In addition to its conceptual interest, the cluster state or one-way model of quantum computation \cite{1} appears to yield a highly desirable route to quantum computing for a variety of technologies \cite{2,3,4,5,6,7,8}, not least due to the clear cut distinction between the creation and consumption of entanglement. While cluster state computation always requires a level of “classical” feed-forward – wherein settings of single-qubit measurement devices need to be switched according to outcomes obtained previously on other devices – all current proposals for building cluster states with probabilistic gates \cite{5,6,7,8,9} rely on larger amounts (by several orders of magnitude) of the much more problematic “active switching” type of feed-forward. This type of coherent feed-forward involves the quantum systems being routed into different possible interactions with other quantum systems, based on success or failure of various entangling gates. In addition to the need for implementing such switching in a way that preserves coherence (so as to enable success of the subsequent two-qubit gates), availability of quantum memory \cite{10} capable of storing the systems while they await such switching then also becomes of major concern. This is particularly true for the linear optical quantum computational paradigm \cite{6,7,11,12,13}, and it is within this framework that most of our results will be phrased, although they apply to any technologies making use of probabilistic gates.

In this work we demonstrate that it is possible to dispense with all of the active switching, once very small initial pieces of cluster state have been obtained. Given such small clusters, every qubit is only ever involved in one probabilistic two-qubit gate, followed by one single-qubit measurement. The principal idea is to use the probabilistic gates to combine small such pieces of cluster according to a lattice geometry specially chosen such that occurrence of a percolation phenomenon is assured \cite{14}. On the percolated lattice a pattern of single qubit measurements can then be efficiently determined by an offline classical computation, and universal quantum computation is attainable \cite{15}. Remarkably, it is possible to achieve this complete removal of active feed-forward at essentially no cost. More precisely, the resources required induce at most a sublinear overhead per qubit on the resources which would be required if we had perfect deterministic gates with which to build the cluster state. We will also present strong numerical evidence the actual per qubit overhead can be reduced to logarithmic in the cluster size.

In the second half of this work we will show how the initial entangled states required can be as small as 4-qubit cluster states, which have already been prepared in down conversion experiments \cite{16}. We will begin, however, by discussing in detail the conceptually simpler procedure involving percolation using a cubic lattice and an initial resource of 7-qubit star-shaped cluster states (equivalent to 7-qubit GHZ states).

The technique we use to deal with the randomness of the cluster states produced by all the percolation phenomena we study, is that of coarse graining an underlying lattice \( U \) into blocks which correspond to logical qubits, and form a renormalized lattice \( M \), which can be described as a graph with vertices comprising the blocks, and edges denoting connections between crossing clusters in neighboring blocks, see Fig. 1.

This is clearly reminiscent of the concept of static renormalization for bond percolation discussed in Refs. \cite{14,17}. Here, we are, however, not interested in the percolating properties of the renormalized lattice. Instead, we want to build the cluster state. We will also present strong numerical evidence the actual per qubit overhead can be reduced to logarithmic in the cluster size.

For concreteness we focus on \( M = [1, L]^2 \) for some length \( L \), that is, the renormalized lattice is a 2d square lattice. We consider bond percolation, so a bond is present (“open”) with probability \( p \). Unfortunately, the square lattice itself has a critical bond-percolation probability of \( p_c = 1/2 \), marking the
so starting from a 3d cubic lattice, for which $p^*$ context at hand, namely fusion gates [7] operating with a success probability of at most $\varepsilon$, which form the covering lattice.

For the purposes of implementing a quantum computation, it suffices to use the 6-qubit graph states [2] depicted (i.e., the complete graph $K_6$) which form the covering lattice.

The arrival of an infinite open cluster [14]. Thus, in our primary context at hand, namely fusion gates [7] operating with a success probability of at most $\varepsilon$, see also Fig. 2 creating bonds with fusion gates will not result in enough crossing clusters on average. This can be overcome, however, by taking $U \subset \mathbb{Z}^3$, so starting from a 3d cubic lattice, for which $p_c = 0.249$. We will identify each vertex $x \in M$ with a block of size $(2k)^3$. We can now meaningfully define an event $X_x(k)$ of $x \in M$ being “occupied”. With this we mean that there exists a crossing open cluster within the block, so a connected path on the graph connecting each pair of faces on opposite sides, at least in the first and second dimension [14]. Moreover, this crossing cluster is connected to each of the crossing clusters of the blocks associated with sites $y$ adjacent to $x$. We show the following:

Renormalized cubic lattices: Let $p > p_c$. Then for any $\mu > 0$, the probability $P_p(L)$ of having $X_x(k)$ satisfied for all $x \in M$ with $k = L^\mu$ fulfills

$$\lim_{L \to \infty} P_p(L) = 1.$$  

In other words, with a sublinear overhead $k = O(L^\mu)$, one can create a cubic lattice $M = [1, L]^2 \times \mathbb{Z}^3$ out of $U$ using bond percolation. Moreover, this preparation is asymptotically certain (in the same sense as in Refs. [14]), despite the underlying elements being probabilistic. The value of $k$ specifies to what extent we “dilute” the superlattice $M$ compared to $U$.

To see the validity of (1), we introduce a series of blocks of the underlying lattice $U$, which, in addition to the blocks of $M$ include blocks overlapping with those (see dashed lines in Fig. 1). For any $y \in [2, 2L]^2 \times \mathbb{Z}^3$, let $A_y(k) = [y_1 k, y_1 k + 2k - 1] \times [y_2 k, y_2 k + 2k - 1] \times [1, 2k]$. Each vertex $x \in M$ is identified with $y = 2x$. To show that $X_{X_x}(k) = 1$ (almost certainly) for all $x \in M$ for large $L$, we make use of statements on crossing clusters in cubic lattices, as well as of a convenient tool in percolation theory, the FKG inequality: Let $C$ and $D$ be two increasing events, i.e., events that “become more likely” for increasing $p$. Then the FKG inequality states that $P_p(C \cap D) \geq P_p(C)P_p(D)$ [14]. In other words, increasing events are positively correlated.

Let us denote with $C_y(k)$ the event that $A_y(k)$ has a left-to-right crossing cluster in the first dimension, i.e., an open path having vertices $a$ and $b$ satisfying $a_1 = y_1 k$ and $b_1 = y_1 k + 2k - 1$. Now there exists a constant $\gamma > 0$, only dependent on $p$, such that

$$P_p(C_y(k)) \geq 1 - \exp(-\gamma k^2)$$  

for $k \geq 3$ [14]. We only need to “connect these vertices”. The blocks $A_y(k)$ and $A_z(k)$ are overlapping for $\text{dist}(y, z) = 1$. Now take a site $z \in [2, 2L - 1] \times [2, 2L]$, and take a site $w$ with $z_1 = y_1 + 1$, and $z_2 = y_2$. Let $D_{w}(k)$ be the event that there is a left-to-right crossing cluster in $A_y(k)$, and $D_{z}(k)$ the event that there is such a cluster in $A_z(k)$. Both events are increasing events, and therefore, we can use the FKG inequality: intuitively, if in $A_y(k)$ there is already a crossing cluster, then this crossing cluster is already half way through $A_z(k)$, and hence renders a crossing cluster there more likely. Consider the overlap between two adjacent blocks, $B_{y}(k) = A_y(k) \cap A_z(k)$. We can define the following event: For $p \in [0, p_c]$, we define $E_{y}(k)$ as the event that never occurs, for $p \in (p_c, 1)$ it is the event of having at most a single left-to-right crossing cluster in this overlap $B_{y}(k)$. This is an increasing event [18]. Hence, the probability of having simultaneously a left-to-right crossing cluster in $A_y(k)$, one in $A_z(k)$, and exactly one in $B_{y}(k)$ can be estimated using the FKG inequality. There exist constants $c, a > 0$, only dependent on $p$ such that the probability of having the event $E_{y}(k)$ satisfies [18]

$$P_p(E_{y}(k)) \geq 1 - (2k)^6 a \exp(-ck).$$

So, using again the FKG inequality, one finds that the probability, $F_{y}(k)$, of having two crossing clusters in $A_y(k)$ and $A_z(k)$ which are actually connected as

$$P_p(F_{y}(k)) \geq (1 - \exp(-\gamma k^2))^2 (1 - (2k)^6 a \exp(-ck)).$$

This procedure can be iterated, using FKG in each step. To find connections in the other direction, we can again make use of the argument on having at most a single crossing cluster, but now using $[1, 3k]$ in the third dimension, in order to be able to apply the results of Ref. [18]. This gives an overall probability of having $X_{X_x}(k)$ for each $x \in [1, L]^2$ as

$$P_p(L, k) \geq (1 - \exp(-\gamma k^2))^{3L^2 - 2L} \times (1 - (2k)^6 a \exp(-ck))^{L(L-1)} \times (1 - (3k)^6 a \exp(-c3k/2))^{L(L-1)}.$$  

Now, there clearly exists an integer $k_0$ such that

$$P_p(L, k) \geq (1 - (3k)^6 a \exp(-c3k/2))^{5L^2}.$$  

for all $k \geq k_0$. Let us set $k = L^\mu$ for $\mu > 0$. Then, it is straightforward to show that in fact $\lim_{L \to \infty} \left(1 - (3k)^6 a \exp(-c3k/2)\right)^{5L^2} = 1$, using that for any $c, f > 0$, we have that $\lim_{n \to \infty} \left(1 - en^{a/2} \exp(-fn^{a/2})\right)^b = 1$. This means that by making use of a sublinear overhead, we arrive...
at an asymptotically certain preparation of the renormalized lattice.

This gives rise to an overall resource requirement of \(O(L^3) \times L^2 = O(L^{5+3\mu})\) 7-qubit states to build a fully connected cluster state that (almost certainly) consists of \(L \times L\) blocks, and requires no rerouting. As long as \(p > p_c\), this scaling will hold. Obviously, heralded losses (lossy optical components and imperfect detectors in the optics case) can be incorporated using the gap between the gate’s ideal success probability and the critical value \(p_c\). This should be compared to the \(O(L^2)\) qubits we would require if we had perfect deterministic gates with which to build the cluster.

To finally utilize the renormalized blocks some classical computation is needed, and we need to ascertain that it is efficient in the system size. One first has to find the crossing clusters, e.g., by the Hoshen-Kopelman-algorithm \[19\] requiring \(O(k^3)\) steps and \(O(k^2)\) additional classical memory per block. Scanning the surface for suitable sites on the border between neighboring blocks needs \(O(k^2)\) steps. However we require more than simply identifying the crossing cluster, and so next we must identify intersecting paths through this cluster. Instead of the 4-way-junctions of a square lattice we now explain how to identify T-junctions which is conceptually slightly simpler and still allows for universal quantum computation. Three qubits on the block’s border that have been chosen before have to be connected. This may be achieved by finding paths between them on the surface of the crossing clusters. After identification of suitable paths through the lattice, one can implement a quantum computation in a fairly obvious manner by pushing quantum information down the paths with \(\sigma_z\) measurements and removing unwanted qubits with \(\sigma_x\) measurements. Alternatively, measurements can be chosen such that the selected paths collapse to single qubits, and unneeded sites are measured out leaving a square lattice cluster with which to compute in the standard fashion. The former method opens up the interesting possibility of correcting for errors “on the fly”, since there in general will exist a very large number of paths crossing any given block, hence edges are redundantly available, and any identified errors (losses for example), may perhaps be avoided by suitable changing the flow of information during one-way computation.

At this stage we have used 7-qubit clusters on a cubic lattice, see Fig. 2. We now turn to various methods for reducing the size of this initial resource. The first one is quite general, and will apply to any lattice. We see from Fig. 2 that a qubit is left on each successfully formed bond. One interesting observation is that this qubit may be measured out, relaxing the requirement of photon number resolving detectors to dichotomic detectors. However, one might also use this to construct the covering lattice \[14\] of the original lattice, by connecting these sites with all perimeter sites from the neighboring stars, and removing the stars’ central qubits (Fig. 2(b)). From percolation theory it is known \[14\] \[20\] that the critical bond percolation probability of a lattice equals the critical site percolation probability of the covering lattice (for which a site is “open” with a certain probability \(p\)). Thus by using 6-qubit clusters (with the connectivity of the complete graph \(K_6\) as shown) the covering lattice can be built by fusion of neighboring corner qubits. Obviously, these percolation processes are equivalent for computational purposes, because a path between two arms of one star in the original lattice exists iff the fusion processes involving these two arms were successful, and a path between two corner qubits in the covering lattice exists iff the fusion attempts on the equivalent two qubits were successful.

A quite different method (somewhat more specific to linear optics applications) can further reduce the size of the initial states required on the cubic lattice to 5-qubit star clusters. This method (explained in Fig. 3) involves generating the 7-qubit star clusters by judiciously fusing two “central” qubits of each of the 5-qubit stars, while simultaneously effecting the Type-I fusion operations on the bonding qubits (i.e., no feed-forward required). Crucially, the central fusion operations can be applied in parallel and succeed with probability 3/4, while the bond fusions still succeed with probability 1/2. These two probabilities lie above the mixed site/bond percolation threshold for the cubic lattice \[21\]. A key observation is that even if the central (“site”) fusion fails, the bond fusions can still be attempted as usual, since the single qubits resulting from the failure are in the state \(|+\rangle^{\otimes 6}\), and fusion gates involving them will succeed or fail with probability 1/2. Hence, the site and bond generation processes are independent and do not require active switching. Thus, we can be assured the percolation will proceed as desired.

A more general approach to decrease the size of the initial resources is the following: Instead of using the cubic lattice, we switch to the 3d lattice with the lowest vertex degree, namely the diamond lattice which has vertex degree 4, and a bond percolation threshold of \(p_c = 0.389\). While percolating on the diamond lattice directly would require 5-qubit star clusters, by percolating on the covering lattice (as explained above) we even further reduce the initial resources required to 4-qubit tetrahedral graph states \[2\]. These consist of triangles and are thus not two-colourable. However, due to the structure of the diamond lattice and especially when identify-
Bonds between neighboring sites exist iff the crossing clusters mon face. Depending on of the corresponding blocks are connected through the com-
ranged in two dimensions as described above. The blocks are
an edge being open, the probability
ing T-junctions by surface paths, the resulting graph state s can
the diamond lattice of size
this lattice suffices for our purposes. In fact we find that the
amond lattice, we have turned to a numerical assessment that
still be reduced to universal cluster states.

We provided a proof that to prepare an
L × L cluster state, asymptotically with certainty, even with this very restricted
set of tools, a scaling in the number of resources of
4
asymptotically with certainty, even with this very restric-
ted amount of coherent feed-forward required; specifically the re-
turn computing beyond the one-way computer as proposed in
Ref. [23] gives rise to further improvements concerning re-
source requirements. The presented ideas in this work open up a new way to deal with randomness of probabilistic gates in quantum computing.

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[15] If the entangling gates need any feed-forward like in the optics case, this will be incorporated into the one-way computation measurement scheme.


