, $w_2 = w - j + \bar{j} + j_q - \bar{j}_q$ and $w_3 = w + \bar{j} - \bar{j}_q$.

References

math.qa/0503410]
[26] L. Frappat, P. Sorba, A. Sciarrino, Dictionary on Lie algebras and superalgebras, Academic
Press (London, 2000); [hep-th/9607161]
[34] B. Sutherland, Phys. Rev. B 12 (1975) 3795.
V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Quantum inverse scattering method and
355;