The Isgur-Wise function in the BPS limit

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Abstract

From sum rules in the heavy quark limit of QCD, using the non-forward amplitude, we demonstrate that if the slope $\rho^2 = -\xi'(1)$ of the Isgur-Wise function $\xi(w)$ attains its lower bound $\frac{3}{4}$ (as happens in the BPS limit proposed by Uraltsev), the IW function is completely determined, given by the function

$$\xi(w) = \left(\frac{2}{w+1}\right)^{3/2}.$$
In the leading order of the heavy quark expansion of QCD, Bjorken sum rule (SR) \cite{1,2} relates the slope of the elastic Isgur-Wise (IW) function $\xi(w)$, to the IW functions of the transition between the ground state $j^P = \frac{1}{2}^-$ and the $j^P = \frac{1}{2}^+ \cdot \frac{3}{2}^+$ excited states, $\tau_{1/2}^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, at zero recoil $w = 1$ ($n$ is a radial quantum number). This SR leads to the lower bound $-\xi'(1) = \rho^2 \geq \frac{1}{4}$. A new SR was formulated by Uraltsev in the heavy quark limit \cite{3}, involving also $\tau_{1/2}^{(n)}(1)$, $\tau_{3/2}^{(n)}(1)$, that implies, combined with Bjorken SR, the much stronger lower bound

$$\rho^2 \geq \frac{3}{4}$$

(1)

A basic ingredient in deriving this bound was the consideration of the non-forward amplitude $B(v_i) \to D^{(n)}(v') \to B(v_f)$, allowing for general $v_i$, $v_f$, $v'$ and where $B$ is a ground state meson. In refs. \cite{4,5,6} we have developed, in the heavy quark limit of QCD, a manifestly covariant formalism within the Operator Product Expansion (OPE), using the matrix representation \cite{7} for the whole tower of heavy meson states \cite{8}. We did recover Uraltsev SR plus a general class of SR that allow to bound also higher derivatives of the IW function,

$$(-1)^L \xi^{(L)}(1) \geq \frac{(2L + 1)!!}{2^{2L}}$$

(2)

The general SR obtained from the OPE can be written in the compact way \cite{4}:

$$L_{\text{Hadrons}}(w_i, w_f, w_{if}) = R_{\text{OPE}}(w_i, w_f, w_{if})$$

(3)

where the l.h.s. is the sum over the intermediate $D$ states, while the r.h.s. is the OPE counterpart. This expression writes, in the heavy quark limit \cite{4}:

$$\sum_{D=P,V} \sum_n \text{Tr} \left[ \overline{B}_f(v_f) \Gamma_f D^{(n)}(v') \right] \text{Tr} \left[ \overline{D}^{(n)}(v') \Gamma_i B_i(v_i) \right] \xi^{(n)}(w_i) \xi^{(n)}(w_f)$$

$$+ \text{Other excited states} = -2\xi(w_{if}) \text{Tr} \left[ \overline{B}_f(v_f) \Gamma_f P'_+ \Gamma_i B_i(v_i) \right]$$

(4)

where $w_i = v_i \cdot v'$, $w_f = v_f \cdot v'$, $w_{if} = v_i \cdot v_f$. $P'_+ = \frac{1 + v'}{2}$ is the positive energy projector on the intermediate $c$ quark and the $B$ meson is the pseudoscalar ground state $(j^P, J^P) = \left( \frac{1}{2}^-, 0^- \right)$, where $j$ is the angular momentum of the light cloud and $J$ the spin of the bound state. The heavy quark currents considered in the preceding expression are $\overline{h}_{\nu'} \Gamma_i h_{\nu}$, $\overline{h}_{\nu'} \Gamma_f h_{\nu'}$ and $B(v)$, $D(v)$ are the $4 \times 4$ matrices representing the $B$, $D$ states \cite{7,8}. 

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The domain for the variables \((w_i, w_f, w_{if})\) is \([4]\): \(w_i \geq 1, w_f \geq 1,\)

\[
(w_i w_f - \sqrt{(w_i^2 - 1)(w_f^2 - 1)}) \leq w_{if} \leq w_i w_f + \sqrt{(w_i^2 - 1)(w_f^2 - 1)}
\]  

(5)

In\([4]\) the following SR were established. Taking \(\Gamma_i = \frac{\gamma_i}{v_i}\) and \(\Gamma_f = \frac{\gamma_f}{v_f}\) and \(w_i = w_f = w\) one finds the so-called Vector SR

\[
(w + 1)^2 \sum_{L \geq 0} \frac{L + 1}{2L + 1} S_L(w, w_{if}) \sum_n \left[\tau_{L+1/2}^{(L)(n)}(w)\right]^2
\]

\[+
\sum_{L \geq 1} S_L(w, w_{if}) \sum_n \left[\tau_{L-1/2}^{(L)(n)}(w)\right]^2 = (1 + 2w + w_{if}) \xi(w_{if})
\]

(6)

and for \(\Gamma_i = \frac{\gamma_i}{v_i} \gamma_5\) and \(\Gamma_f = \frac{\gamma_f}{v_f} \gamma_5\) one finds the Axial SR

\[
\sum_{L \geq 0} S_{L+1}(w, w_{if}) \sum_n \left[\tau_{L+1/2}^{(L)(n)}(w)\right]^2 + (w - 1)^2 \sum_{L \geq 1} \frac{L}{2L - 1} S_{L-1}(w, w_{if})
\]

\[-
\sum_n \left[\tau_{L-1/2}^{(L)(n)}(w)\right]^2 = - (1 - 2w + w_{if}) \xi(w_{if})
\]

(7)

In the precedent equations the IW functions \(\tau_{L+1/2}^{(L)(n)}(w)\) correspond to the transitions \(\frac{1}{2}^{-}\to j = L \pm \frac{1}{2}\) and the function \(S_L(w, w_{if})\) is given by the Legendre polynomial

\[
S_L(w, w_{if}) = \sum_{0 \leq k \leq L/2} C_{L,k} \left(w^2 - 1\right)^{2k} \left(w^2 - w_{if}\right)^{L-2k}
\]

(8)

with

\[
C_{L,k} = \frac{(-1)^k (L)!^2}{(2L)!^2} \frac{(2L - 2k)!}{k!(L-k)!(L-2k)!}
\]

(9)

Differentiating \(n\) times both SR \((6), (7)\) with respect to \(w_{if}\) and going to the border of the domain \(w_{if} = 1\), one gets, the bounds \((2)\).

On the other hand, Uraltsev \([9]\) has proposed a special limit of HQET, namely the so-called BPS limit, that implies \(\rho^2 = \frac{3}{4}\). We have demonstrated \([10]\), using the above SR, that if the slope reaches its lower bound \((1)\), as happens in the BPS limit, then all derivatives reach their lower bounds \((2)\), and then the Isgur-Wise function is completely fixed, namely

\[
\xi(w) = \left(\frac{2}{w + 1}\right)^{3/2}
\]

(10)

The motivation to introduce the BPS limit \([9]\) has been the rather close values obtained from experiment in inclusive \(B\) decay for the fundamental parameters \(\mu^2_\pi\)
and $\mu^2_G$:

$$
\mu^2_\pi = \frac{-< B(v)|O^{(b)}_{\text{kin},v}|B(v)>}{2m_B}
$$

$$
\mu^2_G = \frac{< B(v)|O^{(b)}_{\text{mag},v}|B(v)>}{2m_B}
$$

(11)

i.e. the matrix elements of the operators that appear in the $1/m_Q$ perturbation of the HQET Lagrangian,

$$
O^{(Q)}_{\text{kin},v} = \overline{h}_v(Q)(iD)^2h_v^{(Q)}
$$

$$
O^{(Q)}_{\text{mag},v} = \frac{g_s}{2} \overline{h}_v(Q)\sigma_{\alpha\beta}G^{\alpha\beta}h_v^{(Q)}
$$

(12)

In terms of $^{1/2}_- \to ^{1/2}_+, ^{3/2}_+$ Isgur-Wise functions at zero recoil $\tau_j^{(n)}(1)$ and level spacings $\Delta E_j^{(n)}$ ($j = ^{1/2}_1, ^{3/2}_2$), these quantities read [11]

$$
\mu^2_\pi = 6 \sum_n \left[ \Delta E_{3/2}^{(n)} \right]^2 \left[ \tau_3^{(n)}(1) \right]^2 + 3 \sum_n \left[ \Delta E_{1/2}^{(n)} \right]^2 \left[ \tau_1^{(n)}(1) \right]^2
$$

(13)

$$
\mu^2_G = 6 \sum_n \left[ \Delta E_{3/2}^{(n)} \right]^2 \left[ \tau_3^{(n)}(1) \right]^2 - 6 \sum_n \left[ \Delta E_{1/2}^{(n)} \right]^2 \left[ \tau_1^{(n)}(1) \right]^2
$$

(14)

The inequality $\mu^2_\pi \geq \mu^2_G$ holds, and one has found empirically, from the inclusive decay $\overline{B}_d \to X_c \ell \nu_\ell$, that $\mu^2_\pi$ and $\mu^2_G$ are rather close [12] $\mu^2_\pi \approx 0.4$ GeV$^2$, $\mu^2_G \approx 0.35$ GeV$^2$.

The value of $\mu^2_G \approx 0.35$ GeV$^2$ is obtained from the heavy-light mesons hyperfine splitting (see for example ref. [11]), while the value $\mu^2_\pi \approx 0.4$ GeV$^2$ comes from the fit to inclusive $\overline{B}_d \to X_c \ell \nu_\ell$ decay moments.

Uraltsev has suggested a dynamical hypothesis that implements the limiting condition of $\mu^2_\pi$ and $\mu^2_G$ being equal, the so-called BPS approximation, $\mu^2_\pi = \mu^2_G$.

Let us underline that our main purpose is a mathematical one within the heavy quark limit of QCD. Namely, the determination of the form of the Isgur-Wise function in the heavy quark limit by adding one dynamical assumption, the BPS condition. Let us consider the pseudoscalar $B$ meson at rest, $v = (1,0,0,0)$. The equation of motion of HQET in the heavy quark limit implies $iD^0 h_v^{(b)}|B(v)> = 0$, where $D^\mu$ is the covariant derivative and $h_v$ is the heavy quark field. Uraltsev has proposed a new more specific constraint, valid only for the pseudoscalar ground state meson $B$, the so-called BPS constraint $(\vec{\sigma} \cdot i \vec{D}) h_v^{(b)}|B(v)> = 0$ that amounts to the vanishing of the smaller components of the heavy quark field within the pseudoscalar $B$ meson.
It will be convenient in the following to write these two conditions in a covariant way, for any value of \( \nu \). These equations then read,

\[
(iD \cdot \nu)h^{(b)}_\nu |B(\nu)| > 0 \quad , \quad \gamma \xi i \not D h^{(b)}_\nu |B(\nu)| > 0 \tag{15}
\]

From \( i \not D i \not D = (iD)^2 + \frac{g}{2} \sigma_{\alpha\beta} G^{\alpha\beta} \) this implies the equality \( \mu^2 = \mu_G^2 \).

Using the formalism of Leibovich et al. \[13\], we have demonstrated in \[10\], using translational invariance, the equations of motion and the BPS condition (15) that the slope and the curvature of the IW function satisfy:

\[
- \xi'(1) = \frac{3}{4} \quad , \quad \xi''(1) = \frac{15}{16} \tag{16}
\]

i.e. the lower bounds [2] are saturated.

One can demonstrate by induction that in general the \( L \)-th derivative attains its lower bound [2]

\[
(-1)^L \xi^{(L)}(1) = \frac{(2L + 1)!!}{2^{2L}} \tag{17}
\]

We will assume the relation for \( L - 1 \),

\[
(-1)^{L-1} \xi^{(L-1)}(1) = \frac{(2L - 1)!!}{2^{2(L-1)}} \tag{18}
\]

and use (6) and (7) to demonstrate for \( L \).

Let us differentiate the SR (6), (7) \( M \) times relatively to \( w_{i\not f} \). Using (8)-(9), we need

\[
\left[ \frac{\partial^M}{\partial w_{i\not f}^M} S_L(w, w_{i\not f}) \right]_{w_{i\not f} = 1} = F_{L,M}(w) \tag{19}
\]

where \( F_{L,M}(w) = R_{L,M}(w^2 - 1)^{L-M} \), with

\[
R_{L,M} = (-1)^M \sum_{0 \leq k \leq (L-M)/2} (-1)^k \frac{(L)!^2}{(2L)!} \frac{(2L - 2k)!}{k!(L-k)!(L-M-2k)!} \tag{20}
\]

One obtains then two equations respectively for the vector and axial SR. In the Vector case we obtain two useful relations for \( M = L, \ w = 1 \) and for \( M = L - 1 \) differentiating once relatively to \( w \) and taking \( w = 1 \). Similarly, for the Axial case we obtain three useful relations, taking \( M = L - 1 \) and \( w = 1 \), \( M = L \) and \( w = 1 \), and \( M = L \), differentiating once relatively to \( w \) and making \( w = 1 \).
To proceed with the proof by induction, we assume \( \tau_{L-1-1/2}^{(L-1)(n)}(1) = 0 \), that implies (18). One obtains

\[
(-1)^L \xi^{(L)}(1) = \frac{(2L + 1)!!}{2^{2L}} + \frac{2L + 1}{4L} L! \sum_n \left[ \tau_{L-1/2}^{(L)(n)}(1) \right] ^2
\]

that imply \( \tau_{L-1/2}^{(L)(n)}(1) = 0 \) and (17), as we wanted to demonstrate. Since (17) are the successive derivatives of (10), assuming natural regularity properties, in the BPS limit the Isgur-Wise function is given by expression (10).

In conclusion, we have demonstrated in this paper that if the heavy quark limit of QCD is supplemented with a dynamical assumption, namely the BPS approximation proposed by Uraltsev, the Isgur-Wise function is completely determined, given by the expression

\[
\xi(w) = \left( \frac{2}{w + 1} \right)^{3/2}
\]

This is a mathematical result that comes from the heavy quark limit of QCD plus the BPS condition introduced by Uraltsev. The comparison with data is not straightforward, since \( 1/m_Q \) and radiative corrections have not been taken into account. Indeed, the function that has to be extrapolated at \( w = 1 \) to obtain \(|V_{cb}|\) is the form factor \( h_{A_1}(w) \), and moreover the two ratios of form factors \( R_1(w) \), \( R_2(w) \) are involved, that become \( R_1(w) = R_2(w) = 1 \) in the heavy quark limit, considered in this paper. In a recent BaBar paper, the fit to \( h_{A_1}(w) \) gives a slope \( \rho_{A_1}^2 = 1.14 \) [16]. This is far away from the heavy quark limit result with the BPS condition \( \rho^2 = 0.75 \). However, to make a proper comparison, radiative corrections to the heavy quark plus BPS limit should be considered [17], and the constraints on the slope from Voloshin SR, that result in an upper bound on \( \rho^2 \) that is close to the BPS limit, should also be taken into account [18]. This discussion deserves a delicate and detailed discussion that will be done elsewhere.

In conclusion, we have obtained an explicit expression for the Isgur-Wise function \( \xi(w) \) by implementing the heavy quark limit of QCD with a dynamical assumption, namely the BPS condition proposed by Uraltsev, coming from the condition \( \mu_{G}^2 = \mu_{\pi}^2 \) or, equivalently, from \( \rho^2 = \frac{3}{4} \).
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References


