Hidden Quantum Gravity in 4d Feynman diagrams: Emergence of spin foams

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We show how Feynman amplitudes of standard QFT on flat and homogeneous space can naturally be recast as the evaluation of observables for a specific spin foam model, which provides dynamics for the background geometry. We identify the symmetries of this Feynman graph spin foam model and give the gauge-fixing prescriptions. We also show that the gauge-fixed partition function is invariant under Pachner moves of the triangulation, and thus defines an invariant of four-dimensional manifolds. Finally, we investigate the algebraic structure of the model, and discuss its relation with a quantization of 4d gravity in the limit $G_N \to 0$.

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I. INTRODUCTION

One of the main challenges faced by anybody thinking seriously about quantum gravity is the fact that such a theory should be understood and formulated in a background independent manner. That is, classical space-time should emerge as a low energy approximation of a more fundamental - and yet unknown - description of quantum spacetime; and a vacuum selection principle should be designed in order to give a dynamical understanding for the emergence of the particular type of space-time we live in among all possibilities. It also means that one should be able to have a proper handle on the observables of quantum gravity, which are intrinsically non local because of background independence - or more precisely diffeomorphism invariance. This basic and fundamental challenge is however at odds with our well established current understanding of fundamental physics formulated in terms of local quantum field theory living on a fixed background. This schizophrenic state of affair seems to force a painful choice between the questions we want to address and the fundamental techniques at our disposal.

If one takes the point of view that understanding quantum gravity in a background independent manner is the key to success, one is led to first devise a new set of appropriate tools and techniques well tailored to this problem. There has been, in the recent years, a large body of work in that direction. Such works have led to the conclusion, or more appropriately the hypothesis, that the proper tools are, at the kinematical level, given by spin networks as developed in loop quantum gravity; and at the dynamical level, given by the so-called spin foam models. Spin foam models give a well defined framework allowing to address the dynamical problem of quantizing gravity in a background independent manner, and provide a description of quantum space-times in a purely algebraic and combinatorial way [1]. The state of development is such that one can now propose, for Euclidean 4-dimensional pure gravity, well defined and finite quantum gravity transition amplitudes, which are independent of any triangulation or undesirable discrete structure [2].

Is it a satisfying state of affair? For a specialist working along this line of thought, there are many reasons to be satisfied with all the new developments; however the answer is clearly no. The answer is negative since it is not yet possible to convincingly argue that, when the Newton constant $G_N$ is treated as a small parameter, this set of amplitudes reproduces local quantum field theory; or that, when $\hbar$ is treated as a small parameter, one recovers the dynamics of general relativity.

Some recent progress have been achieved recently concerning the later problem in the context of spin foam [3] and in the Hamiltonian framework [4], but still more work is needed.

This means that we cannot yet falsify the spin foam hypothesis; it is for us a serious shortcoming of all the developments in background independent approach to quantum gravity, since any physical theory should expose itself to falsifiability tests.

These serious problems are related to the fact that it is extremely hard, if not impossible [2], to construct proper observables having a clear physical meaning in the context of background independent pure gravity without matter fields\(^1\). This is not really surprising, given that one can argue that space-time geometry is a mathematical abstraction devised to account in a simple way for all the relations between dynamical objects moving in space-time. For a physicist every phenomenon should be described in terms of observable quantities and physical processes, that is in an operational manner. From this point of view it is clear that, without matter fields to probe the spacetime geometry, it is very hard to really ‘observe’ our quantum spacetime and address properly the ‘semiclassical’ issues raised previously.

In order to promote these statements into physics, we need to propose explicit quantum gravity observables when matter fields are present. Remarkably, such observables are very easy to construct and have been right in front of our eyes for a long time: they are simply the Feynman diagrams. In order to be precise, let us define what we mean. A closed\(^2\) Feynman diagram is a purely combinatorial data, namely an abstract graph whose edges are colored by representations of the Poincaré group (mass and spin), ends of edges are labeled by representations of the Lorentz group contained in the Poincaré group representation, and vertices are labeled by intertwiners of the Lorentz group [6]. In the simplest case (spin zero), we often look only at the subclass of Feynman diagrams with all Lorentz representations equal to the trivial one, and we don’t talk about this label - this is enough if we only want to describe perturbative expansion of non-derivative interactions.

Given such data, denoted by $\Gamma$, and a space-time manifold equipped with a metric $g$, we can compute the Feynman amplitude $I_\Gamma(g)$. Of course, what we are interested in is a sum of Feynman diagrams, for instance those of fixed valency and given degree, as generated by a field theory at given order of perturbation theory. In order to keep the

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\(^1\) Note that recently a new proposal has been made to overcome this difficulty [3]. The idea in this work is to compute an overlap between the spin foam amplitude and a semi-classical coherent state in order to extract the graviton propagator from these models. This can be successfully done in a restricted class of configurations, one of the challenge being to extend this new strategy to general configurations.

\(^2\) We restrict ourselves in this paper to closed Feynman diagrams.
exposition simple we will refer to individual Feynman diagrams, keeping in mind this important remark. The object of interest is the quantum gravity observable

$$\bar{I}_\Gamma(l_p) = \int \mathcal{D}g \, e^{\frac{i}{\hbar} S(g)} I_\Gamma(g), \tag{1}$$

where the integral is over the space of metrics, \(l_p\) is the Planck length and \(S(g)\) is the gravity action. Of course we have to be able to make sense of this path integral, and spin foam models aim to give a background independent way of computing this amplitude. No definite proposal is available yet in this framework. However, even in the absence of a definite proposal, we strongly claim that we can still propose a falsifiability test of the spin foam hypothesis as a valid candidate for a theory of quantum gravity.

The main point is that whatever the quantum gravity amplitude is, one should be able to recover from it usual field theory when quantum gravity effects are negligible; that is, \(\bar{I}_\Gamma(l_p)\) should admit a perturbative expansion

$$\bar{I}_\Gamma(l_p) = I^{(0)}_\Gamma + l_p I^{(1)}_\Gamma + l_p^2 I^{(2)}_\Gamma + o(l_p^2). \tag{2}$$

Moreover, the first term \(I^{(0)}_\Gamma\) in the expansion should be the evaluation of usual Feynman diagram in the gravity vacuum state, namely flat space - or de-Sitter or Anti-de-Sitter if a cosmological constant is included in the gravity action. This is a mandatory constraint on any proposal for a background independent approach to quantum gravity if we want to make the link with experiments and the highly successful effective field theory point of view.

This provides a non-trivial first step falsifiability test on the spin foam hypothesis. Indeed, this hypothesis implies\(^3\) that \(\bar{I}_\Gamma(l_p)\), and hence \(I^{(0)}_\Gamma\), should be written as a combinatorial state sum model depending on the choice of a triangulation \(\Delta\) adapted to the Feynman graph, and of a coloring of the faces and edges of \(\Gamma\) by Lorentz group representations; that is

$$\bar{I}_\Gamma(l_p) = \sum_{j_f,j_e,j_v} \prod_f A_f(j_f) \prod_e A_e(j_e,j_f) \prod_v A_v(j_e,j_f) O_\Gamma(j_f,j_e) \tag{3}$$

where \(f,e,v\) denote faces, edges and vertices of the 2-complex \(\Delta\) dual to the triangulation, and \(A_f,A_e,A_v\) are local face, edge and vertex amplitudes depending on the spins which are summed and represent quantum gravity fluctuations. \(O_\Gamma\) is an observable characterizing the coupling of Feynman diagram from insertion of matter. To any field theorist familiar with Feynman graphs, this seems to be a structure quite removed from anything a Feynman integral looks like.

The second consequence of the spin foam hypothesis follows from the work\(^3\) where a new background independent approach to quantum gravity perturbation theory was proposed in the language of spin foam model. In this approach, the starting point is to write 4d gravity as a perturbation of a topological \(BF\) theory based on the de-Sitter group for positive cosmological constant. The perturbation parameter \(G_N\Lambda\) is dimensionless and the perturbation theory transmutes gauge degree of freedom into physical degrees of freedom in a controlled way, order by order. In particular this means that the theory becomes topological in the limit \(G_N \rightarrow 0\). It has also been shown in this context that the coupling to matter particles can be explicitly performed by computing expectation value of Wilson lines observables\(^3\) which are the most natural gauge invariant observables in this formulation.

The main consequence of interest to us from these works is the fact that, not only Feynman diagram amplitudes should be written as expectation values of certain natural observables in a spin foam model, but, moreover, the corresponding model should be a topological spin foam model based on a Poincaré \(BF\) theory.

So in summary, the spin foam hypothesis implies that usual Feynman graph can be expressed as the expectation value of certain observables in a topological spin foam model based on the Poincaré group. The validity of such a statement is for us a non-trivial check in support of the spin foam hypothesis. The check is fourfold: first, spin foam should arise naturally in Feynman integrals; second, the spin model should agree with the structure predicted by\(^3\); third, it should confirm the idea that the limit \(G_N \rightarrow 0\) is a limit where gravity becomes topological; and fourth the Feynman diagram observables should be understood as a Wilson lines (or more generally spin networks) expectation value in this spin foam model.

In this paper, we show that the first three conditions are indeed satisfied. We will take a very conservative approach not relying on any hypothesis about quantum gravity dynamics. Instead, we will carefully study the structure of Feynman integrals, and show that they can indeed be written as the expectation value of certain observables in an

\(^3\) We refer the reader to\(^3\) for introduction and review on spin foam models.
explicit topological spin foam model. The idea of our derivation is to consistently erase the information about flat space geometry from the Feynman integral and encode this information in terms of a choice of quantum amplitudes that should be summed over, and which dynamically determine flat space geometry. In doing so, a triangulation, and a specific spin foam model living on it, are naturally found; this allows us to express usual field theory amplitude in a background independent manner. The idea that spin foam models code, in a background independent manner, the integration measure viewed by Feynman diagrams was formulated for the first time in \cite{9} and \cite{10} in the context of 3d-gravity.

An analysis similar to the one done here has already been performed in 3d \cite{12}, where it has been shown that the corresponding spin foam model is constructed in terms of 6j symbols of the 3d Euclidean group for flat space. The deformation of this spin foam model using quantum group naturally leads to a formulation of Feynman diagram coupled to 3d quantum gravity amplitudes \cite{10, 11}. This corresponds to a deformation of field theory carrying a deformed action of the Poincaré group.

The strategy followed here is, to some extent, analogous to the one followed by Polyakov \cite{13}, when he showed that Feynman diagrams can be rewritten as worldline integrals. Such an interpretation leads to powerful partial resummation of Feynman diagrams \cite{14}. Moreover, it leads to a natural proposal for deforming the structure of quantum field theory in terms of a dimensionfull parameter, that is to consider worldsheet (instead of worldline) integrals - hence string theory.

In our case we have written 4d Feynman diagram in terms of a specific spin foam model, and this hopefully opens a new way to think about consistent dimensionfull deformations of field theory structure. This reformulation of Feynman graph amplitudes does not a priori simplify the computation of usual Feynman diagrams, but, since it is based on a topological field theory, it allows to give a natural generalization of the definition of Feynman amplitudes in the context where the underlying manifold admits a non-trivial topology.

The paper is organized as follows. In section \textbf{II} we show how, from a reformulation of Feynman integrals in terms of relatives distances between vertices, the background geometry can be induced dynamically in the amplitudes. This analysis will allow us to formulate our main statement, namely that Feynman amplitudes are the evaluation of observables for an explicit spin foam model. The section \textbf{III} focuses on a careful study of this model: the complete identification of its symmetries and the expression of the gauge-fixed partition function. The proof of our statement is then given in section \textbf{IV} first the gauge-fixed model is shown to be invariant under Pachner moves; then for any graph $\Gamma$, the so-called Feynman graph observable is defined, which breaks part of the symmetry of the model, promoting gauge degrees of freedom into dynamical degrees of freedom; finally the expectation value of this observable is shown to reproduce the Feynman amplitude associated the graph $\Gamma$. Eventually, in section \textbf{V} the algebraic and physical interpretation of the Feynman graph spin foam model is discussed.

\section*{II. DYNAMICAL GEOMETRY IN FEYNMAN AMPULUTIDES}

We restrict our study to the case of closed Feynman diagrams that arise in the context of QFT in flat Euclidean space-time. A Feynman amplitude for a scalar field takes the form

$$I_{\Gamma} = \int_{\mathbb{R}^n} d^4x_1 \cdots d^4x_n \mathcal{O}_\Gamma(\{\vec{x}_i - \vec{x}_j\}), \quad \mathcal{O}_\Gamma = \prod_{(ij) \in \Gamma} G^F(\vec{x}_i - \vec{x}_j)$$

($\Gamma$ is the Feynman graph, $\vec{x}_i, i = 1, \cdots n$ denotes the positions in $\mathbb{R}^4$ of the $n$ vertices of the graph. The product is over all edges of $\Gamma$ and $G^F$ is the Feynman propagator$^4$). In this section we show how to write this amplitude as a sum over labels living on a specific triangulation of a 4-dimensional ball, of the product of propagators. This result is established in every dimension in our previous work \cite{12}, and here we summarize the main arguments leading to it. We then describe, through a simple example, how flat geometry can be dynamically implemented in \cite{4}.

The usual way to express QFT amplitudes involves Lebesgue measures in $\mathbb{R}^4$, which explicitly carry information about flat geometry. The product of propagators depending only on the distances between vertices of the graph, the integrand in \cite{11} is invariant under the action of the Euclidean group $ISO(4) = SO(4) \times \mathbb{R}^4$. A first idea here is to gauge out this symmetry and express the integral in terms of the invariant measure, acting on the space of functions of the relatives distances $l_{ij} = |\vec{x}_i - \vec{x}_j|$. The invariant measure can be constructed out of the Lebesgue measure as

\footnotetext{4}{We will work with a regulated form of the Feynman propagator in order to avoid divergences. It is also understood that the volume of $\mathbb{R}^4$ is divided out from this integral: this is achieved by not integrating over one of the $x_i$.}
follows. Consider first the case of four points, which are the vertices of a tetrahedron in $\mathbb{R}^4$. The relative position of these points is fully specified by the edge lengths of the tetrahedron. The Lebesgue measure splits then into the Haar measure $d^4\alpha d\Lambda$ of the Poincaré group and a product of $l_{ij}dl_{ij}$:

$$d^4x_1 \cdots d^4x_4 = d^4\alpha d\Lambda \prod_{i<j} l_{ij}dl_{ij},$$

(5)

With an additional point $x_5$, one can form a 4-simplex $\sigma$, and write the Lebesgue measure in terms of the four edge lengths $l_{i5}$ as

$$d^4x_5 = \sum_{\epsilon} \frac{\prod_{i=1}^4 l_{i\epsilon}dl_{i5}}{V(l_{ij})}$$

(6)

In this formula $V$ is the volume$^5$ of the simplex, and $\epsilon \in \{\pm 1\}$ labels its orientation. The simplex constitutes the simplest triangulation $\Delta_1$ of a 4-ball, without any internal face. For general values of $n = 4 + k$, the invariant measure is obtained recursively: if $x_1, \cdots, x_{4+k}$ are the vertices of a triangulation $\Delta_p$ of a 4-ball, without any internal face, on can choose a tetrahedral face of $\Delta_p$ and connect an additional point $x_{4+(p+1)}$ to its four vertices, to form a new triangulation $\Delta_{p+1}$, and compute the Lebesgue measure by using $\mathbb{P}$. Eventually, the measure is expressed in terms of the edge lengths and orientations of the 4-simplices of the triangulation $\Delta_k$.

$$d^4x_1 \cdots d^4x_{4+k} = d^4\alpha d\Lambda \sum_{\epsilon \in \{\pm 1\}^k} \prod_{e \in \Delta_k} l_e dl_e \prod_{\sigma \in \Delta_k} \frac{1}{V_\sigma}$$

(7)

With this formula we can express the Feynman integrals purely in term of edge lengths of $\Delta_k$, up to an overall $SO(4)$ volume factor that we drop out from now on.

$\Delta_k$ is a triangulation of a 4-ball, with $4+k$ vertices, such that all vertices, edges and faces lie on the boundary. Let us emphasize that any triangulation of this kind can be built recursively by the procedure described above, once an ordering of the vertices is chosen. This can be seen by noticing that such triangulations are those for which the dual 1-skeleton is a connected 5-valent tree (containing no loops), with open ends. Now in order to draw such a tree, one first chooses one of its vertex $v_0$, called the ‘root’ of the tree, and draws the four edges meeting at $v_0$; one of these edges connects $v_0$ to a second vertex $v_1$, and one draws the three additional edges meeting at $v_1$, and so on. By using the duality between tree and triangulation (namely, vertices of the tree are dual to 4-simplices, and edges are dual to tetrahedra) we see that the building procedures of trees on one hand, and triangulations $\Delta_p$ on the other, are identical.

Given such an abstract triangulation $\Delta_k$, the data \{$l_e, \epsilon_e$\} of edge and simplex labels defines a flat geometry on the triangulation, that is, specifies the relative position of the vertices in $\mathbb{R}^4$. Consequently, distances between vertices which are not connected by any edge of $\Delta_k$ are well defined functions of the labels. We denote symbolically $l^\epsilon_e$ these functions; the ‘prime’ means that the edge $e'$ does not belong to the triangulation. The Feynman amplitude is finally given in terms of the invariant measure by

$$I_\Gamma = \int \prod_{e \in \Delta_k} l_e dl_e \sum_{\epsilon \in \{\pm 1\}^k} \prod_{\sigma \in \Delta_k} \frac{1}{V_\sigma} \mathcal{O}_\Gamma(l_e, l^\epsilon_e(l_e))$$

(8)

where the product of propagators depend both on the edge labels and the ‘missing’ distances $l^\epsilon_e$. In this expression the overall factor corresponding to the gauge volume has been dropped.

This formula is not enough since there is still an explicit flat geometry dependence encoded in the functions $l^\epsilon_e$. In order to go further, lets consider the following example where the Feynman graph $\Gamma$ forms the 1-skeleton of a 5-simplex, with 6 vertices and 10 edges connecting all these vertices together, as shown in Fig[4]. We denote by $\sigma_j$ the 4-simplex obtained by dropping the point $j$, by $V_j$ its volume and $\epsilon_j$ its orientation. The two 4-simplices $\sigma_0, \sigma_5$, sharing a tetrahedron, triangulate a 4-ball, in such a way that all faces belong to the boundary, and that all vertices are connected to each other except for 0 and 5. The product of propagators depends on the edge lengths $l_{ij}, (ij) \neq (05)$

$^5$ For a flat D-simplex $\sigma$ in $\mathbb{R}^D$, with vertices $\vec{x}_i$, $V_\sigma$ denotes the square root of the determinant $\det(\vec{l}_i, \vec{l}_j)$, with $\vec{l}_i = \vec{x}_i - \vec{x}_1$, $i = 2 \cdots D+1$, and it therefore equals $D!$ times the volume of the simplex. Abusing terminology the quantities $\mathcal{V}$ will be called ‘volume’ in all the paper. For the particular case $D = 2$, the simplex is a triangle $F$, the volume is an area and the notation $A_F$ is used instead of $\mathcal{V}$. 
FIG. 1: A 5-simplex defines a complex of two 4-simplices \( \sigma_0, \sigma_5 \) sharing a tetrahedral face \([1234]\), as well as a complex of four 4-simplices \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) sharing an edge \((05)\).

of the triangulation, as well as the distance between the points 0 and 5 which, as emphasized above, is a function of lengths \( l_{ij} \) and orientations \( \epsilon_0, \epsilon_5 \).

For that particular case one can specify the dependence on the orientations. Indeed, it is possible to choose conventions\(^6\) on orientations such that, when two 4-simplices embedded in \( \mathbb{R}^4 \) and sharing a tetrahedron \( \tau \) have identical orientations, then the points opposite to the common tetrahedron in each 4-simplex do not belong to the same half-space defined by the hyperplane spanned by \( \tau \). It is easy to convince oneself that if all \( l_{ij}, (ij) \neq (05) \) are fixed, then \( l_{05} \) can take two values \( l_{05}^{+}, l_{05}^{-} \), with \( l_{05}^{-} < l_{05}^{+} \) and that the sign ‘±’ coincides with the product \( \epsilon_0 \epsilon_5 \).

According to (8), the Feynman amplitude for this graph reads

\[
I_\Gamma = \int \prod_{(ij) \neq (05)} l_{ij} \mathrm{d}l_{ij} \sum_{\epsilon_0, \epsilon_5} 1 \frac{1}{V_0 V_5} \mathcal{O}_\Gamma(l_{ij}, l_{05}^{\epsilon_5}(l_{ij})) \tag{9}
\]

The form of the function \( l_{05}^{\epsilon} \) encodes the flatness of the geometry. Our goal is to show that this function can be replaced by a free label \( l_{05} \), and that the flat geometry can be induced dynamically. The keypoint of the proof is the remarkable identity of measures

\[
\sum_{\epsilon_0, \epsilon_5} \delta(l_{05} - l_{05}^{\epsilon_5}) \frac{1}{V_0 V_5} = \sum_{\epsilon_1, \epsilon_2, \epsilon_3} l_{05} \frac{A_{045}}{V_1 V_2 V_3} \delta(\omega_{045}^{\epsilon}) \tag{10}
\]

where \( A_{045} \) is (2 times) the area of the triangle \([045]\). In this identity all lengths \( l_{ij}, i, j = 1 \cdots 4 \) are fixed, the length \( l_{05} \) being free to fluctuate; the measures can be used to integrate functions \( f(l_{05}) \) of this label. The delta function in the right hand side is the \( 2\pi \)-periodic delta function; its argument is the deficit angle of the face 045

\[
\omega_{045}^{\epsilon} = \sum_{i=1}^{3} \epsilon_i \theta_{045}^{\epsilon}
\]

where \( \theta_{045}^{\epsilon} \) is the dihedral angles of the face \([045]\) in the 4-simplex \( \sigma_i \). This deficit angle, considered as a function of \( l_{05} \) (and the orientations), is the curvature, in the sense of Regge calculus, carried by the face. It vanishes modulo \( 2\pi \) if and only if the complex of four simplices \( \sigma_1, \cdots, \sigma_4 \) can be mapped in \( \mathbb{R}^4 \), in such a way to give the orientation \( \epsilon_j \) to the simplex \( \sigma_j \). By symmetry of the role of the four points 1 \( \cdots \) 4, similar identities hold for any permutation of \((1234)\). \((10)\) is the four dimensional analogue of the identity used in \([12]\). Also the proof is similar, namely consists in identifying, thanks to the formula \((A1), (A2)\), the two sides of the equalities to the functional \( 4l_{05} \delta(G) \), where \( G \equiv V^2 \) is the square of the volume of the 5-simplex \([0 \cdots 5]\).

\(^6\) For more details we refer the reader to the appendix of \([12]\).
Plugging (10) into the Feynman integral (9) promotes the lengths $l_{05}^{e_0}$ to a free label; the price to pay is a constraint which plays the role of a projector on the space of flat geometries. If one expands the delta function as a sum $2\pi \delta(\omega) = \sum_{s \in \mathbb{Z}} e^{i s \omega}$, the Feynman amplitude takes the following form:

$$I_\Gamma = \frac{1}{2\pi} \int \prod \frac{dl_{IJ}d\epsilon_I d\epsilon_J}{\epsilon_{e_1, e_2, e_3}} A_{045} \prod \frac{e^{i s_{045} \epsilon_{e_1, e_2, e_3}}}{V_1 V_2 V_3} O_\Gamma (l_{IJ})$$  \hspace{1cm} (11)

The integral is over all edges connecting the vertices 0, \cdots, 5. It represents therefore a sum over all (not necessarily flat) geometries of the simplicial complex.

All these considerations serve as a motivation for our main statement, which will be established later: the Feynman amplitude (8) can be written as the expectation value of an observable

$$I_\Gamma = \langle \mathcal{O}_\Gamma (l_e) \rangle_\Delta, \quad \mathcal{O} = \prod_{e \in \Gamma} G^F (l_e)$$  \hspace{1cm} (12)

for the spin foam model:

$$Z_\Delta = \frac{1}{(2\pi)^{|F|}} \int \prod_{e \in \Delta} l_e dl_e \prod_{F \in \Delta} A_F \sum_{\{s_F, \epsilon_\sigma\}} \left( \prod_{\sigma} \frac{e^{i \epsilon_\sigma S_\sigma (s_F, l_e)}}{V_\sigma} \right)$$  \hspace{1cm} (13)

$\Delta$ is a triangulation of a closed 4D-manifold and $\Gamma$ is embedded into the one-skeleton of $\Delta$. Edges are labelled by positive numbers $l_e$, summed over a domain where triangular inequalities are satisfied; faces are labelled by integers $s_F$. We denote by $\epsilon_\sigma = \pm 1$ the orientation of the simplex $\sigma$. The measure involves a product of area $A_F (l_e)$ of all faces $F$, while the action for each simplex is similar to the 4d Regge action [13].

$$S_\sigma (s_F, l_e) = \sum_{F \in \sigma} s_F \theta_F^\sigma (l_e),$$  \hspace{1cm} (14)

where $\theta_F^\sigma (l_e)$ is the interior dihedral angle of the face $F$ in $\sigma$. $|F|$ denotes the number of faces of the triangulation.

As we will see, the evaluation (12) is independent of the choice of the triangulation which contains $\Gamma$ as a subgraph. Also, the model (10) is a state-sum version of the 4-manifold invariant constructed by Korepanov in the remarkable work [16, 17, 18]. The observable $\mathcal{O}_\Gamma$ is the product of propagators in which the distances are replaced by the labels $l_e$ living on the graph $\Gamma$. The equality (12) is obtained if one restricts to trivial topologies, that is, if $\Delta$ triangulates the 4-sphere $S^4$.

Notice that, although the structure of the integrand in (11) is similar to that of the model (13), they are not identical. In particular, a product of the area of all the faces, a product of the volume of all the simplices and a sum of the labels of all the faces are taken over in the latter, while, in the former, only the area of the face $\{045\}$ appears, only the label of this same face is summed over, and the volume $V_4$ is missing. The reason of such differences is the following: the integral (11) has to be interpreted as expectation value for the gauge-fixed model, and therefore hides a gauge-fixing term and a Faddeev-Popov determinant. Hence, what we learn from the study of the example is actually twofold: not only it leads to the explicit proposal (13) for our statement, but it also indicates that, in order to prove this statement, we definitely need to identify the symmetries of the model and find the gauge-fixing prescriptions.

In the following, as suggested by the previous remark, we will start by dwelling further into the meaning of the integration measure which appears in (13). As is customary for theories with gauge symmetries, this measure should be understood as the naive measure modulo gauge transformation, and defined using a Faddeev-Popov gauge-fixing procedure: we will study in detail the symmetries of our model and construct explicitly the gauge-fixed measure. This analysis will allow us to define unambiguously the model. We then prove the statement (12) in the subsequent section.

## III. SYMMETRIES AND GAUGE FIXING

The total action of the model reads

$$S_\Delta [s_F, l_e] = \sum_\sigma \epsilon_\sigma S_\sigma (s_F, l_e) = \sum_F s_F \omega_F (l_e)$$  \hspace{1cm} (15)

involving the deficit angle of each face of the triangulation. In ways similar to the 3d case [12], symmetries mapping classical solutions to classical solutions induce divergences in (13). Taking the view that these symmetries are gauge symmetries, we want to write down explicitly the gauge-fixed model, free of these naive divergences. The face labels $s_F$ are treated here as continuous variables.
A. Classical solutions and zero modes

The equations of motions corresponding to the action (15) are given by

\[ 0 = \frac{\delta S}{\delta F} = \omega_F'(l) \quad \forall F \]  

(16)

\[ 0 = \frac{\delta S}{\delta l_e} = \sum_{\sigma} \epsilon_{\sigma} \left( \sum_{F \subset \sigma} sF \frac{\delta \theta_F}{\delta l_e} \right) = \sum_{F} sF \frac{\delta \omega_F}{\delta l_e} \quad \forall e \]  

(17)

The first equation expresses the flatness condition. A set of labels \( \{ l_\epsilon^0 \} \) is solution if the simplicial complex \( \Delta \) can be locally mapped in \( \mathbb{R}^4 \). Solutions of the second equation are provided by the Schlafli identity for the flat 4-simplex, which implies:

\[ \sum_{F \subset \sigma} A_F \frac{\delta \theta_F^\sigma}{\delta l_e} = 0 \]  

(18)

Thus, we see that \( s_F^\epsilon = \alpha A_F (l_\epsilon) \) are solutions of the equations of motion, \( \alpha \) being an arbitrary constant. Remarkably, by inserting this solution into (15) one recovers the Regge action of discrete 4d gravity

\[ S_R = \alpha \sum_{F} A_F \omega_F'(l_\epsilon). \]

In the following a solution \( (l_\epsilon^0, s_F^\epsilon = A_F (l_\epsilon^0)) \) is called a Regge solution.

We want to study the zero modes of the model, namely infinitesimal deformations \( \delta l_\epsilon, \delta s_F \) of the labels which belong to the kernel of the Hessian \( \delta^2 S \), computed on shell. The system of equations which characterize this kernel is

\[ \sum_{e} \frac{\delta \omega_F}{\delta l_e} \delta l_\epsilon = 0 \quad \forall F \]  

(19)

\[ \sum_{F} \frac{\delta \omega_F}{\delta l_{e'}} \delta s_F + \sum_{F,e} s_F \frac{\delta^2 \omega_F}{\delta l_e \delta l_{e'}} \delta l_\epsilon = 0 \quad \forall e' \]  

(20)

where the label \( \epsilon \) has been dropped for clarity. One can reorganize the left hand side of the second equation by using derivatives of the Schlafli identity, which yields

\[ 0 = \sum_{\sigma} \epsilon_{\sigma} \left[ \sum_{F \subset \sigma} A_F \frac{\delta \theta_F^\sigma}{\delta l_e} \right] = \sum_{F} \frac{\delta A_F}{\delta l_{e'}} \frac{\delta \omega_F}{\delta l_e} + \sum_{F} A_F \frac{\delta^2 \omega_F}{\delta l_e \delta l_{e'}} \]  

(21)

Equation (20) can then be written as

\[ \sum_{F} \frac{\delta \omega_F}{\delta l_{e'}} \delta (s_F - A_F) + \sum_{F,e} (s_F - A_F) \frac{\delta^2 \omega_F}{\delta l_e \delta l_{e'}} \delta l_\epsilon = 0 \]  

(22)

where the symbol \( \delta A_F \) denotes the variation \( \sum_{e} \frac{\delta A_F}{\delta l_e} \delta l_e \) of the area. Now if one restricts to fluctuations around Regge solutions, one obtains eventually the equations satisfied by the zeros modes

\[ \sum_{e} \frac{\delta \omega_F}{\delta l_{e'}} \delta l_\epsilon = 0 \quad \forall F \]  

(23)

\[ \sum_{F} \frac{\delta \omega_F}{\delta l_e} \delta s_F = 0 \quad \forall e \]  

(24)

These equations characterize two independent symmetries \( l_\epsilon \rightarrow l_\epsilon + \delta l_\epsilon \) and \( s_F \rightarrow s_F + \delta s_F \) which we now identify and then gauge-fix.
The symmetry of edge labels $l_e$ is similar to the one arising in 3d and has a simple geometrical interpretation. Starting from a stationary point of the action, the variations $\delta l_e$ satisfying (23) are those for which the deficit angles remain invariant. In other words, given a set of labels $\{l_e\}$ that define a flat geometry, solutions of (23) are such that the set of labels $\{l_e + \delta l_e\}$ also define a flat geometry. Since the geometry of the starting configuration is flat, one can embed the neighborhood of each vertex in $\mathbb{R}^4$. Variations of the labels that do not modify the geometry arise from infinitesimal moves of the vertices in $\mathbb{R}^4$; and therefore they are generated by 4-vectors $\vec{\alpha}_v$ attached to each vertex of the triangulation.

Given a vertex $v$ and an infinitesimal 4-vector $\vec{\alpha}_v$ associated to it, the corresponding variations of the labels are defined to be 0 for each edge $e$ that does not touch the vertex $v$, and, for each edge $e$ touching $v$, to be the projection of $\vec{\alpha}_v$ in the direction defined by $e$, namely

$$\delta l_e(\vec{\alpha}_v, l_e) \equiv -\vec{\alpha}_v \cdot \vec{l}_e$$

where $\vec{l}_e$ is the vector represented by the edge $e$ when $v$ is placed at the origin.

The gauge-fixing is performed by fixing the value of a subset of labels and by taking into account the Fadeev-Popov determinants. Intuitively, in order to eliminate the four components of the gauge parameter at each vertex $v$, we need to fix the labels $(l_j)$ of four edges sharing $v$. If one chooses these four edges so that they belong to a same simplex $\sigma$ touching $v$, the corresponding determinant can then be read out from the relation between the Lebesgue measure $d^4\alpha_v$ and the variations $\delta l_j$ computed in section (6)

$$d^4\alpha_v = 2 \prod_{j=1}^4 l_j dl_j / V_\sigma$$

where the factor 2 is due to the summation over the values of the orientation for $\sigma$.

The gauge-fixing procedure goes as follows. We first choose 5 vertices that form a 4-simplex $\sigma_0$ in $\Delta$, and then assign to every other vertex $v$ of the triangulation a 4-simplex $\sigma_v$ to which this vertex belongs. Each of these vertices provides four edges $e_v^i, \ldots, e_v^4$, namely the four edges of $\sigma_v$ that meet at $v$. We impose the assignment $A_l = (\sigma_0, \{\sigma_v\}_{v \neq \sigma_0})$ to satisfy the admissibility condition that $e_v^i \neq e_{v'}^i$ for every couple $(v, v')$ of distinct vertices that are not in $\sigma_0$; this condition insures that no edge is picked up more than once in the procedure. Such an assignment can be constructed recursively by using a maximal tree in the 1-skeleton dual of the triangulation [12].

Given an admissible assignment $A_l$, we say that a 4-simplex $\sigma$ belongs to $A_l$ if either $\sigma = \sigma_0$ or $\sigma = \sigma_v$ for some vertex $v$ not in $\sigma_0$, and that an edge $e$ belongs to $A_l$ if either $e \in \sigma_0$ or $e \in \sigma_v$ and $e$ admits $v$ as one of its vertices. The gauge-fixing terms and Fadeev-Popov determinants associated to the symmetry (23) read then

$$\delta_{GF}^{A_l} = \prod_{e \in A_l} \delta(l_e - l_e^o), \quad D_{FP}^{A_l} = \frac{1}{2^{|v|-3} \prod_{e \in A_l} l_e} \left( \prod_{\sigma \in A_l} V_{\sigma} \right) \prod_{e \in A_l} l_e$$

where $|v|$ is the number of vertices of $\Delta$ and $l_e^o$ arbitrary fixed values of the labels.
C. Face symmetry

We now want to deal with the symmetry of the face labels $s_F$. We will identify generators as being 3-vectors $\vec{\beta}_e \in \mathbb{R}^3$ attached to the edges of the triangulation.

Variations $\delta s_F$ that satisfy (24) can be described as follows. Let $e$ be an edge of the triangulation. For a given configuration of the $l$'s labels inducing a flat geometry, the complex of 4-simplices sharing $e$ is mapped in $\mathbb{R}^4$ and one can consider the vector space $e^\perp \simeq \mathbb{R}^3$ orthogonal to the straight line spanned by the edge. Next, we attached to $e$ an infinitesimal 3-vector $\vec{\delta}_e \in e^\perp$, and consider the translation in $\mathbb{R}^4$ of the edge $e$ by $\vec{\delta}_e$, that is, the translation of all the points of $e$ by the same vector $\vec{\delta}_e$.

This translation deforms each triangle $F$ to which $e$ belongs. In particular, it modifies the height $h_F(l_e)$ associated to $e$ in $F$, that is, the height of the point opposite to the edge $e$ in $F$. The variation of this height induced by the translation of $e$ reads

$$\delta h_F(\vec{\delta}_e, l_e) = -\vec{\delta}_e \cdot \frac{\vec{h}_F}{|\vec{h}_F|}$$

(26)

where $\vec{h}_F$ is the vector represented by the height when its intersection with $e$ is placed at the origin. Then for any face $F$ of the triangulation, we define the transformation $s_F \rightarrow s_F + \delta s_F$ generated by $\vec{\beta}_e$ as

$$\delta_e s_F = \begin{cases} l_e \frac{4}{3} \delta h_F(\vec{\beta}_e, l_e) & \text{if } F \supset e \\ 0 & \text{if not} \end{cases}$$

(27)

In order to show that the variations $\delta_e s_F$ defined above satisfy (24), let us mention a preliminary geometrical result. Let $\sigma$ be one of the 4-simplices to which the edge $e$ belongs. The data of labels and orientation allows us to map $\sigma$ in $\mathbb{R}^4$. We consider the orthogonal projection $P : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ onto the space $e^\perp$. $P$ maps $e$ onto a vertex $v_e$, the three triangles $\{F_i, i = 1, 2, 3\}$ meeting at $e$ onto a triplet of edges $\{e_i\}$ meeting at $v_e$, and the simplex $\sigma$ itself onto a tetrahedron $\tau$ to which $e_i$ and $v_e$ belong, as shown in Fig.2 Note that the length of the edge $e_i$ equals the height $h_i$ associated to $e$ in the triangle $F_i$. Then the result is the following: for all $i$, the 4d dihedral angle $\theta_{F_i}^\tau$ of the face $F_i$ in $\sigma$ equals the 3d dihedral angle $\theta_{F_i}^\tau$ of the edge $e_i$ in $\tau$. This correspondence is geometrically clear: the dihedral angle of $F_i$ in $\pi$ minus the angle between the normal vectors to the two tetrahedra meeting at $F_i$. They are orthogonal to the edge $e$, thus the dihedral angle is given by the angle between their orthogonal projection. These normal vectors project onto normals of the faces meeting at $v$, so that their angle is $\pi$ minus the dihedral angle of the projected tetrahedron.

As we have seen, the transformation generated by a vector $\vec{\beta}_e$ attached to the edge $e$ affects only the labels $s_F$ of the faces $F$ to which $e$ belongs. Therefore, in order to show the equality (24), one can restrict the sum to the faces

---

7 The unnatural prefactor $l_e \frac{4}{3}$ is chosen in order to simplify the computation of the determinant. Geometrically it would have made more sense to define $\delta_e s_F \equiv l_e h_F$ since, on-shell, we identify $s_F$ with an area. The key remark is that if $\delta_e s_F$ satisfies (24) then $\delta_e s_F \equiv f(l_e)\delta s_F$ satisfies (24) as well. Accordingly, we are free to redefine the transformation of $s_F$ at our convenience, using any $f$. The final result is independent of this choice.
sharing $e$. Thus, we need to show
\[ \sum_{F \ni e} \frac{\delta \omega_F}{\delta l_{e'}} \delta s_F = 0 \quad \forall e'. \] (28)

Let $F$ be a face such that $F \supset e$. We consider the collection $C_F = \{ \sigma^j \}$ of all 4-simplices around $F$: $F \subset \sigma^j$. For each $j$, $\sigma^j$ is embedded in a copy of $\mathbb{R}^4$ and one can define, as before, the projection $P^j$ onto the sub-space $e^j$, the vertex $v^j_e \equiv v_e$ image of the edge $e$, and the tetrahedron $\tau^j$ image of the simplex $\sigma^j$. We also denote by $e^j_F \equiv e_F$ the common image of $F$ by the projections $P^j$. The collection $\{ \sigma^j \}$ form a complex of 4-simplices around the face $F$, and we see that the projections $P^j$ define a complex $\{ \tau^j \}$ of tetrahedra around the edge $e_F$. Now thanks to the correspondence between 3d and 4d dihedral angles mentioned above, the deficit angles $\omega_F$ of the face $F$ and $\omega_{e_F}$ of the edge $e_F$ equal each other:
\[ \omega_F = \sum_j \epsilon_j \theta^j_F = \sum_j \epsilon_j \theta^j_{e_F} \equiv \omega_{e_F}. \] (29)

The deficit angle $\omega_F$ is a function of the edge lengths $\{ l_{e'} \}$ of the 4d complex $C_F$, while the deficit angle $\omega_{e_F}$ depends on the labels $\{ l_{e'} \}$ of the 3d complex $C_{e_F}$. $l_{e''}$ are well defined function of the $l_{e'}$: $l_{e''} \equiv l_{e''}(l_{e'})$. Therefore, given an edge $e'$, differentiating (29) with respect to the label $l_{e'}$ yields
\[ \frac{\delta \omega_F}{\delta l_{e'}} = \sum_{e''} \frac{\delta \omega_{e_F}}{\delta l_{e''}} \frac{\delta l_{e''}}{\delta l_{e'}}. \] (30)

Then, let us first mention that the matrix $\left( \frac{\delta \omega}{\delta l_{e'}} \right)_{e,e'}$ is symmetric, since it is the Hessian of the 3d Regge function
\[ S^{(3)}_R = \sum_e l_e \omega_e. \]

Secondly, let us recall that the length $l_{e_F}$ of the edge $e_F$ equals the height $h_{e_F}$ associated to $e$ in the triangle $F$. Consequently, multiplying (30) by the variation $\delta s_F = l_e^j \delta h_{e_F}$ and summing over $F \supset e$, lead to the following equalities, holding for every $e'$ of the triangulation:
\[ \sum_{F} \frac{\delta \omega_F}{\delta l_{e'}} \delta s_F = \sum_{F,e'} \frac{\delta \omega_{e_F}}{\delta h_{e_F}} \frac{\delta l_{e'}}{\delta l_{e''}} \delta s_F = l_e^j e' \sum_{e''} \left( \sum_{F \ni e} \frac{\delta \omega_{e_F}}{\delta h_{e_F}} \frac{\delta l_{e''}}{\delta l_{e'}} \right) \delta l_{e''} = l_e^j e' \sum_{e''} \left( \sum_{F \ni v_e} \frac{\delta \omega_{e_F}}{\delta h_{e_F}} \frac{\delta l_{e''}}{\delta l_{e'}} \right) \delta l_{e''}. \] (31)

Now, on-shell, all the deficit angles vanish; all the spaces $e^j$, as well as the projection $P^j$, can be identified to each other $e^j \equiv e^i$ and $P^j \equiv P^i$; and the complex of tetrahedra $\tau_j$ can be mapped in $\mathbb{R}^3$. The term in parenthesis turns out to be the variation of the 3d deficit angle induced by a displacement, by a 3-vector $\beta_e \equiv e^i \cong \mathbb{R}^i$, of the vertex $v_e$; therefore it vanishes, and (24) is proved. Hence, we have identified the parameters $\beta_e$ as generating the symmetry of the face labels.

The gauge-fixing is performed by fixing a subset of the face labels while taking into account the Faddeev-Popov determinants. By analogy with the edge symmetry, in order to eliminate the three components of the gauge parameter at each edge $e$, we would need to fix the labels $(s_F)$ of three faces sharing $e$. Lets consider a simplex $\sigma$ touching $e$ and the three faces $(F_i)$ of $\sigma$ that meet at $e$. The ‘dimensional reduction’ performed in the previous proof associates a tetrahedron $\tau$ to the simplex $\sigma$, and three edges to the faces $F_i$: the lengths of these edges are also the heights $h_i$ associated to $e$ in $F_i$. The determinant that corresponds to the fixing of the labels $(s_{F_i})$ can then be read out from the relation between the Lebesgue measure $d^3 \beta_e$ and the variations $\delta h_i$
\[ d^3 \beta_e = \frac{\prod_{i=1}^3 h_i dh_i}{V_r} = \frac{\prod_{i=1}^3 A_{F_i} \delta s_{F_i}}{V_3} \]
FIG. 2: The orthogonal projection of a simplex $\sigma \supset e$ onto the 3-space $e^\perp$, maps the edge $e$ to the vertex $v_e$ and the three faces $F_i$ adjacent to $e$ to three edges $e_i$ meeting at $v_e$. The dashed line represents the height associated to $e$ within the triangle $F_i$.

where $V_\tau$ is ($3!$ times) the volume of the tetrahedron $\tau$, and where we made use, for the second equality, of $h_F = l_e$, $dh_F = l_e^2 \delta_{s_F}$, $V_\sigma = l_e V_\tau$, and $A_{F_i} = l_e h_{F_i}$. The determinant reads then

$$D^\sigma = \frac{V_\sigma}{\prod_{i=1}^3 A_{F_i}}. \quad (33)$$

There is no factor 2 in these expressions, since here we work with fixed orientations for the simplices.

Still by analogy with the previous symmetry, we would want to assign, to each edge $e$ of the triangulation, a simplex $\sigma_e$ to which $e$ belongs. Each of these simplices provides three faces $F_i^e$, $F_i^e$, namely the three faces of $\sigma_e$ sharing $e$. Suppose that there exists an assignment $A_s = \{\sigma_e\}$ such that $F_i^e \neq F_j^{e'}$ for every couple $(e, e')$ of distinct edges - this admissibility condition insures that no face is picked up more than once in the procedure. Then the gauge-fixing terms and Faddeev-Popov determinant associated to the symmetry (24) are expected to be

$$\delta_{GF}^{A_s} = \prod_{F \in A_s} (2\pi)\delta_{s_F}, \quad D_{FP}^{A_s} = \prod_{F \in A_s} \frac{V_\sigma}{\prod_{F \in A_s} A_F} \quad (34)$$

where by definition $F \in A_s$ if $F \in \sigma_e$ and $F \supset e$ for some edge $e$.

A subtlety arises here however: it is in general impossible to choose an assignment $A_s$ satisfying the admissibility condition. This can be seen by the following argument. Although this condition seems quite analogous to the admissibility condition for the assignment $A_l$ associated to the edge symmetry, there is major difference between the comportment of these two conditions under refinement of the triangulation, i.e under a $(1, 5)$ Pachner move. Such a move consists in a subdivision of a simplex $\sigma_0$ into five 4-simplices $\sigma_1, \ldots, \sigma_5$, providing one additional vertex, as well as five additional edges and ten additional faces. Extending an admissible assignment $A_l$ requires to assign four edges to the new vertex, which can be picked up beyond the five new edges. Extending an admissible assignment $A_s$, on the other hand, would require to assign to each of the five new edges a triplet of faces, without any repetition of the faces. This means that $3 \times 5 = 15$ faces, adjacent to one of the new edges, are needed, whereas only ten are at our disposal. An over-counting of the faces, and, thus, of the labels that have to be fixed, seems therefore to be unavoidable. This problem, treated in the next section, is a reflection of the fact that the symmetry (24) generated by $\vec{\beta}_e$ is a reducible symmetry [19]. It turns out, indeed, that the gauge parameters $\vec{\beta}_e$ are not independent, which leads to an overestimation of the number of gauge degrees of freedom.
D. Reducibility

In this part we show that the action of the symmetry \( s_F \to s_F + \delta s_F(\vec{\beta}_e) \) on the labels is not free, that is, there exists non trivial transformations \( \vec{\beta}_e \to \vec{\beta}_e + \delta \vec{\beta}_e \) of the gauge parameters, such that

\[
\delta s_F(\vec{\beta}_e + \delta \vec{\beta}_e) = \delta s_F(\vec{\beta}_e).
\]

This symmetry of the gauge parameters is characterized infinitesimally by the following equations

\[
\sum_e \delta s_F \delta \vec{\beta}_e = 0 \quad \forall F
\]

which provide dependence relations between the gauge parameters. As we will see, such transformations of the parameters \( \vec{\beta}_e \) are generated by elements \( (\sigma_v)_{\mu\nu} \) of the Lie algebra \( so(4) \) living on the vertices of the triangulation.

An analogous result can be found in Korepanov’s work [18]. Hence, the ‘true’ degrees of freedom for the symmetry \( \{\vec{\beta}_e\}_{e \in \Delta} \) are no longer lists \( \{\vec{\beta}_e\}_{e \in \Delta} \) of 3-vectors but rather \( \text{orbits} \) of such lists \( \text{modulo} \) an action - which we will specify - of the Lie algebra elements.

Transformations (35) can be described as follows. Let \( v \) be a vertex of \( \Delta \) and an element \( \sigma_{\mu\nu} \in so(4) \) associated to it. For a configuration \( \{\ell_e\} \) of the \( \ell \)'s labels inducing a flat geometry, the complex of 4-simplices sharing \( v \) can be mapped in \( \mathbb{R}^4 \), in such a way that \( v \) is placed at the origin. The edges meeting at \( v \) form then vectors \( \vec{l}_e \in \mathbb{R}^4 \). For any edge \( e \) of the triangulation, we define the transformation \( \vec{\beta}_e \to \vec{\beta}_e + \delta \vec{\beta}_e \) generated by \( \sigma_{\mu\nu} \) as

\[
\delta_v \vec{\beta}_e = \begin{cases} 
\frac{2}{|e|} \sigma \left( \begin{array}{c} l_a \\ l_b \\ e \\ \ell_e \\
\end{array} \right) & \text{if } e \supset v \\
0 & \text{if not} 
\end{cases}
\]

where \( \sigma \left( \begin{array}{c} l_a \\ l_b \\ e \\ \ell_e \\
\end{array} \right) = \sigma_{\mu\nu} \frac{\vec{e}}{l_e} \) is the image of the unitary vector \( \frac{\vec{e}}{l_e} \) by the operator which represents \( \sigma_{\mu\nu} \in \mathbb{R}^4 \). Let see why this transformation satisfies (35). Given a face \( F \) to which the vertex \( v \) belongs, let \( e_a, e_b \) be the edges of \( F \) meeting at \( v \), and \( l_a, l_b \) their lengths. According to (27) and (37), the variation \( \delta s_F \) induced by the combined action of \( \delta_v \vec{\beta}_e \) and \( \delta_v \vec{\beta}_b \) can be written as

\[
\delta s_F \equiv \delta_a s_F + \delta_b s_F = -\frac{1}{l_a} \sigma(\vec{l}_a) \cdot \frac{\vec{h}_a}{h_F} - \frac{1}{l_b} \sigma(\vec{l}_b) \cdot \frac{\vec{h}_b}{h_F}
\]

where \( h_F^a, h_F^b \) are the two heights associated to \( e_a \) and \( e_b \) in \( F \). Note that it is always possible to find a scalar \( \alpha \) such that \( \vec{l}_b = \vec{l}_a + \alpha \vec{e} \) and, since \( \sigma \) is skew symmetric and \( A_F = l_a h_a = l_b h_b \), we have

\[
\delta_a s_F = \sigma(\vec{l}_a) \cdot \vec{l}_b, \quad \delta_b s_F = \sigma(\vec{l}_b) \cdot \vec{l}_a, \quad \delta_a s_F + \delta_b s_F = 0
\]

Hence, we see that the label \( s_F \) transforms trivially under the action of \( \delta_v \vec{\beta}(\sigma) \), which shows (35).

The proper way to deal with the gauge fixing of the reducible symmetry is, first, to fix the symmetry of gauge parameters, acting at the vertices of the triangulation, and thus reduce the gauge degrees of freedom to an independent set of gauge parameters; and second, to eliminate the remaining gauge degrees of freedom by the usual Faddeev-Popov procedure. Notice that, by taking into account the reducibility of the symmetry, the over-counting problem highlighted at the end of the last section no longer holds: under a refinement of the triangulation (move (1, 5)), the action of an \( so(4) \)-element \( \sigma_{\mu\nu} \) (6 components) attached to the new vertex has to be fixed beforehand; it is expected to reduce the number of gauge parameters to 15 − 6 = 9. Then, the remaining gauge degrees of freedom are eliminated by fixing the labels of 9 of the 10 new faces.

In order to get an intuition of what the precise gauge-fixing prescriptions should be, let us again consider a vertex \( v \) of the triangulation and an element \( \sigma_{\mu\nu} \in so(4) \) associated to it. We suppose that the neighborhood of \( v \) is mapped in \( \mathbb{R}^4 \), the vertex being placed at the origin. The element \( \sigma_{\mu\nu} \) generates a transformation of the gauge parameters \( \vec{\beta}_e \) living on the edges meeting at \( v \). These edges define vectors \( \vec{l}_e \in \mathbb{R}^4 \) and we see, with (37), that the variations of the gauge parameters is related to the displacement \( \delta \vec{l}_e \equiv \sigma(\vec{l}_e) \) of these vectors under the infinitesimal rotation \( 1_{4 \times 4} + \sigma_{\mu\nu} \).

Now we know how to eliminate rotational degrees of freedom that act on 4-vectors: given four edges \( \vec{l}_1, \cdots, \vec{l}_4 \) meeting
at $e$ and belonging to a simplex $\sigma_v$, this elimination consists in fixing the direction of $\vec{l}_1$, restricting $\vec{l}_2$ to a fixed plane and $\vec{l}_3$ to a fixed hyperplane. In other words, the gauge-fixing of the action of $\sigma_{\mu\nu}$ can be performed by fixing the value of the three components of $\vec{\beta}_1$, two components of $\vec{\beta}_2$ and one component of $\vec{\beta}_3$. If these six components are denoted by $\{\beta^k\}$, the determinant $\Delta_2$ that results from the gauge-fixing of this ‘first-stage’ symmetry, is the determinant of a square matrix whose elements are derivatives of the $\beta^k$ with respect to the six independent components $\{\sigma^i\}$ of $\sigma_{\mu\nu}$:

$$
\Delta_2 = \det \left[ \frac{\partial \beta^k}{\partial \sigma^i} \right]
$$

Once the symmetry (37) is fixed, one can use the six remaining components $\{\beta^j\}$ of the gauge parameters living on $\vec{l}_1, \vec{l}_2, \vec{l}_3$ to fix the labels $s_F$ of the six faces of $\sigma_v$ that share the vertex $v$. The determinant $\Delta_1$ that results from the gauge-fixing of this ‘first-stage’ symmetry, is the determinant of a square matrix whose elements are derivatives of the $s_F$ with respect to the $\beta^j$:

$$
\Delta_1 = \det \left[ \frac{\partial s_F}{\partial \beta^j} \right]
$$

The factors $\Delta_1$ and $\Delta_2$ are explicitly computed in Appendix B for a suitable choice of gauge-fixing conditions.

This analysis shows how to treat the face symmetry acting at the edges of a simplex $\sigma_v$ that share a vertex $v$, while taking into account its reducibility. Namely, this partial gauge-fixing is performed by fixing the variables $s_F$ labeling the faces of $\sigma_v$ that meet at $v$, and by inserting of Faddeev-Popov determinant

$$
D^{\sigma_v} = \Delta_1 \Delta_2^{-1}
$$

The form of this determinant is a typical feature of a reducible symmetry. The first term $\Delta_1$ arises from the integration of usual fermionic ghosts, while the second term $\Delta_2^{-1}$ arises from the integration of bosonic ghosts for ghosts [19].

The product takes the remarkably simple form, given by (B13):

$$
D^{\sigma_v} = \frac{V_{\sigma_v}}{\prod_F A_F}
$$

where $V_{\sigma_v}$ is the volume of the simplex $\sigma_v$, and where the product is over the faces of the simplex which meet at $v$.

Let us suppose that the previous procedure is applied for every vertex $v$ of the triangulation. That is for each vertex we choose a 4-simplex $\sigma_v$ and fix the value of $s_F$ for all faces of $\sigma_v$. The symmetry is therefore fully reduced, the gauge parameters living on the edges which belongs to none of the $\sigma_v$ act now freely on the labels $s_F$ that are not fixed. Given such an edge $e$, the symmetry generated by $\vec{\beta}_e$ is treated by fixing the variables of three faces $F_e^1, F_e^2, F_e^3$ adjacent to $e$. As emphasized in section III C, if one chooses these three faces so that they belong to the same simplex $\sigma_v$, the corresponding determinant reads

$$
D^{\sigma_v} = \frac{V_{\sigma_v}}{A_{F_e^1} A_{F_e^2} A_{F_e^3}}
$$

Repeating this operation for every edge $e \notin \bigcup_{v \in \Delta} \sigma_v$ completes the full gauge-fixing of the symmetry (37).

The general gauge-fixing procedure of the reducible face symmetry is then as follows. We first define an admissible assignment $A_s^{(1)} = (\sigma_0, \{\sigma_v\}_{v \not\in \sigma_0})$ as in section III B using a maximal tree in the 1-skeleton dual to the triangulation [12]. A convenient choice is $A_s^{(1)} = A_t$. As before we say that an edge $e$ belongs to $A_s^{(1)}$ if either $e \in \sigma_0$, or $e \in \sigma_v$ for some $v$ and $e$ admits $v$ as one of its vertices; likewise we say that a face $F$ belongs to $A_s^{(1)}$ if either $F \in \sigma_0$, or $F \in \sigma_v$ for some $v$ and $F$ admits $v$ as one of its vertices. We then assign to every additional edge $e \notin A_s^{(1)}$ a 4-simplex $\sigma_e$; it can be checked that, by construction, $\sigma_e \notin A_s^{(1)}$. Each of these edges provides three faces $F_e^1, F_e^2, F_e^3$, namely the faces of $\sigma_v$ to which $e$ belongs. We say that a face $F$ belongs to $A_s^{(2)}$ if $F \in \sigma_e$ and $F$ admits $e$ as one of its edges. The assignment $A_s^{(2)} = \{\sigma_e\}_{e \notin \sigma_v}$ is said admissible if $F_e^1 \neq F_e^3$ for every couple $(e, e')$ of distinct edges that are not

---

9 Whether or not there exists a systematic way to construct such admissible assignments, is a question that is left open here.
in $A^{(1)}_s$. Eventually we define $A_s = A^{(1)}_s \cup A^{(2)}_s$: a simplex $\sigma$ or a face $F$ belong to $A_s$ if they belong to one of the assignments $A^{(1)}_s, A^{(2)}_s$. The assignment $A_s$ is said admissible if both $A^{(1)}_s$ and $A^{(2)}_s$ are admissible.

Given an admissible assignment $A_s$, the gauge-fixing term and Faddeev-Popov determinant, associated to the symmetry $|2\pi| l_s l_F$ read then

$$
\delta_{GF}^A = \prod_{F \in A_s} (2\pi) \delta_{s_F, s_F'}, \quad \tilde{D}_{GF}^A = \prod_{v} D^{\sigma_v} \prod_{e} D^{\sigma_e} = \prod_{\sigma \in A_s} V_{\sigma} \prod_{F \in A_s} A_F.
$$

The gauge fixing described above allows us to fix the value of $l_s, s_F$ on a subset of edges and faces. By removing the summation over $s_F$, we remove a redundant factor $\delta(\omega_F)$ which naively makes the partition function divergent, this naive divergence being, as we have just shown, the expression of a gauge symmetry acting around Regge solutions. It is important to note however that the constraints $\omega_F = 0$ act not only as constraints on the continuous edge labels, but also on the discrete orientation labels $\epsilon$. That is, given a set of $l_s$ which describes a flat space geometry, there is only a restricted choice of orientations $\epsilon$ which allows the realization of this flat space geometry in terms of an oriented triangulation. Thus if the $\epsilon$ is not chosen appropriately it is not possible to satisfy the flatness constraint $\omega_F = 0$ and the corresponding partition function is in fact zero. The gauge fixing of $s_F$ removes this necessary restriction on the orientations and an additional gauge fixing factor acting on the orientation label should be added. The additional gauge fixing factor acting on orientations has a natural topological interpretation: in order for the triangulation to describe a manifold, one should insure that the link around any internal edge as the topology of a 2-sphere. As shown in the appendix, this condition can be implemented by demanding that the solid angle $\Omega^e_{\epsilon}$ around any edge is equal to $4\pi$, where $\Omega^e = \sum_{\sigma \supset e} \epsilon_{\sigma} \Omega^e_{\sigma}$ and $\Omega^e_{\sigma}$ is the solid angle of the edge $e$ within the 4-simplex $\sigma$. This is realized by adding to the face gauge fixing $\{12\}$ a term $\prod_{e \in A_s} \Theta(\Omega^e_{\epsilon})$ and define

$$
D_{GF}^A \equiv \tilde{D}_{GF}^A \prod_{e \in A_s} \Theta(\Omega^e_{\epsilon})
$$

where $\Theta$ is a characteristic function defined to be constant, with value 1, on $4\pi \mathbb{Z}$ and 0 elsewhere.

### E. The gauge-fixed model

For a given triangulation $\Delta$, the fully gauge-fixed partition function is defined by choosing admissible assignments $A_s$ and $A_l$ and by inserting gauge-fixing terms and Faddeev-Popov determinants $|2\pi| l_s l_F$ in the integral.

$$
Z_{\Delta} = \frac{1}{(2\pi)^{|F|}} \int l_s l_F \prod_{F \in \Delta} A_F \sum_{\{s_F, l_s\}} \left( \prod_{\sigma} \frac{e^{l_s s_F (s_F, l_s)}}{V_{\sigma}} \right) \delta_{GF}^A \tilde{D}_{GF}^A D^{A_l} D^{A_s}.
$$

This analysis completes the definition of the spin foam model $\{13\}$. The next section is devoted to proving that the Feynman integral $\{8\}$ is equal to the evaluation of an observable for this model.

### IV. FEYNMAN DIAGRAMS AS SPIN FOAM AMPLITUDES

In this part, we want to show our main statement $\{12\}$. To do so, we first establish that the gauge-fixed partition function is independent of the triangulation - and thus defines an invariant of 4d-manifold. Then given a Feynman graph $\Gamma$, we will consider the so-called Feynman graph observable, which breaks part of the gauge symmetry, promoting gauge degrees of freedom to dynamical degrees of freedom. We properly define the expectation value of this observable and show that it coincides with the Feynman amplitude $\{8\}$ associated to the graph $\Gamma$.

#### A. Topological invariance

Showing that the model does not depend on the triangulation, and depends only on the piecewise linear topology of the underlying manifold, amounts to check that it is invariant under 4d Pachner moves and under change of the admissible assignments $A_s, A_l$. The invariance under change of admissible assignments is a priori insured by the fact that these assignments arise from the gauge fixing of a gauge symmetry of the theory; this is the usual BRST symmetry. Indeed the gauge fixing terms and Faddeev-Popov determinants are constructed in order to define the right
measure of integration over the orbit space of configuration modulo gauge transformation. It would be nevertheless interesting to give a formal and direct proof of this invariance: we leave this as an open problem and now focus on the invariance under Pachner moves. In this proof we can however show that, whenever a gauge fixing is needed in order to define the Pachner move, the result is independent under the gauge fixing choice.

We consider six 4-simplices \( \sigma_0, \ldots, \sigma_5 \) which triangulate the boundary of a 5-simplex, \( \sigma_i \) being the 4-simplex where the vertex \( i \) is omitted. The volume of \( \sigma_i \) is denoted by \( V_i \); the action term \( S_{\sigma_i}(s_F, l_e) \), defined in (14) and associated to the simplex \( \sigma_i \), is simply written \( S_i \).

We first investigate the behavior of (44) under the move \((3,3)\) :
\[
\sigma_1, \sigma_2, \sigma_3 \rightarrow \sigma_0, \sigma_4, \sigma_5,
\]
which erases the face \([045]\) and provides the face \([123]\). The triangulations arising in both sides of the move do not contain internal faces or edges therefore no gauge fixing is needed in this case.

The invariance of \( Z_{\Delta}^{GF} \) under this move is due to the following \((3,3)\) identity proven in appendix A
\[
\sum_{\epsilon_1, \epsilon_2, \epsilon_3, s_{045}} \frac{A_{045}}{V_1} \frac{e^{\epsilon_1 S_1}}{V_2} \frac{e^{\epsilon_3 S_3}}{V_3} = \sum_{\epsilon_0, \epsilon_4, \epsilon_5} \frac{A_{123}}{V_0} \frac{e^{\epsilon_0 S_0}}{V_4} \frac{e^{\epsilon_4 S_4}}{V_5} \frac{e^{\epsilon_5 S_5}}{V_5}
\]
\[(45)\]

We then consider the move \((2,4)\) : \( \sigma_0, \sigma_5 \leftrightarrow \sigma_1, \sigma_2, \sigma_3, \sigma_4 \), which provides (or erases) the edge \((05)\), as well as the four faces \([05i], i = 1 \cdots 4 \) adjacent to \((05)\). The complex with 4 4-simplices contains one internal edge \((05)\) and four internal faces \([05i]\) whose labels should be summed over. Three of the summation over faces variables can be gauge fixed due to the symmetry carried by the edge \((05)\) and acting on faces.

The invariance under this move follows from the gauge-fixed hexagonal identity
\[
\sum_{\epsilon_0, \epsilon_5} \frac{e^{\epsilon_0 S_0}}{V_0} \frac{e^{\epsilon_5 S_5}}{V_5} = \frac{1}{(2\pi)^4} \int dl_{05} d_{05} \sum_{\{s_i\}} \prod_{i=1}^4 A_i \prod_{i=1}^4 \frac{e^{\epsilon_i S_i}}{V_i} \delta_G^{(2,4)} D_{FP}^{(2,4)}
\]
\[(46)\]

where \( s_i \equiv s_{05i} \) and \( A_i \equiv A_{05i} \) denote label and area of the face \([05i]\). The quantities \( \delta_G^{(2,4)} \) and \( D_{FP}^{(2,4)} \) read
\[
\delta_G^{(2,4)} = (2\pi)^3 \prod_{i=1}^3 \delta_{s_i, s_{05i}}^p, \quad D_{FP}^{(2,4)} = \frac{V_4}{A_1 A_2 A_3} \Theta(\Omega_{05i})
\]
\[(47)\]

where the \( s_{05i}^p \) are any fixed values.
These terms can be understood as follows in the case of the move $2 \to 4$: the gauge fixing assignments of the triangulation containing 2 simplices need to be extended in order to accommodate for the internal edge (05) added in the move. According to our prescription, one needs to choose one 4-simplex ($\sigma_4$ say) and three faces [05i], $i = 1, 2, 3$ of $\sigma_4$ which are adjacent to (05). Following (12), the extension of the gauge fixing assignment to ($\sigma_4$, [05i]) implies that the gauge-fixing term and Faddeev-Popov determinant are then multiplied by (12). Note that, by symmetry, the identity (10) is independent of the choice of extension of the gauge fixing assignment.

We eventually examine the change of the model under refinement of the triangulation, namely the move (1, 5) : $\sigma_0 \longleftrightarrow \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$, which creates (resp. erases) the vertex 0, five edges (0j), $j = 1 \cdots 5$ and ten faces [0ij], $i, j = 1 \cdots 5$ meeting at 0.

The invariance under this move is a consequence of the gauge-fixed (1, 5) identity

$$
\sum_{s_0} e^{s_0 \cdot S_0} V_0 = \frac{1}{(2\pi)^6} \sum_{s_i} \int \prod_{j=1}^5 dl_{ij} l_{ij} \prod_{i,j} A_{ij} \prod_{j=1}^5 V_j \delta^{(1,5)}_{GF} D^{(1,5)}_{FP}
$$

(48)

where $s_{ij} \equiv s_{0ij}$ and $A_{ij} \equiv A_{0ij}$ denote label and area of the face [0ij]. The quantities $\delta^{(1,5)}_{GF}$ and $D^{(1,5)}_{FP}$ read

$$
\delta^{(1,5)}_{GF} = (2\pi)^9 \prod_{(ij) \neq (45)} \delta_{s_{ij}, s'_{ij}} \prod_{i=1}^4 \delta(l_{0i} - l'_{0i}), \quad D^{(1,5)}_{GF} = \left( \frac{\nu_5}{2 \prod_{i=1}^5 l_{0i}} \right) \left( \prod_{i,j=1}^5 A_{ij} \right) \prod_{j=1}^5 \Theta(\Omega'_{0j})
$$

(49)

where the $l'_{0i}, s'_{ij}$ are any fixed values. These terms are those by which the gauge-fixing terms and Faddeev-Popov determinants (25) and (12) are multiplied when the assignment $A_1$ is extended to the new vertex 0, the simplex $s_5$ associated to it, and the four edges (0i) of $s_5$ that meet at 0; when the assignment $A_5^{(1)}$ is extended to the new vertex 0, the simplex $s_5$ associated to it, and the six faces [0ij] of $s_5$ that share 0; and when the assignment $A_5^{(2)}$ is extended to edge (05), the simplex $s_4$ associated to it, and the three faces [0ij] of $s_4$ which are adjacent to (05). The role of the terms (49) is to give the edge geometry acting at the new vertex 0, as well as the reducible face symmetry that acts at the new edges (0i) - while taking into account the symmetry of the gauge parameters acting at the vertex 0.

Let us stress that in the (2, 4) and (1, 5) identities, once we take into account the additional gauge fixing terms $\Theta(\Omega'_{05})$ and $\prod_{j=1}^5 \Theta(\Omega'_{0j})$, we can restrict the sum of orientations to be only over $\epsilon_1, \epsilon_2, \epsilon_3$: the values of $\epsilon_4$ and $\epsilon_5$ are then functions of these orientations and the length $l_{05}$, so that on shell namely when $\omega'_{05} = 0$, the other deficit angles vanish as well. This is explained in appendix and in more detail in [12].

The derivation of these identities is given in appendix A. In particular, the keystone of the proof of identity (2, 4) is shown to be the equality of measures (10). As a consequence, the hexagonal identity (40) is not only a relation between geometrical quantities, but it is an equality of measures which allows to integrate a function of the label $l_{05}$ on the RHS and a function of the other labels and orientations - via the value on shell $l'_{05}(l_{ij})$, see section 11 and Appendix - on the LHS. This remark will be crucial for the demonstration of the statement (12).

**B. Observables and partial gauge-fixing**

We consider a Feynman graph $\Gamma$ and a triangulation $\Delta$ of the 4-sphere $S^4$ in which $\Gamma$ is embedded. We define the Feynman graph observable as a function of the labels $l_e$ living on the edges of $\Gamma$, given by

$$
\mathcal{O}_\Gamma(l_e) = \prod_{e \in \Gamma} G^F(l_e)
$$

(50)
where $G^F$ is the Feynman propagator. The function $O_T$ in not gauge invariant; its insertion breaks the symmetry of the labels $l_v$ which acts at the vertices of the graph $\Gamma$ and thus modifies the gauge-fixing procedure. The evaluation of this observable is defined to be

$$\langle O_T \rangle_\Delta = \frac{1}{(2\pi)^{k'}} \int_{\Delta} \prod_{e \in \Delta} dl_e l_e \prod_{F \in \Delta} A_F \sum_{\{s_F, l_e\}} O_T(l_e) \left( \prod_{s} \frac{e^{i s_F w_F}}{\nu_s} \right)$$

(51)

where the label $GF$ means that the integral is partially gauge-fixed: namely, the face symmetry is fully gauged fixed as before, but only the edge symmetry acting at the vertices of $\Gamma \setminus \Delta$, is fixed. In order to fix these gauge symmetries, we first choose as before an admissible assignment $A_\Gamma = A_{(1)}^{(1)} \cup A_{(2)}^{(2)}$. We then specify an admissible assignment $A_t = \{\sigma_v\}_{v \in \Gamma}$, so that a simplex is assigned to every vertex that does not belong to the graph $\Gamma$. One can conveniently choose $A_t$ to be the subset of simplices $\sigma_v \in A_{(1)}^{(1)}$ associated to the vertices of $\Delta \setminus \Gamma$. Inserting the gauge fixing terms (51) and (52) fully fixes the face symmetry and partially fixes the edge symmetry; the gauge degrees of freedom that are not eliminated couple to the Feynman graph observable, thereby they are promoted to dynamical degrees of freedom.

We want to show that the evaluation (51), where the gauge fixing is partially performed, equals the Feynman amplitude $I_T$ associated to the graph $\Gamma$. Recall that, in (5), the amplitude is expressed as a quantity computed on a triangulation $\Delta_k$ of a 4-ball $B$, of a special type: every vertex, edge or face of $\Delta_k$ lie on the boundary. The 1-skeleton dual to $\Delta_k$ is a four-valent tree $T_k$ with open ends. Since $S^4$ can be obtained by gluing two 4-balls with reversed orientation along their common boundary $S^3$, we can then construct from $\Delta_k$ a triangulation of $S^4$ denoted by $D\Delta_k \equiv \Delta_k \cup S^3 \Delta_k$.

The reasoning, identical to the 3d case, is then as follows. Both triangulations $\Delta$ and $D\Delta_k$ of $S^4$ contain the vertices of the graph $\Gamma$; they can therefore be constructed out of each other by a sequence of Pachner moves which do not remove the vertices of $\Gamma$. Now according to the previous analysis, the quantity $\langle O_T \rangle_\Delta$ is invariant under these moves, provided that the variable $l_e$, living of each edge $e$ erased by a $(2,4)$ move is replaced by its value ‘on shell’ - this is where the remark of the end of the previous part acquires its importance. Consequently:

$$\langle O_T \rangle_\Delta = \langle \hat{O}_T \rangle_{D\Delta_k}, \quad \text{with} \quad \hat{O}_T \equiv O_T(l_e, l_e')(l_e).$$

(52)

The values $l_e', l_e''$ are fully specified by the edge labels of $D\Delta_k$ and the orientations; they are the Euclidean distances, in any embedding of $\Delta_k$ in $\mathbb{R}^4$, between vertices that are not connected by the edges of the triangulation.

Now with a mechanism explained in (12), the tree $T_k$ dual to the triangulation $\Delta_k$, once given a root and an orientation, can be used to define an admissible assignment $A_{(1)}^{(1)} = \{\sigma_0, \{\sigma_v\}_{v \in \sigma_0}\}$: the simplex $\sigma_0$ is dual to the root of the tree, and the $\sigma_v$ are defined recursively by following the branches of the tree according to its orientation. It is then easy to see that every edge of the triangulation belongs to $A_{(1)}^{(1)}$; therefore $A_{(2)}^{(2)} = \emptyset$ and the face symmetry is fully fixed by inserting the factors (12) associated to the assignment $A_{(1)}$. The edges, faces and 4-simplices of $A_{(1)}$ are then the edges, faces and 4-simplices of $\Delta_k$. Furthermore, since every vertex of $D\Delta_k$ belongs by construction to the graph $\Gamma$, there is no remaining edge symmetry once the symmetry-breaking observable $O_T$ is inserted into the partition function. Therefore the gauge fixing is performed by plugging

$$\delta_{GF}^{A_{(1)}^{(1)}} = \prod_{F \in \Delta_k} (2\pi)\delta_{s_F, s_F'}, \quad D_{GF}^{A_{(1)}^{(1)}} = \frac{\prod_{\sigma \in \Delta_k} \nu_{\sigma} \prod_{e \in \Delta_k} \Theta(\Omega_{e}^{(1)})}{\prod_{F \in \Delta_k} A_F \prod_{e \in \Delta_k} \Theta(\Omega_{e}^{(1)})}$$

(53)

Since $\Delta_k$ has no internal vertex, edge or face, $\Delta_k$ and $D\Delta_k$ possess the same number of vertices, edges and faces, which all lie on the boundary of the ball $B$, whereas the number of 4-simplices in $D\Delta_k$ is $2k$. Each 4-simplex in the interior of $B$ has a copy in the exterior of $B$; the two copies share their edges, and consequently have the same volume. The orientation of a 4-simplex $\sigma$ within $B$ and that of its copy are denoted by $\epsilon_\sigma$ and $\epsilon'_\sigma$. Taking into account the gauge fixing terms (53), the evaluations (52) read

$$\langle \hat{O}_T \rangle_{D\Delta_k} = \int \prod_{e \in D\Delta_k} l_e dl_e \sum_{\{s_F\}} \prod_{e \in \Delta_k} A_F \sum_{e \in \{\pm\}^{2k}} e^{i \sum_{\sigma \in \Delta_k} s_\sigma \epsilon_\sigma}_e \prod_{\sigma \in D\Delta_k} \nu_{\sigma} O_T(l_e, l_e'(l_e)) \delta_{GF}^{A_{(1)}^{(1)}} D_{GF}^{A_{(1)}^{(1)}}$$

(54)

$$= \int \prod_{e \in \Delta_k} l_e dl_e \sum_{e, e' \in \{\pm\}^k} \prod_{\sigma \in \Delta_k} \Theta(\Omega_{e}^{(1)}) e^{i \sum_{\sigma \in \Delta_k} s_\sigma \epsilon_\sigma}_e \prod_{\sigma \in \Delta_k} \Theta(\Omega_{e}^{(1)})$$

(55)

10 Note that, for triangulations of the type of $\Delta_k$, such an embedding always exists.
Indeed, the spherical Schl"afli identity, unlike the flat one, admits a second term proportional to the spherical volume of the simplex.

The Gram determinants reduce to Euclidean volumes in the flat limit, see Appendix A.

One can now convince oneself that this condition imposes \( \epsilon_\sigma + \epsilon'_\sigma = 0 \) for every 4-simplex of \( \Delta_k \). This finally shows our statement.

\[
\langle \mathcal{O}_\Gamma \rangle_\Delta = \int \prod_{e \in \Delta_k} l_e dl_e \sum_{e \in \{ \pm 1 \}^k} \prod_{\sigma \in \Delta_k} \frac{1}{V_\sigma} \mathcal{O}_\Gamma(l_e, l'_e(l_e)) = I_\Gamma
\]

The conclusion of this analysis is therefore that QFT Feynman amplitudes are obtained by inserting the partially symmetry-breaking observables \( \langle O \rangle \) into the topological spin foam model \( \langle \mathcal{O}_\Gamma \rangle \).

C. Feynman diagrams on homogeneous spaces

Let us briefly mention how the results established above can directly be extended to spherical and hyperbolical space-times. The Feynman amplitude of a graph \( \Gamma \) embedded in the unit 4-sphere \( \mathcal{S}^4 \) takes the form

\[
I_\Gamma = \int_{\mathcal{S}^3} du_1 \cdots du_N \prod_{\langle ij \rangle \in \Gamma} G_m(l_{ij})
\]

\( du_j \) is the normalized measure on the 3-sphere. The integrand is a product of propagators, which are functions of the dimensionless spherical distances \( l_{ij} \in [0, \pi] \) between the vertices, and invariant under the action of the group \( SO(5) \).

Following the strategy used for the flat case, this amplitude is first expressed in terms of the invariant measure

\[
\sum_{e \in \{ \pm \}^k} \prod_{\Delta_k} \sin l_e dl_e \prod_{\sigma \in \Delta_k} \frac{1}{V_\sigma}
\]

\( \Delta_k \) is the spherical analogue of the triangulation defined in section II. \( V_\sigma \) is the square root of the Gram determinant \( \det [\cos l_{ij}] \) associated to the simplex \( \sigma \).

As shown in appendix, all the geometrical identities for flat simplices also hold for spherical simplicies, provided that all ‘volumes’ and ‘area’ are replaced by the square root of the Gram determinants \( \langle O \rangle \). The analysis of Feynman graphs on the unit sphere \( \mathcal{S}^4 \) leads then to the emergence of the spin foam model

\[
Z_\Delta = \frac{1}{(2\pi)^F!} \int \prod_{e \in \Delta} dl_e \sin l_e \prod_{F \in \Delta} A_F \sum_{\{s_F, \epsilon_\sigma\}} \left( \prod_{\sigma} e^{\epsilon_\sigma S_\sigma(s_F, l_e)} \frac{1}{V_\sigma} \right)
\]

with an action term for each 4-simplex which reads:

\[
S_\sigma = \sum_F s_F \theta^\sigma_F(l_e)
\]

where \( \theta^\sigma_F \) is the spherical interior dihedral angle of the face \( F \) in \( \sigma \). The Feynman amplitude \( I_\Gamma \) is then the expectation value of the partially symmetry-breaking observable \( \langle \mathcal{O}_\Gamma(l_e) \rangle = \prod_{e \in \Gamma} G_m(l_e) \) for the model \( \langle \mathcal{O}_\Gamma \rangle \), computed on any triangulation \( \Delta \) of \( \mathcal{S}^4 \) which contains \( \Gamma \) as a subgraph. Analogous results for hyperbolical space are obtained by working on the hyperboloid and by replacing all the angles by hyperbolic angles. The cosmological constant \( \Lambda \) is explicitly introduced by re-scaling the spherical lengths \( l_e \rightarrow L_e = l_e/\sqrt{\Lambda} \).

Notice that the global action of the model no longer admits Regge solutions \( l^0_e, s_F = 0, A_F = \) - where \( A_F \) is the spherical area of the face \( F \) - unless\(^{12} \) \( \alpha = 0 \); therefore the analysis of the symmetries has to be done by restricting to fluctuations around degenerate solutions \( s_F = 0 \). However, the Regge solutions can be reintroduced by adding to the action \( \langle \mathcal{O}_\Gamma \rangle \) a term \( 3\alpha \text{Vol}_\rho \) proportional to the spherical volume of the simplex. It can be checked that all the results mentioned above hold with this modified action as well.

\(^{11} \) The Gram determinants reduce to Euclidean volumes in the flat limit, see Appendix A.

\(^{12} \) Indeed, the spherical Schl"afli identity, unlike the flat one, admits a second term proportional to the spherical volume of the simplex.
V. ALGEBRAIC STRUCTURE: DISCUSSION

In the previous sections we have shown that the state-sum model \([13]\) naturally emerges from Feynman amplitudes of ordinary QFT, and provides dynamics for the background geometry. In the case of 3d Feynman diagrams, an identical analysis had led us to a dynamical model whose algebraic structure has been fully dissected \([12]\): the model turned out to be the spin foam quantization of a BF theory related to 3d quantum gravity. Similar results are expected in 4d. Namely, it should be possible to show that the state-sum \([13]\) can be understood in terms of an underlying algebraic structure. Moreover this state sum model should arise as a limit of a model of quantum gravity in the regime where usual QFT takes place. In this part we give some insights into the investigation of the algebraic interpretation of the model.

A. A new kind of spin foam model

It is worth introducing, for each simplex \(\sigma \in \Delta\), the following 20j-symbol

\[
\{ l_{e_1}, \ldots, l_{e_{10}} \} \equiv \sum_{\epsilon} e^{i\epsilon S_{\sigma}(l_e, s_F)} \frac{2\cos S_{\sigma}(l_e, s_F)}{V_\sigma(l_e)} \tag{62}
\]

which depends on ten variables \(l_e \in \mathbb{R}^+\) labeling the edges, and ten variables \(s_F \in \mathbb{Z}\) labeling the faces. With our notations \(F_1\) denotes the face opposite to the edge \(e_i\) in \(\sigma\), while the quantity \(S_{\sigma}\) is the action term \([14]\) associated to the simplex \(\sigma\). We also define the measures

\[
\int d\mu_e \equiv \int l_e dl_e \quad \text{and} \quad \int d\nu_F \equiv \frac{1}{2\pi} \sum_{s_F} A_F \tag{63}
\]

for each edge \(e\) and face \(F\) of the triangulation. Note that the face measure \(\nu_F\) depends, via \(A_F\), on the labels of the three edges that bound \(F\). The model \([13]\) takes then the form of a sum over the labels, with the measures \([63]\), of a product of 20j-symbols:

\[
Z_{\Delta} = \int \prod_e d\mu_e \prod_F d\nu_F \prod_{\sigma} \left\{ l_{e_1}, \ldots, l_{e_{10}} \right\} \tag{64}
\]

where \(e_i^\sigma\) and \(F_i^\sigma\) label edges and faces of the simplex \(\sigma\).

Let us mention two important properties satisfied by the symbols \([62]\). The first property is the orthogonality relation

\[
\int d\mu_{e_1} d\nu_{F_{10}} \left\{ l_{e_1}, \ldots, l_{e_{10}} \right\} \left\{ l_{e_1}, \ldots, l'_{e_{10}} \right\} = 2\pi \frac{\delta_{s_{F_1} s'_{F_1}} \delta(l_{e_{10}} - l'_{e_{10}})}{A_{F_1}} \tag{65}
\]

which involves two symbols with identical labels except for the face \(F_1\) and the edge \(e_{10}\): \(s_{F_1} \neq s'_{F_1}\) and \(l_{e_{10}} \neq l'_{e_{10}}\). An identical relation holds for every pair \((F, e)\) of face and edge that are not opposite to each other. The proof of this identity is identical to the derivation of the orthogonality relation for the Poincaré 6j-symbol written in \([12]\) and we do not repeat it here. The second property is the \((3, 3)\) identity \([65]\)

\[
\int d\nu_{\{123\}} \prod_{i=4,5,6} \left\{ l_{e_i}, \ldots, l_{e_{10}} \right\} = \int d\nu_{\{456\}} \prod_{j=1,2,3} \left\{ l_{e_j}, \ldots, l_{e_{10}} \right\} \tag{66}
\]

between the symbols of six 4-simplices which triangulate the boundary of a 5-simplex \([1 \cdots 6]\). The 4-simplex \(\sigma_i\) is the one obtained by dropping the point \(i\), and \([ijk]\) is the face whose vertices are \(i, j, k\). In this relation we have denoted by \(l_{e_i}^\sigma, s_{F_i}^\sigma\) the labels of edges and faces of \(\sigma_i\) - keeping in mind that, of course, \(l_{e_i}^\sigma = l_{e_i}'\) (resp. \(s_{F_i}^\sigma = s_{F_i}''\)) if \(\sigma_i\) and \(\sigma_j\) share the edge \(e\) (resp. the face \(F\)). The identity \([66]\) insures, together with gauge-fixed hexagonal and \((1,5)\) identities, the topological invariance of \(Z_{\Delta}\).

Although constructing models out of symbols, attached to each simplex and function of the coloring, is a common feature of the spin foam approach, let us emphasize that the structure revealed in \([64]\) is quite unusual. The basic ingredient of a spin foam model is, indeed, a 2-complex whose faces are colored by representations of a group and edges by intertwiners. If this 2-complex is the 2-skeleton \(\Delta_2\) dual to a triangulation, one can equivalently work with a labeling of the triangular and tetrahedral faces of \(\Delta\). Hence, in the usual picture, unlike for our model, no edge labels are involved. The structure of \([64]\) is however reminiscent of that of 2-category state-sum models, based on representation theory of categorical groups \([20, 21, 22]\). We expect, more precisely, our model to be related to the Poincaré 2-group representation theory \([23]\).
B. A duality relation

We would like to mention an intriguing duality relation between the symbol (62) and the vertex amplitude of the Barrett-Crane spin-foam model for 4d quantum gravity [24]. In this model the vertex amplitude takes the form of a 10j-symbol which depends on ten variables labeling simple representations of $SO(4)$. We consider here the Barrett-Crane 10j-symbol associated with the Poincaré group instead of $SO(4)$. The representations are labelled by their mass $m$ and their spin $s$, and the simple representations are those for which $s = 0$. Let $\sigma \equiv [12345]$ be a 4-simplex whose edges $(ij)$ carry Poincaré simple representations $(m_{ij}, 0)$. With the technology introduced in [25], we know that the corresponding 10j-symbol can be expressed as the Feynman graph evaluation

$$\int_{\mathbb{R}^4} \prod_{i=1}^5 dx_i \prod_{i<j} K_{m_{ij}}(x_i, x_j)$$

(67)

where the kernel $K_m(x, y)$ is the Hadamard propagator in $\mathbb{R}^4$, $(\Delta + m^2)K_m = 0$, $K_m(x, x) = 1$, which only depends on the distance $|x - y|$. Note however that the quantity (67), which corresponds to the contraction, according to the geometry of the 4-simplex, of five Poincaré intertwiners attached to the vertices, is divergent. We get in fact the correct definition of the 10j-symbol by gauge fixing the $ISO(4)$ symmetry in the integral and, thus, working with the invariant measure. This invariant measure has been computed in section III in terms of the distances $l_{ij} = |x_i - x_j|$ and the volume of the simplex:

$$d\mu(l_{ij}) = \frac{\prod_{i<j} dl_{ij}}{V_{\sigma}(l_{ij})}.$$  

We see therefore that the Barrett-Crane 10j-symbol can be expressed in terms of a Fourier transform of the symbols (62) for zero spin

$$\{ (m_{ij}, 0) \}_{BC} = \int \prod_{i<j} dl_{ij} K_{m_{ij}}(l_{ij}) \left\{ \begin{array}{ccc} l_{12} & \cdots & l_{45} \\ 0 & \cdots & 0 \end{array} \right\}$$

(68)

up to normalization factors. This relation is reminiscent of the duality relations arising in 3 dimensions and studied in [26]. We expect this duality relation to admit a generalization to the case of non trivial spins.

C. Gravity and BF theory

If we take the view, detailed in the introduction, that the dynamical model [13] is the limit $G_N \to 0$ of the quantum gravity amplitude, then this model is expected to be a spin foam quantization of classical gravity in this sector. In the work [27], it is shown that gravity action including an Immirzi parameter can be written as an $SO(5)$ gauge theory

$$S = \int \left( B^{ij} \wedge (R_{ij}(\omega) - \frac{1}{l^2} \epsilon_i \wedge \epsilon_j) + \frac{1}{l} B_i \wedge d_\omega \epsilon^i - \frac{\beta}{2} B^{ij} \wedge B_{ij} - \frac{\beta}{2} B^i \wedge B_i - \frac{\alpha}{4} B_{ij} \wedge B_{kl} \epsilon^{ijkl} \right)$$

(69)

where $\epsilon^i$ is the frame field, $\omega^{ij}$ the spin connection and $R_{ij}(\omega)$ its curvature, $B^{ij}, B^i$ are 2-form fields valued respectively in the adjoint and vectorial representation of $SO(4)$. $l$ is the cosmological length scale and $\alpha, \beta$ dimensionless parameters expressed in term of the Newton constant $G_N$, the cosmological constant $\Lambda$ and Immirzi parameter $\gamma$:

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G_N \Lambda}{3(1 - \gamma^2)}, \quad \beta = \frac{\gamma G_N \Lambda}{3(1 - \gamma^2)}. \quad (70)$$

Now when $G_N \to 0$, the theory becomes topological. Indeed in this limit, if $\gamma$ is fixed, we have $\alpha, \beta \to 0$ and therefore the action (69) reduce to a $SO(5)$ BF theory

$$S = \int B_{ij} \wedge (R^{ij}(\omega) - \frac{1}{l^2} \epsilon^i \wedge \epsilon^j) + \frac{1}{l} B_i \wedge d_\omega \epsilon^i.$$  

(71)

13 This kernel is a Bessel function $K_m(|x|) = \frac{2}{m|x|} \text{BesselJ}(1, m|x|)$, with $\text{BesselJ}(1, a) \equiv \frac{1}{2} \int_{-1}^1 du \sqrt{1 - u^2} e^{iau}$. 

Note that we could consider gravity coupled to matter, described by (69) with an additional matter action $S_m(\epsilon, \phi)$, which depends on the frame fields and on matter fields denoted collectively by $\phi$: we see that the limit $G_N \to 0$ does not affect the matter sector. The equations of motion of (71) coming from the variation of the $B$ fields are
\[ R^{ij} = \frac{\Lambda}{3} \epsilon^i \wedge \epsilon^j, \quad d_w e^i = 0. \] (72)
The unique solution to these equations is the deSitter, anti deSitter or flat space, depending on the value of the cosmological constant.

The topological models (13, 60) are thus expected to be a spin foam quantization of (71), and the QFT Feynman amplitudes to be related to Wilson lines observables for this theory [8].

VI. CONCLUSION

The goal of this paper was to bridge the gap between the language of spin foam models, which provide a well defined framework to address the dynamical issue of quantum gravity in a background independent way, and the usual language of quantum field theory. The significance of the results we have obtained is twofold. Firstly, it gives a background independent perspective to standard field theory, in a way that is in agreement with the spin foam hypothesis. Indeed, in the formulation we have proposed, background geometry is dynamical and the dynamics is governed by a spin foam model. Furthermore, this model is revealed to be topological, which confirms the idea that gravity becomes topological in the limit $G_N \to 0$. The second interest of our results is that they provide a falsification test for any candidate for the quantum gravity amplitude; we indeed claim that it must reduce to the spin foam model (13) in a suitable semi-classical limit. This requirement represents strong constraints on the physically viable proposals for quantum gravity models.

Further investigations regarding the structure of the model are needed, in order to understand the algebraic origin of the quantum weights. Some indications have been given in the last section, which have to be studied in more detail. The conjecture of an interpretation in terms of 2-categories needs to be investigated; possible links with the Barrett-Crane model have to be explored; and the connection with the continuum approach deserves to be precisely established. In our view, dissecting the algebraic structure of the Feynman graph spin foam model is a key step, hopefully allowing us a new way to propose and study possible dimensionful deformations of usual field theory and their relations with quantum gravity models.

We have restricted our analysis to closed Feynman diagrams: it will be extremely interesting to extend it to the case of open diagrams. The main issue is to reexpress the dependence of Green functions on the positions in terms of boundary spin networks.

Another direction of investigation concerns the nature of the Feynman graph observable itself, which should be eventually understood as a natural observable from the spin foam point of view.

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APPENDIX A: PACHNER MOVES IDENTITIES

In this appendix we describe the main steps leading to (45, 46, 48), which provide the topological invariance of the model.

Geometry of the simplex. The key ingredient for the proof of the identities is a 4d extension of geometrical equalities established in [12]. We consider a spherical 5-simplex $(e_0, \cdots, e_5)$, and denote by $l_{ij} \in [0, \pi]$ its lengths and $G = \det [\cos l_{ij}]$ its Gram determinant. Let $V_j$ be the square root of the Gram determinant associated to the 4-simplex $\sigma_j$ obtained by dropping the vertex $j$, and $\epsilon_j$ its orientation. Then derivatives of $G$, with respect to the lengths on one hand, and to deficit angles on the other hand, are related to the quantities $V_j$ in the following way:
\[ \left. \frac{\partial G}{\partial l_{ij}} \right|_{l_{ij}^\pm} = \mp 2 \sin l_{ij} V_j V_j \] (A1)
\[ \left. \frac{\partial G}{\partial \omega_{ijk}^\pm} \right|_{l_{ij}^\pm} = -2 \epsilon_i \epsilon_j \epsilon_k \frac{V_i V_j V_n}{A_{ijk}} \] (A2)
(ijklmn) is any permutation of (012345); \(\omega_{ijk}^\epsilon\) is the deficit angle of the face \([ijkl]\) which depends on the lengths and orientations \(\epsilon_l, \epsilon_m, \epsilon_n\); \(A_{lmn}\) is the square root of the Gram matrix associated to the triangle \([ijk]\). All lengths are supposed fixed except for \(l_{ij}\); and \((l_{ij}^\epsilon, \epsilon_l, \epsilon_m, \epsilon_n)\) is solution of \(G = 0\) and \(\omega_{ijk}^\epsilon = 0 \mod 2\pi\), with \(l_{ij}^\epsilon < l_{ij}^\epsilon\). These equalities can be easily derived by using the technology introduced in the appendix of [12], and we do not give more details here.

Note that the flat counterpart of this result is recovered in the limit \(l_{ij} \to 0\) while ratios \(l_{ij}/l_{kl}\) are kept fixed; in this limit the Gram matrices reduce to the square of usual Euclidean volumes and we have the following correspondence:

\[
V(\epsilon_0, \cdots, \epsilon_D) \sim D! V(l_{ij}), \quad \sin l_{ij} \sim l_{ij}
\]  

(A3)

In the following we work with flat simplices. For any 4-simplex \(\sigma\) with edge lengths \(l_{ij}\), and for any face \(F\) of \(\sigma\), \(V_F\) denotes \(4!\) times the Euclidean volume of \(\sigma\) and \(A_F\) denotes 2 times the Euclidean area of \(F\). For simplicity, in all the paper, those quantities are respectively called ‘volume’ of the simplex and ‘area’ of the face.

**Identity (3,3).** We consider a length configuration \(l_{ij}^\epsilon\) for the 5-simplex, which is a solution of \(G(l_{ij}) = 0\); and \(\epsilon \equiv (\epsilon_j)\) an orientation configuration such that \(\omega_{ij}^\epsilon = 0\) for every face \(F\) of the 5-simplex. The formula (A2) yields a relation between the functionals \(\delta(G)\) and \(\delta(\omega_{ijk}^\epsilon)\), which holds in the neighborhood of the solution \(l_{ij}^\epsilon\):

\[
2\delta(G) = \frac{A_{045}}{V_1 V_2 V_3} \delta(\omega_{045}^\epsilon) = \frac{A_{123}}{V_0 V_4 V_5} \delta(\omega_{123}^\epsilon)
\]  

(A4)

where \(\epsilon\) denotes \((\epsilon_0, \epsilon_4, \epsilon_5)\) for the first deficit angle and \((\epsilon_1, \epsilon_2, \epsilon_3)\) in the second one. The (3,3) identity arises from (A1) by using the fact that, for \(l_{ij} = l_{ij}^\epsilon\), the actions of the (3,3) move are related by

\[
\sum_{j=0}^5 \epsilon_j S_j^o = \sum_{F \neq [054], [123]} s_F \omega_F^\epsilon = 0 \mod 2\pi.
\]  

(A5)

where the subscript \(o\) means that the actions are evaluated for \(s_{054} = s_{123} = 0\). Note that the identity also holds if all orientations are switched \(\epsilon_j \to -\epsilon_j\). Taking into account this remark, we get, from (A1) and (A5):

\[
A_{054} \frac{e^{\epsilon_0 S_0^o} S_1^o}{V_1} e^{\epsilon_4 S_2^o} e^{\epsilon_5 S_3^o} \delta(\omega_{054}^\epsilon) = A_{123} \frac{e^{-\epsilon_0 S_0^o}}{V_0} \frac{e^{-\epsilon_4 S_2^o}}{V_4} \frac{e^{-\epsilon_5 S_3^o}}{V_5} \delta(\omega_{123}^\epsilon) \quad \forall \epsilon = \pm 1
\]  

(A6)

We then want to sum this equality over \(\eta\); now the delta functions act as constraints on orientations in such a way that summing over \(\eta\) amounts to summing over the values of \(\epsilon_1, \epsilon_2, \epsilon_3\) on the left hand side and over the values of \(\epsilon_0, \epsilon_4, \epsilon_5\) on the right hand side of the equation. Eventually, we get the (3,3) identity by extending this analysis to all solutions of \(G = 0\), and expanding the \(2\pi\)-periodic delta functions in series.

\[
\sum_{\epsilon_1, \epsilon_2, \epsilon_3} \sum_{s_{054}} A_{045} \frac{e^{\epsilon_0 S_0^o} S_1^o}{V_1} \frac{e^{\epsilon_4 S_2^o} S_3^o}{V_2} \frac{e^{\epsilon_5 S_3^o}}{V_3} = \sum_{\epsilon_0, \epsilon_4, \epsilon_5} \sum_{s_{123}} A_{123} \frac{e^{-\epsilon_0 S_0^o}}{V_0} \frac{e^{-\epsilon_4 S_2^o}}{V_4} \frac{e^{-\epsilon_5 S_3^o}}{V_5}
\]  

(A7)

**Identity (2,4).** Formula (A1) with \((ij) = (05)\) and (A2) with \((ijk) = (123)\) allows one to compute the derivative \(\partial \omega_{045}^\epsilon / \partial l_{05}^\pm\), which provides the equalities of measures

\[
\frac{\delta(l_{05} - l_{05}^\pm)}{V_0 V_5} = \frac{l_{05}}{V_0 V_2 V_3} A_{045} \delta(\omega_{045}^\pm) \quad \forall \epsilon = \pm 1
\]  

(A8)

where \(\epsilon^\pm\) are values of orientations that satisfy \(\omega_{045}^\pm(l_{05}^\pm) = 0\). By using again a relation, which holds when deficit angles vanish, between the actions

\[
\sum_{j=0}^5 \epsilon_j^o S_j^o = 0 \mod 2\pi
\]  

(A9)

where the subscript \(o\) means that \(s_{045} = 0\), we get

\[
\frac{e^{\epsilon_0 S_0^o} S_1^o}{V_0} \frac{e^{\epsilon_4 S_2^o} S_3^o}{V_3} \delta(l_{05} - l_{05}^\pm) = l_{05} A_{045} \prod_{i=1}^3 \frac{e^{\epsilon_i S_i^o} S_i^o}{V_i} \delta(\omega_{045}^\pm)
\]  

(A10)
Now one can sum over contributions of $\eta$ on the left and the contributions over $\epsilon_1, \epsilon_2, \epsilon_3$ on the right - since $\{\eta \epsilon^\pm, \eta = \pm 1\}$ are the only solutions of $\omega^\pm_{045}(t^\pm_{05}) = 0$. By also summing over values $t^\pm_{05}$ and expanding the delta function, the equality can be written in the following integral form

$$\sum_{\epsilon_0, \epsilon_5} \frac{e^{\epsilon_0 S_0} e^{\epsilon_5 S_5}}{V_0 V_5} = \frac{1}{2\pi} \sum_{\epsilon_i} \int dl_{05} l_{05} \sum_{s_{045}} A_{045} \prod_{i=1}^3 \frac{e^{\epsilon_i S_i}}{V_i} e^{\epsilon_4 S_4}$$

(A11)

The RHS of this expression contains a sum over values of three orientations; the value of $\epsilon_4$ is a function of $\epsilon_i$ defined to be such that (A9) holds. We will see below how to restore a summation over the values of an independent variable $\epsilon_4$. The identity (2, 4) is then obtained by reorganizing terms in the integrand and inserting the trivial ‘gauge-fixing’ identity

$$1 = \frac{1}{(2\pi)^3} \sum_{\{s_i\}} \prod_{i=1}^3 2\pi \delta_{s_{0i}, s_{0i}}$$

(A12)

We get:

$$\sum_{\epsilon_0, \epsilon_5} \frac{e^{\epsilon_0 S_0} e^{\epsilon_5 S_5}}{V_0 V_5} = \frac{1}{(2\pi)^3} \sum_{\epsilon_i} \int dl_{05} l_{05} \sum_{\{s_i\}} A_{i} \prod_{i=1}^4 \frac{e^{\epsilon_i S_i}}{V_i} \delta^{(2, 4)} \tilde{D}^{(2, 4)}$$

(A13)

where

$$\delta^{(2, 4)} = (2\pi)^3 \prod_{i=1}^3 \delta_{s_i, s_i}^\circ \quad \text{and} \quad \tilde{D}^{(2, 4)} = \frac{V_4}{A_1 A_2 A_3}$$

Notice that we could have inserted in (A10) a function $f(l_{05})$ of the label $l_{05}$; thus, (A13) has to be understood as an identity of measures allowing one to integrate a function of the free label $l_{05}$ on the RHS and a function $f(l_{05}^\epsilon \epsilon^\circ l_{ij})$ of the other labels and the orientations on the LHS.

**Identity (1, 5).** The derivation of (1, 5) is similar to the previous one, the starting point being the following equalities of measures

$$\frac{\delta(l_{05} - l^\pm_{05})}{V_0} = \frac{l_{05} A_{045}}{V_1 V_2 V_3} V_3 \delta(\omega^\eta_{045}) \quad \forall \eta = \pm 1$$

(A14)

that arises from (A8) - since the volume $V_5$ does not depend on $l_{05}$. We easily get

$$\sum_{\epsilon_0} \frac{e^{\epsilon_0 S_0}}{V_0} = \frac{1}{(2\pi)^5} \sum_{\epsilon_i} \int \prod_{j=1}^5 dl_{0j} \sum_{\{s_{0ij}\}} A_{0ij} \prod_{j=1}^5 \frac{e^{\epsilon_j S_j}}{V_j} \delta^{(1, 5)} \tilde{D}^{(1, 5)}$$

(A15)

where

$$\delta^{(1, 5)} = (2\pi)^5 \prod_{(ij) \neq (45)} \delta_{s_{ij}, s_{ij}^\circ} \prod_{i=1}^4 \delta(l_{0i} - l_0^0), \quad \text{and} \quad \tilde{D}^{(1, 5)} = \left(\frac{V_5}{2 \prod_{i=1}^4 l_{0i}}\right) \left(\frac{V_5}{\prod_{i=1}^4 A_{ij}}\right) \left(\frac{V_4}{A_{51} A_{52} A_{53}}\right)$$

On the RHS of (A15), the sum over values of three orientations $\epsilon_1, \epsilon_2, \epsilon_3$ is taken over, the values of $\epsilon_4, \epsilon_5$ being those for which a relation similar to (A9) is satisfied. The issue of promoting these two orientations to independent variables is discussed below.

**Orientations.** We want to write (2, 4) and (1, 5) identities where a summation of all orientations is taken over. To do so, as mentioned in section (III), we add an additional gauge-fixing term which plays the role of a constraint for the orientations. In order to describe this term, let us first define the total algebraic solid angle at an edge $e$. Let $p_e$ be a point of the edge $e$. Each simplex $\sigma$ to which $e$ belongs is mapped in a copy of $\mathbb{R}^4$; a unit 2d-sphere, surrounding $p_e$, in the hyperplane $e^\perp$ orthogonal to the edge intersects $\sigma$ along a spherical triangle. The angles of this triangle are the dihedral angles $\theta^\tau_{\sigma}$ of the faces of $\sigma$ meeting at $e$, and its area, denoted by $\Omega^\tau_{\sigma, e}$, is the solid angle seen at $e$ within $\sigma$. The spherical angles associated to all the 4-simplices sharing $e$ triangulate a surface, called the link $L_e$ of the edge $e$. The total algebraic angle is then defined to be

$$\Omega^\tau_e(l_e) = \sum_{\sigma \ni e} \Omega^\tau_{\sigma, e}$$

(A16)
We interpret the restriction on orientations as a gauge fixing. It is however clear that this restriction deserves a better formulation and

\[ \frac{1}{2\pi} \left[ \Omega^\sigma_e + \sum_{F \supset e} (2\pi - \bar{\omega}_F) \right] = \chi(L_e) \]  

(A17)

where \( \chi(L) = |\sigma| - |\tau| + |F| \) is the Euler characteristic of the surface \( L_e \), \(|\sigma|, |\tau|\) and \(|F|\) being the number of 4-simplices, tetrahedra and faces touching the edge \( e \), or equivalently the number of triangles, edges and vertices of the triangulation of the link \( L_e \). If \( \Delta \) triangulates a manifold, every link \( L_e \) is homeomorphic to a 2-sphere, and therefore \( \chi(L_e) = 2 \). We also consider the function \( \Theta(x) \) defined to be constant, with value 1, on \( 4\pi \mathbb{Z} \), and 0 elsewhere. The sum over orientations \( \epsilon_4 \) in (1,4) identity and over orientations \( \epsilon_4, \epsilon_5 \) in (1,5) identity is then restored by inserting the additional gauge-fixing factors

\[ \Theta(\Omega^\epsilon_{0\beta}) \text{ and } \prod_{j=1}^{5} \Theta(\Omega^\epsilon_{0j}) \]  

(A18)

in (A13) and (A14) respectively. One can check that these terms act on shell as Kronecker symbols for the orientations. They are introduced\(^{14}\) to replace the necessary constraints on orientations removed by the gauge fixing of the \( s \)'s labels.

**APPENDIX B: COMPUTATION OF DETERMINANTS**

In this part, following Korepanov\(^{18}\), we present explicit computations of Faddeev-Popov determinants associated to the gauge-fixing of the reducible face symmetry. Since the contributions of the simplex of \( A_4^{(2)} \) are well understood - they are given by (33) - we will focus on the contribution \( \Delta_{F,v}^{\sigma_v} \) of a simplex \( \sigma_v \in A_4^{(1)} \) assigned to a vertex \( v \). We will show, for a suitable gauge-fixing condition, the relation

\[ \Delta_{F,v}^{\sigma_v} = \frac{\mathcal{V}_{\sigma_v}}{\prod_{F \supset v} A_F}. \]  

(B1)

\( \mathcal{V}_{\sigma_v} \) is the volume of the simplex and the product is over the area of the six faces sharing \( v \).

**Gauge-fixing condition.** We want to fix the symmetry of six labels \( s_F \) living on the faces of \( \sigma_v \) that touch the vertex \( v \), generated by four 3-vectors \( \vec{\beta}_c \) attached to the edges of \( \sigma_v \) meeting at \( v \); to do so, we primarily need to fix the symmetry of the gauge parameters \( \vec{\beta}_c \), generated by a \( so(4) \)-element \( \sigma_{\mu\nu} \) attached to \( v \). The actions of these symmetries are respectively given by

\[ \delta_c s_F = -l_e^{-\frac{i}{2}} \beta_c \cdot \vec{h}_F \quad \frac{\vec{h}_F}{h_F} \]  

(B2)

\[ \delta_v \vec{\beta}_c = l_e^{-\frac{i}{2}} \sigma(\vec{r}_e) \]  

(B3)

All the vectors are defined in a given embedding of the simplex in \( \mathbb{R}^4 \). \( \vec{h}_F \) is the vector represented by the height of the point opposite to \( e \) within \( F \) when its intersection with \( e \) is placed at the origin; \( \sigma(\vec{u}) \) is the displacement of a vector \( \vec{u} \) by the infinitesimal rotation \( I_{3\times 3} + \sigma_{\mu\nu} \). The actions of \( \sigma_v \) are denoted by \( 0, 1, 2, 3, 4 \), where \( 0 \) is the vertex \( v \). Recall that the parameter \( \vec{\beta}_i \) attached to the edge (0i) belongs to the 3-dimensional space \( \vec{I}_i^2 \) orthogonal to the straight line spanned by (0i). In order to express the gauge-fixing condition, we define, for each \( i = 1, 2, 3 \), a convenient basis \( (\vec{x}_i, \vec{y}_i, \vec{z}_i) \) of \( \vec{I}_i^2 \) as follows. Given a cyclic permutation \((ijk) \) of (123), we choose \( \vec{x}_i \) in the plane spanned by the triangle \([0ij]\) and such that \( \vec{x}_i \cdot \vec{l}_j > 0 \); next we choose \( \vec{y}_i \) in the space spanned by the tetrahedron \([0ijk]\), orthogonal to the plane \([0ij]\) and such that \( \vec{y}_i \cdot \vec{I}_k > 0 \); we eventually choose \( \vec{z}_i \) to be orthogonal to the tetrahedron \([0ijk]\) and such that \( \vec{z}_i \cdot \vec{l}_4 > 0 \). - note that the three \( \vec{z}_i \) axis coincide. The components of \( \vec{\beta}_i \) in these basis are denoted by \( \beta_i^x, \beta_i^y, \beta_i^z \).

\(^{14}\) We interpret the restriction on orientations as a gauge fixing. It is however clear that this restriction deserves a better formulation and a deeper physical understanding.
The gauge-fixing prescriptions are the following. We first take advantage of the six independent components of $\sigma_{\mu\nu}$ to fix the values of the three components $\beta_{1}^{21}, \beta_{1}^{32}, \beta_{1}^{31}$ of $\tilde{\beta}_{1}$, two components $\beta_{3}^{22}, \beta_{3}^{22}$ of $\tilde{\beta}_{2}$ and one component $\beta_{3}^{33}$ of $\tilde{\beta}_{3}$. The Faddeev-Popov determinant $\Delta_{2}$ which arises from the gauge-fixing of this ‘second-stage’ symmetry is the determinant of the Jacobian $6 \times 6$ matrix associated to the function $\beta(\sigma_{\mu\nu})$. The six remaining parameters, namely $\beta_{2}^{32}, \beta_{3}^{33}, \beta_{3}^{33}$ and the three components of $\tilde{\beta}_{4}$, are then used to fix the values of six variables $s_{01J}, I, J = 1 \cdots 4$, labelling the faces that share the vertex 0. The Faddeev-Popov determinant $\Delta_{1}$ which arises from the gauge-fixing of this ‘first-stage’ symmetry is the determinant of the $6 \times 6$ Jacobian matrix associated to the function $s_{01, J}(\beta)$. The Faddeev-Popov determinant corresponding to the full reducible symmetry reads then

$$\Delta_{FP}^{2x} = \Delta_{1}^{-1} \Delta_{2}^{-1} \quad \text{(B4)}$$

**Value of $\Delta_{1}$.** The elements of the Jacobian matrix associated to the function $s_{01, J}(\beta)$ are derivatives of the labels $s_{012}, s_{013}, s_{023}, s_{041}, s_{042}, s_{043}$ with respect to the parameters $\beta_{2}^{32}, \beta_{3}^{33}, \beta_{3}^{33}$ and the three components of $\tilde{\beta}_{4}$; these derivatives can be read out from (B2). The matrix is block triangular, since the variations $\delta s_{012}, \delta s_{013}, \delta s_{023}$ do not depend on $\tilde{\beta}_{4}$. Therefore the desired determinant splits in two factors. We already know how to compute one of them: it is the Faddeev-Popov determinant which corresponds to the gauge-fixing, using the vector $\tilde{\beta}_{4}$, of the labels $s_{04i}$ of three faces meeting at (04), and given by the formula (B3):

$$\frac{\mathcal{V}_{\sigma_{x}}}{\mathcal{A}_{04i} \mathcal{A}_{042} \mathcal{A}_{041}} \quad \text{(B5)}$$

The other factor is the determinant of a triangular matrix, since $\delta s_{012}$ depend neither on $\beta_{3}^{33}$ nor on $\beta_{3}^{33}$, and $\delta s_{013}$ is independent of $\beta_{3}^{33}$. Therefore it reduces to the product of the diagonal elements:

$$\left(\frac{\partial s_{012}}{\partial \beta_{2}^{32}}\right) \left(\frac{\partial s_{013}}{\partial \beta_{3}^{33}}\right) \left(\frac{\partial s_{023}}{\partial \beta_{3}^{33}}\right)$$

Now using (B2) and our definition of the axis $\vec{x}_{2}, \vec{x}_{3}, \vec{y}_{3}$, one can convince oneself that the first derivative equals $l_{2}^{-\frac{2}{3}}$ times the cosine of the angle $\alpha_{2}$ between the faces [012] and [032], the second one is simply $l_{3}^{\frac{2}{3}}$, and the third one equals $l_{3}^{-\frac{2}{3}}$ times the sine of the angle $\alpha_{3}$ between the faces [013] and [023]. The second factor therefore reads

$$l_{2}^{-\frac{2}{3}} l_{3}^{-\frac{1}{3}} \cos \alpha_{2} \sin \alpha_{3} \quad \text{(B6)}$$

Finally the desired determinant is the product of (B5) and (B6):

$$\Delta_{1} = l_{2}^{-\frac{2}{3}} l_{3}^{-\frac{1}{3}} \frac{\mathcal{V}_{\sigma_{x}}}{\mathcal{A}_{04i} \mathcal{A}_{042} \mathcal{A}_{041}} \cos \alpha_{2} \sin \alpha_{3} \quad \text{(B7)}$$

**Value of $\Delta_{2}$.** In order to compute this second determinant let us first define a convenient basis $(e_{1}, \cdots, e_{4})$ of $\mathbb{R}^{4}$ in which the matrix elements of the rotation $\sigma_{\mu\nu}$ will be expressed. We choose the orthonormal basis such that $e_{1} = \frac{\vec{l}_{1}}{l_{1}}$: $e_{2}$ belongs to the plane spanned by $\vec{l}_{1}, \vec{l}_{2}$, with $e_{2} \cdot \vec{l}_{2} > 0$; $e_{3}$ belongs to the space spanned by $(\vec{l}_{1}, \vec{l}_{2}, \vec{l}_{3})$, with $e_{3} \cdot \vec{l}_{3} > 0$; and $e_{4}$ orthogonal to $e_{1}, e_{2}, e_{3}$ and satisfying $e_{4} \cdot \vec{l}_{4} > 0$. The matrix elements of $\sigma_{\mu\nu}$ in this basis are denoted by $\sigma_{IJ}$. The elements of the Jacobian matrix associated to the function $\beta(\sigma_{IJ})$ are derivatives of the components $\beta_{1}^{21}, \beta_{2}^{22}, \beta_{3}^{33}, \beta_{3}^{33}, \beta_{3}^{33}, \beta_{3}^{33}$ with respect to $\sigma_{14}, \sigma_{24}, \sigma_{34}, \sigma_{12}, \sigma_{13}, \sigma_{23}$; these derivatives can be read out from (B3). The matrix is block diagonal, since for $i = 1, 2, 3$, $\delta \beta_{i}^{31}$ only depend on $\sigma_{14}, \sigma_{24}, \sigma_{34}$, while the three other variations $\delta \beta_{i}^{21}, \delta \beta_{i}^{22}, \delta \beta_{i}^{33}$ do not depend on $\sigma_{14}$. Therefore, again, the determinant splits in two factors. We already know how to compute the first factor: if we consider the 3-vector $\vec{\sigma}$ of span $\{e_{1}, e_{2}, e_{3}\} \simeq \mathbb{R}^{3}$ whose components are $\sigma_{i4}$, then (B3) yields:

$$\delta \left[ l_{i}^{-\frac{2}{3}} \beta_{i}^{31} \right] = \vec{\sigma} \cdot \frac{\vec{l}_{i}}{l_{i}} \quad \text{(B8)}$$

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15 modulo a sign, which is irrelevant here since we are only interested in the absolute value of the determinants.
for $i = 1, 2, 3$. The LHS is thus the variation of the edge lengths under an infinitesimal move, by the vector $\vec{\sigma}$, of the vertex 0 within $\mathbb{R}^3$. The first factor reads then \[\text{(B9)}\]:

$$\det \left[ \left( \frac{\delta \beta_{ij}^x}{\delta \beta_{ij}^x} \right)_{ij} \right] = (l_1 l_2 l_3)^{\frac{2}{3}} \frac{\nu_r}{l_1 l_2 l_3}$$

where $\nu_r$ is the 3d-volume of the tetrahedron spanned by $\vec{l}_1, \vec{l}_2, \vec{l}_3$. Let $\theta_{ij}$ be the angle between the edges $(ik)$ and $(jk)$ within the triangle \[\text{(12)}\]; also, let $\alpha_3$ be the angle between the faces $[013]$ and $[023]$. By using the relations

$$\nu_r = l_1 l_2 l_3 \sin \theta_{13} \sin \theta_{23} \sin \alpha_3, \quad A_{0ij} = l_i l_j \sin \theta_{ij}$$

\[\text{(B9)}\] can be written as

$$l_1^{-\frac{2}{3}} l_2^{-\frac{1}{3}} l_3^{-\frac{1}{3}} A_{012} A_{023} \sin \alpha_3$$

The second factor is the determinant of a triangular matrix, since $\delta \beta_1^x$ only depends on $\sigma_{12}$ and $\delta \beta_1^y$ does not depend on $\sigma_{23}$. It therefore reduces to the product of the diagonal elements:

$$\left( \frac{\partial \beta_{12}^x}{\partial \sigma_{12}} \left( \frac{\partial \beta_{13}^y}{\partial \sigma_{13}} \left( \frac{\partial \beta_{23}^x}{\partial \sigma_{23}} \right) \right) \right)$$

Now using \[\text{(B3)}\] and the definition of the axis $\vec{x}_1, \vec{y}_1, \vec{y}_2$, one can convince oneself that, up to a sign, each of the first and second derivatives equals $l_1^2$, while the third one equals $l_2^2$ times the sine of $\theta_{12}$ times the cosine of the angle $\alpha_2$ between the faces $[012]$ and $[032]$. The second factor therefore reads

$$l_1^2 l_2^3 \sin \theta_{12} \cos \alpha_2 = l_1^2 l_2^3 A_{012} \cos \alpha_2$$

\[\text{(B11)}\]

where we have used, again, the relation $A_{012} = l_1 l_2 \sin \theta_{12}$ and between ‘area’ and angle in the triangle \[\text{(12)}\]. Finally the desired determinant is the product of \[\text{(B10)}\] and \[\text{(B11)}\]:

$$\Delta_2 = l_2^2 l_3^3 A_{012} A_{013} A_{023} \cos \alpha_2 \sin \alpha_3$$

\[\text{(B12)}\]

**Value of $\Delta_{FP}^{\nu_r}$**: The Faddeev-Popov determinant is finally obtained by taking the quotient of \[\text{(B7)}\] by \[\text{(B12)}\]; and the statement \[\text{(B1)}\] is proved.

$$\Delta_{FP}^{\nu_r} = \frac{\nu_{0r}}{\prod_{F \supseteq v} A_F}$$

\[\text{(B13)}\]


