AN INVESTIGATION INTO THE USE OF ITERATION METHODS FOR THE
ANALYSIS OF AXIALLY SYMMETRIC AND SHEET BEAM ELECTRODE SHAPES
WITH AN EMITTING SURFACE

by


Abstract

The techniques for analyzing an axially symmetric or two dimensional electrode system with an emitting surface have been studied with a computer programme. The programme is very easy to use, and performs in one pass the steps which have usually previously been followed on analogue and digital devices separately. The programme alternately solves Poisson's equation by a Relaxation Technique, traces trajectories, recomputes the space-charge distribution, and again performs a relaxation process. This procedure continues until satisfactory convergence has been achieved.

The ease of use of the computer programme has allowed the investigation of different methods of speeding up the convergence. The results are illustrated with two specific examples.
1. Introduction

In most procedures for analyzing specific electrode systems in which a beam comes from a cathode, the same basic methods are used. First in the given electrode system, without space-charge, or with an assumed space-charge distribution, Poisson's equation is solved. Then, trajectories are traced, and, by use of the continuity of charge equation, a new set of space-charge source terms are evaluated. Poisson's equation is then solved again and the process continued. In the past the solution of Poisson's equation has usually been performed on an analogue device \(^1,2\), while the trajectory tracing has either been performed numerically or by an analogue or digital computer attached to the analogue device. Relaxation methods have long been used to solve Poisson's equation, and it is clearly more convenient if the whole cycle of computation can be performed automatically on a digital computer \(^3,4,5,6\). Unfortunately, such computations for practical problems require a computer with a large fast memory. The programme used in this investigation was written for an IBM 709 computer with a fast memory of 32,000 words. The details of the programme are given in Refs. 6 and 7 and will be discussed here only as far as they effect the methods used.

The essence of the programme is its simplicity. It is only necessary to enter the coordinates and potentials of the points on the boundary, the number of rays which will be traced in the beam, and certain information related to the relaxation net, the convergence criteria, and the printing; the machine does the rest automatically. For this reason it is possible to investigate the methods of solution — even for large numbers of iterations — much more exhaustively than by other devices.

While similar computer programmes have been described before, there have been almost no attempts to compare calculated solutions with theoretical or experimental ones to a high accuracy. Those comparisons which have been made are usually for fairly trivial solutions such as rectilinear flow from planar, circular, or spherical cathodes. In this paper comparisons are made for a more sophisticated flow.

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In section 2 the equations to be solved are presented, while in section 3 the method of solution is briefly discussed. In section 4 the different initial conditions are given, while in section 5 the accuracy criteria are enumerated.

In section 6 the restrictions and speed of the programme are discussed. In section 7 the detailed numerical analyses of two electrode systems are discussed. One is an axially symmetric, high convergence, perveance 2.2, Pierce-type gun; the other is a sheet-beam gun in a crossed magnetic field. Based mainly on these studies, some general conclusions are drawn in section 8 about the analysis of electrode systems for this type of problem. Some of these conclusions depend on the particular computer programme used; others apply to all methods which use discrete analogs — i.e. computer and resistance-network methods. Yet a third set of conclusions are common to all methods which use iteration techniques — i.e. are valid even if perfect analogs, such as perfect electrolytic tanks, are used to solve Poisson's Equation.

2. Statement of the Problem

The electrode system or boundary enclosing the region of interest is sketched in Fig. 1. We will assume in our discussion that the problem is two-dimensional and that there is a constant magnetic field in the third direction. The argument continues without change in an axially symmetric system. The boundary is made up of a cathode and any number of other conductors. It is assumed that the current will so adjust itself, that the electric field at the cathode is zero. It is then necessary to satisfy Poisson's equation, the Lorentz force law, and the equation of charge continuity. These may be written in the form:

\[
\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} = - \frac{\rho}{\varepsilon_0}
\]  

(2.1)
\[
\begin{align*}
\frac{dv_x}{dt} &= \eta (\frac{\partial \phi}{\partial x} - v_y B) \\
\frac{dv_y}{dt} &= \eta (\frac{\partial \phi}{\partial y} + v_x B)
\end{align*}
\]

(2.2)

and

\[
I = \text{const. in the beam ;}
\]

(2.3)

Here \( \phi \) is the potential, \( \rho = \) the space-charge density, \( v_x, v_y \), the components of the velocity, \( t \) is time, \( I \) is the total current in the beam, \( \eta \) is the numerical value of the ratio of charge to mass of the particle, and \( \varepsilon_0 \) is the dielectric constant of free space.

In Eq. (2.1) the boundary conditions are that the potential is known on a closed boundary; in Eq. (2.2), \( v_x \) and \( v_y \) are zero at the cathode. In Eq. (2.3) the constant value of \( I \) is obtained from the factor of proportionality in the relation, valid near the cathode for space-charge-limited flow,

\[
\phi \sim C \xi^{4/3}
\]

(2.4)

where \( \xi \) is the distance from the cathode. It may be shown that \( C^{3/2} \) and \( I \) are proportional\(^2\).

Equations (2.1) - (2.3) together with the above equation are sufficient to define a unique solution. The procedure of solution is basically that described previously in Refs. 1-7.

3. Methods of Solution

Each of Eqs. (2.1) and (2.2) may be put in the form of a difference equation, and then solved by an iterative technique. If the value of potential, charge density etc. is \( \phi^{(n)} \), \( \rho^{(n)} \) etc. after the \( n \)th iteration, Eq. (2.1) can be written
\[ \frac{\partial^2 \phi^{(n)}}{\partial x^2} + \frac{\partial^2 \phi^{(n)}}{\partial y^2} = -\rho^{(n)}/\varepsilon_0. \]  

(3.1)

If we divide the cathode into segments \( A_0 \rightarrow A_m \) and call the part of the beam coming from \( A_i \) to \( A_{i+1} \) the \( i \)th beam, then the trajectory of the centre of the \( i \)th beam after the \( n \)th iteration can be written

\[
\begin{align*}
\frac{dx_i^{(n)}}{dt} &= v_{xi}^{(n)}, \\
\frac{dy_i^{(n)}}{dt} &= v_{yi}^{(n)} \\
\frac{dv_{xi}^{(n)}}{dt} &= \eta(\partial \phi^{(n)}/\partial x - v_{xi}^{(n)} B) \\
\frac{dv_{yi}^{(n)}}{dt} &= \eta(\partial \phi^{(n)}/\partial y + v_{yi}^{(n)} B)
\end{align*}
\]  

(3.2)

From the form of Eq. (3.2), with constant \( B \), it is seen that the programme is appropriate to sheet beams with a crossed, constant, magnetic field or to axially-symmetric problems with an azimuthal magnetic field. However, it can easily be modified for other more general problems.

The equation of charge continuity is satisfied by keeping the current in the \( i \)th beam constant. We consider all the current in the \( i \)th beam as concentrated at its centre; therefore we will denote the \( i \)th beam as the \( i \)th ray. The interior region is divided into a rectangular mesh as shown in Fig. 1. Then the current density in the \( p \)th cell after the \((n+1)\)th iteration is \( \rho_{p}^{(n+1)} \) where

\[
\rho_{p}^{(n+1)} = \sum_i \Gamma_{i}^{(n)} \mathcal{L}_{i p}^{(n)} v_{i p}^{(r)} ,
\]  

(3.3)

where \( v_{i p}^{(n)} \) is the speed of the \( i \)th ray in the \( p \)th cell after the \( n \)th iteration, \( \Gamma_{i}^{(n)} \) is the total current in the \( i \)th beam on the \( n \)th iteration, and \( \mathcal{L}_{i p}^{(n)} \) is the length of the \( i \)th ray in the \( p \)th cell on the \( n \)th iteration. It may be shown theoretically that \( v_{i p}^{(n)} \) is proportional to \( \sqrt{\rho_{p}^{(n)}} \).
Explicitly \( I_i^{(n+1)} \) can be computed from the expression 6)

\[
I_i^{(n+1)} = (4 \sqrt{2/9}) \varepsilon_0 \xi \left( \sqrt{\rho(n)} \right)^3 / \xi^2 \]

(3.4)

where \( A_i \) is the area of the cathode contributing to the \( i \)th ray, \( \xi \) is the distance normal to the cathode and the suffix \( i \) means the ratio \( \sqrt{\rho(n)} / \xi^2 \) is evaluated near the centre of the cathode of the \( i \)th ray.

Equations (3.1) - (3.3) allow us to set up an iteration procedure. First we solve Eq. (3.1) by a relaxation technique assuming \( \rho_i^{(n)} \) known, this allows us to find \( \varphi_i^{(n)} \). By tracing trajectories, from Eq. (3.2), \( V_i^{(n)} \), \( I_i^{(n)} \) are found. From the behaviour of \( \rho_i^{(n)} \) near the cathode, \( I_i^{(n)} \) may be found and then, using Eq. (3.3), \( \rho_i^{(n+1)} \) deduced. If the required accuracy has not been achieved, the whole cycle may be repeated.

The general procedure needs little comment; however, a little will be said about the iteration procedures used, the convergence criteria, and the initial conditions which may be used.

4. Over- and Under Relaxation and Convergence

At two stages in the use of the equations of section 3, instead of the non-linear differential equations of section 2, iterations methods must be used. The digital computer can only solve difference equations, and the differential equation of Eq. (2.1) can be approximated by the difference equation 8), for known \( \rho \),

\[
R \equiv D[\varphi] - \rho / \varepsilon_0 = 0
\]

(4.1)

Here \( D[\varphi] \) is the difference operator equivalent to the Laplacian, and \( R \) is related to the residue. The usual method of solving Eq. (4.1) by iteration techniques is to set up the iteration procedure.
\[ \psi^{(m)} = \psi^{(m-1)} + R^{(m)}. \]  

(4.2)

In Eq. (4.2), \( \psi^{(m)} \) is the potential after the \( m \)th application of Eq. (4.1). It is well known that convergence can be speeded up by over-relaxation \(^8\). In this method, Eq. (4.2) is replaced by

\[ \psi^{(m)} = \psi^{(m-1)} + \beta R^{(m)}. \]  

(4.3)

Here \( \beta \) is a constant, between 1 and 2. Various methods of computing \( \beta \) have been suggested. The method used in our computer programme is completely automatic; it is described elsewhere \(^8\).

However, in our problem it is not Eq. (2.1), with known \( \rho \), which is to be solved, but Eq. (3.1). If the result of the \( m \)th application of Eq. (4.1) to the potential of Eq. (3.1) is \( \psi^{(m,n)} \), and \( R^{(m,n)} \) is similarly defined, then Eq. (4.3) becomes

\[ \psi^{(m,n)} = \psi^{(m-1,n)} + \beta R^{(m,n)}. \]  

(4.4)

Each application of Eq. (4.4) is termed a minor iteration cycle. Equation (4.4) is applied until the maximum change in potential is less than some preset value \( \epsilon \), i.e.

\[ \text{Max} \left| 1 - \frac{\psi^{(m,n)}}{\psi^{(m-1,n)}} \right| < \epsilon. \]  

(4.5)

The estimated maximum error in \( \psi^{(m,n)} \), as solution to Eq. (3.1), is now proportional to \( \epsilon \).

The \( \psi^{(m,n)} \) satisfying Eq. (4.5) is now made the first guess of a new major iteration cycle, i.e. the definition is now made

\[ \psi^{(0,n+1)} \equiv \psi^{(m,n)}. \]  

(4.6)
Also with this \( \rho^{(0, n+1)} \), a new set of trajectories are computed. Using Eq. (3.3), a new set of \( \rho, \rho^{(n+1)}_p \) may be deduced, and Eq. (3.1) solved for this new \( \rho \). Now the use of the over-relaxation helped i.e. the use of Eq. (4.2) instead of Eq. (4.3) improved the convergence of the solution of Eq. (4.1). In a similar way if the left-hand-side of Eq. (3.3) is termed \( \rho^{(n+1)}_p \), an under-relaxation process

\[
\rho^{(n+1)}_p = \rho^{(n)}_p + \gamma (\rho^{(n+1)}_p - \rho^{(n)}_p),
\]

(4.7)

may be used. Here \( \gamma \) is a constant, less than or equal to unity, and is termed the under-relaxation coefficient for major iterations. If \( \gamma = 1 \), Eq. (4.7) is identical to Eq. (3.3).

Alternately Eq. (3.3) can be used directly, but the under-relaxation procedure applied to the current in each ray. In this case if the left-hand side of Eq. (3.4) is termed \( I^{(n+1)}_i \), the current in the \( i^{th} \) ray is computed from the expression

\[
I^{(n+1)}_i = I^{(n)}_i + \gamma (I^{(n+1)}_i - I^{(n)}_i).
\]

(4.8)

Both under-relaxation of charge density, using Eq. (4.7), and of cathode current, with Eq. (4.8), were tried. Both worked satisfactorily, but the method of Eq. (4.8) converged slightly faster.

For the convergence of the major iteration cycle to \( \epsilon \), we used the criterion that Eq. (4.5) should be satisfied after one minor iteration cycle, i.e. the whole process was taken to converge to \( \epsilon \) if

\[
\max \left| 1 - \rho^{(1,n)}/\rho^{(0,n)} \right| < \epsilon.
\]

(4.9)

Since at first the space-charge densities might change considerably between major iterations, the \( \epsilon \) during the first major iteration was taken
to be .05, and then was progressively halved each subsequent major iteration until it was equal to the desired value. For subsequent iterations the desired value of ε was used. This easing of accuracy requirements in the early major iterations, considerably speeds up the computation.

5. The Initial Conditions

Normally both ϕ and ρ are initially set zero everywhere inside the region of interest. However, if a similar problem has been solved before with the same mesh there is provision for feeding in the ϕ and ρ from a magnetic tape which has been saved from a previous run.

At the first major iteration, there is provision for feeding in a guessed value of cathode current density. This provision is useful since computing I_1(1) from a Laplacian solution ϕ(1) may be much more inaccurate than using a guessed value.

At the first minor iteration β is put unity. Thereafter it is evaluated automatically as described in Ref. 8. At the first major iteration, γ is put unity. Thereafter a constant value, specified in the input, is used.

The velocity, electric field, etc. of a ray near the cathode are found from assuming one has a set of planar diodes near the cathode. Equation (2.2) is integrated up to one mesh diameter from the cathode, using the appropriate series expansion. Throughout the integration, a constant time interval is used.

6. Restrictions and Speed of the Programme

There are several restrictions on the number of mesh points which may be treated. The principal one is that the total number of mesh points must be less than about 3000.
The time taken for running problems obviously depends on the number of mesh points, number of rays considered, and the number of iterations required. All times quoted are for an IBM computer. The time per mesh point to set up the grid is 1.8 min, while for one minor iteration is .045 min/mesh point, and the time per step of ray tracing .7 min. In a typical problem to be described in the next section, there were 600 mesh points inside the electrode system, and 90 integration steps per trajectory; 13 rays were used. 11 major iteration cycles and 66 minor iterations were required to converge to 1% accuracy. Under these conditions the total computer time required was about 13 minutes. This particular problem was started with no assumed knowledge of the potential or space-charge distribution inside the electrode system.

This example illustrates that most of the time is usually spent in the ray tracing part. For this reason it is important to reduce the number of major iteration cycles to a minimum. In this respect the situation is identical when electrolytic tanks or resistance networks are used to solve Poisson's Equation.

7. The Two Problems Studied Numerically

Two electrode systems were analysed in detail. The electrode systems are shown in Figs. 1 and 2. Figure 1 shows an axially symmetric gun, with an electrode system due to Frost et al. This gun has been tested experimentally and reported to have a perveance of 2.2 and area convergence of 30°. The gun of Fig. 2 is a sheet beam, crossed-field, gun which has been analysed exactly by several authors (e.g. Ref. 10). These two electrode systems will be called Guns 1 and 2 in the subsequent discussion. Both the guns are discussed because they have different features. Gun 1 has a comparatively rectilinear flow, and is axially symmetric. Its solution is more dependent on the "under-relaxation coefficient" $\gamma$ of Eq. (4.8) than that of Fig. 2. However, its exact solution is unknown, and therefore the information to be gleaned from analysing this system can only give relative results.
Gun 2, being analysable exactly, allows a direct investigation of the errors incurred by using different parameters in the computer programme. For this reason, most of the absolute comparison will be with Gun 2.

a) The Axially Symmetric Gun

The actual trajectory for the beam edge found by the programme is shown in Fig. 1. In the scale of Fig. 1, a mesh of $\Delta r = \Delta z = 0.4$ was used; the results were hardly altered by using a smaller mesh. The final computed perveance varied between 2.05 and 2.15, depending on the number of rays, mesh size, guessed initial perveance, etc. This compares with a measured value of 2.29). The calculated convergence at the plane of the grid gap was 110; the measured perveance was 3009). Since the beam is still converging at the anode of Fig. 1, the calculated convergence seems to agree satisfactorily with the measured.

In an experimental gun, it is difficult to estimate the current density variation across the cathode. Taking the electrode system of Fig. 1, the computed relative current density variation across the cathode is shown in Fig. 3, taking different numbers of rays across the cathode. It is seen that the current density in the centre of the beam is only 0.35 of that at the edge. A better gun could clearly be designed by making the gun a little bit more elliptical, i.e. bringing the cathode centre nearer the anode.

With 15 rays, and the $\Delta r, \Delta z$ used, there was one ray per cell at the cathode. From Fig. 3, it is clear that taking substantially fewer rays had little effect on the cathode current density: the effect on the trajectories was negligible.

In order to converge at all to 1 o/o accuracy, it was necessary that the number of rays be sufficient to have at least one ray per cell at the cathode. If the number of rays were increased beyond this, no appreciably greater speed of convergence or change in beam pattern, resulted.
Starting with no knowledge of the potential or space-charge distribution, the analysis leading to Figs. 1 and 3 took 10 mins on an IBM 709 computer. However, in order to analyse a modified electrode system -- i.e. the effects of changing the cathode -- the potential and space-charge of Fig. 1 can be used as a first guess (it already exists on magnetic tape in the correct form). Instead of the 75 minor iterations required to obtain Fig. 1, the modified electrode system only required 10-15 to converge to the same accuracy -- reducing the computation time substantially.

It is instructive to investigate the dependence of the convergence on the under-relaxation coefficient \( \gamma \) of Eq. (4.8). In Fig. 4 are given plots of the computed permeance versus the number of major iterations -- with the value of \( \gamma \) as a parameter. The convergence was quickest for a \( \gamma \) of 0.75. The cases with \( \gamma = 0.75 \) converged to an accuracy of 1 o/o, as determined by Eq. (4.9), in 5 iterations; the other cases had not converged to this accuracy after 8 iterations. In most cases the convergence to 1 o/o accuracy was only possible by choosing a \( \gamma \) between 0.7 and 0.9.

b) The Crossed-Field Gun

The exact theoretical electrode system for this case was used. This electrode system, together with the theoretical equipotentials and the beam edges, is shown in Fig. 2. Superposed on the exact solutions are computed equipotentials and beam edges. This computation is for a mesh \( \Delta x, \Delta y \) of 0.3 and 25 rays. It is seen that the theoretical and computed trajectories and computed trajectories and equipotentials are almost identical except right near the anode for one edge.

Several measures of the accuracy of the solution may be considered. First, in Fig. 5, plots are given of the percentage error in potential as a function of distance along the central trajectory, with the number of the iteration as parameter. The normalised theoretical potential as a function
of arc length is given in Fig. 6. One overall measure of accuracy is the root-mean-square error of the potential, as averaged over most of the trajectory. Here the part very near the cathode was excluded, since small absolute errors in potential would greatly distort the mean error. The effects of errors near the cathode are discussed in the next two paragraphs. In Fig. 7 this mean percentage error of potential is plotted versus the number of major iterations. The ultimate accuracy attained (after 20 iterations) was 0.15 o/o. These figures all refer to the central trajectory, the accuracies on the edges were a factor of three worse.

A second measure of the accuracy of the solution is the behaviour of the beam near the cathode. Here there are two criteria, the total pervance, and variation of current density along the cathode. In Fig. 8 are shown plots of the ratio of total pervance to theoretical pervance as a function of the number of iterations. The different curves refer to different mesh sizes. It is seen that the final pervance approaches the theoretical one as the mesh size tends to zero. In Fig. 9 are drawn the same curves for one particular mesh size but with different under-relaxation coefficients $\gamma$. All curves are terminated when the computation has converged for $\varepsilon = 0.01$ in Eq. (4.5). It is seen that although convergence is faster and smoother for $\gamma = 0.8$, convergence is achieved almost as well for $\gamma = 0.6$ and $\gamma = 1.0$. Also shown in Fig. 9 is a second curve with $\gamma = 0.8$ and a different initial guess to the pervance density. It can be seen, and additional computation verified, that the two curves asymptote to substantially different total perveances.

The variation of current density across the cathode for different mesh sizes (and one ray per mesh cell at the cathode) is shown in Fig. 11. It is seen that the cathode current density distribution approaches the theoretical as the mesh size decreases -- but that the current density errors at one edge remain serious. In Fig. 11 are given curves similar to those of Fig. 10, but now one mesh size is chosen and the number of rays across the cathode is varied. One ray per cell at the cathode implies, in this case, 13 rays in all.
It is seen that for 7 or 5 rays there is little additional error in the cathode current density, though for 3 rays the picture somewhat changes.

A third measure of the accuracy of the solution is to compare the theoretical and computed trajectories. In Fig. 12 are shown the edge and central trajectories for various mesh sizes. It is seen that the trajectories approach the theoretical ones as the mesh size tends to zero. The outside trajectory just misses the anode, for $\Delta x = \Delta y = .3$ or .5, so that this trajectory seems to be considerably in error. In actual fact, this missing of the anode by a part of the beam in this situation may well result physically if there are small errors near the cathode. None of the interior rays, even for the 13 or 25 ray cases, miss the cathode. To see the effect of reducing the number of rays, Fig. 13 shows trajectories similar to those of Fig. 12 for constant mesh size and different numbers of rays at the cathode. Again thirteen rays correspond to one per cathode cell. The difference between using 7, 13, or 19 rays cannot be seen on the scale of these graphs.

Conclusions

The type of computer programme described in Refs. 3-7 has clearly supplanted the use of electrolytic tanks and resistance networks for designing high permeance electron and ion guns. The programmes are capable of analysing complicated electrode systems in a comparatively short time on a fast computer to any practical accuracy. Moreover, the solutions reached are probably those the experimental beam will satisfy. Since the numerical process must converge to the final solution, if a theoretical flow is unstable for small errors, the iteration process will not converge to that solution. From Fig. 10, for example, it seems quite likely that the cathode current in a gun with the electrode system of Fig. 1 would not have constant current density, but would draw more current from the leading edge. Similarly, from Fig. 12, it seems quite likely that a few particles, from the trailing edge
of the cathode, would miss the anode. These results have been experimentally observed; for example in Ref. 10 (where this electrode system was experimentally tested) it was found necessary to tilt the cathode for maximum transmission.

Much computer time can be saved by using care in how this type of programme is used. The discrete nature of the way space-charge is introduced (cf. Eq. 3.3) is only approximate. However, Figs. 4, 11, 13 show that satisfactory results can be obtained with remarkably few rays -- particularly if a constant perveance gun is being designed. However, if a few rays are used, the convergence criteria of Eq. (4.5) may be difficult to fulfil for a small $\varepsilon$. The choice of an under-relaxation coefficient of about 0.8 may greatly speed up convergence.

Inspection of Figs. 4, 8, and 9 shows that a first guess for the perveance some 30 o/o lower than the actual one leads to more rapid convergence. This is due to the variation of Laplacian versus Langmuir potentials near a cathode. In addition, the final solution is somewhat dependent on the initial guess for the perveance density, and seems better if the initial guess is somewhat too low.

A uniqueness theorem exists to prove that there is only one solution of Eq. (2.1) with given $\rho$ and $\phi$ on the boundary. However, there is no uniqueness theorem known to the authors that there is only one solution of Eqs. (2.1) - (2.4) with given $\phi$ on the boundary. In fact the numerical evidence seemed to show that the methods could lead to convergence to slightly different solutions. Whether this is only dependent on the combination of difference methods used has not been decided.

A combination of the computations made with small numbers of rays and the time estimates of section 6 indicate that it is usually most economical of computer time to make preliminary computations to comparatively crude
accuracy, with small numbers of rays (and large mesh sizes). The results are then very good approximations for more elaborate and accurate computations --- even with perturbed boundaries.

Let us give some examples of this. It required 80 minor and 6 major computations to achieve 1.5 o/o accuracy with 5 rays with one set of parameters of the electrode system of Fig. 2. It only required a further 8 minor and 3 major iterations to achieve 1.5 o/o with 19 rays. To achieve 1.5 o/o with 19 rays directly required 70 minor and 6 major iteration cycles. This resulted in a saving of about 40 o/o in machine time. Even greater savings arise when the potential of one electrode system is used as a guess for a slightly different one. The computer time is often reduced by a factor of two or three.

Most of the above considerations apply whether or not computer programmes are used. They also apply when electrolytic tanks or resistance networks are used to solve Poisson's equations. However, the feeding in of initial guesses is simpler when the process has been followed on a computer, because the previous results should already be available on magnetic tape.

It is often desirable to use meshes in which \( \Delta x \neq \Delta y \), because greater discrimination is required in one direction than another. For example with the electrode system of Fig. 2, the cathode is along the x-axis, so that the behaviour near, and perpendicular to, the cathode is more critical. Therefore it is more satisfactory to use a mesh-spacing in the x-direction which is two or three times as coarse as that in the y-direction. With an electrolytic tank this problem would not arise, but with a resistance network it must be foreseen in the construction of the network. In a computer programme, the modification can be made with one number.

In general the programmes of this type are immensely useful for gun design, and can be used by engineers on a production basis. However, if
economy of machine time is a consideration, care should be taken in the choice of parameters used. By choosing an unnecessarily large number of rays, trajectory steps, or an unnecessarily small mesh, one can increase the required machine time by an order of magnitude. As the experience from the production use of these programmes becomes available, more of these parameters can be chosen automatically. It is not even difficult to imagine that they could be used to adjust their boundaries automatically to ensure criteria such as that the cathode current density be constant.
References:


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9. The percentage error in total perveance of the beam from the electrode system of Fig. 2 plotted versus the number of major iterations. The different curves refer to different values of the under-relaxation $\Delta x = \gamma$ of Eq. (4.8). These curves were for a mesh size of $\Delta y = 1.0$ so that the final perveance was always 12 o/o greater than the theoretical. The two curves for $\gamma = 0.8$ refer to different initial guesses to the perveance density.

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13. Theoretical and calculated trajectories for the beam from the electrode system of Fig. 2. The different trajectories refer to taking different numbers of rays along the cathode.
Fig. 1.

Linear Potential Variation Assumed
Beam Forming Electrode
Beam Edge
Cathode
Trajectories
Beam Edge
Anode

0 1.0 2.0 3.0 4.0 5.0 Z axis
Fig. 3

DISTANCE ALONG CATHODE

RELATIVE CURRENT DENSITY

4 RAYS
8 RAYS
15 RAYS

-1.0
-0.5
0
0.5
1.0
Fig. 6.

THEORETICAL NORMALISED POTENTIAL $\phi_{th.}$

NORMALISED DISTANCE ALONG TRAJECTORY.

 Signature: G.S.
Fig. 7.

Mean percentage error of potential versus number of iterations.
Fig. 8.

- $\Delta x = \Delta y = 0.3$
- $\Delta x = \Delta y = 0.5$
- $\Delta x = \Delta y = 1.0$
Fig. 9.

Percentage Error in Perveance

Number of Iterations

γ = 1.25
γ = 1.0
γ = 0.8
γ = 0.6
γ = 0.8
Fig. 10.

\[ \Delta x = \Delta y = 1.0 \]
\[ \Delta x = \Delta y = 0.5 \]
\[ \Delta x = \Delta y = 0.3 \]
PERCENTAGE ERROR IN CURRENT DENSITY.

DISTANCE ALONG CATHODE IN UNITS.

Fig. 11.

13 Rays
19 Rays
7 Rays
5 Rays
3 Rays
Fig. 12.

Theoretical

$\Delta x = \Delta y \cdot 3$

$\Delta x = \Delta y \cdot 5$

$\Delta x = \Delta y \cdot 10$

0 2 4 6 8 10 12 14 16 18 20 22 24

CATHODE

6 5 4 3 2 1