Renormalization Group Approach towards the QCD Phase Diagram

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Abstract

The idea of the functional renormalization group and one-loop improved renormalization group flows are reviewed. The associated flow equations and nonperturbative approximations schemes for its solutions are discussed. These techniques are then applied to the strong interaction in the framework of an effective quark meson model which is introduced in great detail. The renormalization group analysis of the two flavor quark meson model is extended to finite temperature and quark chemical potential which allows for an analysis of the chiral phase diagram beyond the mean field approximation.

1 Introduction

Quantum Chromodynamics (QCD) is the quantum field theory of strongly interacting matter. Many of its vacuum features have been tested experimentally over a wide range of momentum scales. At high momenta or small distances the asymptotic freedom of non-Abelian gauge theories can be used to apply perturbative methods for the computation of physical observables. Due to the running of the strong gauge coupling the situation becomes significantly more complicated at smaller momentum scales or larger distances. Here perturbation theory breaks down and nonperturbative methods are called for. There are no analytical methods starting from first principles which allow to treat QCD at larger distances, where the strong gauge coupling becomes large and perturbation theory fails. The main reason for this difficulty is that QCD describes qualitatively different physics at different length or energy scales.

The situation is further complicated at finite temperature and/or baryon density. For instance, at very high temperatures perturbative calculations are plagued by serious infrared divergences. Furthermore, it is generally expected that at high enough temperature and densities hadronic matter attains a state in which chiral symmetry is almost restored and its fundamental degrees of freedom, the quarks and gluons, are no longer confined. The system undergoes a phase transition from the ordinary hadronic phase to a chirally restored and deconfined quark gluon plasma (QGP). Furthermore, recent theoretical studies reveal an increasing richness in the structure of the phase diagram of strongly interacting matter.

Therefore nonperturbative methods are indispensable to obtain quantitatively reliable results because one has to deal with large couplings and a perturbative expansion of physical observables fails. One such nonperturbative method is given by the renormalization group (RG). The RG method represents an effi-
cient way to describe critical phenomena and phase transitions. It can be used to characterize universal and non-universal aspects of second-order as well as first-order phase transitions and is very well adapted to reveal the full phase diagram of strongly interacting matter. In the context of the phase diagram the RG-method has been applied in [1, 2, 3, 4] using a two-flavor quark-meson model, which captures essential chiral aspects of QCD.

In the following we will introduce the concept of the effective average action and its associated renormalization group equation. Since it is clearly impossible to review here the many facets of the renormalization group developments we will concentrate on the so called proper-time renormalization group approach [5, 6].

2 Renormalization group methods

The renormalization group deals with the effect of a scale change in a theory. The central issue is the understanding of the macroscopic physics at large distances or at low momenta in terms of the underlying fundamental microscopic interaction. In order to understand the evolution from the microscopic to the macroscopic scales one has to consider the quantum or statistical fluctuations on all scales in between. The general RG idea is to treat the fluctuations not all at once but successively from scale to scale [7].

This RG idea combined with functional methods yields the so called functional RG. By means of functional methods the computation of generating functionals of correlation functions becomes feasible. All important physical information is contained in the correlation functions once the fluctuations have been integrated out. Instead of evaluating correlation functions by averaging over all fluctuations at once, only the change of the correlation functions induced by an infinitesimal momentum shell of fluctuations is considered. This goes along Wilson's philosophy of integrating out modes momentum shell by momentum shell.

From a technical point of view this means one has to work with functional differential equations - the so called RG or flow equations - instead of functional path integrals which is usually the case in standard quantum field theory. The differential structure of the RG equations has a larger versatility and offers several advantages compared to an integral formulation. It is analytically and numerically better accessible and more stable. This is of great interest in non-Abelian gauge theories such as QCD. During the evolution from the microscopic to the macroscopic scales these theories turn from a weak to a strong coupling becoming thus nonperturbative at macroscopic scales. In this sense RG methods provide an analytical and powerful tool to investigate nonperturbative phenomena of quantum field theories and statistical physics.

There are various RG methods known in the literature [8, 9, 10, 11, 12, 13, 14]. One particular formulation of RG flows is based on the concept of the effective average action $\Gamma_k$, which is a simple generalization of the standard effective action $\Gamma$, the generating functional of the one-particle irreducible (1PI) Green functions [15]. The generalization is achieved by implementing an infrared cutoff scale $k$ in the functional integral that defines the effective action. $\Gamma_k$ is
then obtained by integrating over all modes of the quantum fields with Euclidean momenta larger than the infrared cutoff scale, i.e. $q^2 > k^2$. In the limit $k \to 0$, the infrared cutoff is removed and the effective average action becomes the full quantum effective action $\Gamma$ containing all fluctuations. For any finite infrared cutoff $k$ the integration of quantum fluctuations is only partially done. The influence of modes with momenta $q^2 < k^2$ is not included.

On the other hand, in the limit $k \to \infty$ (or to some finite ultraviolet cutoff $\Lambda$) the effective average action matches the bare or classical action, which does not contain any fluctuations. Hence, the knowledge of the $k$-dependent $\Gamma_k$ allows to interpolate from the bare action in the ultraviolet to the full effective action in the infrared. As the scale $k$ is lowered more and more quantum fluctuations are taken into account. As a consequence $\Gamma_k$ can be viewed as a microscope with varying resolution length scale $\sim 1/k$. It averages the pertinent fields over a $d$-dimensional volume with size $1/k^d$ and permits to explore the system on larger and larger length scales. This is similar to a block-spin transformation on the sites of a coarse lattice where more and more spin-blocks are averaged over.

The dependence of the effective average action $\Gamma_k$ on the scale $k$ is governed by the flow equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left( \frac{\partial_t R_k}{\Gamma_k^{(2)}[\phi] + R_k} \right).$$

(1)

Here, $\Gamma_k^{(2)}[\phi]$ stands for the second functional derivative with respect to some given field $\phi$. The trace involves an $d$-dimensional integration over momenta (or coordinates) as well as a summation over internal indices (e.g. flavor, color and/or Dirac indices). It is convenient to introduce the logarithmic variable $t = \ln(k/\Lambda)$ which induces the equivalence of the partial derivatives $\partial_t \equiv k \partial/\partial k$.

The infrared cutoff function which regularizes the generating functional is denoted by $R_k$ and depends on the IR scale $k$. This regulator term, defined in momentum space, can be viewed as a momentum-dependent mass term. The regulator function is not completely arbitrary but has to satisfy certain conditions in order to provide an IR and UV regularization for the flow equation. Since it screens the IR modes in a mass-like fashion it should always be a positive function regarding its arguments. Furthermore, it has to vanish for $k \to 0$ removing the regulator from the flow. This condition ensures that the full effective action is recovered in this limit. Finally, it has to diverge for $k \to \infty$ (or for $k \to \Lambda$). This limit ensures that the correct initial condition $\lim_{k \to \Lambda} \Gamma_k = S_{\Lambda}$ in the ultraviolet is reached where $S_{\Lambda}$ denotes some bare action. This last condition acts as a delta functional in the defining integral representation of $\Gamma_k$ which is dominated by the stationary point of the action. This limit justifies the use of a saddle-point approximation which filters out the classical field configurations and the bare action.

The flow equation (1) is an exact equation in the sense that it can be derived from first principles and is often labeled as Exact RG (ERG). It has a simple one-loop structure but is not of one-loop order because it depends on the full field-dependent inverse average propagator $\Gamma^{(2)}_k[\phi] + R_k$. This one-loop structure
has some advantages because technical complications due to overlapping loop integrations do not arise.

The solution of the flow equation corresponds to a trajectory in the theory space, the space of all possible action functionals which are spanned by all possible invariant field operators. The starting point of the trajectory in this space is the bare action $S_\Lambda$ and the end point the full effective action $\Gamma$. As already mentioned, the explicit form of the regulator function $R_k$ is not fixed uniquely. Different choices for $R_k$ will lead to different trajectories thus reflecting the RG scheme dependence of the flow. Of course, the end point of all trajectories in the theory space is independent of the choice of the explicit form of the regulator $R_k$.

General methods for the solution of the functional flow equation are rare. Nevertheless, one can turn the flow equation into a coupled system of non-linear ordinary partial differential equations for infinitely many couplings by expanding the effective average action in terms of invariants. In order to reduce the infinite system to a numerically manageable size one needs to truncate the most general form of $\Gamma_k$. This can be accomplished in various systematic approximation schemes. One such approximation scheme is the 'operator expansion' which constructs the effective action from operators with increasing mass dimension. An example for this type of truncation is the gradient expansion where $\Gamma_k$ is expanded in numbers of derivatives. For a scalar field theory with one real field one obtains

$$\Gamma_k[\phi] = \int d^4x \left\{ V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + O(\partial^4) \right\}. \quad (2)$$

The first (constant) term $V_k$ corresponds to the scalar effective potential and the first correction includes the field-dependent wave function renormalization $Z_k$. In general, it is not guaranteed that the chosen truncation scheme converges. One has to control the truncation error separately as for any expansion.

The flow of the effective average action, Eq. (1), is not the only exact one. It is closely related to other well-known exact flows like e.g. Wegner-Houghton flows [16] or Callan-Symanzik flows [17, 18].

Another class of RG flow equations can be obtained by approximations to Exact RG flows. An example of such an approximation is the so-called Proper-time RG (PTRG) flow which we want to investigate in the following. Compared to the ERG flow the proper-time flow has a numerically simpler and physically more intuitive representation and yields in the lowest-order gradient expansion smooth and analytical threshold functions.

2.1 Proper-time RG flow

Proper-time flows in their original formulation are one-loop improved RG flows [3, 4, 6]. They can be obtained by an RG improvement in an one-loop flow equation [19, 20, 21, 22]. Since the PTRG flows do not depend linearly on the full field-dependent propagator they cannot be exact flows [23]. Furthermore one can show that the (non-exact) PTRG flow starts to deviate from standard perturbation theory already at two-loop order [24].
Nevertheless, within a background field formalism a generalized PTRG flow can be formulated which is exact. This is achieved by adding further terms to the standard PTRG flow [25]. In the following an ad hoc derivation of the standard one-loop improved PTRG flow is presented. The discussion is restricted to an one-component scalar theory with a general interaction. The generalization to other theories is straightforward.

### 2.1.1 Derivation of the standard PTRG flow

The starting point for the derivation of an one-loop improved RG flow is the one-loop effective action

\[
\Gamma^{\text{1-loop}}_\Lambda[\phi] = S_{\text{class}}[\phi] + \frac{1}{2} \text{Tr} \ln S^{(2)}_{\text{class}}[\phi],
\]

where the trace denotes a sum over all momenta. The classical action is labeled by \(S_{\text{class}}[\phi]\) and its second field-derivative by \(S^{(2)}_{\text{class}}[\phi] = \delta^2 S_{\text{class}}[\phi]/\delta \phi \delta \phi\). The ill-defined logarithm is regularized by means of a Schwinger proper-time representation

\[
\Gamma^{\text{1-loop}}_\Lambda[\phi] = S_{\text{class}}[\phi] - \frac{1}{2} \int_0^\infty d\tau f_k(\tau) \text{Tr} \exp(-\tau S^{(2)}_{\text{class}}[\phi]),
\]

where the regulator function \(f_k(\tau)\) provides an IR cutoff \(k\). The lower (upper) limit of the proper-time integral has inverse mass dimension two and regularizes the UV (IR) region of the theory. Since the momenta and the proper-time variable \(\tau\) are coupled, high momenta correspond to small proper-time values and vice versa. Taking the \(k\)-derivative of Eq. (4) the standard PTRG flow equation is obtained as

\[
\partial_t \Gamma_k[\phi] = -\frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \partial_\tau f_k(\tau) \text{Tr} \exp(-\tau \Gamma^{(2)}_k[\phi]),
\]

where the classical action \(S^{(2)}_{\text{class}}\) on the right hand side has been replaced by the scale-dependent effective action \(\Gamma^{(2)}_k\). This represents the RG improvement [6].

Similar to the ERG flow the regulator function \(f_k\) has to satisfy some conditions: Since the flow should start from the initial action \(\Gamma_\Lambda\) in the UV it is required that \(\lim_{k \to \Lambda} f_k(\tau) = 0\). The second condition \(\lim_{\tau \to \infty} f_k(\tau) = 0\) regularizes the IR since the upper proper-time limit coincides with lower momenta. Finally, at the end of the evolution, the condition \(\lim_{\tau \to 0} f_k(\tau) = 1\) reduces the proper-time regularization to the usual Schwinger proper-time regularization.

One class of regulator functions \(f^{(n)}_k(\tau)\) with \(n \geq 0\), which fulfill all these required conditions can be expanded in terms of incomplete \(\Gamma\)-functions and have the structure

\[
\partial_\tau f^{(n)}_k(\tau) = -2(\tau k^2)^{n+1} \frac{e^{-\tau k^2}}{\Gamma(n+1)}. \tag{6}
\]
The $k^2$-term in the exponential acts as a mass term and controls the IR behavior. Inserting this basis set of regulators in Eq. (5) and performing the proper-time integration yields the flow

$$\partial_t \Gamma_k[\phi] = \text{Tr} \left( \frac{k^2}{\Gamma^{(2)}_k[\phi] + k^2} \right)^{n+1}.$$  \hspace{1cm} (7)

For $n > 0$ the PTRG flow cannot be an exact flow since it does not depend linearly on the full propagator. For $n = 0$ the flow depends linearly on the full propagator and one could conjecture that it is an exact flow. In fact, it has the form of a Callan-Symanzik flow, but it is not exact. Furthermore, the flow looks like an ERG flow with a mass insertion $R_k = k^2$ but is not precisely an ERG flow. In contrast to the ERG flow, the momentum integration is not regularized in the UV because the condition $\lim_{k \to \Lambda} R_k \to \infty$ is not fulfilled in this case. All momenta contribute to the flow and an additional UV renormalization would be required in contrast to the ERG flow. In addition, for this flow the Wilsonian single momentum-shell interpretation becomes questionable.

Nevertheless, the standard PTRG flow is a well-defined approximation to a first-principles flow like the ERG flow [26]. It is an approximation to a background field ERG flow where additional terms proportional to $\partial_t \Gamma^{(2)}_k$ are neglected and the difference between fluctuation fields and background fields has been omitted. With these additional terms the generalized PTRG flow becomes exact. So far, applications of the PTRG flow are typically based on flows of the form of Eq. (7) and further approximations thereof. The standard PTRG flow with the lowest-order gradient expansion truncation has a simple polynomial momentum dependence which will lead to simple flow equations. However, it is still an open issue how these additional flow terms proportional to $\partial_t \Gamma^{(2)}_k$ which make the flow exact affect the results obtained with the standard PTRG flow. First results in this respect for an $O(N)$ symmetric scalar theory indicate a minor influence of these addition flow terms at least at criticality in three dimensions [21, 27].

In the following the standard PTRG flow will be used to investigate strongly interacting matter which is described by QCD.

### 2.2 Applications

In QCD the quarks and gluons represent the microscopic degrees of freedom, whereas the macroscopic degrees of freedom are the observed color neutral particles like the mesons, baryons and/or glueballs. Hence there must be a transition from the microscopic to the macroscopic degrees of freedom. Therefore, in QCD the relevant degrees of freedom change with the scale $k$ which is suitable for an RG treatment. In order to apply RG techniques to QCD an initial starting effective action has to be formulated.

When constructing effective models for these macroscopic degrees of freedom one usually relies on the guiding symmetries of QCD because a first-principle derivation from QCD is still missing. One important symmetry of QCD is the local $SU(N_c)$ color invariance which is related to confinement. This symmetry
cannot be used here since the observed hadronic spectrum consists of color blind states. We will concentrate on the chiral dynamics of QCD and consider QCD with only two light quark flavors. To a good approximation the masses of these two flavors are small compared to the other quark flavors. For vanishing current quark masses the classical QCD Lagrangian does not couple left- and right-handed quarks. Omitting the axial anomaly and baryon number conservation it exhibits a global chiral invariance under the $SU_L(N_f) \times SU_R(N_f)$ symmetry group where $N_f$ denotes the number of quark flavors. In the observed hadron spectrum only the vector-like subgroup $SU_V(N_f)$ is realized which implies a spontaneous symmetry breaking of the chiral $SU_L(N_f) \times SU_R(N_f)$ symmetry down to the $SU_V(N_f)$ symmetry. This symmetry breaking predicts for $N_f = 2$ the existence of three light parity-odd Goldstone bosons, the three pions $\vec{\pi}$. Their comparably small masses are a consequence of the explicit chiral symmetry breaking due to the finite current quark masses.

At scales above $\sim 2$ GeV the dynamics of the relevant degrees of freedom appear to be well described by perturbative QCD. At somewhat lower scales quark condensates, bound states of quarks and gluons emerge and confinement sets in.

To each such non-perturbative phenomenon one can associate an appropriate scale. Focusing on the physics of scalar and pseudoscalar mesons and assuming that all other bound states are integrated out it appears that all these scales are rather well separated from each other. The compositeness scale $k_\phi$, where mesonic bound states are formed due to the increasing strength of the strong interaction is somewhere below 1 GeV. The chiral symmetry breaking scale $k_\chi$ at which the chiral quark condensate develops a non-vanishing value will be below the compositeness scale, typically around 500 MeV. The last scale where confinement sets in, is related to the Landau pole in the perturbative evolution of the strong coupling constant and is of the order of $\Lambda_{QCD} \sim 200$ MeV.

For scales between the ranges $k_\chi \leq k \leq k_\phi$ the most relevant degrees of freedom are quarks and mesons and their dynamics in this regime is dominated by the strong Yukawa coupling $g$ between them. This picture legitimates the use of a quark meson model only if one assumes that the dominant QCD effects are included in the meson physics. Below the scale $k_\chi$ the strong coupling $\alpha_s$ increases further and quark degrees of freedom will confine. Getting closer to $\Lambda_{QCD}$ it is not justified to neglect those QCD effects which certainly go beyond the meson dynamics. Of course, gluonic interactions are expected to be crucial for an understanding of the confinement phenomenon. But due to the increase of the constituent quark masses towards the IR the quarks decoupled from the further evolution of the mesonic degrees of freedom. As long as one is only interested in the dynamics of the mesons one expects that the confinement on the mesonic evolution has only little influence even for scales below $\Lambda_{QCD}$. Hence there are good prospects that the meson physics can be described by an effective quark meson model [28, 29, 30].

The Ansatz for the effective action at the compositeness scale $k_\phi$ is given by

$$\Gamma_{k=k_\phi} = \int d^4x \left\{ \bar{q} \left( i\gamma^\mu g(\sigma + i\gamma_5 \vec{\tau} \vec{\pi}) \right) q + \frac{1}{2} (\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 - U(\sigma, \vec{\pi}) \right\} \quad (8)$$
where the purely mesonic potential is defined as

\[ U(\sigma, \vec{\pi}) = \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma. \]

The quark fields are denoted by \( q \) and \( g \) is the Yukawa coupling which describes the interaction strength of the isoscalar-scalar \( \sigma \) and the three isovector-pseudoscalar pion fields \( \vec{\pi} \) with the quarks. The value for \( g \) is usually fixed to the value of the constituent quark masses \( M_q \) which are of the order of 300 MeV in the vacuum. Here we have neglected the \( SU(2) \)-isospin violation and consider averaged quark masses. The explicit symmetry breaking parameter \( c \) in the mesonic potential controls the value of the pion mass. For \( c = 0 \) the chiral limit is obtained and the pion mass vanish. In the chiral limit the action is invariant under global chiral \( SU(2)_L \times SU(2)_R \) symmetry transformations. The last two parameters of the model, \( \lambda \) and \( v^2 \), are fixed to the mass of the sigma field and the pion decay constant \( f_\pi \).

This effective action at the compositeness scale \( k_\phi \) emerges from short distance QCD in basically two steps. Starting from QCD in the UV one first computes an effective action which only involves quarks. This corresponds to an integration over the gluonic degrees of freedom in a quenched approximation. This will generate many effective nonlocal four and higher quark vertices and a nontrivial momentum dependence of the quark propagator. In the second step decreasing the scale further these four and higher quark interactions will cause the formation of mesonic bound states. Thus, at the compositeness scale not only quarks but also composite fields, the mesons, are present and interact with each other. The four quark interactions have been replaced by mesonic fields. It is obvious that for scales below the compositeness scale a description of strongly interacting matter in terms of quark degrees of freedom alone would be rather inefficient.

The Ansatz for the effective action (8) incorporates a truncation to four quark interactions. Higher quark interactions have been suppressed at the compositeness scale. This leads to a purely quadratic mesonic potential at \( k_\phi \) with a positive mass term. This implies that the chiral symmetry is restored at the compositeness scale and the scalar expectation value \( \langle \sigma \rangle \) will vanish [3].

The effective action also corresponds to a gradient expansion to lowest order (cf. Eq. (2) where wave-function renormalizations are neglected. It serves as the required initial action for the RG evolution. The evolution is started at the compositeness scale where the mesons are assumed to be formed due to strong gluonic interactions. During the evolution towards the IR this effective action for the quark meson model is used as a truncation for an approximate solution of the RG flow. Instead of expanding the mesonic potential in a polynomial form for the fields we will be more general here and allow for arbitrary higher mesonic \( O(4) \)-symmetric self-interactions in the potential. We will solve numerically the flow equations for the full effective potential. But the running of the Yukawa coupling will be neglected.

The dynamics in the beginning of the scale evolution just below the compositeness scale is almost entirely driven by quark fluctuations. These fluctuations rapidly drive the squared scalar mass term in the action to negative values.
This then immediately leads to a potential minimum away from the origin such
that the vacuum expectation value \( \langle \sigma \rangle \) becomes finite. This happens at
the chiral symmetry breaking scale \( k_\chi < k_\phi \), not far below \( k_\phi \). The reason for this
behavior lies in the suppression of the meson contributions. All meson masses
are much larger than the constituent quark masses around these scales and are
therefore further suppressed during the evolution. Furthermore, the quarks are
strongly coupled to the mesons since the Yukawa coupling is relatively large.
Below \( k_\chi \) the systems stays in the regime with spontaneous chiral symmetry
breaking. Around scales of the order of the pion mass the evolution of the po-
tential minimum stops. The reason for this stability of the vacuum expectation
value is that the quarks acquire a relatively large constituent mass \( M_q \). These
heavy modes will decouple from the further evolution once the scale drops below
\( M_q \). The evolution is then essentially driven by the massless Goldstone bosons
in the chiral limit. Of course, for non-vanishing pion masses the evolution of the
model is effectively stopped around scales \( k \sim m_\pi \). Quarks below such scales
appear to be no longer too important for the further evolution of the mesonic
system. Because of confinement effects quarks should anyhow no longer be in-
cluded for scales below \( A_{QCD} \). The final goal of such an evolution is to extract
phenomenological quantities like meson masses, decay constants etc. in the IR.
This can be achieved in a straightforward computation from the effective action
\( \Gamma = \lim_{k \to 0} \Gamma_k \).

So far we have considered the vacuum fluctuations that contribute to the flow
of the effective average action \( \Gamma_k \). Using certain phenomenological quantities in
the IR as input we have fixed all model parameters at the compositeness scale.
The extension of the presented RG formalism to a system in thermal equilibrium
system with finite net-baryon or net-quark number density is straightforward.
In such systems the effective average action plays the role of the grand canonical
potential \( \Omega \) and depends also on the temperature \( T \) and on one averaged quark
chemical potential \( \mu_q \) if again isospin symmetry is assumed.

In the Imaginary-time or Matsubara formalism a finite temperature \( T \) re-
sults in (anti-)periodic boundary conditions for (fermionic) bosonic fields in
the compact Euclidean time direction with radius \( 1/T \). This leads to a replacement
of the zeroth component of the momentum integration with discrete (even) odd
Matsubara frequencies for (fermions) bosons. Thus, for finite temperature a
four-dimensional quantum field theory can be interpreted as a three-dimensional
system plus an infinite tower of Matsubara modes for each degree of freedom.
Since all Matsubara frequencies are proportional to the temperature all massive
Matsubara modes will decoupled from the dynamics of the system at high tem-
perature. One therefore expects an effective three-dimensional theory with the
bosonic zero modes as the only relevant degrees of freedom. This leads to the
phenomenon of dimensional reduction.

The implementation of a finite quark chemical potential in the quark me-
son model is also straightforward. One has to add a term proportional to
\( i \mu_q \int d^4x \bar{q}(x) \gamma_0 q(x) \) to the Ansatz \( 3 \) for the effective average action. This Ansatz
together with the choice of a half-integer regulator function \( f_k^{(n)} \), see Eq. (10),
where the variable \( n \) is replaced by \( 3/2 \) is used as the input to derive the proper-
time flow equations. The choice of a half-integer power in the regulator function
corresponds to a three-dimensional momentum cutoff and allows for an analytic evaluation of the Matsubara sums. This is one advantage of using the proper-time flow with this regulator compared to other RG flows. Finally, one obtains for the scale-dependent grand canonical potential the proper-time flow equation

\[ \partial_t \Omega_k(T, \mu) = \frac{k^5}{12\pi^2} \left[ \frac{3}{E_\pi} \coth \left( \frac{E_\pi}{2T} \right) + \frac{1}{E_\sigma} \coth \left( \frac{E_\sigma}{2T} \right) \right. \\
\left. - \frac{2N_c N_f}{E_q} \left\{ \tanh \left( \frac{E_q - \mu_q}{2T} \right) + \tanh \left( \frac{E_q + \mu_q}{2T} \right) \right\} \right] \]

with the pion- \( E_\pi = \sqrt{k^2 + 2\Omega_k} \), the \( \sigma \)-meson \( E_\sigma = \sqrt{k^2 + 2\Omega_k + 4\phi^2\Omega_k'} \) and quark energies \( E_q = \sqrt{k^2 + g^2\phi^2} \). The primed potential denotes the \( \phi^2 \)-derivative of the potential, i.e., \( \Omega'_k := \partial \Omega_k / \partial \phi^2 \) and correspondingly the higher derivatives. The potential for this model depends on the expectation value of the square of the chiral 4-component field \( \phi^2 \) which coincides with \( \langle \sigma \rangle^2 \) since \( \langle \vec{\pi}^2 \rangle = 0 \). The scale-dependent effective meson masses are defined as \( m_{\sigma,k}^2 = 2\Omega_k' + 4\phi^2\Omega_k'' \) and \( m_{\pi,k}^2 = 2\Omega_k' \) where the potential has to be evaluated at the global scale-dependent minimum \( \phi^2 = \phi_0^2 \). The dynamically generated constituent quark mass \( M_q = g\phi_0 \) is proportional to the minimum since the running of the Yukawa coupling is not considered here.

The summations of the fermionic (bosonic) Matsubara sums yield analytical functions which represent the corresponding mass threshold functions for finite temperature and quark chemical potential. The threshold functions are an important nonperturbative ingredient generally appearing in flow equations. They control the smooth decoupling of massive modes from the evolution once the IR cutoff scale \( k \) drops below the corresponding mass. The different degrees of freedom contribute in an additive way to the flow. One recognizes the three degenerate pion, the sigma and the quark/antiquark threshold functions in the square brackets. The flow equation has an overall scale factor \( k^4 \) which reflects the correct dimension of the effective potential in \( d = 4 \) dimensions. This can be seen explicitly by rewriting all mass threshold functions in a dimensionless form. The fermionic contributions enter with a negative sign due to the fermion loop and have a degeneracy factor of \( (2s + 1)N_c N_f \) with \( s = 1/2 \). The quark chemical potential enters only in the quark/antiquark threshold functions with the appropriate sign as it should be. It influences the bosonic part of the flow equation only implicitly through the meson masses. The flow for the vacuum can be deduced immediately of the full flow (9) by examining the limits \( T = 0, \mu_q = 0 \) of the finite temperature and quark chemical potential threshold functions. In this limit all hyperbolic functions tend to one and only the vacuum threshold functions remain.

The flow equation (9) constitutes a coupled, highly non-linear, partial differential equation which can be integrated numerically in principle in two ways: either one discretizes the unknown potential \( \Omega_k \) on a \( \phi^2 \)-grid or expands the potential in powers of \( \phi^2 \) around its minimum \( \phi_0^2 \). The advantage of the potential expansion is that only a finite set of coupled flow equations has to be solved, depending on the chosen expansion order. In particular, this yields the beta
functions for the couplings of the potential. For each higher order of the potential expansion, however, a new coupled beta function is introduced increasing the numerical effort drastically. A further drawback is that the potential is only known around the minimum $\phi^2_0$ once the system has been solved \cite{22,19}. This is different for the grid solution: Here, the potential is not only known around the minimum but also for arbitrary $\phi^2$. This is of importance, for example, in a first-order phase transition where two degenerate minima of the potential emerge. In this case the knowledge of all local minima is required to describe the phase transition correctly. This is cumbersome in a potential expansion, except for some simple potentials. With an additional explicit symmetry breaking term in the potential every potential minimum has always a finite value because the symmetry is never restored exactly. A precise determination of the critical temperature of a first-order transition is very difficult within an expansion scheme around only one potential minimum.

Since we have opted for the grid solution, the field $\phi^2$ has been discretized for a general potential term on a regular grid. Thus, for each grid point a flow equation is obtained which leads finally to a coupled closed system. As initial condition at the chiral symmetry breaking scale $k_\chi$ we use a symmetric potential Ansatz. All model parameters are fixed in such a way that they match the physical IR vacuum quantities accordingly.

For zero chemical potential and for two massless quark flavors a second-order phase transition is found in which the spontaneously broken chiral symmetry is restored. The phase transition and fixed point structure belongs to the $O(4)$-universality class with the corresponding critical exponents. The value for the critical temperature is a non-universal quantity and depends on the model parameters. For finite pion masses, i.e. with an explicit symmetry breaking term in the potential, the second-order phase transition is washed out and turns into a smooth crossover. This results in a shift of the “pseudo-critical” temperature defined as the inflection point of the order parameter towards larger values.

For finite chemical potential, the $O(4)$-universal second-order phase transition persists up to a tricritical point which is a critical point where three phases coexist. The curvature of the second-order transition line $T_c(\mu)$ has a negative slope. Thus, the location of the tricritical point must be below the value of the critical temperature at zero chemical potential. The tricritical point belongs to a trivial Gaussian fixed point with mean-field critical exponents. The precise location of this tricritical point is in general not known and again depends on the model parameters \cite{31}. Thus, the existence of this point, the shape of the transition line and its universality class are predictions within the underlying quark meson model. At higher chemical potential and smaller temperatures the phase transition changes initially to a single first-order phase transition. For smaller temperatures we observe a splitting of the transition line and two phase transitions emerge. The left transition line represents a first-order transition down to the $T = 0$ axis. At this transition the order parameter jumps not to zero but to a finite value. The chiral symmetry remains spontaneously broken and is only restored for higher chemical potentials which then produce the second (right) transition line. At this right transition line we initially find a second first-order transition where the order parameter jumps to zero and
chiral symmetry is restored. But for smaller temperatures close the \( \mu \)-axis the order parameter tends smoothly to zero. It seems that this transition is again of second-order. If this is true we infer that there must be a second tricritical point in the phase diagram in the chiral limit.

As the pions become massive the tricritical point turns into a critical end point (CEP). For temperatures below the CEP a first-order curved transition line is found which persists down to the \( \mu \)-axis. For each value of the explicit symmetry breaking parameter (finite pion mass), there is a corresponding CEP. In an extended, three-dimensional \((T, \mu, m_\pi)\) phase diagram these points arrange into a critical line. The static critical behavior of this line falls into the universality class of the Ising model in three dimensions corresponding to the one-component scalar \( \phi^4 \)-theory in three dimensions. As the pions become massive no symmetry remains which would require the order parameter to have more than one component. On the other hand, in the chiral limit we have a chiral \( SU(2)_L \times SU(2)_R \) symmetry restoration which is isomorphic to the \( O(4) \) symmetry. In this case the order parameter should be made up of four components, one sigma and three pions.

In the chiral limit below the splitting point, the right second-order transition turns into a crossover for finite quark masses. Analogously, the second tricritical point, if it exists, should turn into a critical point. Some remnants of this critical point can indeed be seen in the vacuum expectation value and in the behavior of the meson masses. But a detailed analysis of this point is postponed to a future work.

3 Summary

In this lecture note we have presented an brief introduction to the functional renormalization group methods with a focus on the flow equation for the effective average action. We specialized and discussed in great detail the standard proper-time flow and its relation to the effective average action. In the second part we used this proper-time RG to explore the QCD phase diagram within an linear quark meson model for two quark flavors. This model captures essential features of QCD such as the spontaneous breaking of chiral symmetry in the vacuum and can therefore yield valuable insight into the critical behavior associated with chiral symmetry.

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