Geometry of $N = 4, d = 1$ nonlinear supermultiplet

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Abstract

We construct the general action for $N = 4, d = 1$ nonlinear supermultiplet including the most general interaction terms which depend on the arbitrary function $h$ obeying the Laplace equation on $S^3$. We find the bosonic field $B$ which depends on the components of nonlinear supermultiplet and transforms as a full time derivative under $N = 4$ supersymmetry. The most general interaction is generated just by a Fayet-Iliopoulos term built from this auxiliary component.

Being transformed through a full time derivative under $N = 4, d = 1$ supersymmetry, this auxiliary component $B$ may be dualized into a fourth scalar field giving rise to a four dimensional $N = 4, d = 1$ sigma-model. We analyzed the geometry in the bosonic sector and find that it is not a hyper-Kähler one. With a particular choice of the target space metric $g$ the geometry in the bosonic sector coincides with the one which appears in heterotic $(4, 0)$ sigma-model in $d = 2$. 
1 Introduction

The one-dimensional theories (mechanics) with extended supersymmetries possess a number of specific features which make them selected among their higher-dimensional counterparts. In this respect the existence of nonlinear off-shell supermultiplets in \( d = 1 \) is an impressive example. The simplest examples of such multiplets are nonlinear \((3, 4, 1)\) [1, 2] and nonlinear \((2, 4, 2)\) [2] supermultiplets. These nonlinear supermultiplets have the same component contents as their linear analogs while the transformation properties of their components are highly nonlinear under supersymmetry. As a result, the geometry of the bosonic target space of the sigma models constructed from nonlinear supermultiplets should be rather different from those in the linear cases. The supersymmetric mechanics constructed with nonlinear \((2, 4, 2)\) supermultiplet has been considered in details in \([3]\), while no such an exhaustive study was undertaken so far for nonlinear \((3, 4, 1)\) supermultiplet. The basic aim of this paper is to give a detailed description of the \( N = 4 \) supersymmetric mechanics with nonlinear \((3, 4, 1)\) supermultiplet. We construct the most general sigma-model type action for this nonlinear supermultiplet and extend it by the most general Fayet-Iliopoulos (FI) term. Being equipped with a general FI term we, in full analogy with linear \((3, 4, 1)\) supermultiplet, perform the dualization of an auxiliary component into a fourth physical boson, thus finishing with nonlinear \((4, 4, 0)\) supermultiplet. This new multiplet belongs to a new class of nonlinear supermultiplets which contain a functional freedom in the defining relations \([4, 5, 6, 7]\). We construct the sigma-model action for this supermultiplet and explicitly demonstrate that the bosonic target space geometry is not a hyper-Kähler one.

2 \( N=4 \) nonlinear multiplet and its action

The nonlinear \( N = 4 \) supermultiplet is a \( d = 1 \) analog of the \( N = 2, d = 4 \) nonlinear multiplet \([8, 9, 10]\). It can be described by 4 by 4 matrix variables \( N^{ai} \) \((i = 1, 2; a = 1, 2)\) obeying the constraints \([2]\)

\[
N^{ia}N_{ia} = 2, \quad N^{a(i}D^jN^{k)}_a = 0, \quad N^{a(i}\overline{D}^jN^{k)}_a = 0, \quad (2.1)
\]

where \( N = 4, d = 1 \) spinor derivatives are defined by

\[
D^i = \frac{\partial}{\partial \theta_i^1} + i\theta_i^2\partial_t, \quad \overline{D}_i = \frac{\partial}{\partial \theta_i^2} + i\theta_i^1\partial_t, \quad \{D^i\overline{D}_j\} = 2i\delta^i_j\partial_t. \quad (2.2)
\]

This very symmetric description of the nonlinear supermultiplet by matrix superfields \( N^{ia} \) is not very useful in practice. The preferable one is by the following representation of \( N^{ia} \) \([2]\):

\[
N^{11} = \frac{e^{-i\phi}}{\sqrt{1 + \Lambda\overline{\Lambda}}}, \quad N^{21} = \frac{e^{i\phi}}{\sqrt{1 + \Lambda\overline{\Lambda}}}, \quad N^{12} = -\frac{e^{-\phi}}{\sqrt{1 + \Lambda\overline{\Lambda}}}, \quad N^{22} = \frac{e^{\phi}}{\sqrt{1 + \Lambda\overline{\Lambda}}}. \quad (2.3)
\]

The representation \((2.3)\) solves the algebraic constraint in \((2.1)\) in terms of independent \( N = 4 \) superfields \( \phi, \Lambda, \overline{\Lambda} \) which, as a consequence of the differential constraints in \((2.1)\), should obey

\[
D^1\Lambda = -\Lambda D^2\Lambda, \quad \overline{D}^2\Lambda = \Lambda\overline{D}^1\Lambda, \quad D^2\overline{\Lambda} = \overline{\Lambda}D^1\overline{\Lambda}, \quad \overline{D}^1\overline{\Lambda} = -\overline{\Lambda}\overline{D}^2\overline{\Lambda},
\]

\[
iD^1\Phi = -D^2\Lambda, \quad i\overline{D}^2\Phi = \overline{D}^1\overline{\Lambda}, \quad iD^2\Phi = -D^1\overline{\Lambda}, \quad i\overline{D}^1\Phi = \overline{D}^2\overline{\Lambda}. \quad (2.4)
\]
From (2.4) one may immediately reveal the component field content of the nonlinear supermultiplet which includes three physical \( \phi, \lambda, \bar{\lambda} \) and one auxiliary \( A \) bosonic fields and four fermions \( \psi_a, \bar{\psi}^a \) defined as

\[
\phi = \Phi|, \quad \lambda = \Lambda|, \quad \bar{\lambda} = \bar{\Lambda}|, \quad A = (D^1 \overline{D}_1 - \overline{D}_1 D^1) \Phi|,
\]

\[
\psi_1 = \frac{1}{2} \overline{D}_1 \Phi|, \quad \psi_2 = -\frac{1}{2} \overline{D}_2 \Phi|, \quad \bar{\psi}^1 = -\frac{1}{2} D^1 \Phi|, \quad \bar{\psi}^2 = \frac{1}{2} D^2 \Phi|,
\]

where \( | \) as usual means \( \theta_i = \bar{\theta}^i = 0 \). Under \( N = 4 \) supersymmetry these components transform as follows:

\[
\begin{align*}
\delta \phi &= 2 \left( \epsilon_1 \bar{\psi}^1 - \epsilon_2 \bar{\psi}^2 - \bar{\epsilon}^1 \psi_1 + \bar{\epsilon}^2 \psi_2 \right), \\
\delta \psi_1 &= -\frac{1}{2} \epsilon_1 \left( i \dot{\phi} + \frac{1}{2} A \right) - \frac{1}{2} \epsilon_2 \left( 2 \dot{\bar{\lambda}} + 4i \psi_1 \bar{\psi}^2 + i \bar{\lambda} \dot{\phi} + \frac{1}{2} \lambda A \right), \\
\delta \psi_2 &= \frac{1}{2} \epsilon_2 \left( i \dot{\phi} - \frac{1}{2} A \right) + \frac{1}{2} \epsilon_1 \left( 2 \dot{\lambda} - 4 \bar{\psi}_2 \bar{\psi}^1 - i \lambda \dot{\phi} + \frac{1}{2} \bar{\lambda} A \right), \\
\delta A &= -4i \left( \epsilon_1 \dot{\bar{\psi}}^1 + \epsilon_2 \dot{\bar{\psi}}^2 + \bar{\epsilon}^1 \dot{\psi}_1 + \bar{\epsilon}^2 \dot{\psi}_2 \right).
\end{align*}
\]

One may check that, despite the presence of nonlinear terms, the transformations (2.6) perfectly close to span \( N = 4 \) supersymmetry, the general off-shell action

\[
S = \int dt d^2 \theta d^2 \bar{\theta} L(\Phi, \Lambda, \bar{\Lambda}),
\]

where \( L(\Phi, \Lambda, \bar{\Lambda}) \) is an arbitrary real function of the superfields \( \Phi, \Lambda, \bar{\Lambda} \), is invariant under \( N = 4, d = 1 \) supersymmetry. Being rewritten in terms of the components (2.5), the action (2.7) reads

\[
S = \int dt g(1 + \lambda \bar{\lambda}) \left[ \left( \phi + i \frac{\dot{\lambda} \bar{\lambda} - \lambda \dot{\bar{\lambda}}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4 \lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \frac{1}{4} \bar{A}^2 + \text{fermions} \right],
\]

where the metric \( g \) is defined through the prepotential \( L(\phi, \lambda, \bar{\lambda}) \) entering (2.7) as

\[
g = (1 + \lambda \bar{\lambda}) L_{\lambda \bar{\lambda}} + L_{\phi \phi} + i \lambda L_{\lambda \phi} - i \bar{\lambda} L_{\bar{\lambda} \phi}
\]

and the new auxiliary field \( \bar{A} \) reads

\[
\bar{A} = A + 2 \frac{d}{dt} \log(1 + \lambda \bar{\lambda}).
\]

For the sake of brevity we omitted in the action (2.8) all fermionic terms, which may be easily reconstructed, if needed.

From now on, we have two options to go further. Firstly, one may immediately exclude the auxiliary field \( \bar{A} \) from the action (2.8). The resulting action reads

\[
S = \int dt g(1 + \lambda \bar{\lambda}) \left[ \left( \phi + i \frac{\dot{\lambda} \bar{\lambda} - \lambda \dot{\bar{\lambda}}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4 \lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \text{fermions} \right].
\]
It is clear that the terms in square brackets are just the sigma model action of the principal chiral field on $SU(2)$ \cite{2}. The curvature of this three dimensional space is equal to $3/2$. Thus, the full action (2.11) describes the $N = 4$ supersymmetric extension of the particle moving on $S^3$ deformed by the conformal factor $g(1 + \lambda \bar{\lambda})$.

Alternatively, due to transformation properties of $A$ (2.6), one may dualize this components into a fourth scalar field as

$$\tilde{A} = 2\dot{y}.$$ (2.12)

Plugging (2.12) back in (2.8) we get the four-dimensional $N = 4$ sigma model with the following bosonic sector:

$$S_{\text{bosonic}} = \int dt \; g(1 + \lambda \bar{\lambda}) \left[ \left( \dot{\phi} + i \frac{\dot{\lambda} \bar{\lambda} - \lambda \bar{\lambda}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4\lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \dot{y}^2 \right].$$ (2.13)

In the simplest case when

$$g(1 + \lambda \bar{\lambda}) = 1 \quad \Rightarrow \quad L(\Phi, \Lambda, \bar{\Lambda}) = \log(1 + \Lambda \bar{\Lambda})$$ (2.14)

the action (2.13) is just the action of $SU(2) \times U(1)$ sigma model. Let us note that the corresponding target space is conformally flat. Indeed, one may easily check that the action

$$S_1 = \int dt \; e^{-y} \left[ \left( \dot{\phi} + i \frac{\dot{\lambda} \bar{\lambda} - \lambda \bar{\lambda}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4\lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \dot{y}^2 \right]$$ (2.15)

describes the particle in flat four-dimensional space. But the metric $g = \frac{e^{-y}}{(1 + \lambda \bar{\lambda})}$ is unreachable because in our construction $g$ is a function depending on $(\phi, \lambda, \bar{\lambda})$ only.

Another particular choice of the metric $g = e^{a\phi}$ produce the action

$$S_2 = \int dt \; e^{a\phi} (1 + \lambda \bar{\lambda}) \left[ \left( \dot{\phi} + i \frac{\dot{\lambda} \bar{\lambda} - \lambda \bar{\lambda}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4\lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \dot{y}^2 \right].$$ (2.16)

One may check that in this case all components of the curvature tensor are proportional to $(a^2 + 1)$. Therefore, within our model with any choice of real function $g(\phi, \lambda, \bar{\lambda})$ the flat metric is unreachable.

In the next section we will construct a more general action for the four-dimensional sigma model, but in any case there are no chances for its bosonic manifold to be a hyper-Kähler one. Indeed, any more sophisticated four dimensional sigma model action will contain (2.15) as a particular solution. But within our approach the metric corresponding to the action (2.15) is not even conformally a hyper-Kähler one. Thus the same will be true for any of its generalizations. Let us remind that in the linear case \cite{4, 5} the analog of the action (2.11) corresponds to a flat target space which is a rather particular case of a hyper-Kähler geometry. Thanks to this fact there exists its generalization for a general case of a hyper-Kähler geometry with one isometry.
3 Towards N=4 four-dimensional sigma model

In this section we will construct the potential terms for the nonlinear supermultiplet and show that there is another dualization which gives rise to a four dimensional sigma-model with a geometry in the bosonic sector which depends on two functions and is different from the hyper-Kähler one.

The simplest way to get the potential terms is to add the Fayet-Iliopoulos term to the action (2.8)

\[ \tilde{S} = S + m \int dt \tilde{A}. \]  

After excluding the auxiliary field \( \tilde{A} \) in the action (3.1) we have the following potential term:

\[ S_{pot} = - \int dt \frac{m^2}{g(1 + \lambda \bar{\lambda})}. \]  

Clearly, this potential terms is not the most general one. In order to have more possibilities for the interaction one should construct a more general bosonic field \( B \) which transforms as a full time derivative with respect to \( \mathcal{N} = 4 \) supersymmetry (2.6). Starting from the most general Ansatz one may check that the most general real combination of dimension \( cm^{-1} \) that is composed of the nonlinear supermultiplet components and transforms as a total time derivative has the following form:

\[ B = fA + b\lambda + \bar{b}\bar{\lambda} + \dot{c}\phi + a (\bar{\psi}^i \psi_1 - \bar{\psi}^2 \psi_2) + a_1 \bar{\psi}^2 \psi_1 + a_2 \psi^1 \psi_2, \]  

where real dimensionless coefficients \( (b, \bar{b}, c, a, a_1, a_2) \) are expressed through the function \( f \) depending on \( (\phi, \lambda, \bar{\lambda}) \) as

\[ a = -8i \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \quad a_1 = -8i f_\lambda + 8\lambda \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \quad a_2 = 8i f_\lambda + 8\bar{\lambda} \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \quad b_\lambda - \bar{b}_{\bar{\lambda}} = -4i \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \quad c_\lambda - b_{\phi} = -2i f_\lambda - 4\lambda \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \quad c_{\bar{\lambda}} - \bar{b}_{\phi} = 2i f_\lambda - 4\bar{\lambda} \frac{f_{\phi}}{1 + \lambda \bar{\lambda}}, \]  

while \( f \) obeys the following equation:

\[ f_{\phi\phi} + (1 + \lambda \bar{\lambda}) f_{\lambda \lambda} + i\lambda f_{\phi \lambda} - i\bar{\lambda} f_{\lambda \phi} = 0. \]  

With all these equations satisfied, the new auxiliary component \( B \) transforms under \( \mathcal{N} = 4, d = 1 \) supersymmetry as follows:

\[ \delta B = \frac{d}{dt} \left[ -4i f (c_i \bar{\psi}^i + \bar{c}^i \psi_i) + b \delta \lambda + \bar{b} \delta \bar{\lambda} + c \delta \phi \right], \]  

where \( \delta \lambda, \delta \bar{\lambda}, \delta \phi \) are defined in (2.6).

Let us note that the coefficients \( (b, \bar{b}, c) \) are defined up the to following gauge transformation:

\[ \delta b = v_\lambda, \quad \delta \bar{b} = v_{\bar{\lambda}}, \quad \delta c = v_\phi \quad \Rightarrow \quad \delta B = \dot{v}, \]  

where \( v \) is an arbitrary function of \( (\phi, \lambda, \bar{\lambda}) \). Clearly, the new auxiliary component \( \dot{B} = B + \dot{v} \) will also transform through a full time derivative. Just this freedom is reflected in the equations for \( (b, \bar{b}, c) \) which are invariant under (3.7).
In principle, the equations (3.3), (3.4), (3.5) give a complete solution for our problem. But, when working in one dimensional space, one may drastically simplify these expressions. Let us carefully explain all steps in this simplification.

First of all our basic Ansatz is too general for one dimension. Indeed, without loss of generality one may write

$$ c = \partial_{\phi} \tilde{c} \quad \Rightarrow \quad c\dot{\phi} \rightarrow \frac{d}{dt} \tilde{c} - \tilde{c}_\lambda \dot{\lambda} - \tilde{c}_{\bar{\lambda}} \dot{\bar{\lambda}}. $$

(3.8)

Obviously, one may discard the full time derivative $\frac{d}{dt} \tilde{c}$, which does not affect the transformation properties of $B$ through the full time derivative. Moreover one may cancel two additional terms in (3.8) by a proper redefinition of $b$ and $\bar{b}$. Therefore from the beginning one may set $c = 0$. This choice will restrict the gauge freedom (3.7) till residual transformations with a function $\tilde{v}(\lambda, \bar{\lambda})$ depending on $(\lambda, \bar{\lambda})$ only.

Next, introducing the new function $h$ as

$$ h_\phi \equiv f $$

(3.9)

one may integrate last two equations in (3.4)

$$ b_\phi = 2i h_{\phi\lambda} + 4 \lambda \frac{h_{\phi\bar{\lambda}}}{1 + \lambda \bar{\lambda}}, \quad \bar{b}_\phi = -2i h_{\phi\bar{\lambda}} + 4 \lambda \frac{h_{\phi\lambda}}{1 + \lambda \lambda} $$

(3.10)

to get

$$ b = 2i h_\lambda + 4 \lambda \frac{h_\phi}{1 + \lambda \lambda} + \tilde{b}(\lambda, \bar{\lambda}), \quad \bar{b} = -2i h_{\bar{\lambda}} + 4 \lambda \frac{h_\phi}{1 + \lambda \lambda} + \bar{\tilde{b}}(\lambda, \bar{\lambda}). $$

(3.11)

Finally, representing $\tilde{b}(\lambda, \bar{\lambda})$ and $\bar{\tilde{b}}(\lambda, \bar{\lambda})$ as

$$ \tilde{b}(\lambda, \bar{\lambda}) = \partial_\lambda b_0 + i \partial_{\bar{\lambda}} b_1, \quad \bar{\tilde{b}}(\lambda, \bar{\lambda}) = \partial_{\bar{\lambda}} b_0 - i \partial_\lambda b_1 $$

(3.12)

one may see that the terms with $b_1$ may be absorbed into a new function $h$ by a redefinition of $h \rightarrow h + b_1$, while the terms with $b_0$ are just gauge transformations (3.7), and we may choose the gauge $b_0 = 0$.

Plugging these $(b, \bar{b})$ (3.11) (with $\tilde{b}(\lambda, \bar{\lambda}) = \bar{\tilde{b}}(\lambda, \bar{\lambda}) = 0$) into the last equation in (3.4) for them

$$ \bar{b}_\lambda - b_\lambda = -4i \frac{h_{\phi\bar{\lambda}}}{1 + \lambda \lambda} $$

(3.13)

we will get

$$ h_{\phi\bar{\lambda}} + (1 + \lambda \bar{\lambda}) h_{\lambda\lambda} + i \lambda h_{\lambda\phi} - i \bar{\lambda} h_{\bar{\lambda}\phi} = 0. $$

(3.14)

Thus, we have the following solution for the most general auxiliary component $B$ which transforms through a full time derivative under $N = 4$ supersymmetry:

$$ B = h_\phi A + b_\lambda + \bar{b}_{\bar{\lambda}} + a_1 (\bar{\psi}^1 \psi_1 - \bar{\psi}^2 \psi_2) + a_2 \bar{\psi}^2 \psi_1 + a_2 \bar{\psi}^1 \psi_2, $$

(3.15)

where

$$ a = -8 \frac{h_{\phi\bar{\lambda}}}{1 + \lambda \bar{\lambda}}, \quad a_1 = -8i h_{\phi\bar{\lambda}} + 8 \lambda \frac{h_{\phi\lambda}}{1 + \lambda \lambda}, \quad a_2 = 8i h_{\phi\lambda} + 8 \bar{\lambda} \frac{h_{\phi\bar{\lambda}}}{1 + \lambda \lambda}, \quad b = 2i h_\lambda + 4 \lambda \frac{h_\phi}{1 + \lambda \lambda} \quad \bar{b} = -2i h_{\bar{\lambda}} + 4 \lambda \frac{h_\phi}{1 + \lambda \lambda}.$$

(3.16)
and $h$ obeys the Laplace equation on $S^3$ (3.14).

Now, with the newly defined auxiliary field $B$ we may add to the action (2.8) a new generalized Fayet-Iliopoulos term
\[
\hat{S} = S + m \int dt B.
\] (3.17)

After excluding the auxiliary field $A$ we will have the following interaction terms in the components action (as usual, we consider only the bosonic part of the action):
\[
\hat{S}_{\text{pot}} = \int dt \left[ -\frac{m^2 h^2}{g(1 + \lambda \bar{\lambda})} + 2im \left( h_{\lambda} \dot{\lambda} - h_{\bar{\lambda}} \dot{\bar{\lambda}} \right) + 2mh_{\phi} \frac{\partial h(\lambda \bar{\lambda})}{1 + \lambda \bar{\lambda}} \right].
\] (3.18)

Thus, we see that, with the newly defined Fayet-Iliopoulos term, we succeeded in constructing the most general action which describes interactions with electric and magnetic fields and depends on the function $h$ which obeys equation (3.14). Let us observe that even for the $SU(2) \times U(1)$ sigma-model, which corresponds to the choice $g = 1/(1 + \lambda \bar{\lambda})$, we have a function freedom in the interaction terms. It will be interesting to analyze the possible self-interactions by choosing a proper solution of the Laplace equation (3.14).

Another trick we may do is to dualize the auxiliary field $B$ instead of $A$, as follows:
\[
B = \dot{u} \quad \Rightarrow \quad \tilde{A} = \frac{1}{h_{\phi}} \left[ \dot{u} - b \dot{\lambda} - \bar{b} \dot{\bar{\lambda}} - a (\bar{\psi}^1 \psi_1 - \bar{\psi}^2 \psi_2) - a_1 \bar{\psi}^2 \psi_1 - a_2 \bar{\psi}^1 \psi_2 \right].
\] (3.19)

Plugging the expression for $A$ (3.19) into the action (2.8), we get a four dimensional sigma-model action with the following bosonic part:
\[
\hat{S} = \int dt \left[ \left( \dot{\phi} + i \frac{i \dot{\bar{\lambda}} - \dot{\lambda}}{1 + \lambda \bar{\lambda}} \right)^2 + \frac{4 \lambda \bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} + \frac{1}{4h_{\phi}^2} \left( \dot{\bar{\psi}}^1 \psi_1 - \ddot{\psi}^1 \psi_1 - \dot{\bar{\psi}}^1 \psi_1 - \ddot{\psi}^2 \psi_1 - \dot{\bar{\psi}}^2 \psi_2 - \ddot{\psi}^2 \psi_2 \right) \right].
\] (3.20)

The action (3.20) depends on two functions: the arbitrary metric $g(\phi, \lambda, \bar{\lambda})$ and the auxiliary function $h(\phi, \lambda, \bar{\lambda})$, which obeys the Laplace equation on $S^3$. Thus, $N = 4$ supersymmetry in $d = 1$ leaves a lot of freedom in the sigma-model action. It is interesting that if we choose
\[
g = 2 \frac{h_{\phi}}{1 + \lambda \bar{\lambda}}
\] (3.21)

the action (3.20) will exhibit the same target space geometry which appears in the heterotic $(4,0)$ sigma-model in $d = 2$ [11]. The only simplification is that in $d = 1$ one may completely solve the equations (3.4) and write explicitly the action in terms of the harmonic function $h$ only.

**Conclusion**

In this paper we considered the most general action for the $N = 4, d = 1$ nonlinear supermultiplet. We explicitly constructed the most general interaction terms which depend on an arbitrary function $h$ obeying the Laplace equation on $S^3$. Of course, in $d = 1$ we have more freedom in the action, as compared to higher dimensions. This freedom is reflected
in the arbitrary metric $g$ which appear in the action. In order to get these results we
found the most general bosonic field $B$ which depends on the components of nonlinear
supermultiplet and transforms as a full time derivative under $N = 4$ supersymmetry. The
most general interaction is generated just by a generalized Fayet-Iliopoulos terms build
from this auxiliary component.

Being transformed through a full time derivative under $N = 4, d = 1$ supersymmetry,
this auxiliary component $B$ may be dualized into a fourth scalar field. In such way
we constructed a four dimensional $N = 4, d = 1$ sigma-model which possesses as a basic
three-dimensional manifold the $SU(2)$ sigma-model. We analyzed the four-dimensional
geometry in the bosonic sector and found that it is not a hyper-Kähler one. With a
particular choice of the metric $g$ (3.21), the geometry in the bosonic sector coincides with
the one which appears in the heterotic $(4,0)$ sigma-model in $d = 2$.

One of the interesting problems for further studying is to explicitly construct and
analyze the potential terms, corresponding to some interesting specific solution of the
Laplace equation obeyed by the function $h$. We expect that some cases will correspond
to integrable systems, in full analogy with $N = 4, d = 1$ hyper-Kähler sigma models.
Another intriguing question concerns the $N = 8, d = 1$ nonlinear supermultiplet which is
a direct dimensional reduction of $N = 2, d = 4$ nonlinear supermultiplet [8, 9, 10]. It has
been known for a long time that starting from $N = 2, d = 4$ nonlinear supermultiplet in
$N = 1, d = 4$ superspace on may dualize the scalar supermultiplet into a chiral one [9].
The resulting metric of the new four-dimensional manifold will be the hyper-Kähler one.
So, the same should be also true for $N = 8, d = 1$ nonlinear supermultiplet. But at the
same time, the action for $N = 8, d = 1$ nonlinear supermultiplet should admit a reduction
to the $N = 4, d = 1$ case, which as we see does not support hyper-Kähler geometry. We
hope to resolve this paradox in a future publication.

Finally, we would like to stress that one dimensional extended supersymmetry is much
simpler then the four-dimensional one. So, the analysis of the possible sigma-model
geometries is greatly simplified as compared to higher dimensional cases. Indeed, the
search for a generalized auxiliary component of the supermultiplet which may be then
dualized into an additional scalar component is equivalent in a some sense to performing
a Legendre transform in $d = 4$ [10], being much simpler, though. Moreover, all $N = 4$
and most of $N = 8, d = 1$ supermultiplets are off shell, what also simplified the analysis.
Unfortunately, the dualization of the auxiliary component will always produce a manifold
with an isometry. Nevertheless, the dualizations procedure can be generalized to give
manifolds without any obvious isometry. We hope to report the corresponding results
elsewhere.

**Acknowledgements**

The authors are grateful to A. Shcherbakov and A. Beylin for useful discussions. S.K.
would like to thank the INFN–Laboratori Nazionali di Frascati for the warm hospitality
extended to him during the course of this work.

This work was partially supported by the European Community’s Marie Curie Research
Training Network under contract MRTN-CT-2004-005104 Forces Universe, by INTAS under
contract 05-7928 and by grants RFBR-06-02-16684, DFG 436 Rus 113/669/03.
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