1. INTRODUCTION

Quantum Chromodynamics (QCD) exhibits dynamical chiral symmetry breaking and confinement properties. Both these expected features of QCD are related with the global symmetries of the Lagrangian. However, the relation of the spontaneous chiral symmetry breaking and confinement still remains an open issue. While the chiral symmetry is exact in the limit of massless quarks the confinement phenomena being related to the $Z(3)$ center symmetry is an exact only for infinitely heavy quarks. In order to obtain a unified picture of confinement and chiral symmetry breaking several effective chiral models have been studied. Recently an extension of the Nambu–Jona-Lasinio (NJL) model was proposed and developed to address this question.

The NJL model describes interactions of constituent quark fields. It exhibits a global SU(3) symmetry that is a replacement of a local gauge SU(3) color transformation of the QCD Lagrangian. Thus, color confinement is lost in the NJL dynamics. Recently, color degrees of freedom were introduced in the NJL Lagrangian through an effective gluon potential expressed in terms of Polyakov loops. Such a potential was constructed to preserve the $Z(3)$ symmetry of the gluon part of the QCD Lagrangian and it has an origin from the recently developed effective models with Polyakov loops as dynamical fields. An extended NJL Lagrangian (PNJL model) contains unified properties of QCD related with $Z(3)$ and the chiral symmetries. The interaction of quarks with effective gluon fields in the PNJL model is included through covariant derivatives. Furthermore, due to the symmetries of the Lagrangian, the PNJL model belongs to the same universality class as that expected for QCD. Thus, such a model can be considered as a testing ground to study the phase structure and critical phenomena related with deconfinement and chiral phase transitions. This is particularly interesting since there are still limitations in applicability of lattice gauge theory (LGT) to describe QCD thermodynamics at large quark densities.

Recently, it was shown that the PNJL model formulated at finite temperature and finite quark chemical potential is capable to reproduce some thermodynamical observables calculated within the LGT. The properties of the equation of state, in-medium modification of the meson masses as well as the validity and applicability of the Taylor expansion coefficients used in LGT were recently addressed within the PNJL model. In Ref. 1, the model was extended to a system with finite isospin chemical potential and pion condensation was studied.

In this paper we will consider the properties of different fluctuations in the near vicinity to the phase transition. In particular, we focus on the behavior of the net–quark number fluctuations and the susceptibilities of the order parameters. The susceptibilities of the Polyakov loop and its conjugate as well as the chiral condensate will be introduced and analyzed. The relation between the chiral and deconfinement phase transitions will be quantified through susceptibilities. Our calculations are performed within the mean field approximation and in the extended PNJL model that includes a non local four–fermion interactions as a regulator of momentum loop integrals.

The paper is organized as follows: In Section 2 the PNJL model is formulated and its thermodynamics is introduced. In Section 3 the influence of the Polyakov loop degrees of freedom on different thermodynamical quantities is discussed. In Section 4 and Section 5 we introduce the concept of fluctuations of order parameters and focus on their properties. Our concluding remarks and discussions are given in Section 6.

2. TWO-FLAVOR NJL MODEL WITH THE POLYAKOV LOOP

Several methods have been proposed to couple the effective chiral models with the color gauge fields. In this extension of the Nambu–Jona-Lasinio (NJL) Lagrangian through its coupling with the Polyakov loop has been considered. In the PNJL...
model formulated for three colors \((N_c=3)\) and two flavors \((N_f=2)\) the quark fields are coupled to a temporal background gauge field. The PNJL Lagrangian extended to the non-local four-fermion interactions is given by

\[
\mathcal{L} = \bar{\psi}(i\not\!D - \hat{m})\psi + \bar{\psi}\gamma_0\psi - \mathcal{U}(\Phi[A], \bar{\Phi}[A]; T) + \frac{G_S}{2} \left[ (\bar{q}(x)q(x))^2 + (\bar{q}(x)\gamma_5\vec{\tau}q(x))^2 \right],
\]

where \(\hat{m} = \text{diag}(m_u, m_d)\) is the current quark mass whereas \(\hat{\mu} = \text{diag}(\mu_u, \mu_d)\) is the quark chemical potential and \(\vec{\tau}\) are Pauli matrices. We assume isospin symmetry and take \(m_u = m_d \equiv m_0\) and \(\mu_u = \mu_d \equiv \mu\).

The above Lagrangian is formulated with non-local interactions that are controlled by the form factor. Such an extension of the model is implemented to avoid the problems of the ultraviolet singularities that appear in the loop integrations. The form factor \(F(x)\) in the coordinate space for the non-local current-current interaction reads:

\[
q(x) = \int d^4y F(x-y)\psi(y) .
\]

A possible choice of this regulator in the momentum space is introduced in the following form \([18]\):

\[
f^2(p) = \frac{1}{1+ (p/\Lambda)^{2\alpha}},
\]

where \(f(p)\) is the Fourier transform of the form factor \(F(x)\) and \(p\) is the three–momentum.

The strength of the interaction among constituent quarks in \((2.1)\) is parameterized by the coupling constant \(G_S\) which carries dimensions of the length square. In the pure NJL sector, the model is controlled by four parameters: the coupling constant \(G_S\), the current quark mass \(m_0\) and the constants \(\alpha\) and \(\Lambda\) which characterize the range of the non-local interactions. In the vacuum the values of those parameters are determined for a given \(\alpha\) to reproduce the experimental values of the pion decay constant \(f_\pi = 92.4\) MeV and the pion mass \(m_\pi = 135\) MeV as well as the dynamical quark mass \(M_p=0\) is \(335\) MeV at vanishing momentum. For \(\alpha = 10\) the model parameters are found to be: \(\Lambda = 684.2\) MeV, \(G_S\Lambda^2 = 4.66, m_0 = 4.46\) MeV with the quark condensate \((\bar{\psi}\psi)^{1/3} = -256.2\) MeV \([19]\). The relevant parameters of the model that are used in our calculations are summarized in Table \[I\].

The interaction of an effective gluon field with quarks is implemented in the PNJL Lagrangian \((2.1)\) through a covariant derivative

\[
D_\mu = \partial_\mu - iA_\mu, \quad A_\mu = \delta_{\mu0}A^0,
\]

where the color SU(3) gauge coupling constant \(g\) is absorbed into the gauge field \(A_\mu = gA^\mu_\lambda \frac{\lambda^\alpha}{2}\) with \(\lambda^\alpha\) being the Gell-Mann matrices.

The effective potential \(\mathcal{U}\) of the gluon field in \((2.1)\) is expressed in terms of the traced Polyakov loop \(\Phi\) and its conjugate \(\bar{\Phi}\)

\[
\Phi = \frac{1}{N_c} \text{Tr}_c L, \quad \bar{\Phi} = \frac{1}{N_c} \text{Tr}_c L^\dagger,
\]

where \(L\) is a matrix in color space related to the gauge field by

\[
L(x) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau A_4(x, \tau) \right],
\]

with \(\mathcal{P}\) being the path (Euclidean time) ordering, and \(\beta = 1/T\) with \(A_4 = iA_0\).

In the heavy quark mass limit QCD has the \(Z(3)\) center symmetry which is spontaneously broken in the high-temperature phase. The thermal expectation value of the Polyakov loop \(\langle \Phi \rangle\) acts as an order parameter of the \(Z(3)\) symmetry. Consequently, \(\langle \Phi \rangle = 0\) at low temperature in the confined phase and \(\langle \Phi \rangle \neq 0\) in the high temperature limit corresponding to deconfined phase. For the SU(3) color gauge group the Polyakov loop matrix \(L\) satisfies \(LL^\dagger = 1, \det L = 1\) and can be diagonalized as

\[
L = \text{diag} \left( e^{i\sigma}, e^{i\sigma'}, e^{-i(\sigma + \sigma')} \right).
\]

Thus, it is clear that in general \(\Phi\) in Eq. (2.6) is not identical to \(\bar{\Phi}\).

The effective potential \(\mathcal{U}(\Phi, \bar{\Phi}; T)\) of the gluon field is expressed in terms of the Polyakov loops so as to preserve the \(Z(3)\) symmetry of the pure gauge theory \([20]\). We adopt an effective potential in the following form \([3]\) #1:

\[
\frac{\mathcal{U}(\Phi, \bar{\Phi}; T)}{T^4} = -\frac{b_2(T)}{2} \bar{\Phi}\Phi - \frac{b_3}{6}(\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4}(\Phi\bar{\Phi})^2,
\]

with

\[
b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3.
\]

The coefficients \(a_i\) and \(b_i\) are fixed to reproduce the equation of state that is obtained in the pure-gauge lattice QCD where at \(T_0 = 270\) MeV the system exhibits a first order deconfinement phase transition. The values of these parameters are listed in Table \[II\].

In the mean field approximation the Lagrangian \((2.1)\) is rewritten as

\[
\mathcal{L} = \bar{\psi}(i\not\!D - M(x) + \bar{\mu}\gamma_0)\psi - \frac{(m_0 - M(x))^2}{2G_S} - \mathcal{U}(\Phi, \Phi; T),
\]

#1 One might consider that dynamical quarks could change the form of \(\mathcal{U}\), e.g., an additional term proportional to \(\exp[\pm \mu/T]\) can contribute. However, such a term could be considered as double counting of quark degrees of freedom since in the PNJL model they are already included through the fermionic sector. The effects of dynamical quarks in the gluon potential are partly included through the equations of motion.
where in momentum space the dynamical quark mass $M(x)$ is related with the current quark mass $m_0$, the quark momentum distribution function $f(p)$ and the chiral condensate $\langle \bar{\psi} \psi \rangle$ as

$$M_p = m_0 + (M - m_0) f^2(p),$$

$$M = m_0 - G_S \langle \bar{\psi} \psi \rangle,$$

where $M$ denotes the dynamical quark mass at $p = 0$.

From the mean field Lagrangian \[2.10\] one obtains the thermodynamic potential in the following form

$$\Omega(M, \Phi, \bar{\Phi}; T, \mu) = U(\Phi, \bar{\Phi}; T) + \frac{(m_0 - M)^2}{2 G_S}$$

$$- 6N_f \int \frac{d^3p}{(2\pi)^3} [E_p - E_p(M_p = m_0)]$$

$$- 2N_f T \int \frac{d^3p}{(2\pi)^3} \left\{ \text{Tr}_c \ln \left[ 1 + L e^{-E^{(+)}/T} \right] \right\}$$

$$+ \text{Tr}_c \ln \left[ 1 + L^+ e^{-E^{(-)/T}} \right],$$

where we have introduced $E^{(\pm)} = E_p + \mu$ for the particle $(\pm)$ and anti-particle $(\mp)$ with $E_p = \sqrt{p^2 + M_p^2}$ being a quasiparticle energy. The third term in the thermodynamic potential \[2.12\] that corresponds to the vacuum contribution is modified by subtracting the term $E_p(M_p = m_0)$ under the momentum integral to avoid the ultraviolet singularities.

By taking the trace in a color space, the thermodynamic potential given in the Eq. \[2.12\] is further reduced into

$$\Omega(M, \Phi, \bar{\Phi}; T, \mu) = U(\Phi, \bar{\Phi}; T) + \frac{(m_0 - M)^2}{2 G_S}$$

$$- 6N_f \int \frac{d^3p}{(2\pi)^3} [E_p - E_p(M_p = m_0)]$$

$$- 2N_f T \int \frac{d^3p}{(2\pi)^3} \ln \left[ g^{(+)}(M, \Phi, \bar{\Phi}; T, \mu; p) \right]$$

$$+ \ln \left[ g^{(-)}(M, \Phi, \bar{\Phi}; T, \mu; p) \right],$$

with

$$g^{(+)}(M, \Phi, \bar{\Phi}; T, \mu; p)$$

$$= 1 + 3 \left( \Phi + \bar{\Phi} e^{-E^{(+)}/T} \right) e^{-E^{(+)/T}} + e^{-3E^{(+)}/T},$$

$$g^{(-)}(M, \Phi, \bar{\Phi}; T, \mu; p)$$

$$= 1 + 3 \left( \Phi + \Phi e^{-E^{(-)/T}} \right) e^{-E^{(-)/T}} + e^{-3E^{(-)/T}}.$$  \[2.14\]

In spite of the complex structure of the Polyakov loop the thermodynamic potential is real. This is because due to the symmetry related with a color rotation, the imaginary part of $\Omega$ vanishes after performing the functional integral [see Appendix A].

An interesting feature of the PNJL model described by the thermodynamic potential \[2.13\] is its qualitative behavior in the low temperature phase. In the limit of $\Phi, \bar{\Phi} \to 0$ expected at low temperature the contribution of one- and two-quark states to $g^{(\pm)}$ are suppressed and only three–quark states $\sim \exp(-3E^{(\pm)/T})$ survive. In this sense the PNJL model mimics the confinement of quark within "baryons", the leading states contributing to the thermodynamics in the limit of vanishing temperature. Such a behavior of the PNJL model is qualitatively similar to that expected in QCD thermodynamics. Thus, the PNJL model is better suited for describing the low temperature QCD phase than the standard NJL model.

In the mean field approximation the dynamical quark mass $M$ and the expectation values of the Polyakov loop $\Phi$ and $\bar{\Phi}$ are obtained from the stationary conditions

$$\frac{\partial \Omega}{\partial M} = \frac{\partial \Omega}{\partial \Phi} = \frac{\partial \Omega}{\partial \bar{\Phi}} = 0,$$  \[2.15\]

to minimize the thermodynamic potential with respect to $M$, $\Phi$ and $\bar{\Phi}$. 
The above conditions give the following set of the coupled gap equations:

\[
M = m_0 + 6G_S N_f \int \frac{d^3p}{(2\pi)^3} \frac{M_0 f^2(p)}{E_p} \left[ 1 - \frac{(\Phi + 2\Phi e^{-E^{(+)}/T} + e^{-2E^{(+)}/T})e^{-E^{(+)}/T}}{g^{(+)}(M, \Phi, \Phi; T, \mu; p)} \right. \\
\left. - \frac{(\Phi + 2\Phi e^{-E^{(-)}/T} + e^{-2E^{(-)}/T})e^{-E^{(-)}/T}}{g^{(-)}(M, \Phi, \Phi; T, \mu; p)} \right],
\]

\[
b_2(T)\Phi + b_3\Phi^2 - b_4(\Phi\Phi)\Phi = -\frac{12N_f}{T^3} \int \frac{d^3p}{(2\pi)^3} \left[ e^{-E^{(+)}/T} \frac{e^{-E^{(+)}/T}}{g^{(+)}(M, \Phi, \Phi; T, \mu; p)} + e^{-E^{(-)}/T} \frac{e^{-E^{(-)}/T}}{g^{(-)}(M, \Phi, \Phi; T, \mu; p)} \right],
\]

\[
b_2(T)\Phi + b_1\Phi^2 - b_4(\Phi\Phi)\Phi = -\frac{12N_f}{T^3} \int \frac{d^3p}{(2\pi)^3} \left[ e^{-E^{(+)}/T} \frac{e^{-E^{(+)}/T}}{g^{(+)}(M, \Phi, \Phi; T, \mu; p)} + e^{-E^{(-)}/T} \frac{e^{-E^{(-)}/T}}{g^{(-)}(M, \Phi, \Phi; T, \mu; p)} \right].
\]

One can see that the gap equation (2.17) agrees with (2.16) at \(\mu = 0\). We note that by taking \(\Phi = \bar{\Phi} = 1\) the Eq. (2.18) is reduced to that obtained in the standard NJL model without any coupling to the color SU(3) gauge field. Note that in the Polyakov loop sector of the gap equations (2.17) and (2.18) one should take the limit of both \(\Phi, \bar{\Phi} \to 1\) and \(T \to \infty\) for the Polyakov loop to decouple from the dynamics.

### 3. Thermodynamic Quantities at Finite Quark Chemical Potentials

The thermodynamics of the PNJL model under mean-field approximation is characterized by the potential \(\Omega(\Phi, \Phi, M)\) introduced in Eq. (2.13). Figs. 1 and 2 show the properties of the gluonic sector \(\mathcal{U}\) and that of the PNJL thermodynamic potential \(\Omega\) in the limit of high temperature where the chiral symmetry is restored and the dynamical quark mass \(M\) vanishes. Consequently, only the Polyakov loop and its conjugate are relevant classical fields that quantify the structure of \(\Omega\). For vanishing quark chemical potential \(\Phi\) and \(\bar{\Phi}\) are equivalent, thus thermodynamics is characterized by only one variable \(\Phi\) that for a given temperature is obtained as the solution of the gap equation (2.17). In Fig. 1 we show the \(\Phi\) dependence of \(\mathcal{U}\). This potential exhibits the expected structure in the Z(3) symmetry broken phase with a minimum at the finite value of \(\Phi\) that is always smaller than the unity. The influence of quarks on the PNJL model thermodynamics is shown in Fig. 2 where the \(\Phi\) dependence of an effective potential \(\Omega\) is calculated for the same value of the temperature \(T = 0.5\) GeV as in Fig. 1. Comparing Figs. 1 and 2 its is clear that due to quark interactions with an effective gluon field there is a shift of the position of the minimum of the potential to a larger value of \(\Phi > 1\). In addition, the strength of the potential is also modified showing that there is a stronger attractive interaction in a system when the quark field are coupled to gluons.

The effective quark mass \(M\) and the traced Polyakov loop \(\Phi\) are the order parameters of chiral and Z(3) symmetries respectively. Thus, the \(T\) and \(\mu\) dependence of the order parameters is used to identify the phase diagram of the model. Figs. 3 and 4 show \(\Phi\) and \(M\) at \(\mu = 0\) for different values of \(T\) in the chiral limit. The phase change of the model is clearly seen in these figures. At the temperature \(T \approx 242\) MeV the dynamical quark mass vanishes indicating the position of the chiral phase transition.

The Polyakov loop potential \(\mathcal{U}\) introduced in the PNJL Lagrangian preserves the invariance under the \(Z(3)\) sym-
symmetry. However, due to the interactions of the Polyakov loop with quarks this symmetry is explicitly broken in the model. Thus, the Polyakov loop is not anymore an order parameter of a global Z(3) symmetry. Consequently $\Phi$ is non-vanishing at the low temperature. Nevertheless, as seen in Fig. 3 the Polyakov loop exhibits an abrupt change in the narrow temperature range around $T \approx 0.2$ GeV. Such a behavior of $\Phi$ could be related with a remnant of "deconfinement" transition.

In Fig. 4 the dynamical quark mass $M$ obtained in the PNJL model is also compared with that calculated in the NJL model. It is clear that coupling of quarks to an effective gluon field shifts the chiral phase transition temperature to the higher value. This is a consequence of attractive interactions in the system. In the presence of the Polyakov loop there is stronger "binding" of the constituent quarks in the chirally broken phase.

The non-locality of the quark interactions introduced in the PNJL model also affects the position of the chiral phase transition temperature. Suppressing the form factor in the PNJL Lagrangian results in a lower transition temperature. With the parameters used in our actual calculations the reduction of $T_c$ is in the range of 15 MeV if in the thermal part of the PNJL model the momentum cutoff is not included.

Introducing a finite quark chemical potential is expected to modify the behavior of the order parameters. The $\mu$ dependence of $M$, $\Phi$ and $\bar{\Phi}$ are shown in Figs. 5 and 6. There is a clear shift of the chiral critical point to lower temperatures with increasing $\mu$ as is seen in the $T$-dependence of $M$ for different values of $\mu$. On the level of the Polyakov loop expectation value the essential influence of the finite chemical potential results in splitting between $\Phi$ and $\bar{\Phi}$. At a fixed temperature the $\Phi$ and $\bar{\Phi}$ show the opposite response to a change in $\mu$. As seen in Fig. 5 the Polyakov loop expectation value $\Phi$ is decreasing whereas $\bar{\Phi}$ is increasing with $\mu$. Such a behavior is expected due to the relation of the order parameter and its conjugate with the free energy of a quark and an antiquark respectively [21]. At the non-zero net–quark number density it is easy to screen a static antiquark to form virtual $q\bar{q}$ state whereas the heavy static quark can only be screened with diquark to form a colorless $qqq$ state.

At finite quark chemical potential the order of the chiral phase transition can be changed. This is illustrated in Fig. 7 where the phase diagram of the PNJL model is shown. At low temperatures and large $\mu$ the transition is first order and terminates at the tricritical point (TCP). With the actual value of the model parameters summarized in Table 1.11 the TCP appears at $(T_c=157, \mu_c=266)$ MeV. Above the TCP temperature the transition is second order. Such a structure of the phase diagram is also expected in QCD based on universality arguments [22].

To study the influence of the Polyakov loop on the thermodynamics, we consider observables that are related with the conservation of the net–quark number such as the quark number density $n_q(T, \mu)$ and its susceptibility $\chi_q(T, \mu)$. Both these observables are obtained as derivatives of $\Omega$ with respect to $\mu$. The quark number density is obtained from

$$n_q = -\frac{\partial \Omega}{\partial \mu}, \quad (3.1)$$

With the thermodynamic potential of the PNJL model [24,25] one finds

$$n_q = 6N_f \int \frac{d^3p}{(2\pi)^3} \left[ \frac{e^{-E^+(p)/T}}{g^+(p)} \times \left( \Phi + 2\bar{\Phi} e^{-E^+(p)/T} + e^{-2E^+(p)/T} \right) - \frac{e^{-E^-(p)/T}}{g^-(p)} \left( \bar{\Phi} + 2\Phi e^{-E^-(p)/T} + e^{-2E^-(p)/T} \right) \right]. \quad (3.2)$$

The temperature dependence of the net–quark number density calculated from Eq. 3.2 for different values of $\mu$ is shown in Figs. 8 and 9. The PNJL model results are also compared with the prediction of the NJL model. It is clear from this figure that there is a substantial change
in the behavior of \( n_q \) when the interaction of quarks with the Polyakov loop is included.

In the region above the chiral phase transition the standard NJL model shows a strong decrease of \( n_q/T^3 \) with an increasing temperature. Such a behavior is related with the momentum cutoff effects that reduce the phase space of quarks at large momenta [23]. There is also an increase of the quark density when approaching \( T_c \) due to the decreasing effective quark mass towards the chiral transition. The PNJL model shows different properties of the quark density: First, there is no suppression of density in the high temperature since the momentum cutoff
is replaced by the non-local dynamics. Second, there is a suppression of the density below $T_c$ due to the presence of interactions with the gluon field and the contribution of three-quark "bound" states.

The quark number susceptibility measures the response of the quark number density to any changes in the quark chemical potential. This observable is of particular interest to verify the existence and the position of the tricritical point in the phase diagram. This is due to an expected divergence of $\chi_q$ at TCP related with the coupling of the quark density correlator to the scalar-isoscalar sigma field [24, 25, 26]. The net-quark number susceptibility is defined by

$$\chi_q = \frac{\partial \eta_q}{\partial \mu}.$$ (3.3)

In the PNJL model the dynamical quark mass $M$ and the Polyakov loops $\Phi, \bar{\Phi}$ implicitly depend on $\mu$. Thus, besides an explicit $\mu$-dependent contribution through Eq. (3.3) there are also terms proportional to $\mu$-derivatives of the effective condensates. In the PNJL model one obtains

$$\chi_q = \chi_q^{(0)} + T^2 A_M^{(\mu)} \frac{\partial M}{\partial \mu} + T^3 A_{\Phi}^{(\mu)} \frac{\partial \Phi}{\partial \mu} + T^3 A_{\bar{\Phi}}^{(\mu)} \frac{\partial \bar{\Phi}}{\partial \mu}$$

$$\equiv \chi_q^{(0)} + \chi_q^{(M)} + \chi_q^{(\Phi)} + \chi_q^{(\bar{\Phi})},$$ (3.4)

where $\chi_q^{(0)}$ corresponds to the 0-th order contribution defined by

$$\chi_q^{(0)} = \frac{6N_f}{T} \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{e^{-E^{(+)}(p)/T}}{(g^{(+)}(p))^2} \left\{ \Phi + 4\Phi e^{-E^{(+)}(p)/T} \right. 

+ 3 \left( 1 + \Phi \bar{\Phi} \right) e^{-2E^{(+)}(p)/T} + 4\Phi e^{-3E^{(+)}(p)/T} 

+ \bar{\Phi} e^{-4E^{(+)}(p)/T} \right] + \left( \bar{\Phi}, \Phi; -\mu \right) \right].$$ (3.5)

The functions $A^{(\mu)}(M, \Phi, \bar{\Phi}; T, \mu)$ and the $\mu$-derivatives of the condensates are introduced in Appendix [8].

The quark number susceptibility at vanishing and at finite $\mu$ and for different temperatures is shown in Figs. [10] and [11] for the PNJL and for the NJL models. At high temperature $\chi_q$ approaches the ideal gas limit, $\chi_q/T^2 \simeq 2$. In the chiral limit, the quark number susceptibility has a discontinuity at the chiral phase transition at finite $\mu$ as found in the Landau-Ginzburg theory [22].

The influence of the quark-gluon interactions on the quark number susceptibility shown in Fig. [10] is similar as discussed in the context of the net-quark density. There is a reduction of the quark fluctuations below $T_c$ in the PNJL model relative to the fluctuations obtained in the NJL model. The PNJL model results on the $T$ and $\mu$ dependence of $\chi_q$ and $\eta_q$ coincides with that obtained recently in terms of LGT calculations. Thus, the PNJL model provides a better description of the QCD thermodynamics near the phase transition than the NJL studies.

At the TCP the quark number susceptibility diverges as shown in the left panel of Fig. [12]. The divergence comes from the non-trivial contributions in Eq. (3.4). The right panel of Fig. [12] clearly shows that the singular behavior of $\chi_q$ is caused by the divergence of $\chi_q^{(M)}$. Both $\chi_q^{(\Phi)}$ and $\chi_q^{(\bar{\Phi})}$ have sharp peak structures at $T_c$, however their orders of magnitude are much smaller than $\chi_q^{(M)}$. It should be noted that the chiral susceptibility $\chi_{mm}$ diverges along the second order transition line and at the TCP as

$$\chi_{mm} \sim \frac{1}{a(T, \mu) + 3M^2b(T, \mu)},$$ (3.6)

where $a(T, \mu)$ and $b(T, \mu)$ are coefficients of the thermodynamic potential expanded around the phase transition point,

$$\Omega(T, \mu) \sim \Omega_0(T, \mu) + \frac{1}{2} a(T, \mu) M^2 + \frac{1}{4} b(T, \mu) M^4.$$ (3.7)

The TCP is characterized by a flatness of the potential, i.e. $b(T, \mu) = 0$ [see e.g., Ref. [24]]. On the other hand, $b(T, \mu)$ is finite for the second order phase transition whereas $a(T, \mu)$ vanishes. Taking into account that

![Figure 10: The quark number susceptibility $\chi_q/T^2$ in the chiral limit as a function of temperature $T$. The dash-dotted line denotes the result obtained in the NJL model. The lines correspond to the results at $\mu = 0$.](image1)

![Figure 11: The quark number susceptibility $\chi_q/T^2$ in the chiral limit as a function of temperature $T$. The lines correspond to the results in the PNJL model at $\mu = 0, 100, 150, 200$ MeV from below.](image2)
The fluctuations introduced in the Eq. (4.2) can be now

\[ \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0} = \langle \phi(x) \phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle . \]

(4.2)

From the generating functional \( W[J] \) an effective action is introduced through the Legendre transformation

\[ \Gamma[\phi] = -W[J] - \int d^4x J(x) \phi(x) . \]

(4.3)

The fluctuations introduced in the Eq. (4.2) can be now expressed as

\[ \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0} = \left( \frac{\delta^2 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} \right)^{-1} . \]

(4.4)

It is clear from the above equation that in the absence of the source terms in the effective potential the susceptibilities can be calculated from derivatives of the potential with respect to classical fields related with the order parameters.

In the case of PNJL model there are three different classical fields \( \hat{\phi} = (M, \Phi, \hat{\Phi}) \) that correspond to the order parameters; the effective quark mass, the Polyakov loop and its complex conjugate. Consequently, to define susceptibilities in this case Eq. (4.4) has to be general-
ized. We start from the chain rules,
\[ \delta_{ij} = \frac{\delta \phi_i}{\delta \phi_j} = \frac{\delta}{\delta \phi_j} \delta J_k = \frac{\delta J_k}{\delta \phi_j} \frac{\delta^2 W}{\delta \phi_k \delta J_k} = \frac{\delta^2 \Gamma}{\delta \phi_k \delta J_k \delta J_k}, \]
where \( \vec{J} = (J_M, J_\phi, J_\Phi) \) is the vector of the source fields related with the order parameters.

It is clear from Eq. (4.5) that different susceptibilities of the order parameters should be introduced as an inverse matrix of the second derivatives of the effective action with respect to classical field expectation values. Following the Ref. [3] we introduce an inverse dimensionless matrix
\[ \hat{C} = \left( \begin{array}{ccc} C_{mm} & C_{ml} & C_{ml} \\ C_{ml} & C_{ll} & C_{ll} \\ C_{ml} & C_{ll} & C_{ll} \end{array} \right), \]
where each component is given by
\[ C_{mm} = \frac{1}{T^2} \frac{\partial^2 \Omega}{\partial M^2}, \quad C_{ll} = \frac{1}{T^2} \frac{\partial^2 \Omega}{\partial \Phi^2}, \]
\[ C_{ml} = \frac{1}{T^2} \frac{\partial^2 \Omega}{\partial M \partial \Phi}, \quad C_{ll} = \frac{1}{T^2} \frac{\partial^2 \Omega}{\partial \Phi^2}. \]

Through Eq. (4.6) a set of susceptibilities is defined by
\[ \chi_{ij} = \left( \hat{C}^{-1} \right)_{ij}, \quad i, j = \{ m, l, \bar{l} \}. \]

The \( \chi_{mm} \) is the chiral susceptibility for \( \langle \tilde{\psi} \psi \rangle \) and \( \chi_{ll} \) and \( \chi_{\bar{l}l} \) are the susceptibilities for \( \Phi \) and \( \Phi \) order parameters. The off-diagonal terms correspond to the mixed susceptibilities.

In the pure gluon sector the susceptibilities are related with the fluctuations of \( \Phi \) and \( \Phi \) fields. Under the \( Z(3) \) transformation, invariant objects are: \( \tilde{\Phi} \Phi, \Phi^3, \Phi^3 \) and their combinations. The off-diagonal component \( \chi_{ll} \) is a susceptibility for the \( Z(3) \) invariant combination \( |\Phi| = \sqrt{\Phi \Phi} \), while the diagonal pieces \( \chi_{l\bar{l}, l\bar{l}} \) are those for the \( Z(3) \) non-invariant terms \( \Phi, \Phi \).

The chiral susceptibility is very narrow near the chiral phase transition temperature and diverges at the chiral critical point \( T_{ch} \). The Polyakov loop susceptibility shows different behavior: it is broader near \( T_{ch} \) and is finite for all values of temperature. This is due to an explicit breaking of the \( Z(3) \) symmetry related with the presence of the quark fields in the PNJL Lagrangian. However, \( \chi_{ll} \) still shows a peak structure that can be considered as an indication of the remnant of deconfinement transition in this model.#2 The peak

\[ \chi_{ll} = \frac{1}{2} (\chi_{ll} + \chi_{l\bar{l}l}) |_{\Phi=\Phi}. \]

which corresponds to fluctuations of the real part of the Polyakov loop. This observable was also used in LGT calculations to identify the position of deconfinement transition in the QCD medium.

Due to the presence of dynamical quarks in the PNJL model the \( Z(3) \) symmetry is explicitly broken and the Polyakov loop is not any more a definite order parameter. However, the \( Z(3) \) symmetry is a guiding principle in constructing the present model and in fact the expectation values of \( \Phi \) and \( \Phi \) were shown in Figs. [3] and [5] to be suppressed in the low temperature phase. Thus, the off-diagonal susceptibilities \( \chi_{ml} \) and \( \chi_{l\bar{l}l} \) are also expected to be suppressed at low temperatures since they are not invariant under the \( Z(3) \) transformation.

5. SUSCEPTIBILITIES AND THE PHASE TRANSITION IN THE PNJL MODEL

The fluctuations are sensitive observables to the phase transitions. Thus, a behavior of the fluctuations is a promising probe of the phase structure in a system. The phase transition trajectories can be identified from the response of the fluctuations to the changes in the thermal parameters. In this section we focus on the phase structure of the PNJL model to study associated susceptibilities with the order parameters. Their definitions were introduced in section [4].

5.1. Susceptibilities at vanishing quark chemical potential

Fig. [13] shows the chiral \( \chi_{mm} \) and Polyakov loop \( \chi_{l\bar{l}l} \) susceptibilities calculated at \( \mu = 0 \) in the PNJL model in the chiral limit. The chiral susceptibility is very narrow near the chiral phase transition temperature and diverges at the chiral critical point \( T_{ch} \). The Polyakov loop susceptibility shows different behavior: it is broader near \( T_{ch} \) and is finite for all values of temperature. This is due to an explicit breaking of the \( Z(3) \) symmetry related with the presence of the quark fields in the PNJL Lagrangian. However, \( \chi_{ll} \) still shows a peak structure that can be considered as an indication of the remnant of deconfinement transition in this model.#2 The peak

\[ \chi_{ll} = \frac{1}{2} (\chi_{ll} + \chi_{l\bar{l}l}) |_{\Phi=\Phi}. \]

The definition of deconfinement transition is usually applied within LGT studies of QCD thermodynamics.
position of $\chi_{\bar{u}u}$ appears at $T \simeq 217$ MeV and is lower than $T_{ch} \simeq 242$ MeV. Thus, deconfinement phase transition occurs at lower temperature than that of the chiral phase transition. The separation between the two peaks is roughly 20 MeV in the local model and increases to 40 MeV in the non-local model.

There is also an interesting feature of the $\chi_{\bar{u}u}$ susceptibility, seen in Fig. 13 that is directly connected with the interference with the chiral phase transition. Besides a broad peak structure in $\chi_{\bar{u}u}$ there is also an abrupt and narrow change of this susceptibility at $T = T_{ch}$. Such a structure confirms that in the PNJL model and with the parameters used in the actual calculations deconfinement transition being identified as a broad peak position in $\chi_{\bar{u}u}$ sets in earlier than chiral phase transition in the net-quark neutral system.

5.2. Susceptibilities at a finite quark chemical potential

Discussing the properties of the chiral order parameter it was clear that at a finite chemical potential there is a shift of the position of the chiral phase transition towards lower temperature. This property of the PNJL model is consistent with the recent LGT findings. In addition due to broadening of the Polyakov loop expectation value with increasing $\mu$ one expects that also deconfinement transition will be modified when increasing $\mu$. Clearly, the change of the position of deconfinement and chiral transition can be quantified by considering a behavior of susceptibilities of the order parameters that were introduced in the previous section.

Figs. 13 and 15 show the change of the $T$-dependence of different susceptibilities with the chemical potential. With increasing $\mu$ the peak position of chiral and Polyakov loop susceptibilities are clearly shifted towards lower $T$ and approach each other as seen in Fig. 14. At $\mu_0 \simeq 185$ MeV both peaks coincide that indicates that chiral and deconfinement transitions appear at the same temperature. For the larger $\mu > \mu_0$ the Polyakov loop susceptibility $\chi_{\bar{u}u}$ has a sharp peak at $T_{ch}$ and a broad bump above $T_{ch}$. Such a behavior was also observed for a $\chi_{\bar{u}u}$ susceptibility in the PNJL-like model in Ref. 3. The peak at $T_{ch}$ in $\chi_{\bar{u}u}$ is clearly due to an interference with the chiral phase transition. On the other hand, the bump structure may correspond to the pseudo-critical point related with the deconfinement transition. This structure is also seen in $\chi_{\bar{u}u}$ at the tricritical point that is located at $(T_c = 157, \mu_c = 266)$ MeV. Furthermore, even in the parameter range where the first order chiral transition sets in such a structure of $\chi_{\bar{u}u}$ holds as seen in Fig. 14.

In the previous section we have introduced an average susceptibility $\bar{\chi}_{\bar{u}u}$ that corresponds to correlations of the real part of the Polyakov loop. In Fig. 15 the temperature variation of $\bar{\chi}_{\bar{u}u}$ for different values of the quark chemical potential is shown. It is clear from this figure that a behavior of $\bar{\chi}_{\bar{u}u}$ is similar to that of $\chi_{\bar{u}u}$ having a similar bump structure at some temperature. However, an exact determination of the peak position of $\bar{\chi}_{\bar{u}u}$ shows that it does not coincide with $T_{dec}$ determined from $\chi_{\bar{u}u}$. Thus, in the Polyakov loop sector the determination of the position of deconfinement through susceptibilities is
The recent LGT results both at the vanishing and at finite quark chemical potential show that deconfinement and chiral symmetry restoration appears in QCD along the similar critical line [28]. In general it is possible to choose the PNJL model parameters such that the critical temperatures of chiral and deconfinement transition coincide at $\mu = 0$. The corresponding result of the phase diagram is shown in Fig. [17] and the parameters used in the actual calculations are summarized in Table [I]. Clearly the boundary line related with the chiral symmetry restoration do not coincide. There is only one common point in the phase diagram where both transitions appear simultaneously. Such a structure could be expected from the properties of susceptibilities for different values of $\mu$.

The peak positions of $\chi_{nn}$ and $\chi_{ll}$ susceptibilities appear at the transition temperature. Thus, from the behavior of these susceptibilities one can determine the phase boundaries in the $(T, \mu)$–plane. However, the position of the phase boundaries is strongly dependent on the model parameters. Fig. [16] shows the phase diagram for the PNJL model obtained with the parameters from Table [III].

![Figure 16](image1)

**FIG. 16:** The phase diagram of the PNJL model in the chiral limit. The solid (dashed) line denotes the chiral (deconfinement) phase transition respectively. The TCP (bold-point) is located at $(T_c = 157, \mu_c = 266)$ MeV. The parameter set (a) given in Table [III] was used in the calculations.

![Figure 17](image2)

**FIG. 17:** The phase diagram of the PNJL model in the chiral limit. The solid (dashed) line denotes the chiral (deconfinement) phase transition respectively. The TCP (bold-point) is located at $(T_c = 15, \mu_c = 218)$ MeV. The parameter set (b) given in Table [III] was used in the calculations.

The temperature derivatives of effective condensates, thus they are independent on $\mu$. It is conceivable that the effect of dynamical quarks can modify the coefficients of this potential resulting in their $\mu$–dependence. Consequently, the slope of $T_{dec}$ with changing $\mu$ could be steeper [3].

To quantify the deconfinement and chiral transition temperatures it was recently suggested in Ref. [3] to consider the temperature derivatives of effective condensates. The temperature derivatives of the condensates are expressed as combinations of the susceptibilities and some $T$–dependent functions as shown in Eq. [15.3]. Thus, generally the pseudo critical temperature related with decon-

---

#3 Such a modification was studied in Ref. [32] where explicit $\mu$– and $N_f$– dependences of $T_0$ are extracted from the running coupling constant $\alpha_s$ using the argument based on the renormalization group.
finement does not agree with that obtained from the peak position of the corresponding susceptibilities. However, if the order parameters are well defined the susceptibilities of the order parameters show a narrow structure and the peaks of the derivatives could coincide with those of the susceptibilities. Fig. 18 shows the derivatives of order parameters at $\mu = 0$ for different $T_0$. In the previous studies it was shown that the peak positions found from derivatives of the Polyakov loop and the chiral order parameter coincide. Indeed, in the PNJL model with local interactions and choosing $T_0 = 270$ MeV in the effective gluon potential such a coincidence is valid and illustrated in Fig. 18. However, a small change in $T_0$ results in the splitting of the peak positions is also seen in this figure. The parameters in the Polyakov loop sector were fixed from the lattice data in the heavy quark mass limit. Hence, the corresponding transition temperature $T_0 = 270$ MeV was used. In the presence of dynamical quarks the transition temperature drops, thus it is not excluded that the value of $T_0$ may decrease. Choosing $T_0 = 210$ MeV in the local PNJL model results in splitting of peaks of the corresponding derivatives. Therefore, the location of the pseudo-critical points is strongly dependent on the parameters. In the non-local PNJL model considered in this work, there is no coincidence between such peak positions and a peak of $\partial T/\partial T$ even when changing $T_0$. This properties of the actual PNJL model are shown in Fig. 18. A similar structure is also seen in the chiral and Polyakov loop susceptibilities. It is interesting to note that for chiral susceptibilities in the non-local PNJL model the transition temperature obtained from the derivatives of the order parameters and from the corresponding susceptibility is almost identical. This shows that to identify the position of the chiral transition in such a model one can use both methods alternatively. However, the above is not valid for the Polyakov loop correlations where the difference is $\sim 5$ MeV at $\mu = 0$.

Figs. 19 and 20 show further diagonal and off-diagonal susceptibilities of $\hat{\chi}$ at $\mu = 0$ from Eq. (4.3). At $\mu = 0$ the diagonal components $\chi_{uu}$ and $\chi_{ll}$ coincide. These are the susceptibilities for $\Phi$ and $\bar{\Phi}$ that are not invariant under the $Z(3)$ transformation. The $\chi_{uu}$ and $\chi_{ll}$ are suppressed in the low-temperature phase where the $Z(3)$ symmetry is restored. Around $T \sim T_{dec}$ the $\chi_{uu, ll}$ show a rapid drop that is associated with the crossover behavior of $\Phi$ and $\bar{\Phi}$ expectation values. In the high-temperature phase $\chi_{ll, uu}$ are not necessarily suppressed since the $Z(3)$ symmetry is explicitly broken there. The off-diagonal susceptibilities $\chi_{rl, lr}$ are also suppressed in the low-temperature phase and show a clear peak at $T_{bl}$ due to an interference with the chiral phase transition. The off diagonal susceptibilities vanish above $T_{bl}$, because of the vanishing dynamical quark mass in the chirally restored phase.

5.3. Effective potential constraints

The results for the $\chi_{ll}$ susceptibility shown in the Fig. 19 indicate that $\chi_{ll}$ is negative in a broad temperature range. Such a behavior is in contrast with the recent lattice results where $\chi_{ll}$ is always positive in the presence of dynamical quarks [28]. Considering the relation of the $\chi_{ll}$ susceptibility with the free energy of the static quarks the negative structure of $\chi_{ll}$ is not expected either. A possible origin of such behavior could be related with the particular approximation of the effective Polyakov loop potential used in the Eq. (2.8).

The $\chi_{ll}$ susceptibility can be related to the screening masses $m_r$ and $m_i$ of the real and imaginary part of the Polyakov loop correlation functions [33]:

$$\chi_{ll} = \frac{1}{m_r^2} - \frac{1}{m_i^2}. \quad (5.1)$$

In Ref. [33] the ratio

$$\frac{m_i}{m_r} \simeq 3 \quad (5.2)$$

was found in the vicinity to the critical temperature. This result is qualitatively consistent with the perturbative calculation [34]. From the Eqs. (5.1) and (5.2) it is clear that the resulting $\chi_{ll}$ should be positive at $T = T_c$.

In Ref. [33] the screening masses $m_r$ and $m_i$ were calculated with an effective potential that has the same polynomial form as used in the Eq. (2.8) but has different values of the numerical parameters. The effective Polyakov loop potential (2.8) yields

$$m_i < m_r. \quad (5.3)$$

A comparison with Eq. (5.1) show that this inequality implies a negative value of $\chi_{ll}$ at $T_c$.

From the above discussion it is clear that behavior of $\chi_{ll}$ depends crucially on the parameters used in the Polyakov loop potential. The requirement of the $Z(3)$ symmetry of the potential and the choice of the parameters such that to reproduce the lattice results for the equation of state are still not sufficient to provide a correct behavior of the susceptibilities.

Recently an improved effective potential has been suggested with temperature-dependent coefficients [13]

$$\frac{U(\Phi, \bar{\Phi}; T)}{T^4} = -a(T) \frac{\bar{\Phi}\Phi}{2} + b(T) \ln \left(1 - 6\Phi\bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi\bar{\Phi})^2\right). \quad (5.4)$$

where

$$a(T) = a_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2, \quad b(T) = b_3 \left(\frac{T_0}{T}\right)^3, \quad (5.5)$$

The higher order polynomial in $\Phi$ and $\bar{\Phi}$ used in (2.8) is replaced by the logarithmic term, which accounts for the
FIG. 18: Derivatives of the dynamical quark mass $\partial M/\partial T$ (dashed) and the Polyakov loop $\partial \Phi/\partial T$ (solid) in the chiral limit as a function of temperature $T$ for $\mu = 0$.

FIG. 19: The diagonal $\chi_{ii} = \chi_{ii}$ susceptibility in the chiral limit as a function of temperature $T$ for $\mu = 0$.

FIG. 20: The off-diagonal $\chi_{\alpha\ell} = \chi_{\alpha\ell}$ susceptibility in the chiral limit as a function of temperature $T$ for $\mu = 0$.

Haar measure in the group integral [3]. The parameters in [5,4] were fixed to reproduce the lattice results for pure gauge QCD thermodynamics and for the behavior of the Polyakov loop. These parameters are summarized in the Table IV.

Fig. 21 shows the $\chi_{ii}$ susceptibility calculated with the potential from Eq. (5.4). It is clear from this figure that the improved potential yields positive values for the Polyakov loop susceptibilities. In addition the peak positions of the $\chi_{ii}$ and $\chi_{ii}$ susceptibilities almost coincide if the interactions of $\Phi$ and $\bar{\Phi}$ are parameterized through Eq. (5.4). The phase diagram calculated with an improved potential is shown in Fig. 22 to be similar to that obtained in Fig. 16 with the previous choice of the Polyakov loop interactions.

6. SUMMARY AND CONCLUSIONS

We have explored the thermodynamic properties and the critical behavior in a system that exhibits an invariance under $Z(3)$ and chiral symmetry transformations. As a model we have used an extension of the NJL model for three colored and two flavored quarks that are coupled to effective gluon fields described by the Polyakov loops. In this model (PNJL model) a non-local interaction instead of the point-like four-fermion couplings was introduced. The PNJL model carries two essential features of QCD: a spontaneous chiral symmetry breaking
and a “confinement”like property. In addition, due to symmetries of the Lagrangian, the PNJL model belongs to the same universality class as that expected for QCD. Thus, such a model can be considered as a testing ground for the critical phenomena related with the breaking of global $Z(3)$ and chiral symmetries.

Using the mean field dynamics we have discussed within the PNJL model the phase diagram and the order of the phase transition for different values of the parameters. The properties of thermodynamic quantities related with the quark degrees of freedom like quark number density and susceptibility were analyzed in the vicinity of the chiral and deconfinement transitions.

Introducing the concept of susceptibilities related with the three different order parameters in this model we have analyzed their properties and sensitivity to phase transitions. We have shown that there are as many as nine susceptibilities that can be used to identify the phase structure of the model. In particular, considering the quark-antiquark and chiral density-density correlations we have discussed the interplay between chiral symmetry restoration and deconfinement. We have argued that in the actual formulation of the PNJL model there is no coincidence between deconfinement and chiral symmetry restoration.

We have found that within the mean field approximation and with the present form of an effective gluon potential the correlations of the Polyakov loops in the quark–quark channel show unexpected behavior being negative in the broad parameter range. Such a behavior was traced back to the parametrization of the Polyakov loop potential. We have argued that the $Z(N)$-invariance of this potential and its applicability to reproduce the lattice thermodynamics in the pure gluonic sector is still not sufficient to provide correct description of the Polyakov loop fluctuations in the presence of quarks in a medium. We note, however, that the improved potential of ref. 12 yields a positive $\chi_{ll}$, in qualitative agreement with the LGT results.

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**APPENDIX A: FUNCTIONAL INTEGRAL AND PARTITION FUNCTION OF THE MODEL**

The Polyakov loop $L$ is a complex $3 \times 3$ matrix and can be diagonalized as in Eq. (A.7). Thus, the thermodynamic potential (2.19) is a functional of complex variables. However, as a physical observable the imaginary part of $\Omega$ should vanish. In this Appendix we show that the above holds indeed. We follow the method that has been used in the context of a strong coupling QCD and the matrix model 10, 32.

We start with the partition function,

$$Z = \int D\psi D\bar{\psi} DL(\varphi, \varphi') e^{S[\psi, \bar{\psi}, L]}, \quad (A.1)$$

with the action $S$ being divided into three pieces:

$$S[\psi, \bar{\psi}, L] = \int_{0}^{\beta} d\tau \int d^{3}x L[\psi, \bar{\psi}, L] = S_{g}[L] + S_{q}[\psi, \bar{\psi}, L] + S_{int}[\psi, \bar{\psi}]. \quad (A.2)$$

The Polyakov loop effective potential $U$ is included in $S_{g}$. It is quite transparent that the pure gluon part is real. Thus, we focus only on the quark part $S_{q}$. Performing the functional integral over fermion fields one gets,

$$Z_{q} = \int DL(\varphi, \varphi') \text{Det}D(\varphi, \varphi'), \quad (A.3)$$
where $D$ denotes the Dirac operator
\begin{equation}
D(\varphi, \varphi') = \slashed{p} - M - \gamma_0 (\mu + i A_4). \tag{A.4}
\end{equation}

Summing up the Matsubara frequencies, $p_0 = i \omega_n = i(2n + 1)\pi T$ the partition function is obtained as
\begin{equation}
\ln Z_q = \int DL(\varphi, \varphi') \sum_n \int \frac{d^3 p}{(2\pi)^3} \text{Tr} \ln D(\varphi, \varphi') = 2N_f \int DL(\varphi, \varphi') \int \frac{d^3 p}{(2\pi)^3} \left[ \text{Tr}_c \ln \left( 1 + L(\varphi, \varphi') e^{-\beta E^{(+)}} \right) + \text{Tr}_c \ln \left( 1 + L^I(\varphi, \varphi') e^{-\beta E^{(-)}} \right) \right], \tag{A.5}
\end{equation}

taking the trace over color, flavor and Dirac variables in the above equation the imaginary part $I$ of $\ln Z_q$ is proportional to
\begin{equation}
I(M, \Phi, \bar{\Phi}; T, \mu) = \int DL(\varphi, \varphi') \int \frac{d^3 p}{(2\pi)^3} \left\{ \tan^{-1} \left[ \frac{3\text{Im}\Phi \left( 1 - e^{-E^{(+)}/T} \right) e^{-E^{(+)}/T}}{1 + 3\text{Re}\Phi \left( 1 + e^{-E^{(+)}/T} \right) e^{-E^{(+)}/T} + e^{-3E^{(+)}/T}} \right] \right. \\
+ \tan^{-1} \left[ \frac{-3\text{Im}\Phi \left( 1 - e^{-E^{(-)/T}} \right) e^{-E^{(-)/T}}}{1 + 3\text{Re}\Phi \left( 1 + e^{-E^{(-)/T}} \right) e^{-E^{(-)/T}} + e^{-3E^{(-)/T}}} \right] \right\}, \tag{A.6}
\end{equation}

where $\varphi$ and $\varphi'$ dependence were suppressed and $\text{Re}\Phi = \text{Re}\bar{\Phi}$ and $\text{Im}\Phi = -\text{Im}\bar{\Phi}$ were used. Now let us replace the variables $\varphi, \varphi'$ with $\varphi \to -\varphi$ and $\varphi' \to -\varphi'$. The SU(3) Haar measure $DL(\varphi, \varphi')$ is unchanged under these replacements while $\text{Im}\Phi$ changes its sign. Therefore, the first and second terms are separately odd under the change of group variables and vanish after the integration.

The thermal expectation values of complex $\Phi$ and $\bar{\Phi}$ are evaluated as
\begin{equation}
\langle \Phi \rangle = \frac{1}{Z} \int DL(\varphi, \varphi') e^{S_g + S_{\text{int}}} (\text{Re}Z_q \cdot \text{Re}\Phi - \text{Im}Z_q \cdot \text{Im}\Phi),
\end{equation}
\begin{equation}
\langle \bar{\Phi} \rangle = \frac{1}{Z} \int DL(\varphi, \varphi') e^{S_g + S_{\text{int}}} (\text{Re}Z_q \cdot \text{Re}\Phi + \text{Im}Z_q \cdot \text{Im}\Phi), \tag{A.7}
\end{equation}

with
\begin{equation}
Z = \int DL(\varphi, \varphi') e^{S_g + S_{\text{int}}} (\text{Re}Z_q + i \text{Im}Z_q). \tag{A.8}
\end{equation}

It is clear from Eq. (A.8) that the imaginary part of the potential vanishes at $\mu = 0$. Thus, the difference between $\langle \Phi \rangle$ and $\langle \bar{\Phi} \rangle$ comes only from the non-vanishing $\text{Im}Z_q$ at finite $\mu$. This can be also seen in the matrix model for color SU(3) symmetry [36].

**APPENDIX B: DERIVATIVES OF EFFECTIVE CONDENSATES**

In this appendix we summarize the derivatives of effective condensates with respect to $\mu$ and $T$. Taking the $\mu$-derivatives in the coupled gap equations (2.10)-(2.13) one gets
\begin{equation}
\frac{\partial M}{\partial \mu} = \frac{T}{\Lambda} \left( A_{\mu M}^{(\mu)} \chi_{mm} + \frac{T}{\Lambda} A_{\Phi M}^{(\mu)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi l}^{(\mu)} \chi_{ml} \right), \\
\frac{\partial \Phi}{\partial \mu} = \frac{T}{\Lambda^2} \left( A_{\mu M}^{(\mu)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi M}^{(\mu)} \chi_{ll} + \frac{T}{\Lambda} A_{\Phi l}^{(\mu)} \chi_{ll} \right), \\
\frac{\partial \bar{\Phi}}{\partial \mu} = \frac{T}{\Lambda^2} \left( A_{\mu M}^{(\mu)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi M}^{(\mu)} \chi_{ll} + \frac{T}{\Lambda} A_{\Phi l}^{(\mu)} \chi_{ll} \right), \tag{B.1}
\end{equation}
where \( \chi_{ij} \) are defined in Section 4 and the functions \( A^{(\mu)} \) are introduced as

\[
A_M^{(\mu)} = -\frac{6N_f}{T^3} \int \frac{d^3p}{(2\pi)^3} \frac{M_p f^2(p)}{E_p} \left[ e^{-E_+(T)/T} \left( \Phi + 4\Phi \Phi e^{-E_+(T)/T} + 3(1 + \Phi \Phi) e^{-2E_+(T)/T} \right) + 4\Phi e^{-3E_+(T)/T} + \Phi e^{-4E_+(T)/T} \right] - \left( \Phi, \Phi; -\mu \right)
\]

\[
A_{\Phi}^{(\mu)} = \frac{6N_f}{T^3} \int \frac{d^3p}{(2\pi)^3} \left[ e^{-E_+(T)/T} \left( 1 - 3\Phi e^{-2E_+(T)/T} - 2e^{-3E_+(T)/T} \right) \right] - \frac{e^{-2E_-(T)/T}}{(g^-(T))^2} \left[ 2 + 3\Phi e^{-E_-(T)/T} - e^{-3E_-(T)/T} \right],
\]

\[
A_{\Phi}^{(\mu)} = A_{\Phi}^{(\mu)} (\Phi, \Phi; -\mu).
\]

The required temperature derivatives of order parameters are directly obtained from the gap equations as

\[
\frac{\partial M}{\partial T} = \frac{T}{\Lambda} \left( A_M^{(T)} \chi_{mm} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ml} \right),
\]

\[
\frac{\partial \Phi}{\partial T} = \frac{T}{\Lambda^2} \left( A_M^{(T)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ll} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ll} \right),
\]

\[
\frac{\partial \Phi}{\partial T} = \frac{T}{\Lambda^2} \left( A_M^{(T)} \chi_{ml} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ll} + \frac{T}{\Lambda} A_{\Phi}^{(T)} \chi_{ll} \right),
\]

where the functions \( A^{(T)} \) are defined as

\[
A_M^{(T)} = -\frac{6N_f}{T^4} \int \frac{d^3p}{(2\pi)^3} \frac{M_p f^2(p)}{E_p} \left[ e^{-E_+(T)/T} \left( \Phi + 4\Phi \Phi e^{-E_+(T)/T} + 3(1 + \Phi \Phi) e^{-2E_+(T)/T} \right) + 4\Phi e^{-3E_+(T)/T} + \Phi e^{-4E_+(T)/T} \right]
\]

\[
A_{\Phi}^{(T)} = \frac{T}{\Lambda} \frac{\partial \Phi}{\partial T} + \frac{6N_f}{T^4} \int \frac{d^3p}{(2\pi)^3} \left[ e^{-E_+(T)/T} \left( 1 - 3\Phi e^{-2E_+(T)/T} - 2e^{-3E_+(T)/T} \right) \right] - \frac{e^{-2E_-(T)/T}}{(g^-(T))^2} \left[ 2 + 3\Phi e^{-E_-(T)/T} - e^{-3E_-(T)/T} \right] - \frac{18N_f}{T^3} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{e^{-E_+(T)/T}}{g^+(T)} + \frac{e^{-2E_-(T)/T}}{g^-(T)} \right],
\]

\[
A_{\Phi}^{(T)} = A_{\Phi}^{(T)} (\Phi, \Phi; -\mu).
\]

**APPENDIX C: PARAMETERS USED IN THE PNJL MODEL THERMODYNAMICS**

The compilation of parameters used in the model calculations is given in the following tables:

<table>
<thead>
<tr>
<th></th>
<th>( f_s = 92.4 \text{ MeV} )</th>
<th>( m_s = 135 \text{ MeV} )</th>
<th>( M = 335 \text{ MeV} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>non-local NJL model</td>
<td>( \Lambda = 684.2 \text{ MeV} )</td>
<td>( G_S \Lambda^2 = 4.66 )</td>
<td>( m_0 = 4.46 \text{ MeV} )</td>
</tr>
<tr>
<td>local NJL model</td>
<td>( \Lambda = 625.1 \text{ MeV} )</td>
<td>( G_S \Lambda^2 = 4.38 )</td>
<td>( m_0 = 5.31 \text{ MeV} )</td>
</tr>
</tbody>
</table>

**TABLE I:** Set of parameters for the NJL sector [19]. The parameters of non-local NJL model were fixed for \( \alpha = 10 \).

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TABLE II: Set of parameters for the Polyakov-loop effective potential $\Phi$.

<table>
<thead>
<tr>
<th>set (a)</th>
<th>set (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda = 684.2$ MeV</td>
<td>$\Lambda = 684.2$ MeV</td>
</tr>
<tr>
<td>$G_S \Lambda^2 = 4.66$</td>
<td>$G_S \Lambda^2 = 4.05$</td>
</tr>
<tr>
<td>$T_0 = 270$ MeV</td>
<td>$T_0 = 225$ MeV</td>
</tr>
<tr>
<td>$T_{ch}(\mu = 0) = 242$ MeV</td>
<td>$T_{ch}(\mu = 0) = 180$ MeV</td>
</tr>
<tr>
<td>$T_{dec}(\mu = 0) = 217$ MeV</td>
<td>$T_{dec}(\mu = 0) = 180$ MeV</td>
</tr>
</tbody>
</table>

TABLE III: Set of parameters in the chiral limit used in this work and the resultant phase transition temperatures. $\alpha$ was fixed to be $\alpha = 10$.

<table>
<thead>
<tr>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.51</td>
<td>−2.47</td>
<td>15.22</td>
<td>−1.75</td>
</tr>
</tbody>
</table>

TABLE IV: Set of parameters for the improved Polyakov-loop effective potential [13].