Angular momentum conservation for uniformly expanding flows

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(Dated: 3rd November 2006)

Angular momentum has recently been defined as a surface integral involving an axial vector and a twist 1-form, which measures the twisting around of space-time due to a rotating mass. The axial vector is chosen to be a transverse, divergence-free, coordinate vector, which is compatible with any initial choice of axis and integral curves. Then a conservation equation expresses rate of change of angular momentum along a uniformly expanding flow as a surface integral of angular momentum densities, with the same form as the standard equation for an axial Killing vector, apart from the inclusion of an effective energy tensor for gravitational radiation.

PACS numbers: 04.20.Cv, 04.30.Nk

I. INTRODUCTION

Energy, momentum and angular momentum have long been issues in General Relativity, since it is difficult to find definitions with satisfactory properties for a general space-time, though they make sense for weak gravitational fields, with physical properties familiar from flat space-time [1]. For an asymptotically flat space-time, there is an accepted notion of total energy-momentum at infinity, satisfying the Bondi energy-momentum flux equation, whereas angular momentum is more delicate: there is an accepted definition at spatial infinity but not at null infinity, and therefore no accepted flux equation. In the strong-field regime, flux equations in the form of conservation laws have recently been found for black holes, specifically for trapping horizons [2], for which an energy conservation law was recently found [7], [8, 9], with the same form as that for black holes.

In the strong-field regime, flux equations in the form of conservation laws have recently been found for black holes, specifically for trapping horizons [2], for both energy [3, 4] and angular momentum [5, 6].

II. UNIFORMLY EXPANDING FLOWS

General Relativity will be assumed, with space-time metric $g$. Consider a flow of spatial surfaces $S$, i.e. a one-parameter family $(S)$, locally generating a foliated hypersurface $H$. Labelling the surfaces by a coordinate $x$, they are generated by a flow vector $\xi = \partial/\partial x$, which can be taken to be normal to the surfaces, $\bot \xi = 0$, where $\bot$ denotes projection onto $S$. Given a normal vector $v$, $\bot v = 0$, a Hodge duality operation yields a dual normal vector $v^*$ satisfying

$$\bot v^* = 0, \quad g(v^*, v) = 0, \quad g(v^*, v^*) = -g(v, v). \quad (1)$$

In particular,

$$\tau = \xi^* \quad (2)$$

is normal to $H$, with the same scaling as $\xi$ (Fig. 1). The coordinate freedom here is just $x \mapsto \tilde{x}(x)$ and choice of transverse coordinates on $S$, under which all the key formulas will be invariant.

The expansion 1-form

$$\theta = *d + 1 \quad (3)$$

where $*$ denotes the Hodge operator of $S$ and $d$ the exterior derivative in the normal space, yields the expansion

$$\theta_v = \theta(v) = *L_v + 1 \quad (4)$$

along a normal vector $v$, where $L$ denotes the Lie derivative. One can also introduce the unit-expansion vector

$$\eta = g^{-1}(\theta)/g^{-1}(\theta, \theta) \quad (5)$$

where the metric sign convention is that spatial metrics are positive definite. Then $\eta$ and its dual $\eta^*$ have expansions

$$\theta_\eta = 1, \quad \theta_{\eta^*} = 0. \quad (6)$$

Other names for $g^{-1}(\theta)$ and $\eta$ are mean-curvature vector and inverse mean-curvature vector respectively.
A uniformly expanding flow [[7, 8, 9]] is a one-parameter family of surfaces on which both the expansion $\theta_\tau$ of the flow and the dual expansion $\theta_\tau$ are constant on the surfaces:

$$D\theta_\xi = D\theta_\tau = 0.$$  (7)

Equivalently, such flows are given by

$$\xi = a(\eta + c\eta^\ast), \quad Da = Dc = 0$$  (8)

and thereby generalize inverse mean-curvature flows from codimension 1 to codimension 2.

### III. DUAL-NULL FORMALISM

It is convenient to use a dual-null formalism [[10, 11]], describing two families of null hypersurfaces $\Sigma_{\pm}$, intersecting in a two-parameter family of spatial surfaces, including the desired one-parameter family. Relevant aspects of the formalism are summarized as follows. This material is not new, but included to provide a self-consistent treatment.

Labelling $\Sigma_{\pm}$ by coordinates $x^\pm$ which increase to the future, one may take transverse coordinates $x^a$ on $S$, which for a sphere would normally be angular coordinates $x^a = (\vartheta, \varphi)$. Writing space-time coordinates $x^a = (x^+, x^-, x^a)$ indicates how one may use Greek letters ($\alpha, \beta, \ldots$) for space-time indices and corresponding Latin letters ($a, b, \ldots$) for transverse indices. The coordinate basis vectors are $\partial_\alpha = \partial/\partial x^\alpha$ and the dual 1-forms are $dx^\alpha$, satisfying $\partial_\beta(dx^\alpha) = \delta^\alpha_\beta$. Coordinate vectors commute, $[\partial_\alpha, \partial_\beta] = 0$, where the brackets denote the Lie bracket or commutator. Two coordinate vectors have a special role, the evolution vectors $\partial_\pm = \partial/\partial x^\pm$ which generate the dynamics, spanning an integrable evolution space. The corresponding normal 1-forms $dx^\pm$ are null by assumption:

$$g^{-1}(dx^\pm, dx^\pm) = 0.$$  (9)

The relative normalization of the null normals may be encoded in a function $f$ defined by

$$e^f = -g^{-1}(dx^+, dx^-).$$  (10)

The transverse metric, or the induced metric on $S$, is found to be

$$h = g + 2e^{-f}dx^+ \otimes dx^-$$  (11)

where $\otimes$ denotes the symmetric tensor product. There are two shift vectors

$$s_\pm = \perp \partial_\pm$$  (12)

where $\perp$ is extended to denote projection by $h$. The null normal vectors

$$l_\pm = \partial_\pm - s_\pm = -e^{-f}g^{-1}(dx^\mp)$$  (13)

are future-null and satisfy

$$g(l_\pm, l_\pm) = 0, \quad g(l_+ , l_-) = -e^{-f}, \quad l_\pm(dx^\mp) = 1, \quad l_\pm(dx^\mp) = 0, \quad \perp l_\pm = 0.$$  (14)

The metric takes the form

$$g = h_{ab}(dx^a + s_a^b dx^+ + s_a^c dx^-) \otimes (dx^b + s_b^a dx^+ + s_b^c dx^-) - 2e^{-f}dx^+ \otimes dx^-.$$  (15)

Then $(h, f, s_\pm)$ are configuration fields and the independent momentum fields are found to be linear combinations of the following transverse tensors:

$$\theta_\pm = \ast L_\pm 1$$  (16)
$$\sigma_\pm = \perp L_\pm h - \theta_\pm h$$  (17)
$$\nu_\pm = L_\pm f$$  (18)
$$\omega = \frac{1}{2}e^f h([l_-, l_+])$$  (19)

where $L_\pm$ is shorthand for the Lie derivative along $l_\pm$. Adding indices explicitly, the functions $\theta_\pm$ are the null expansions, the traceless bilinear forms $\sigma_{\pm ab}$ are the null shears, the 1-form $\omega_\pm$ is the twist, measuring the lack of integrability of the normal space, and the functions $\nu_\pm$ are the inaffinities, measuring the failure of the null normals to be affine. The fields $(\theta_\pm, \sigma_\pm, \nu_\pm, \omega)$ encode the extrinsic curvature of the dual-null foliation. These extrinsic fields are unique up to interchange $\pm \leftrightarrow \mp$ and diffeomorphisms $x^\pm \leftrightarrow x^{\mp}(x^\pm)$ which relabel the null hypersurfaces. It will also be convenient to use capital Latin letters ($A, B, \ldots$) for normal indices, when not denoted by $\pm$ in the dual-null basis. Then the configuration fields are $(h_{ab}, f, s_A)$, the momentum fields are $(\theta_A, \sigma_{A bc}, \nu_A, \omega)$ and the derivative operators are $(\perp L_A, D_a)$, where $D$ is the covariant derivative operator of $h$.

Returning to a general foliated hypersurface $H$, a normal vector $v$ has components $v^\pm$ along $l_\pm$, so that $v = v^+ l_+ + v^- l_-$, and its normal dual is $v^* = v^+ l_+ - v^- l_-$. In particular, the generating vector is

$$\xi = \xi^+ l_+ + \xi^- l_-$$  (20)

and its dual is

$$\tau = \xi^+ l_+ - \xi^- l_-.$$  (21)

Since $H$ is given parametrically by functions $x^\pm(x)$, the components $\xi^\pm = \partial x^\pm/\partial x$ are constant on the surfaces:

$$D\xi^\pm = 0.$$  (22)

The expansion 1-form $\xi$ is given by

$$\theta = \theta_+ dx^+ + \theta_- dx^-$$  (23)

so that the expansion along a normal vector $v$ can be expressed as

$$\theta_v = \theta_A v^A$$  (24)
and in particular, \( \theta_\xi = \xi^+ \theta_+ + \xi^- \theta_- \) and \( \theta_+ = \xi^+ \theta_+ - \xi^- \theta_- \). The conditions (24) defining a uniformly expanding flow can then be expressed in terms of the null expansions as

\[
D \theta_+ = D \theta_- = 0
\]

unless the flow is null, in which case \( \xi = \pm \tau \) and the two expansion conditions become one.

IV. CONSERVATION OF ENERGY

The energy conservation law [1, 8, 9] will be stated here for later comparison, modifying some notation. Assuming compact \( S \) henceforth, the transverse surfaces have area

\[
A = \oint_S \ast 1
\]

and the area radius

\[
R = \sqrt{A/4\pi}
\]

is often more convenient. The Hawking mass [12]

\[
M = \frac{R}{2} \left( 1 - \frac{1}{16\pi} \oint_S *g^{AB} \theta_A \theta_B \right)
\]

can be used as a measure of the active gravitational mass or energy on a transverse surface. Here units are such that Newton’s gravitational constant is unity.

An energy conservation equation requires a vector playing the role of a stationary Killing vector. For a general compact surface, the simplest definition of such a vector which becomes unit for round spheres in flat space-time is [3, 4]

\[
k = (g^{-1}(dR))^*.
\]

This vector actually was found to be the appropriate dual of \( M \), in the sense of conservation laws for black holes [3, 4] and for uniformly expanding flows [8, 9]. In either case, the energy conservation law can be written as

\[
L_\xi M \cong \oint_S * (T_{AB} + \Theta_{AB}) k^A \tau^B
\]

where \( \Theta \) is an effective energy tensor for gravitational radiation. This determines only the normal-normal components of \( \Theta \), as

\[
\Theta_{\pm \pm} = |\sigma_{\pm}|^2 / 32\pi
\]

(31)

\[
\Theta_{\pm \mp} = e^{-f}|\omega \mp \tilde{D} f|^2 / 8\pi
\]

(32)

where \( |\zeta|^2 = h^{ab} \zeta_a \zeta_b \) and \( |\phi|^2 = h^{ab} h^{cd} \sigma_{ab} \sigma_{cd} \). Further discussion is referred to [3, 4, 8, 9].

V. ANGULAR MOMENTUM

The angular momentum of a surface in a flow, as a functional of an axial vector \( \psi \), is defined following [3, 4] as

\[
J[\psi] = \frac{1}{8\pi} \oint_S * \psi^a \omega_a
\]

(33)

where \( \omega \) is the twist (19). Since the twist encodes the non-integrability of the normal space, it provides a geometrical measure of the twisting around of space-time due to rotational frame-dragging. It is an invariant of a dual-null foliation and therefore of a non-null foliated hypersurface \( H \), so \( J[\psi] \) is also an invariant. The definition was obtained from the Komar integral [13] and shown to recover the standard definition of angular momentum for a weak-field metric [1], with the twist being directly related to the precessional angular velocity of a gyroscope due to the Lense-Thirring effect.

Following [5, 6], the axial vector will be assumed to be transverse, \( \perp \psi = \psi \) (Fig. 2), to be a coordinate vector \( \psi = \partial/\partial \phi \) on \( H \), implying

\[
L_\xi \psi \cong 0
\]

(34)

and to have vanishing transverse divergence:

\[
D_a \psi^a \cong 0.
\]

(35)

The last condition holds if \( \psi \) is an axial Killing vector, and can be understood as a weaker condition, equivalent to \( \psi \) generating a symmetry of the area form rather than of the whole metric, since \( L_\psi(\ast 1) = * D_a \psi^a \). Alternatively, assuming that the integral curves \( \gamma \) of \( \psi \) are closed, it can always be satisfied by choice of scaling of \( \psi \). It implies that \( \psi \) is locally the curl of some function, whose equipotentials are \( \gamma \).

In the situation normally envisaged for angular momentum, \( S \) would have spherical topology and \( \gamma \) would form a smooth foliation of circles, covering the surface except for two poles (Fig. 3), with the coordinate \( \varphi \) identified at 0 and 2\( \pi \). In fact, such conditions play no role in the conservation law to be derived in the next section, which requires only the conditions (34)–(35) on transverse \( \psi \). Note also that the commutator identity [11]

\[
L_\xi (D_a \psi^a) - D_a (L_\xi \psi)^a = \psi^a D_a \theta_\xi
\]

(36)
which then forces
\[ \psi^a D_a \theta \xi \cong 0 \] (37)
is automatically satisfied for uniformly expanding flows \( \mathcal{A} \). This can be seen as a motivation for the divergence-free condition (34), since (34) already forces \( \mathcal{A} (D_a \psi^a) \cong 0 \) for a uniformly expanding flow. This is quite different from the case of large family of conservation laws, corresponding to angular momentum about different axes. In the case of spherical topology, one can choose any two points on an initial surface as poles, and any set of circles \( \gamma \) interpolating smoothly between them, then there will exist a vector \( \psi \) with integral curves \( \gamma \) satisfying (35). The condition (35) then propagates \( \psi \) along the flow.

VI. CONSERVATION OF ANGULAR MOMENTUM

The rate of change of angular momentum (32) along the flow, using (34), is
\[ L_\xi J \cong \frac{1}{8\pi} \oint_S (\theta \xi \psi^a \alpha + \psi^a L_\xi \alpha). \] (38)
The twisting equations
\[ \perp L_\pm \alpha = -\theta \pm \alpha + \frac{1}{2} D_a \mp \theta \pm \frac{1}{2} \theta \pm D_a f \]
\[ \pm \frac{1}{2} h^{cd} D_a \sigma_{\pm ac} \pm 8\pi T_{a \pm} \] (39)
were obtained from the Einstein equations, where \( T_{a \pm} = h_{\alpha \gamma} T_{\alpha \beta} \tilde{\ell}_{\alpha \pm}^2 \) is the transverse-normal projection of the energy tensor \( \mathcal{T} \). They can be used to express
\[ \perp L_\xi \alpha = -\theta \xi \alpha + \tau^B (\frac{1}{2} D_a \nu_B - \frac{1}{2} D_a \theta_B - \frac{1}{2} \theta_B D_a f \]
\[ + \frac{1}{2} h^{cd} D_a \sigma_{B ac} - 8\pi T_{a B}). \] (40)
Then
\[ L_\xi J \cong -\frac{1}{8\pi} \oint_S \star \psi^a \tau (8\pi T_{a B} - \frac{1}{2} h^{cd} D_a \sigma_{B ac} \]
\[ - \frac{1}{3} (D_a \nu_B - D_a \theta_B - \theta_B D_a f)). \] (41)
Now the terms in \( D\nu \) and \( D\theta \) may be removed as total divergences due to (22) and (35). To remove the term in \( \theta Df \) generally requires
\[ \psi^a D_a \theta \tau \cong 0. \] (42)
Clearly this would be generally inconsistent with (37), specifically if \( \theta \xi \) and \( \theta \tau \) have different equipotentials. However, it is consistent in three special cases: (i) for null \( \xi \), since then \( \xi = \pm \tau \); (ii) along a trapping horizon \( \theta_B \cong 0 \), since then \( \theta = \mp \theta_B \); and (iii) for a uniformly expanding flow (7). In each case, this leaves just the \( T \) and \( \sigma \) terms, with the latter expressible in terms of the transverse-normal block
\[ \Theta_{a B} = -\frac{1}{16\pi} h^{cd} D_a \sigma_{B ac} \] (43)
of the effective energy tensor for gravitational radiation. Finally the desired conservation law for angular momentum is obtained as
\[ L_\xi J \cong -\oint_S (T_{a B} + \Theta_{a B}) \psi^a \tau B. \] (44)
Apart from the inclusion of \( \Theta \), this is the standard surface-integral form of conservation of angular momentum, were \( \psi \) an axial Killing vector. The null shears \( \sigma_{\pm ac} \) have previously been identified in the energy conservation law (30) as encoding the ingoing and outgoing transverse gravitational radiation, via the energy densities (31), (32), (36). So the expression (43) implies that gravitational radiation with a transversely differential waveform will generally possess angular momentum density.

VII. CONCLUSION

The conservation laws (30) and (44) take a similar form, expressing rate of change of mass \( M \) and angular momentum \( J \) as surface integrals of densities of energy and angular momentum, with respect to vectors \( k \) and \( \psi \) which play the role of stationary and axial Killing vectors. They also take the same form as the corresponding conservation laws for black holes \([1, 2, 3, 4]\), with the effective energy tensor \( \Theta \) for gravitational radiation also taking the same form (31), (32), (36).

This provides further evidence for the utility of the definition (32) of angular momentum in terms of the twist, and of uniformly expanding flows. However, the question of existence of such flows is generally unresolved except for a time-symmetric hypersurface, where weak flows have been proven to exist \([13, 16]\). Another caveat is that an initial surface should be chosen carefully, in particular not being highly distorted, if the physical interpretation is to be plausible. While no comprehensive prescription is known, it would be reasonable to restrict to topologically spherical surfaces with positive Gaussian curvature. Given a black hole, one could choose a marginal surface to be propagated outward. In an asymptotically flat spacetime, one could propagate inward a suitable surface at null infinity, such as those provided by the recent unified framework for null and spatial infinity \([17]\). Since the Bondi energy flux law can also be written in the conservation form (30), this also offers hope to resolve the issue of angular momentum at null infinity.
Research supported by the National Natural Science Foundation of China under grants 10375081 and 10473007, and by Shanghai Normal University under grant PL609.

[5] S A Hayward, Conservation laws for dynamical black holes [gr-qc/0607081].