Is dark matter present in NGC4736?
An iterative spectral method for finding mass distribution in spiral galaxies

Łukasz Bratek, Joanna Jalocha, and Marek Kutschera
The Henryk Niewodniczański Institute of Nuclear Physics,
Polish Academy of Science, Radzikowskiego 152, 31-342 Kraków, Poland

(Dated: November 8, 2006)

We present Iterative Spectral Method of reconstructing mass distribution in spiral galaxies in an infinitely thin axial symmetric disk approximation. The method, without extra assumptions and free parameters, overcomes all problems encountered in standard use of disk model and utilises only information that is available observationally.

As an example we apply the method to the ringed Sab-type galaxy NGC4736 that breaks sphericity condition at larger radii. We find mass distribution in NGC4736 that agrees perfectly with its high resolution rotation curve (high resolution, however, is not a must in our method). Given the distribution one finds rotational velocity which is the same as the observed rotation curve (!). The obtained surface mass density is consistent with the I-band luminosity profile ($M/L_i=1.2$ in Solar units) and with the amount of hydrogen observed in outermost regions where rotation curve is not measured.

In the framework of Newtonian gravitation, these findings put in question the presence of massive dark halo in this particular galaxy (normally estimated to 2.3 of visible mass: Kent, S. M. 1987, Astron.J., 93(4), 816). Interestingly, we find total mass of the galaxy to be $3.42 \times 10^{10} M_\odot$ which is comparable with masses obtained in MOND model or in metric skew-tensor gravity.

PACS numbers:

I. PRELIMINARY NOTES

Infinitely thin disk model of spiral galaxies [1, 2] seems well established at least for small radii – apart from the disk component ($D$) that is always present, the inner bulge ($B$) with an assumed profile is projected onto the galaxy plane $\mathbb{R}^2$. The problem of determining inner mass distribution is reduced therefore to finding in the galaxy plane the resultant surface mass density from which mass density in $B$ and in $D$ are reconstructed that account for luminosity profiles $\mathbb{R}^2$. According to [3] masses of $B$ and $D$ in galaxy NGC 4736 amount to 12% and 20% of total mass, respectively, while almost 70% of total mass reside in spherical dark halo of the galaxy at larger radii. As so, total mass distribution may be considered almost spherical at large radii, thus Kent’s fit to rotation curve should satisfy the inequality

$$v^2(\rho_1)\rho_1 \leq v^2(\rho_2)\rho_2, \quad \rho_1 \leq \rho_2,$$

and this is really the case, cf. figure 1 which was taken from [3]. In our opinion, however, the fit is not satisfactory at larger radii, although it is surely the best one from the point of view of the assumed mass model.

Interestingly, we find that the inequality is broken by the observed rotation curve, cf. figure 2 in the region $\rho \gtrsim 6\,[\text{kpc}]$ disk component cannot be neglected – the galaxy cannot be dominated there merely by spherical mass distribution. This is the reason why we shall be using disk component also for larger radii (spherical hallo, if required, can always be reproduced from some fraction of the global flat distribution).

Note, that global disk model have already been used in literature [4] but in a way that overestimated density profile at larger radii (an improved but still unsatisfactory approach was presented recently [9]). For example, density profile of galaxy NGC4736 could be reliably determined in [4] out to radius less than 6[kpc] only, while rotation curve is known out to 10.38[kpc] and matter is observed out to $\approx 16$[kpc]. But even in this 6[kpc] interval, as we shall see, it differs much from mass distribution from which observable rotation curve can be reconstructed accurately. In the limit of small radii both the distributions must agree, of course (this is a consequence of cutoff error analysis in disk models which is beyond the scope of the current paper). In the next section we shall find the satisfactory mass distribution by iterations. In addition, to have a complete set of data, apart from rotation curve we use also information about surface density profile in remote regions where rotation curve is not measured and where neutral hydrogen can be still observed. In this way we shall obtain a global flat mass distribution that will account for observations perfectly.

II. APPLICATION OF ITERATIVE SPECTRAL METHOD TO GALAXY NGC4736

1. Data we use

High resolution rotation curve $v_{\text{ex}}(\rho)$ of galaxy NGC 4736 was taken from [5] which is the same as the one...
FIG. 1: This is original figure taken from paper by S. Kent. It shows observed rotation curve (pluses) of galaxy NGC 4736 and the best-fitting solution (solid line). Rotation curve is shown out to 6′ which corresponds to 10.47[kpc] - the Author assumed distance to the galaxy D = 6.0[Mpc]. Currently, the distance is estimated to 5.1[Mpc] (rotation curve is now measured out to 7′ which gives 10.38[kpc]). The separate contributions of the central bulge, disk and halo are shown, respectively, by short-long-dashed, short-dashed and long-dashed lines. Note, that the best fitting curve is not satisfactory close to the end of rotation curve - in the most plausible case for total mass assessment region. Rotational velocity at 10.47[kpc] is overestimated by ≈ 10%, thus one should expect rather ≈ 5.9 × 10^{10} M⊙ R_{< 6}^2 = 4.1 × 10^{10} M⊙ for total mass, not 5.9 × 10^{10} M⊙ as found in [3]. By rescaling this to distance 5.1[Mpc] we obtain ≈ 3.5 × 10^{10} M⊙ (dynamical mass scales as Rv^2 R, where R and v_R are characteristic length and velocity). This is close to total mass of the galaxy we find in the current paper, by integrating surface mass distribution that perfectly reconstructs high resolution rotation curve of the galaxy. This observation shows, that rotation curve modeling similar to [3] are not only subject to high errors [2] and arbitrariness in choosing fitting algorithm, but may also overestimate much the amount of dark matter in (at least some kinds of) spiral galaxies.

available from internet site of Y. Sofue [13] (we stress, however, high resolution is not a must in our method). It was measured out to 7′ which corresponds to radius R = 10.38[kpc] (we assume distance to the galaxy D = 5.1[Mpc] and inclination angle i = 35° as in [2,4]). We define v_R = v_{ex}(R) = 125.6[km/s]. Rotational velocity is measured with the maximum error δv = 11[km/s] (attained at ρ = 5′), and average velocity is 168[km/s], thus, total mass should be determinable with the accuracy better than ≈ 10%. We use also 50° resolution neutral hydrogen surface density published in [2] which not much differs from [3]. X-band surface brightness λ_X, where X = I, V or B are computed from

$$
λ_X \left[ \frac{L_{x,⊙}}{pc^2} \right] = \frac{\cos (i)}{\tan^2 (\text{arcsec})} 10^{0.4 (M_{x,⊙} - μ_X - 5)}
$$

FIG. 2: Negative sphericity test for spiral galaxy NGC4736. The figure shows plot of a function defined by $F = \text{Max} \{ |r_1 v_1|^2/(|r_1 v_1|^2) v_j | : r_1 < r_j \}$. $F > 1$ means breaking sphericity condition. This supports the hypothesis that disk component in the galaxy cannot be neglected even out to distances close to the end of rotation curve. This indicates that for large radii a disk-like component should be still present and cannot be dominated by a spherical mass distribution.

A. Step 1

A first order interpolating function $u^2(x) = v_{ex}^2(Rx)/v_R^2$ was obtained from experimental rotation curve $v_{ex}(ρ)$, cf. figure 3 to calculate spectral coefficients $σ_k$ from integral (15) (spectral coefficients $μ_k$ fall off quickly to zero, eg. $μ_1/μ_{126} ≈ 10^{-5}$). The corresponding (underestimated) surface mass density $σ_0(ρ)$ was calculated from (15). We set $σ_1(ρ) = σ_0(ρ)$ for $ρ < ρ_1$, $σ_1(ρ) = σ_H(ρ)$ for $ρ_1 < ρ < R_H$ and $σ_1(ρ) = 0$ elsewhere, where $σ_H(ρ)$ is azimuthally averaged neutral hydrogen surface density taken from [2] (inclination was corrected from 40° to 35°). The radius where observed $σ_H(ρ)$ ends, cf. line ‘$σ_1$’ in figure 4. Integrated mass of $σ_1(ρ)$ is $M_1 ≈ 3.148 × 10^{10} M⊙$, and should not differ from total mass $M_{TOT}$ more than by $0 < M_{TOT} - M_1 < δM_1$, where $δM_1 < 2 v_{ex}(R) \text{Max} |v_{ex} - v_1|/G ≈ 5.1 × 10^{9} M⊙ ≈ 0.16 M_1$.

By substituting $σ_1(ρ)$ to integral (14) we obtain the corresponding first approximation $v_1(ρ)$ to rotation curve $v_{ex}(ρ)$. As is seen in figure 4 (cf. line $v_1$) $v_1(ρ)$ agrees with $v_{ex}(ρ)$ very well for small radii, where $σ_1(ρ)$ is trustworthy, however, there is still a discrepancy for greater radii to be removed in next iteration steps.
FIG. 3: Observed rotation curve of galaxy NGC4736 (circles) and rotation curve (solid thick line $v_1$) corresponding to surface mass density profile $\sigma_1(\rho)$ (shown in figure 4) found in the fourth iteration. The rotation curve was calculated from equation (A4). Solid thin line $v_\text{m}$ is the corresponding Keplerian asymptote $\sqrt{GM_{\text{TOT}}/\rho}$ with integrated mass of the galaxy $M_{\text{TOT}} \approx 3.42 \times 10^{10} M_{\odot}$. Solid thin line $v_\text{m}$ is the rotational velocity $\sqrt{GM_{\text{TOT}} m(\rho)/\rho}$ corresponding to (normalised) mass function $m(\rho)$ of the galaxy shown in figure 4.

FIG. 4: Surface mass density of galaxy NGC4736 obtained in consecutive iterations $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$ yielding the respective approximations $v_1$, $v_2$, $v_3$, $v_4$ of observed rotation curve, cf. figure 4 and surface mass density of $HI$ ($\sigma_H$ – solid circles). Whole $\sigma_4$ is shown in figure 3.

FIG. 5: Rotation curves obtained in 3 consecutive steps of iteration – the thick solid lines $v_1$, $v_2$ and $v_3$. The thin solid line $v_4$ is rotation curve obtained in the fourth step. Note that even in the third iteration we obtain mass distribution of which rotation curve almost perfectly agrees with observed rotation curve (experimental points are indicated by circles). Whole rotation curve $v_4$ is shown in figure 3.

B. Step 2

Now we shall find a correction $\delta\sigma_1(\rho)$ to $\sigma_1(\rho)$, $\rho < R$, to obtain even better reconstruction of $v_{ex}(\rho)$. In this respect we define $\delta v_1^2(\rho) = v_{ex}^2(\rho) - v_1^2(\rho)$, $\rho < R$. The $\delta\sigma_1(\rho)$ is the source of the correction $\delta v_1^2(\rho)$ and, similarly as $\sigma_1(\rho)$ from $v_{ex}(\rho)$, it can be found from (A4), but now $\sigma(\rho)$ in the equation represents $\delta\sigma_1(\rho)$ and $\delta v_1^2(\rho)/\delta v_1^2(R)$ stands for $u^2(x)$. To determine the corresponding spectral coefficients of (A4), it sufficed here to find the best fit to a first order interpolation of $\delta v_1^2(\rho)$ by the least square method. It is known from the theory of generalised Fourier series 12 that in the theoretical limit of infinite number of terms in the sum (A4), $\sigma_k$’s determined by the least square method would be equal to numbers $\sigma_k$’s from equation (A6).

Having found $\delta\sigma_1$, the corrected surface density in the second approximation $\sigma_2(\rho)$ is obtained by adding $\delta\sigma_1$ to $\sigma_1(\rho)$, with the reservation we set $\sigma_2(\rho) = \sigma_H(\rho)$ for $\rho > \rho_2$, where $\rho_2$ is the smallest radius such that $\sigma_2(\rho_2) = \sigma_H(\rho_2)$, cf. ‘$\sigma_H$’ line in figure 4. Again, the corresponding rotation curve $v_2(\rho)$ can be found by substituting $\sigma_2(\rho)$ to (A4). This time, the original rotation curve $v_{ex}(\rho)$ is reconstructed almost perfectly, cf. ‘$v_2$’ line in figure 4 but it must be corrected a little bit yet in the vicinity of $R$ (Max $(1 - v_2/v_{ex}) \approx 0.02$). The corresponding mass is $M_2 = 3.378 \times 10^{10} M_{\odot}$. Again, this mass is underestimated, but $M_{\text{TOT}} - M_2 < \delta M_2$, where $\delta M_2 < 2Rv_{ex}(R)\text{Max}|v_{ex} - v_2|/G \approx 1.3 \times 10^9 M_{\odot} \approx 0.04 M_2$. 
In the third step, analogously as before, we calculate \( \delta v_3^2(\rho) \) then \( \sigma_3(\rho) \), cf. `\( \sigma_3' \) line in figure 4 to obtain \( v_3(\rho) \), cf. `\( v_3' \) line in figure 5. Integrated mass of \( \sigma_3(\rho) \) is \( M_3 = 3.417 \times 10^{10} M_\odot \) and \( \delta M_3 \approx 4.5 \times 10^8 M_\odot \approx 0.01 M_3 \). The error has not been changed substantially compared with \( \delta M_2 \), so we do the last, fourth step. We obtain \( \sigma_4(\rho) \) and then \( v_4(\rho) \) which are also shown in figures 6 and 5 with mass \( M_4 = 3.424 \times 10^{10} \). Thus total mass of galaxy NGC 4736 in the fourth approximation is \( M_{TOT} = 3.42 \times 10^{10} M_\odot \) (this includes visible HI outside R out to \( R_H \)).

In figure 6 we show mass function \( M_4(\rho)/M_{TOT} \) and compare it with the Keplerian mass function \( \rho v_4^2(\rho)/GM_{TOT} \) corresponding to rotation curve \( v_4(\rho) \).

**D. Discussion**

Reconstruction of mass distribution from rotation curve is satisfactory and consistent with distribution of HI outside \( R \) where rotation curve is not measured but still 21cm observations are possible and the amount of hydrogen can be determined. Total mass of HI is \( M_{HI} = 6.00 \times 10^8 M_\odot = 0.02 M_{TOT} \), while outside \( R \) there is only \( 0.19 M_{HI} \) or \( 0.004 M_{TOT} \) of HI.

Surface mass density of galaxy NGC 4736 in the fourth iteration is shown in figure 8 (the `\( \sigma_4' \) line). The corresponding rotation curve can be calculated in any point from equation 4 and agrees perfectly with experimental rotation curve. In figure 8 are shown also the Keplerian asymptote corresponding to \( M_{TOT} \), and rotation curve \( \sqrt{M_4(\rho)}/\rho \) that would correspond to spherical mass distribution with mass function \( M_4(\rho) \).

As is seen in figure 8 surface mass density is also consistent with infrared band surface luminosity. Note that for the galaxy V and B band surface luminosities are less than \( L_I \), cf. table 1 (\( L_V \approx 0.7 L_I \) and \( L_B \approx 0.5 L_I \)). Moreover, this is the I-band (even brighter would be the K-band) which most accurately maps the amount of radiated energy (Sanders, personal communication); V and B band are affected by dust, moreover, radiation of low-massive stars is illly mapped in the latter bands. Thus, local mass to light ratio should be comparable with local mass density to I-band surface luminosity ratio, and the latter gives the upper bound for such a quantity. For galaxy NGC 4736 global I-band mass to low ratio is small \( M/L_I \approx 1.2 \), as well as, density to luminosity ratio which is of the order of 1 — 2, cf. figure 8 and falls-off at larger radii as one would expect. Also total galaxy mass is surprisingly small. It is comparable with mass of the galaxy predicted in the framework of MOND model which gives \( 3.21 \pm 0.09 \times 10^{10} M_\odot \) or in the framework of metric skew tensor gravity which gives

**FIG. 6:** Surface mass density profile of galaxy NGC 4736 found in the fourth iteration (thick solid line \( \sigma_4 \)). Measured points of the observed 50” resolution density profile of hydrogen HI in the galaxy are shown by solid circles. One can check, by using integral \( \int \) that this mass distribution leads to rotational velocity that agrees perfectly with the observed rotation curve of the galaxy, cf. curve \( v_4 \) in figure 6. To compare with our results we show also the surface mass density of the galaxy taken from \( \int \) (we assume inclination angle 35°), respectively, they are: B-band luminosity (dotted triangles), V-band luminosity (dotted squares) and most important for testing mass distribution the brightest I-band luminosity (dotted circles).

**FIG. 7:** Normalised (with respect to total mass \( M_{TOT} = 3.42 \times 10^{10} M_\odot \)) mass function of galaxy NGC 4736 (thick solid line) compared with (similarly normalised) formal Keplerian mass function \( m(\rho) \) (\( \rho v_4(\rho)/GM_{TOT} \)) where \( v(\rho) \) is the observed rotation curve.
found mass distribution that concords perfectly with its high-resolution rotation curve (high resolution, however, is not a must in our method). What is more, matter distribution we have derived agrees with $I$-band luminosity distribution (in the sense it gives a small mass-to-light ratio) and even with the amount of neutral hydrogen observed in the remote part of the galaxy where rotation curve is not measured. One should expect the ratio be even lower in the $K$-band. Remarkably, we have achieved this consistency for the particular galaxy without the hypothesis of massive dark matter halo and without modifying gravitation.

In our opinion, one should model mass distribution in spiral galaxies with more care, classical modelling of rotation curves (usually based on the ad hoc constant mass to light ratio assumption) not always gives good results and constrains models too much giving little chance of fitting well to observations. As we have shown, spiral galaxies are not always wholly dominated by spherical mass distribution at larger radii, and most importantly, this is the region where rotation curve should be reconstructed accurately in order not to overestimate the required mass distribution too much. For a given rotation curve a simple way for deciding whether or not a disk component may be neglected at large radii is by applying sphericity test or by examining Keplerian mass function.

Our method overcame all problems connected with characteristic for flattened mass distributions data cutoff errors encountered in paper [4], where global disk model was also used but in a way that due to the errors overestimated mass density at larger radii.

It would be interesting to examine other spiral galaxies in a manner similar to that we presented in this paper (we have found primarily that total mass of other spiral galaxies can also be reduced). In particular, most instructive would be to examine galaxies for which additional information about mass distribution independently of rotation curves exist.

APPENDIX A: EQUATIONS WE USE

We denote by $R$ in cylindrical coordinates $(\rho, \phi, z)$ the radius out to which a rotation curve $v(\rho)$ is measured, we define also $v_R = v(R)$. Together with Newton’s constant $G$ these characteristic quantities provide a complete set of physical units in our problem, for example, rotational velocity $v(\rho)$ may be described by a dimensionless function $u(x) = v(xR)/v_R$ where $x = \rho/R$. Under cylindrical symmetry it is convenient to use Fourier-Bessel transforms. In particular, surface mass density $\sigma(\rho)$ in the $z = 0$ plane and it’s transform $\hat{\sigma}(\omega)$ are mutually con-
\[
\sigma(\rho) = \frac{v_R^2}{GR} \int_0^\infty \omega \tilde{\sigma}(\omega) J_0 \left( \frac{\omega \rho}{R} \right) d\omega \quad \Leftrightarrow \\
\tilde{\sigma}(\omega) = \frac{GR}{v_R^2} \int_0^\infty x \sigma(Rx) J_0(\omega x) dx. \quad (A1)
\]

It is straightforward to show that outside the \( z = 0 \) plane (vacuum) gravitational potential is given by

\[
\Phi(\rho, z) = -2\pi v_R^2 \int_0^\infty \tilde{\sigma}(\omega) J_0 \left( \frac{\omega \rho}{R} \right) \exp \left( -\omega \frac{|z|}{R} \right) d\omega,
\]

and is connected with surface mass density by

\[
\sigma(\rho) = \frac{1}{2\pi G} \partial_x \Phi|_{z=0}.
\]

For circular and concentric orbits in a galactic disk we have \( v^2(\rho) = \rho \partial_\rho \Phi(\rho, 0) \) for all \( \rho \). Thus the unknown spectral amplitude \( \tilde{\sigma} \), and hence surface mass distribution, can be found directly from \( v(\rho) \). Indeed, by using above equations we obtain

\[
\tilde{\sigma}(\omega) = \frac{1}{2\pi} \int_0^\infty u^2(x) J_1(\omega x) dx \quad \Leftrightarrow \\
u^2(x) = 2\pi x \int_0^\infty \omega \tilde{\sigma}(\omega) J_1(\omega x) d\omega. \quad (A3)
\]

By using the identities \( \omega^{-1} \partial_x (x J_1(\omega x)) = x J_0(\omega x) \), \( J_0^\infty d\omega J_1(\omega \xi) J_1(\omega x) = (2/\pi) [K(\xi/x) - E(\xi/x)] \xi^{-1} \) for \( 0 < \xi < x \) or \( (2/\pi) (K(x/\xi) - E(x/\xi)) x^{-1} \) for \( 0 < x < \xi \), one can easily show that

\[
v^2(\rho) = 4G\rho \cdot V.P. \left( \int_0^\rho \sigma(\chi) \frac{\chi E \left( \frac{\chi}{\rho} \right)}{\rho^2 - \chi^2} d\chi - \int_0^\infty \sigma(\chi) \left[ \frac{\chi^2 E \left( \frac{\chi}{\rho} \right)}{\rho^2} - \frac{K \left( \frac{\chi}{\rho} \right)}{\rho} \right] d\chi \right), \quad (A4)
\]

1. Utilizing data from a compact support of rotation curve

A function \( u^2(x)/x \) is defined in the unit interval \( x \in (0, 1) \) and attains 1 at \( x = 1 \). Under some assumptions the function can be represented in the interval as a series of orthogonal functions. Although we may choose any complete set of such functions, most appropriate here are cylindric functions. If \( u^2(x)/\sqrt{\pi} \) is integrable in the interval \( x \in (0, 1) \) then \( u^2(x)/x \) can be represented in the interval as a series of Bessel functions [11], in particular

\[
\frac{u^2(x)}{x} = \sum_k \sigma_k J_0(\omega_k x), \quad 0 < x < 1, \quad J_0(\omega_k) = 0, \quad \omega_k > 0. \quad (A5)
\]

The summation is over all consecutive zeros \( \omega_k > 0 \) of the Bessel function \( J_0(x) \). By multiplying both sides by \( x J_1(\omega_m x) \) and by integrating in \( x \in (0, 1) \), it follows that

\[
\sigma_k = \frac{\omega_k}{J_1(\omega_k)} \mu_k, \quad \mu_k = \frac{2}{\omega_k} \int_0^1 u^2(x) \frac{J_1(\omega_k x)}{J_1(\omega_k)} dx. \quad (A6)
\]

It is a matter of approximation how the partial information about distribution of mass encoded in \( \sigma_k \)'s is being used to approximate reality by constructing a particular \( \tilde{\sigma}(\omega) \) (the information is partial as \( v(\rho) \) is unknown for \( \rho > R \)). Different choices correspond to different extrapolations of rotation curve beyond \( R \). To give an example, assume that \( v(\rho) = v_R/2 \) at \( \rho = R \) and \( v(\rho) = 0 \) for \( \rho > R \) \( (u(x) = 0, x > 1 \) then from \( A3 \) we obtain the continuous spectrum

\[
\tilde{\sigma}(\omega) = \frac{1}{2\pi} \sum_k \sigma_k \frac{\omega J_0(\omega x) J_1(\omega_k)}{\omega_k^2 - \omega^2},
\]

(it is continuous as \( J_0(\omega) \approx J_1(\omega_k)(\omega_k - \omega) \) when \( \omega \approx \omega_k \)).

In practice, however, more convenient are discrete spectra. For example, by extending formally \( A3 \) to regions beyond \( \rho = R \), we get

\[
\tilde{\sigma}(\omega) = \frac{1}{2\pi} \sum_k \sigma_k \frac{\delta(\omega - \omega_k)}{\omega}. \quad (A7)
\]

This gives global mass distribution composed of \( R \)-wide rings with distorted images of the internal disk’s surface density

\[
\sigma(\rho) = \frac{v_R^2}{GR} \frac{1}{2\pi} \sum_k \sigma_k J_0(\omega_k x), \quad x = \frac{\rho}{R}, \quad \rho < R. \quad (A8)
\]
This series almost perfectly reconstructs mass distribution (usually for $\rho < 0.6R$) in analytical examples in which rotation curves were deliberately cut off at some $\rho = R$. Close to and beyond $R$ such reconstructed $\sigma$ is not trustworthy and may be even negative. Anyway, one can put $\sigma = 0$ there in a first approximation, then calculate the resulting rotation curve, 'subtract' it from the original one, generate the resulting correction to $\sigma$ and so on, by iterations. As no information about rotation curve is available for $\rho > R$ (the influence is absent under spherical symmetry) the final $\sigma$ will not be exactly the same as true mass distribution for $\rho < R$. Note, however that if in addition to knowing rotation curve for $\rho < R$ also mass distribution is known for $\rho > R$ from elsewhere, such problems can be solved completely by iterations. This feature can be used to find internal mass distribution in spiral galaxies for which disk component can not be neglected at larger radii and for which additional information about mass distribution (e.g. neutral hydrogen) beyond $R$ is available.

[16] Note, that some authors use notation $E(\kappa^2)$ or $K(\kappa^2)$ instead of $E(\kappa)$ or $K(\kappa)$. Therefore, to be precise, we give here definitions of elliptic functions using conventions as in [10]:

$$K(\kappa) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2 \phi}},$$
$$E(\kappa) = \int_0^{\pi/2} d\phi \sqrt{1 - \kappa^2 \sin^2 \phi}.$$