The double Compton emissivity in a mildly relativistic thermal plasma within the soft photon limit

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Abstract

Aims. We wish to provide simple and accurate analytic approximations for the low frequency double Compton (DC) emission coefficient, which are applicable in a broad range of physical situations up to mildly relativistic temperatures and may be useful for checking under which circumstances the double Compton process is important.

Methods. We perform series expansions of the DC emission integral for low energies of the incident photon and electron and compare the derived analytic expressions with the results obtained by numerical integrations of the full DC cross section.

Results. We explicitly derived analytic approximations to the low frequency double Compton emission coefficient for initial monochromatic photons and Wien spectra. We show that combining the analytic approximations given in this paper an accuracy of better than 5\% in a very broad range of temperatures and under various physical conditions can be achieved. Furthermore we show that the double Compton emissivity strongly depends on the ratio of the energies of the incoming photon and electron: for hard photons and cold electrons the emission is strongly suppressed as compared to the case of similar photon and electron energy, whereas in the opposite situation, i.e. hot electrons and soft initial photons, the emission is enhanced. For photons and electrons close to thermodynamic equilibrium the double Compton emissivity increases less rapidly with temperature Lightman-Thorne approximation and the corrections exceed the level of \(\sim 10\%\) for temperatures above \(4\) keV.

Key words. Radiation mechanisms: double Compton scattering — thermal plasmas

1. Introduction

Double Compton (DC) scattering is the process of the form \(\gamma + e \leftrightarrow \gamma + e' + \gamma_1 + \gamma_2\). The DC process is related to Compton scattering, like Bremsstrahlung is related to Coulomb scattering of electrons by heavy ions. It corresponds to the lowest order in the fine structure constant \(\alpha\) to Compton scattering, with one additional photon in the outgoing channel, and like thermal Bremsstrahlung it exhibits an infrared divergence at low frequencies. In spite of these similarities, in the case of Bremsstrahlung the cross section directly depends on the velocity of the electron relative to the charged particle: there is no Bremsstrahlung emission for resting electrons (and ions). On the other hand, for DC scattering the dependence on the velocity of the electron enters only indirectly and can be obtained as a result of special relativistic coordinate transformations. Therefore DC emission occurs whenever there are photons and free electrons, no matter what is the temperature of the electron gas.

It was realized by Lightman (1981), Thorne (1981) and later again by Pozdnyakov et al. (1983) and Svensson (1984) that the DC process in comparison to Bremsstrahlung can become the main source of soft photons in astrophysical plasmas with low baryon density and in which magnetic fields are negligible. It has been shown that for given number densities of the ions, \(N_i\), and photons, \(N_\gamma\), the DC process is expected to dominate over Bremsstrahlung when \(N_i \gtrsim 10N_\gamma (kT_e/m_e c^2)^{5/2}\) is fulfilled. Due to the large entropy of the Universe (there are \(\sim 1.6 \times 10^9\) photons per baryon), at early stages (\(z \gtrsim 5 \times 10^9\), i.e. at high temperatures), the DC process becomes very important for the thermalization of spectral distortions of the cosmic microwave background (CMB) and the evolution of chemical potential distortions after any significant release of energy in the early Universe (Sunyaev & Zeldovich, 1970; Illarionov & Sunyaev, 1975; Danese & de Zotti, 1982; Burigana, Danese & de Zotti, 1991; Hu & Silk, 1993). Other possible environments in which the DC process could be of some importance might be found inside the sources of \(\gamma\)-ray bursts or two temperature accretion disks in the vicinity of black holes, both in close binary systems and active galactic nuclei. Given the relevance of DC emission as a source of soft photons within the context of the thermalization of CMB spectral distortions and hard X-ray sources, it is important to investigate the validity of the approximations usually applied to describe the DC process.

DC emission in an isotropic, cold plasma and for low energy initial photons, i.e. small \(\omega_0 = h\nu_0/m_e c^2\), where \(\nu_0\) is the frequency of the initial photon \(\gamma_0\), was first derived by Lightman (1981) and independently by Thorne (1981) within the soft photon limit. In this approximation it is assumed that one of the outgoing photons (\(\gamma_1\) or \(\gamma_2\)) is soft as compared to the other. Under these assumptions the emissivity of DC increases like \(\propto \omega_0^4\) with photon energy (see Jauch & Rohrlich, 1976, Eq. 11-45).

Again assuming cold electrons and low energy incident photons but making \textit{a priori} no assumption about the emitted photon energy, Gould (1984) obtained an analytic correction factor for the DC emission coefficient of monoenergetic initial...
photons, which extends the Lightman-Thorne approximation beyond the soft photon limit.

However, when the temperature of the electrons increases, or when the energy of the incident photons grows one expects corrections to become important, which have not been included in either of the aforementioned approaches. The corrections due to the motion of the electron can be obtained starting with a resting electron in an anisotropic radiation field and then performing the corresponding transformations into the frame where the electron is moving. In contrast to this, the dependence on the energy of the incident photon is connected with the exact formulae for the DC cross section as computed using the general theory of quantum-electrodynamics. Here we treat the electron-photon interaction due to DC scattering quantum-electrodynamically for moving electrons and perform all the computations directly in the lab frame, where the photon and electron distributions are assumed to be isotropic.

Previously, temperature corrections to the low frequency DC emissivity for an incoming Wien spectrum were discussed by Svensson (1984). It was shown that in the mildly relativistic case the low frequency DC emissivity increases significantly slower with temperature than in the Lightman-Thorne approximation. However, Svensson (1984) only treated one specific case and a more general extension of the Lightman-Thorne approximation is still missing.

In this paper we wish to extend the Lightman-Thorne approach to cases $h\nu_0 \leq m_e c^2$ and $k T_e \leq m_e c^2$ for isotropic initial photon distributions, but still within the soft photon limit. The extension to cases $\omega_0 \sim \omega$ will be left for some future work. For this we study the DC emission integral both numerically and analytically and derive approximations to the DC emission coefficient, which are valid in a very broad range of physical situations up to mildly relativistic temperatures. We show that the DC emission coefficient can be expressed in terms of the Lightman-Thorne emission coefficient times a corrections factor, which can be regarded as very similar to the Bremstrahlung Gaunt-factor, although it has completely different physical nature. As an example, in Fig. we summarize the results for this low frequency DC correction factor in the case of monochromatic initial photons. Clearly one can see that the Lightman-Thorne approximation is accurate in a very limited range of photon energies and electron temperatures. We also investigate the range of applicability of our approximations for initial Wien spectra (Sect. 4.3).

In the following we use $\hbar = c = \kappa = 1$.

2. DC emission for monochromatic photons and thermal electrons in the soft photon limit

DC emission is a result of the interaction between an electron and a photon with \( \text{one additional photon in the outgoing} \) channel

$$e(P) + \gamma(K_0) \rightarrow e(P') + \gamma(K_1) + \gamma(K_2).$$

Here $P = (E, p)$, $P' = (E', p')$ and $K_i = (\nu_i, k_i)$ denote the corresponding particle four-moment vectors. The full DC scattering squared matrix element and differential cross sections for various limiting cases were first derived by Mandl & Skyrme (1952) and may also be found in Jauch & Rohrlich (1976).

In this Section we focus our analysis only on the emission process for isotropic, monochromatic photons, with phase space density $n(\nu) = N_0 \delta(\nu-\nu_0)/\nu_0^3$, and isotropic, thermal electrons of temperature $T_e$. Above $N_0$ in physical units is defined by $N_{\gamma,0} = 8\pi N_0 c^3$, where $N_{\gamma,0}$ is the photon number density. In this case the change of the photon phase space density, $n_2 = n(\nu_2)$, at frequency $\nu_2$ due to DC emission can be written as

$$\frac{\partial n_2}{\partial \nu_2} = \frac{N_0}{\nu_2^3} \int d^3P \int d\Omega_0 \int d\Omega_1 \frac{d\nu_2}{d\nu_0} f \ .$$

Here the DC differential cross section (cf. Jauch & Rohrlich 1976, Eq. 11-38) is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha r_0}{(4\pi)^2} \frac{\nu_1 \nu_2}{g_{\mu\nu}^0} X (P + K_0 - K_2) \cdot K_1,$$

where $X$ may be found in the Appendix and we defined the Muller relative speed of the incident electron and photon as $g_{\mu\nu}^0 = \hat{P} \cdot \hat{K} = 1 - \beta \mu E$, with the dimensionless electron velocity $\beta = |\nu|/c = |p|/E$ and the directional cosine $\mu = \hat{p} \cdot \hat{k}$. Furthermore, $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant, $r_0 = \alpha/m_e \approx 2.82 \times 10^{-13} \text{ cm}$ is the classical electron radius and $\gamma = E/m_e = 1/\sqrt{1-\beta^2}$ denotes the Lorentz-factor of the initial electron.

In Eq. we have neglected induced effects, i.e. we assumed $N_0 \ll \nu_0^3$ and that the electrons are non-degenerate. In this case, the electron phase space density may be described by a relativistic Maxwell-Boltzmann distribution

$$f(E) = \frac{N_e}{4\pi m_e^2 K_2(1/\theta_e)} \ e^{-E/m_e \theta_e},$$

where $K_2(1/\theta_e)$ is the modified Bessel function of second kind (Abramovitz & Stegun 1965), with $\theta_e = T_e/m_e$ and where $N_e$ is the electron number density, such that $N_e = \int f(E) d^3P$. In the low temperature limit ($\theta_e \ll 1$) the relativistic Maxwell Boltzmann distribution is given by the expression 3) which is useful for both analytic and numerical purposes.

In general the DC emission integral 4) has to be performed numerically. Some comments on the numerical approach we use to solve the multidimensional Boltzmann integrals is given in the Appendix. However, in the limit of small temperatures and energies of the initial it is possible to derive various useful analytic approximations, which we shall discuss in the following. In practice we obtained numerical results for the soft photon limit by setting the frequency of the emitted photon to a very small value as compared the the incident photon energy ($\nu_2 \sim \nu_0 \times 10^{-4}$).

2.1. Lightman-Thorne-approximation

The expression for DC emission from monochromatic initial photons and cold electrons as given by Lightman (1981) and independently by Thorne (1981) can be deduced from the emission integral 4) by performing a series expansion of the DC differential cross section 2) in lowest orders of $\nu_0 \ll m_e$ and $\nu_2 \ll m_e$ and setting $\beta = 0$, i.e. assuming that the electrons are initially at rest. In this limit $f(P) = N_0 \delta(\nu)/4\pi \nu^2$ and the integration over $d^3P$ can be carried out immediately. The series expansion in lowest order of $\nu_2 \ll m_e$ is equivalent to using the soft photon approximation for the DC differential cross section, i.e. assuming in addition that $\nu_2 \ll \nu_0$. Due to energetic arguments the scattered photon frequency has to be close to the initial photon frequency ($\nu_1 \sim \nu_0$). Therefore the infrared divergence due to

\[\text{Note that in the following an additional hat above 3- and 4- vectors indicates that they are normalized to the unit time like coordinate of the corresponding 4-vector.}\]
the integration over the phase space volume for the photon $\gamma_1$ is automatically avoided. In the following we use the notations $\mu_{ij} = \hat{k}_i \cdot \hat{k}_j$ and $\phi_{ij}$ for the directional cosines and azimuthal angles between the photons $i$ and $j$, respectively. Then it follows:

$$\frac{d\sigma_{2\text{soft}}^0}{d\Omega} \approx \alpha \frac{\omega_0^2}{4\pi^2 r_0^2} \frac{1}{v_2} \left[ 1 \right.\left. - \mu_{01} - \frac{\Delta \mu^2}{2} \right], \tag{4}$$

with $\Delta \mu = \mu_{02} - \mu_{12}$. Furthermore we have introduced the dimensionless photon energy $\omega = v_1/m_e$.

Now, after aligning the $z$-axis with the direction of the emitted photon all the integrations can be easily performed analytically. Making use of the identity $\mu_{01} = \mu_{02} \mu_{12} + \cos(\phi_{02} - \phi_{12}) (1 - \mu_{12}^2)^{1/2} (1 - \mu_{02}^2)^{1/2}$, where $\phi_{02}$ and $\phi_{12}$ denote the azimuthal angles of the initial and the scattered photon, respectively, this then leads to the Lightman-Thorne-approximation for DC emission spectrum of cold electrons and soft, monochromatic initial photons

$$\frac{\partial n^m_2}{\partial t}_{\text{em.L}} = \frac{4\alpha}{3\pi} N_e N_0 \sigma_T \frac{\omega_0^2}{v_2^2}, \tag{5}$$

with the Thomson scattering cross section $\sigma_T = 8\pi r_0^2/3 \approx 6.65 \times 10^{-25} \text{cm}^2$. The expression (5) is the DC scattering equivalent of the Kramers-formula for thermal Bremsstrahlung. The characteristics of equation (5) can be summarized as follows: the DC emission spectrum increases $\propto \omega^2$ and, as mentioned earlier, exhibits a infrared divergence for $v_2 \to 0$. It is usually assumed that under physical conditions inside an astrophysical plasma emission and absorption balance each other below some frequency, $\nu_{2,\text{min}}$, and that energy conservation in addition introduces some high frequency cutoff, $\nu_{2,\text{max}}$. Therefore the photon production rate increases logarithmically,

$$\frac{\partial n^m_2}{\partial t}_{\text{em.L}} = \frac{4\alpha}{3\pi} N_e N_0 \sigma_T \omega_0^2 \times \ln \left( \frac{\nu_{2,\text{max}}}{\nu_{2,\text{min}}} \right), \tag{6}$$

with the ratio of the upper to the lower frequency cutoff of the DC emission spectrum.

It is obvious that due to the DC process the interacting electrons and high frequency photons can loose some part of their energy due to the emission of soft photons. However, in the absolute majority of applications the energy losses connected with Compton heating and cooling are much larger.

As will be discussed in more detail below, the Lightman-Thorne-approximation only describes the low frequency part of the DC emission spectrum for cold electrons and soft initial photons ($\omega_0/m_e \leq 10^{-3}$) well. For initial photons with slightly higher energies next order terms lead to a significant suppression of the emission at low frequencies relative to the Lightman-Thorne-approximation (see Fig. 1 and the discussion in Sect. 2.2).

### 2.1.1. DC correction factor within the soft photon limit

Below we want to discuss the numerical and analytical results for the DC emission integral (1). For convenience using Eq. (1) and the Lightman-Thorne approximation (5) we therefore introduce the new function $G$ as

$$G_m(\omega_0, \theta_\phi) = \frac{\partial n^m_2}{\partial t}_{\text{em,L}} \bigg|_{\text{DC emission integral (1)}} \tag{7}$$

in order to compare the different cases. This function can be regarded as very similar to the Bremsstrahlung Gaunt-factor, although it has completely different physical origin. Deviations of $G$ from unity are directly related to DC higher order corrections in the energies of the initial photons and electrons.

![Figure 1. DC emission correction factor $G_m^0$ for cold electrons and monochromatic initial photons as a function of $\omega_0 = \nu_0/m_e$.](image)

Also shown are the direct expansion (8), up to different orders in $\omega_0$ and the inverse approximation (9), as indicated respectively. For comparison, we also show the frequency dependence of the total Compton cross section, $\sigma_{\text{KN}}$, as given by the Klein-Nishina-formula and the correction factor $k(\omega_0) = G_m^0/(\sigma_{\text{KN}}/\sigma_T)$ per act of Compton scattering.

### 2.2. Cold electrons and energetic initial photons

We start by expanding the DC differential cross section (2) in lowest order of $\omega_2 = \nu_2/m_e$ and setting $\beta = 0$. Furthermore we expand the resulting expression up to 4. order in $\omega_0$ to take into account higher order corrections in the energy of the initial photon. Carrying out all the integrations in the emission integral (1) yields the correction factor $G_m^0(\omega_0)$ following from the direct series expansion:

$$G_m^0(\omega_0) = 1 - \frac{21}{5} \omega_0 + \frac{357}{25} \omega_0^2 - \frac{7618}{175} \omega_0^3 + \frac{21498}{175} \omega_0^4. \tag{8}$$

Figure 1 shows the full numerical result for $G_m^0(\omega_0)$ in comparison to the analytic approximation (8), taking into account the corrections up to different orders in $\omega_0$. The approximation converges only very slowly and in the highest order considered here it breaks down close to $\omega_0 \sim 0.15$. Due to this behavior of the asymptotic expansion one expects no significant improve-ment when going to higher orders in $\omega_0$, but the monotonic decrease of the emission coefficient suggest that a functional form $G_m^0 = [1 + \sum_{n=1}^4 a_n \omega_0^n]^{-1}$ could lead to a better performance. Determining the coefficients $a_n$ by comparison with the direct expansion (8) one may obtain an inverse approximation for the DC correction factor

$$G_m^{\text{inv}}(\omega_0) = \frac{1}{1 + \frac{21}{5} \omega_0 + \frac{24483}{4375} \omega_0^2 + \frac{2041}{4375} \omega_0^3 + \frac{9603}{4375} \omega_0^4}. \tag{9}$$

As Fig. 1 clearly shows, $G_m^{\text{inv}}(\omega_0)$ provides an excellent description of the numerical result up to very high energies of the initial photon. For comparison, we also show the frequency dependence of the total Compton cross section, $\sigma_{\text{KN}}$, as given by the Klein-Nishina-formula and the correction factor $k(\omega_0) = G_m^0/(\sigma_{\text{KN}}/\sigma_T)$ per act of Compton scattering. Since it is expected that per one Compton scattering a small fraction of electrons and photons undergo DC scattering, this comparison hints
towards the fact that a large part (~ 2/3) of the decrease in the DC emissivity with \( \omega_0 \) is due to the decrease in the probability to Compton scatter photons with larger energies due to quantum-electrodynamical corrections.

2.3. Hot electrons and soft initial photons

We now consider the limit \( \nu_0 \ll m_e \) and \( \nu_2 \ll \nu_0 \). Expanding the DC differential cross section (2) in the lowest orders of \( \nu_0 \) and \( \nu_2 \) one obtains

\[
\frac{d\sigma_{\text{soft}}}{d\Omega} \approx \frac{\alpha e^2}{2\pi^2} \frac{\omega_0^2}{s_{00}} \left[ \gamma^2 a_0 l_1 (\gamma^2 a_0 l_1 - a_{01}) + \frac{a_{01}^2}{2} \right] \\
\times \gamma^2 a_0 l_1 d^2 a_0 - \frac{1}{4} (a_1 a_2 - a_0 a_{12})^2 \\
\times \frac{1}{\gamma_{\text{em}L} A_2^4}.
\]

with the electron and photon scalars \( a_i = \hat{p} \cdot \hat{k}_i = 1 - \beta \mu_{e,i} \) and \( a_{ij} = \hat{k}_i \cdot \hat{k}_j = 1 - \mu_{ij} \). Now we perform all the angular integrations of the Boltzmann emission integral (1) but assume monoenergetic electrons, with velocity \( \beta_0 \). This then leads to

\[
\frac{\partial m}{\partial \Omega}^{m,n} = \frac{1 + \beta_0^2}{1 - \beta_0^2} \times \frac{\partial m}{\partial \Omega}^{m,n,L}.
\]

Here the factor of \( \gamma_0^2 \) can be immediately understood since the initial photons inside the rest frame of the electron will on average have an energy of \( \omega_0^2 \sim \gamma_0 \omega_0 \) and hence the DC emissivity, i.e. \( \omega_0^2 \), will be \( \gamma_0^2 \) times larger than for resting electrons.

From Eq. (11) the correction factor for temperature \( \theta_e \) can be found by averaging over the relativistic Maxwell-Boltzmann distribution (3):

\[
G_m^{\text{nr}}(\theta_e) = \int_{0}^{\infty} \frac{1 + \beta_0^2}{1 - \beta_0^2} f(E_0) \beta_0^2 d\beta_0 \equiv 2 \left\{ \gamma_0^2 \theta_e \right\} - 1 \\
= \left[ 1 + 4 \theta_e^2 \right] K_0(1/\theta_e) + 8 \theta_e \left[ 1 + 6 \theta_e^2 \right] K_1(1/\theta_e) \\
\frac{K_2(1/\theta_e)}{K_2(1/\theta_e)}
\]

2.4. Hot electrons and energetic initial photons

To improve the analytic description of the DC emission we again return to the soft photon limit, expanding the DC differential cross section (2) in the lowest orders of \( \nu_2 \ll m_e \). To include higher order corrections we also expand this expression up to 4. order in \( \omega_0 \). We then carry out all the angular integrations of the Boltzmann emission integral (1) for monoenergetic electrons, with velocity \( \beta_0 \). This leads to

\[
G_m(\omega_0, \beta_0) = \gamma_0^2 \left[ 1 + \beta_0^2 - \frac{21}{5} \left( 1 + \beta_0^2 + \frac{1}{3} \beta_0^2 \right) \gamma_0 \omega_0 \right] \\
+ \frac{357}{25} \left( 1 + \frac{10}{3} \beta_0^2 + \beta_0^2 \right) \gamma_0^2 \omega_0^2 \\
- \frac{7618}{175} \left( 1 + 5 \beta_0^2 + 3 \beta_0^4 + \frac{1}{7} \beta_0^4 \right) \gamma_0 \omega_0^3 \\
+ \frac{21498}{175} \left( 1 + 7 \beta_0^2 + 7 \beta_0^4 + \beta_0^4 \right) \gamma_0^2 \omega_0^4.
\]

Like in the previous case the correction factor for thermal electrons with temperature \( \theta_e \) can be found by performing the 1-
\( \langle G_m \rangle_{\text{th}} \equiv G_m(\omega_0, \theta_e) = \int_0^\infty G_m(\omega_0, \beta_0) f(E_0) p_0^2 \, dp_0 \).  \hspace{1cm} (14)

Unfortunately here no full solution in terms of simple elementary functions can be given, but the integral can be easily evaluated numerically. In the limit of low temperatures one can find the simple approximation

\[
G_m(\omega_0, \theta_e) \approx 1 - \frac{21}{5} \frac{\omega_0}{\theta_e} + \frac{357}{25} \frac{\omega_0^2}{\theta_e^2} - \frac{7618}{175} \frac{\omega_0^3}{\theta_e^3} + \frac{21498}{175} \frac{\omega_0^4}{\theta_e^4}
+ \left[ 6 - \frac{441}{10} \frac{\omega_0}{\theta_e} + \frac{5712}{25} \frac{\omega_0^2}{\theta_e^2} - \frac{34281}{35} \frac{\omega_0^3}{\theta_e^3} \right] \theta_e^2
+ \left[ 15 - \frac{8379}{40} \frac{\omega_0}{\theta_e} + \frac{8568}{5} \frac{\omega_0^2}{\theta_e^2} \right] \theta_e^4
+ \frac{45}{4} - \frac{3969}{8} \omega_0 \theta_e^2 - \frac{45}{4} \theta_e^4 \right] \theta_e^2
\hspace{1cm} (15)
\]

for (14), which clearly shows the connection with the two limits, i.e. Eqn. (8) and (12b), discussed above.

As was shown in Sect. 2.2 an inverse Ansatz for \( G_m \) led to an excellent approximation for the full numerical result in the cold plasma case. In order the improve the performance of the obtained analytic approximation for \( G_m(\omega_0, \beta_0) \) we tried many different functional forms, always comparing with the direct expansion (13). After many attempts we found that

\[
G_m^{\text{inv}}(\omega_0, \beta_0) = \frac{\gamma_0^2 (1 + \beta_0^2)}{1 + \sum_{k=1}^4 f_k(\beta_0) \gamma_0^2 \omega_0^4}, \hspace{1cm} (16a)
\]

with the functions

\[
f_1(\beta_0) = \frac{1}{1 + \beta_0^2} \left[ \frac{21}{5} + \frac{42}{5} \beta_0^2 + \frac{21}{25} \beta_0^4 \right] \hspace{1cm} (16b)
\]

\[
f_2(\beta_0) = \frac{1}{(1 + \beta_0^2)^2} \left[ \frac{84}{25} + \frac{217}{25} \beta_0^2 + \frac{1967}{125} \beta_0^4 \right] \hspace{1cm} (16c)
\]

\[
f_3(\beta_0) = \frac{1}{(1 + \beta_0^2)^3} \left[ \frac{2041}{25} + \frac{1306}{125} \beta_0^2 \right] \hspace{1cm} (16d)
\]

\[
f_4(\beta_0) = \frac{1}{(1 + \beta_0^2)^4} \left[ \frac{9663}{4375} \right] \hspace{1cm} (16e)
\]

provides the best description of the full numerical results.

Figure 3 shows the numerical results for \( G_m(\omega_0, \theta_e) \) in comparison with the integral approximation (14) in combination with the inverse function (16a) and the direct expansion (15) taking into account correction terms up to fourth order. The performance of the integral approximation is excellent in the full range of considered cases, but the direct expansion breaks down at lower and lower temperatures once the initial photon frequency increases. For \( \omega_0 \leq 0.1 \) the direct expansion should be applicable with a few percent accuracy up to \( \theta_e \sim 0.2 \).

In Fig. 4 the dependence of the DC emission correction factor on the energy of the initial photons and the temperature of the electrons is illustrated. This from one can see that the Lightman-Thorne approximation is applicable in a very limited range of photon energies and electron temperatures. On the other hand the approximation based on (16) works in practically the full considered range within very high accuracy. Focusing on the curve \( G_m = 1 \) one can see that for low energies of the initial photon \( (\omega_0 \lesssim 10^{-2}) \) the required electron temperature scales like \( \theta_e \sim 0.7 \omega_0 \). Considering the first order correction terms

\[
\Delta \theta_e = \frac{4 \theta_e}{5 \pi} N_e \frac{\omega_0}{\theta_e} \frac{\theta_e^2}{x_2} \times I_0, \hspace{1cm} (17)
\]

where here we defined the dimensionless photon frequency \( x_2 = \omega_0 / \theta_e = \nu / T_e \). The DC emission factor \( I_0 \) is given by the integral (compare Lightman (1981), Eq. (10b) for \( n(v) \ll 1 \))

\[
I_0 = \int_0^\infty x^4 n(x) \, dx. \hspace{1cm} (18)
\]

over the initial photon distribution \( n(v) \). Stimulated emission was neglected. Here one factor of \( x^2 \) arises from the conversion \( N_0 \to v n(v) \), the other is due to the scaling of the Lightman-Thorne approximation Eq. (5) with \( \omega_0 \). Note that here and below we omit the index ’0’ for the initial photon.

Now including higher order corrections in the energies of the initial photon and electrons but still in lowest order of \( x_2 \), instead of \( I_0 \) one will obtain a more general function \( I \). With this one can define an effective DC emission correction factor in the soft photon limit by

\[
\varepsilon_{\text{soft}} = I / I_0, \hspace{1cm} (19)
\]

We shall now give analytic expressions for \( I \) based on the results obtained in the previous Section.

\[\text{Figure 4. DC emission correction factor for monoenergetic initial photons and thermal electrons. The lines indicate the contours of } G_m = \text{const as obtained by full numerical integrations. Within the shaded regions the approximation based on (16) is accurate to better than 1% and 5%, respectively.}\]
3.1. The effective DC emission correction factor

Here we will give analytic expressions for the effective DC emission correction factor in three different approaches. We will discuss the range of applicability for these expressions for Wien spectra in the next Section.

3.1.1. Approximation based on \( G_m^{\text{inv}} \)

As was shown in Section 2.4, the expression (16) averaged over a thermal electron distribution (compare with Eq. (14)) provides an excellent description of the DC emissivity for monochromatic initial photons. If induced effects are negligible on can simply convolve this result with the corresponding initial photon distribution \( n(\nu) \) to obtain:

\[
I = \int_0^\infty x^4 n(x) \langle G_m^{\text{inv}} \rangle_{\text{th}} \, dx. \tag{20}
\]

Here \( \langle G_m^{\text{inv}} \rangle_{\text{th}} \) denotes the thermally averaged expression (16). Furthermore, in order to included stimulated emission one can replace \( n \rightarrow n[1 + n] \). This treatment is possible in cases when the change of the initial photon energy is very small, i.e. when one can assume \( v_0 \sim v_1 \) and the radiation spectrum is broad, i.e. \( \Delta \nu / \nu \gg 1 \). As will be shown below the approximation (20) works very well in a broad range of different temperatures. However, it involves a 2-dimensional integral which in numerical applications may be too demanding.

3.1.2. Direct expansion of \( G_m \)

In principle one can also replace \( \langle G_m^{\text{inv}} \rangle_{\text{th}} \) in Eq. (20) with the more simple analytic expression (15) derived in the limit of low electron temperature and energy of the initial photon. In this case the DC emission coefficient \( I \) can be written as the direct expansion

\[
I_{\text{exp}} = \sum_{k=0}^{4} I_k \times \theta_k^e. \tag{21}
\]

where the definitions of the integrals \( I_k \) may be found in the Appendix C.2. The first integral \( I_0 \) corresponds to the result obtained earlier by Lightman (1981) and Thorne (1981) in the limit \( n(\nu) \ll 1 \).

3.1.3. Alternative approach

Above we gave the expressions obtained from the direct expansion of \( I \). However, this procedure results in an asymptotic expansion of the effective DC correction factor \( g^{\text{eff}}_{\text{DC}} \) or equivalently \( I \), which in most of the cases is expected to converge very slowly. Therefore, using the direct expansion (21) we again tried to ‘guess’ the correct functional form of the DC emission coefficient \( I \) and thereby to extend the applicability of this simple analytic expression. Examining the general behavior of the results obtained in our numerical integrations showed that for \( \theta_e \leq \theta_0 \) in general the effective DC correction factor \( g^{\text{eff}}_{\text{DC}} = 1/I_0 \) is dropping towards higher electron temperatures. After several attempts we found that this behavior is best represented by the functional form

\[
I_{\text{inv}}^{\text{opt}} = \frac{I_0}{1 + a \theta_e^4}. \tag{22a}
\]

Using Eq. (21), one can immediately deduce

\[
a = 6 - \frac{21}{5} \frac{\theta_f^4}{\theta_e^4} \int x^4 n \, dx. \tag{22b}
\]

As will be shown below, equation (22a) together with (22b) provides a description of the low frequency DC emission coefficient, which for photon distributions close to a Wien spectrum is accurate to better than 5% for temperatures up to \(~25\) keV in the range \(0.2 \leq \rho \leq 50\). Here we defined the ratio of the electron to photon temperature as \( \rho = T_e / T_\gamma \). For a detailed discussion we refer the reader to Sect. 4.2. Since this approximation involves only 1-dimensional integrals over the initial photon distribution in numerical applications it may be more useful than the approximation (20).

4. Results for Wien spectra

4.1. Analytic expressions

For a Wien spectrum \( n(\nu) = N_0 e^{-\nu/\theta_0} \) of temperature \( \theta_0 \), with the condition \( N_0 \ll \nu^2 \) induced terms may be neglected. Above \( N_0 \) in physical units is defined by \( N_{e,0} = 8\pi N_0 / c^2 \), where \( N_{e,0} \) is the photon number density. Then the functions \( I \) are given as

\[
\begin{align*}
I_{0,W} &= 24 N_0 \\
I_{1,W} &= -I_{0,W}[21 - 6 \rho] / \rho \\
I_{2,W} &= I_{0, W}[428.4 - 220.5 \rho + 15 \rho^2] / \rho^2 \\
I_{3,W} &= -I_{0, W}[9141.6 - 6854.4 \rho \\
&\quad+ 1047.375 \rho^2 - 11.25 \rho^3] / \rho^3 \\
I_{4,W} &= I_{0, W}[206380.8 - 205686 \rho + 51408 \rho^2 \\
&\quad- 2480.625 \rho^3 - 11.25 \rho^4] / \rho^4. 
\end{align*}
\]

Here we defined the ratio of the electron to photon temperature as \( \rho = T_e / T_\gamma \). Making use of (22a) and (22b) one finds:

\[
I_{\text{inv}}^{\text{opt}} = \frac{24 N_0}{1 + [21 - 6 \rho] / \theta_0}. \tag{24}
\]

Here we replaced the electron temperature by \( \theta_e \rightarrow \rho \theta_0 \).

4.2. Comparison with numerical results

4.2.1. Case \( T_e = T_\gamma \)

In Fig. 5 the results obtained for incoming Wien spectra with temperature \( \theta_0 = \theta_e \) are shown. With increasing temperature the effective DC correction factor decreases strongly in both cases. This implies that for higher temperatures the efficiency of DC emission is significantly overestimated by the result obtained by Lightman (1981) and Thorne (1981). For example, even at moderate temperatures \( T_e \approx 4 \) keV there is a 10% negative correction due to higher order corrections in the energies of the initial photons and electrons. Since \( I/I_0 < 1 \), with Eq. (15) one can estimate that the main contribution to \( I \) has to come from photons with energies \( \omega_0 > 10^2 \theta_0 \).

Furthermore, one can see that in the considered case the direct formule (21) in fourth order of the electron temperature with (23) are valid up to \( \theta_e \approx 0.05 \), within reasonable accuracy. As mentioned above the convergence of these asymptotic expansions is very slow. However, as will be shown below, the inverse formula (22a) and the corresponding coefficients \( a \) for \( \rho \approx 1 \) provides an approximation which is better than 5% up to \( \theta_e \approx 0.2 \).
temperatures, for which the first order correction to $I$ (cf. Eq. (23)), then two regimes can be distinguished: (i) for $\rho \ll 1$, the ratio $I/I_0$ is monotonically dropping with increasing electron temperature, whereas (ii) for $\rho \gg 1$, it is first increasing and then turning into a decrease at some high temperature. This suggests that $\rho_* \approx 1$ separates the regions where Doppler boosting is compensating the suppression of DC emission due to higher photon energy. Estimating the mean photon energy weighted by $\theta_\gamma I \equiv \int x^5 n(x) dx \int x^4 n(x) dx \approx 5 \theta_\gamma$, and using $\langle \omega \rangle \approx \frac{10}{7} \gamma \theta_\gamma$ for the condition $G_m(\omega_0, \theta) = 1$ (see Eq. (15)) yields $\rho_\gamma \approx 7/2$, which further supports this conclusion.

In general one finds (see Figs. 6) that for $\rho < \rho_\gamma$ the inverse approximation (22a) has a better performance than the direct expansion formula (21) and vice versa for the case $\rho \geq \rho_\gamma$. It is worth mentioning that the direct formula performs very well for $\rho \gg \rho_\gamma$ even in lower orders of the temperature corrections. Combining these two approximations an accuracy of better than $\sim 5\%$ can be achieved in a very broad range of temperatures. If the photons and electrons have similar temperatures then the direct expansion formula (21) and vice versa for the case $\rho \gg \rho_\gamma$. It is worth mentioning that the direct formula performs very well for $\rho \gg \rho_\gamma$ even in lower orders of the temperature corrections. Combining these two approximations an accuracy of better than $\sim 5\%$ can be achieved in a very broad range of temperatures.

In the general case, i.e. $\rho \neq 1$, the approximation given by Svensson (1984) does not reproduce the numerical results, since the strong dependence of the higher order corrections on the ratio of the electron to the photon temperature was not taken into account.

Comparing the result for an incoming Wien spectrum with the approximation given by Svensson (1984),

$$I_0(\theta) \approx \frac{I_{0W}}{1 + 13.91 \theta + 11.05 \theta^2 + 19.92 \theta^3} \quad (25)$$

shows that for $\rho = 1$ Eq. (25) provides a very good fit to the full numerical results. The approximations based on formulae (20) and (24) show a similarly good performance. However, in the more general case, i.e. $\rho \neq 1$, the approximation given by Svensson (1984) as given by Lightman-Thorne approximation as a function of the electron temperature $\theta_e = T_e/m_e$. All the curves where computed for initial photons with a Wien spectrum at temperature $T_\gamma = T_e$. The full numerical results are shown in comparison with the inverse formula from Eq. (24), as indicated respectively. In addition, the approximation formula (25) as given by Svensson (1984) is shown. The approximations based on formulae (20) and (24) show a similarly good performance but for the sake of clarity was not presented here. For further discussion about the performance of the various approximations see Sect. 4.3.

4.2.2. Case $T_e \neq T_\gamma$

Figure 5 illustrates the dependence of the effective DC correction factor on the ratio of the electron to the photon temperature $\rho = T_e/T_\gamma$ for an initial Wien spectra. In general the characteristics of the ratio $I/I_0$ can be summarized as follows: If we define the critical ratio $\rho_* \approx 7/2 \sim 2.33$ of electron to photon temperatures, for which the first order correction to $I$ vanishes (cf. Eq. (23b)), then two regimes can be distinguished: (i) for $\rho \leq \rho_*$ the ratio $I/I_0$ is monotonically dropping with increasing electron temperature, whereas (ii) for $\rho \geq \rho_*$ it is first increasing and then turning into a decrease at some high temperature. This suggests that $\rho$ separates the regions where Doppler boosting is compensating the suppression of DC emission due to higher photon energy. Estimating the mean photon energy weighted by $\theta_\gamma I \equiv \int x^5 n(x) dx \int x^4 n(x) dx \approx 5 \theta_\gamma$, and using $\langle \omega \rangle \approx \frac{10}{7} \gamma \theta_\gamma$ for the condition $G_m(\omega_0, \theta) = 1$ (see Eq. (15)) yields $\rho_\gamma \approx 7/2$, which further supports this conclusion.

In general one finds (see Figs. 6) that for $\rho < \rho_\gamma$ the inverse approximation (22a) has a better performance than the direct expansion formula (21) and vice versa for the case $\rho \geq \rho_\gamma$. It is worth mentioning that the direct formula performs very well for $\rho \gg \rho_\gamma$ even in lower orders of the temperature corrections. Combining these two approximations an accuracy of better than $\sim 5\%$ can be achieved in a very broad range of temperatures. If the photons and electrons have similar temperatures then the inverse formula (22a) is valid even up to $kT \sim 100$ keV. In Section 4.3 we shall discuss the range of applicability for the various approximations given above in more detail.

In Fig. 6 the dependence of the DC emission correction factor on the temperature of the initial photons and the temperature of the electrons is illustrated for Wien spectra. Again one can clearly see that the Lightman-Thorne approximation is accurate in a very limited range of photon energies and electron temperatures. On the other hand, the approximation (20) works in practically the full considered range within very high accuracy. In contrast to the case of monoenergetic initial photons even at low temperatures of the photons the electrons have to be several times hotter in order to achieve $I/I_0 = 1$. As discussed above this is due to the fact that the main emission is coming from photons with $\omega_0 > \text{few} \times \theta_\gamma$.

4.3. Range of applicability for the different approximations

In the previous Sections we have derived different kinds of analytical approximations for the DC emission coefficient. Each of them has advantages and disadvantages. For example the approximation (20) involves a 2-dimensional integral over the photon spectrum and the electron distribution, which is numerically
more expensive than the 1-dimensional integrals for the direct expansion (21) or the inverse approximation (22). One obvious advantage of the inverse formula (22) over the direct expansion (21), is that only the first order corrections are needed to obtain an excellent approximation up to fairly high temperatures. Especially for numerical applications this is important, since higher order derivatives of the spectrum may lead to significant difficulties. However, it is necessary to investigate the range of applicability of each of these approximations in more detail.

Therefore we have studied the accuracy of these approximations for Wien spectra more extensively. For this purpose we have compared the different approximations with the results of the full numerical computation and, for a given \( \rho = T_e/T_\gamma \), determined the lowest values of the electron temperature \( \theta_e \), at which the approximation deviates by more than \( \epsilon \) from the numerical results. In Fig. 8 we illustrate the outcome of these calculations for the approximation (20) based on the results for the monochromatic case, the direct expansion (21) up to fourth order in temperature and the inverse formula from equation (22).

The peaks in the curves for the direct expansion (21) and approximation (20) appear where the relative difference to the results of the full computation in each case changes sign. For \( \rho \leq 1 \) the direct expansion breaks down very fast, while for \( \rho \geq 1 \) it works very well. The inverse formula works best for cases close to \( \rho = 1 \). Whenever induced effects are negligible the approximation based on the results obtained for the monochromatic case is excellent, in particular for \( \rho \geq 1 \).

5. Implications for the thermalization of spectral distortions of the CMB in the early Universe

In the high redshift Universe \( (z \gtrsim 2.9 \times 10^3) \) injection of energy into the medium leads to a \( \mu \)-type distortion of the CMB (Sunyaev & Zeldovich, 1970; Illarionov & Sunyaev, 1975a, b; Danese & de Zotti, 1982; Burigana, Danese & de Zotti, 1991; Hu & Silk, 1993). At low frequencies \( (h\nu \lesssim \text{few} \times 10^{-3}kT_\gamma) \) the production of soft photons by the DC process returns the photon distribution to a Planck spectrum after a very short time, whereas at high frequencies Compton scattering is only able to establish a Bose-Einstein spectrum with constant chemical potential. The limits on the deviations of the CMB spectrum from
a pure blackbody as obtained with CoBE (Fixsen et al. 1996, Fixsen & Mather 2002) constrains the chemical potential today to obey $|\mu| < 9 \times 10^{-3}$, but it is important to note that this small residual distortion can have arisen from a huge energy injection at sufficiently high redshifts ($z \gtrsim 10^6 - 10^7$).

Due to Compton scattering the temperature of the electrons is always very close to the equilibrium temperature in the given radiation field. For a $\mu$-type distortion with $\mu \ll 1$ the difference between the radiation and electron temperature is small and hence for estimates of the DC emission coefficient one may assume $T_e \sim T_\gamma$. For small values of $\mu$ it is also sufficient to use a blackbody spectrum to compute the DC correction factor. The discussion in Sect. 4.2 has shown that for $T_e \sim T_\gamma$ the DC emission due to higher order corrections in the ones of the initial photon and electron is less efficient than in the Lightman-Thorne-approximation. This implies that at high redshifts the thermalization of CMB spectral distortions will also be less efficient. For a temperature of $kT_\gamma \sim 4$ keV, higher order corrections lower the DC emissivity by $\sim 10\%$. This temperature corresponds to a redshift of roughly $z \sim 10^7$. Therefore one can expect that quantum-electrodynamical effects lead to significant corrections of the DC photon production rate for large energy injection at redshifts $z \sim 10^6$, since the integrated effect should be bigger. In addition, higher order temperature corrections to the Compton process may also lower the thermalization efficiency. This then in return will make the constraints placed on energy injection due to exponentially decaying relic particles with short lifetimes ($\tau_X \lesssim \text{few} \times 10^9 \text{s}$) tighter and may affect the results obtained within the standard treatment (Hu & Silk, 1993a) on the level of a few $\times 10^6$.

### 6. Conclusion

Double Compton emission in an isotropic, mildly relativistic thermal plasma was investigated within the soft photon limit. Simple and accurate analytic expression for the low frequency DC emission coefficient have been derived, which extend the Lightman-Thorne approximation up to mildly relativistic temperatures of the medium and should be applicable in a broad range of physical situations. In particular expressions for initially monochromatic photons and Wien spectra were given and discussed in detail.

It has been shown that the DC emissivity strongly decreases with higher mean energy of the initial photons leading to a suppression of the total number of newly created photons as compared to the Lightman-Thorne approach. On the other hand increasing the temperature of the electrons leads to an enhancement of the DC emissivity. If the photons and the electrons have similar temperatures, which is the case in most physical situations close to full thermodynamic equilibrium, then formula (22a) is applicable up to $kT_\gamma \sim 100$ keV with an accuracy of better than a few percent. Since only first order corrections are necessary for this inverse formula (22a), it is generally most suitable for numerical applications.

**References**

Hahn, T., 2004, hep-ph/0404043, download of the CUBA Library available from:
Jauch, J.M., & Rohrlich, F., 1976, Springer-Verlag

Chluba, Sazonov and Sunyaev: The DC emissivity in a mildly relativistic thermal plasma
Appendix A: Expansion of the relativistic Maxwell Boltzmann distribution

The low temperature expansion of (3) leads to

\[
f(E) = \frac{N e^{-\xi}}{(2\pi m_e^2 \theta_e)^{3/2}} \left[ 1 - \theta_e \cdot \left( \frac{15}{8} - \frac{1}{2} \xi^2 \right) + \theta_e^2 \cdot \left( \frac{345}{128} - \frac{15}{16} \xi^2 - \frac{15}{16} \xi^3 + \frac{1}{8} \xi^4 \right) \right]
\]

\[
- \theta_e^3 \cdot \left( \frac{3285}{1024} - \frac{345}{256} \xi^2 - \frac{15}{16} \xi^3 - \frac{25}{64} \xi^4 + \frac{1}{4} \xi^5 - \frac{1}{48} \xi^6 \right)
\]

\[
+ \theta_e^4 \cdot \left( \frac{95355}{32768} - \frac{3285}{2048} \xi^2 - \frac{345}{256} \xi^3 - \frac{855}{128} \xi^4 + \frac{13}{32} \xi^5 + \frac{51}{128} \xi^6 - \frac{1}{16} \xi^7 + \frac{1}{8} \xi^8 \right)
\]

(A.1)

with \( \xi = \eta^2 / 2 \theta_e \) and \( \eta = \rho / m_e \).

Appendix B: Squared matrix element for double Compton

In order to write down the DC differential cross section one has to calculate the squared matrix element \(|M|^2 = \epsilon^6 K^3 X\) describing the DC process. This calculation was first performed by Mandl & Skyrme (1952). In the book of Jauch & Rohrlich (1976) one can find the expression for the squared matrix element of the double Compton process (pp. 235):

\[
X = 2 \left[ (a b - c) \left[ (a + b) (2 x) - (a b - c) - 8 \right] - 2 x \left[ a^2 + b^2 \right] - 2 \left[ a b + c (1 - x) \right] \rho \right]
\]

\[
- 8 c + \frac{4 x}{A B} \left[ (A + B) (1 + x) + x^2 (1 - z) + 2 z - (a A + b B) \left( 2 + \frac{(1 - x) z}{x} \right) \right]
\]

(B.1)

where the following abbreviations have been used

\[
a = \frac{1}{k_0} + \frac{1}{k_1} + \frac{1}{k_2}
\]

\[
b = \frac{1}{k_0'} + \frac{1}{k_1'} + \frac{1}{k_2'}
\]

\[
c = \frac{1}{k_0 k_1} + \frac{1}{k_1 k_2} + \frac{1}{k_2 k_0'}
\]

\[
x = k_0 + k_1 + k_2
\]

\[
z = k_0 k_1' + k_1 k_2' + k_2 k_0'
\]

\[
A = k_0 k_1 k_2
\]

\[
B = k_0' k_1' k_2'
\]

\[
\rho = \frac{k_0}{k_0'} + \frac{k_1}{k_1'} + \frac{k_2}{k_2'}
\]

(B.2a)

(B.2b)

(B.2c)

For the definitions of \(k_i, k_i'\) we used those of the original paper from Mandl & Skyrme (1952):

\[
m_e^2 k_0 = -P \cdot K_0\]

\[
m_e^2 k_1 = +P \cdot K_1\]

\[
m_e^2 k_2 = +P \cdot K_2\]

(B.3a)

(B.3b)

with the standard signature of the Minkowski-metric (+ - - -). Assuming that the outgoing photon \(\gamma(K_2)\) is soft as compared with \(\gamma(K_1)\) one can expand \(X\) into orders of the frequency \(\omega_2\) and keep only the lowest order term, i.e. terms of the order \(O(\omega_2^2)\). Similar expansions for the limits \(\omega_0 \ll 1\) and \(\beta \ll 1\) can be performed. Since the results of these expansions are extremely complex and not very illuminating, we omitted them here.

Appendix C: Numerical solution of the Boltzmann integrals

In order to solve the Boltzmann integrals we implemented two different programs, one based on the NAG routine D01GBF, which uses an adaptive Monte-Carlo method to solve multidimensional integrals, the other using the Vegas routine of the CUBA Library [Hahn 2004]. The latter turned out to be much more efficient since it significantly benefitted from importance sampling. The expense and performance of the calculation critically depend on the required accuracy. For most of the calculations presented here we chose a relative error of the order of \(e^{-10^4}\). For the integrations over different initial photon distributions we typically used \(\omega_{\text{min}} = 10^{-4} \theta_\gamma\) and \(\omega_{\text{max}} = 25 \theta_\gamma\).

\[^4\] Download of the CUBA Library available at: [http://www.feynarts.de/cuba/](http://www.feynarts.de/cuba/)
Integration over the electron momenta

In order to restrict the integration region over the electron momenta for low electron temperature we determined the maximal Lorentz factor, $\gamma_{\text{max}}$, such that the change in the normalization of the electron distribution was less than a fraction of the required accuracy. Furthermore, instead of $p$ we used the variable $\xi = \eta^2 / 2 \theta_e$ with $\eta = p / m_e$.

For high electron temperature it turned out to be more efficient to use the normalization of the electron distribution itself as a variable. For this we defined

$$N(x) = m_e^3 \int_1^x \gamma \sqrt{\gamma^2 - 1} f(\gamma m_e) \, d\gamma,$$  

which for $x \to \infty$ becomes $N \equiv N_e$. In actual calculations one has to invert this equation to find the function $x(N)$. This was done numerically before the integrations were performed for a sufficiently large number of points ($n \sim 512$), such that the function $x(N)$ during the integrations could be accurately represented via spline interpolation. Here it is important to bear in mind, that $N(x)$ is rather steep for $x \sim 0$ and $x \sim 1$.

C.1. Definition of $I_k$

Inserting the direct expansion of $G_m$ in low energies of the initial photon and electron temperature Eq. (15) into Eq. (20) and collecting the different orders in $\theta_e$, one can define the functions $I_i$ of equation (21) as

$$I_0 = D_4$$

$$I_1 = 6 \, D_4 - \frac{21}{5} \frac{\theta_y}{\theta_e} \, D_5$$

$$I_2 = 15 \, D_4 - \frac{441}{10} \frac{\theta_y}{\theta_e} \, D_5 + \frac{357}{25} \left[ \frac{\theta_y}{\theta_e} \right]^2 \, D_6$$

$$I_3 = \frac{45}{4} \, D_4 - \frac{8379}{40} \frac{\theta_y}{\theta_e} \, D_5 + \frac{5712}{25} \left[ \frac{\theta_y}{\theta_e} \right]^2 \, D_6 - \frac{7618}{175} \left[ \frac{\theta_y}{\theta_e} \right]^3 \, D_7$$

$$I_4 = -\frac{45}{4} \, D_4 - \frac{3969}{8} \frac{\theta_y}{\theta_e} \, D_5 + \frac{8568}{5} \left[ \frac{\theta_y}{\theta_e} \right]^2 \, D_6 - \frac{34281}{35} \left[ \frac{\theta_y}{\theta_e} \right]^3 \, D_7 + \frac{21498}{175} \left[ \frac{\theta_y}{\theta_e} \right]^4 \, D_8,$$

where we introduce the integrals $D_i$ as

$$D_i = \int x^i \, n \, dx$$

over the initial photon distribution $n(\nu)$. 