Control Problem for Nonlinear Systems Given by Klein-Gordon-Maxwell Equations with Electromagnetic Field

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Abstract—This paper is aimed at realizing control for systems described by Klein-Gordon-Maxwell (K-G-M) equation. Theoretic approach will be formulated in the framework of variational theory. On the other hand, computational insight using semi-discrete numerical algorithm is consist of finite element method. Lastly, numerical experiments are evident the completely combination of theoretic and computation aspects.

I. INTRODUCTION

A. Physics background

Using the Klein-Gordon-Maxwell electrodynamics, Schrödinger demonstrated that charged particles may be described by real fields. The rationale are considered for the Klein-Gordon-Maxwell electrodynamics, where the sets of solutions with real-valued electron-positron fields. Schrödinger considering interacting scalar charged field and electromagnetic field, and the Klein-Gordon-Maxwell equations of motion (cf. [3], [7], [16]).

B. Problem description

As is well known that the Klein-Gordon-Maxwell equations is extensively studied using numerous methodologies. Such as assigned electromagnetic fields in [1], [2] and [6], the interacting with an electromagnetic fields see references [4], [5], [8], [9], [10], [11] and [13]. Literature survey show some formulations of K-G-M equation for physics consideration. For example, here is the most common description:

\[ D_\mu D^\mu \phi = e^2 \phi + \frac{Q}{c} \frac{\partial^2}{\partial r^2} F_{\mu \nu} = \frac{1}{c^2} \mathcal{I} (\phi D_\mu \phi) \]

where \( e \) is the charge of the field, \( \phi \) is the electromagnetic field tensor, and \( \mathcal{I} \) is the imaginary part. For control problem, a new description will be adopted.

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^3 \). Set \( Q = (0, T) \times \Omega \). The Klein-Gordon-Maxwell systems is described by

\[
\begin{align*}
-\psi_{xx} + e^2 \phi^2 \psi &= -e \omega \phi^2 + u, \\
-\frac{\hbar^2}{m} \phi_{xx} + [m^2 - (\omega + e \psi)^2] \phi &= |\psi|^{p-2} \phi + v,
\end{align*}
\]

(1)

where \( \psi, \phi : \mathbb{R}^3 \rightarrow \mathbb{R} \). Here \( m > 0 \) and \( e > 0 \) are the mass and the charge of the particle respectively, while \( \omega > 0 \) denotes the phase, and \( \hbar \) is the Planck’s constant. The variables of the system are the field \( \psi, \phi \) associated to the particle and the electric potential. Here in (1), \( u \) and \( v \) are control inputs, and it’s meaningful to make the assumption \( m > \omega > 0 \) and \( 2 \leq p < 6 \) for infinitely many radially symmetric solutions having bounded energy. The presence of the nonlinear term simulates the interaction between particles or external nonlinear perturbations. In [8] the regularity are \( \psi \in H^1(\mathbb{R}^3), \phi \in D^{1,2}(\mathbb{R}^3) \), where \( D^{1,2}(\mathbb{R}^3) \) is the completion of \( C^\infty_c(\mathbb{R}^3, \mathbb{R}) \) with respect to the norm of \( \| \phi \|_{D^{1,2}} \equiv \left( \int_{\mathbb{R}^3} |\nabla \phi|^2 dx \right)^{1/2} \). However the more common regularity is considered in our case.

This work is to explore the control problem for Klein-Gordon-Maxwell equations using control theory based on variational approach (cf. [15]). Furthermore, the computational issue is involved for one-dimensional case. The new theoretical contribution of the paper will be carried out to control of two-particles control and its numerical realization.

The contents of this paper is consist of several sections. Section II is to establish the theoretic control theory for Klein-Gordon-Maxwell system. Section III will explore the numerical study of K-G-M equations with finite element approximate. In section IV, the laboratory simulation is carried out for interpreting the established control theory. Section V give concluding remark and future work.

II. CONTROL THEORY FOR K-G-M SYSTEMS

A. Weak solution

Define two Hilbert spaces \( H = L^2(\Omega) \) and \( V = H^1_0(\Omega) \) with usual norm and inner products. Then the pairing \( (V, H) \) is a Gelfand triple space, \( V \hookrightarrow H \hookrightarrow V' \), their embedding are continuous, dense and compact.

Definition 2.1: The Hilbert space \( W(0, T) \) is solution space defined by

\[
W(0, T) = \left\{ (\psi, \phi) : \psi \in L^2(0, T; V), \phi \in L^2(0, T; V'), \phi \in L^2(0, T; V') \right\}.
\]

Definition 2.2: Let \( T > 0 \), the pairing \( (\psi, \phi) \) are weak solutions of (1) when \( \psi, \phi \in W(0, T) \) satisfy

\[
\begin{align*}
\int_0^T \int_{\Omega} \psi \phi_t dx dt &= e^2 \phi^2 \psi \eta dx dt \\
-\int_0^T \int_{\Omega} e \omega \phi^2 \phi dx dt &= \int_0^T \int_{\Omega} \eta dx dt \\
\frac{\hbar^2}{m} \int_0^T \int_{\Omega} \phi_t \phi_{xx} + [m^2 - (\omega + e \psi)^2] \phi \phi dx dt &= \int_0^T \int_{\Omega} \psi |\psi|^{p-2} \phi dx dt \\
\int_0^T \int_{\Omega} (\psi |\psi|^{p-2} \phi) dx dt &= \int_0^T \int_{\Omega} \psi v dx dt
\end{align*}
\]

for all \( \eta, \rho \in C^1(0, T; V) \) and such that \( \eta(T) = \rho(T) = 0 \) a.e. \( t \in [0, T] \).
Theorem 2.3: Given \( \psi_0, \phi_0 \in V \), there then exists a unique weak solution of system (1).

The proof of Theorem 2.3 can be completed referring to Faedo-Galerkin method in [12].

B. Control problem

Let \( u = (u, v) \), and \( \mathcal{U} \) is control space of variables \( u, v \in \mathcal{U} \). The optimal criteria associated with (1) is given by

\[
J(u) = \|\psi(u, T) - z_d^1\|^2_V + \|\phi(u, T) - z_d^2\|^2_V + (u, u)_{\mathcal{U}},
\]

(3)

for all \( u \in \mathcal{U}_{ad} \times \mathcal{U}_{ad} \), where \( z_d^1, z_d^2 \in V \) are desired values of \( \psi(u) \) and \( \phi(u) \), respectively. Let \( \mathcal{U}_{ad} \) be a closed and convex subset of \( \mathcal{U} \), which called the admissible set.

Suppose \( \mathcal{U} = L^2(0, T) \), then Theorem 2.3 deduce that there exists a unique weak solution \( \psi(u), \phi(u) \in W(0, T) \) for any \( u \in \mathcal{U} \), furthermore, by the analogous manipulation as in [15] and [12] to prove Theorem 2.4 and 2.5.

Theorem 2.4: Let \( \psi_0, \phi_0 \in V \). If \( \mathcal{U}_{ad} \) is bounded, then there exists at least one optimal control \( u^* = (u^*, v^*) \) for cost (3) subject to systems (1).

The optimal control \( u^* \) for cost (3) is characterized by optimality system consisting of state equation (1), adjoint equation (4) and necessary condition (5):

\[
\begin{align*}
-p_{xxxx} + e^2 \phi'(u^*)p &= -2[w + ev\psi'(u^*)]\phi(u^*)q + u_0 \quad \text{in } Q, \\
-\frac{5}{2}q_{xxx} + [2m^2 - (\omega + ev\psi'(u^*))^2]q &= 2[w + ev\psi'(u^*)]\phi(u^*)p + \phi(u^*)p^{p-2}q + v_0 \quad \text{in } Q, \\
p(T) = \psi(u^*, T) - z_d^1 \quad \text{in } (0, l), \\
q(T) = \phi(u^*, T) - z_d^2 \quad \text{in } (0, l), \\
p, q \in W(0, T), \\
(u^*, u - u^*)_{\mathcal{U}} + (v^*, v - v^*)_{\mathcal{U}} + \int_Q p(u^*)(u - u^*) \, dx \, dt \quad &+ \int_Q q(u^*)(v - v^*) \, dx \, dt \geq 0 \quad \forall u = (u, v) \in \mathcal{U}_{ad}^2.
\end{align*}
\]

The highlighted point in this section is attempting firstly to seek theoretic control conclusions of quantum optimal control for two particles system described by KG equations.

III. NUMERICAL STUDY

A. Numerical solution

Let \( x_0 = x_1 < \cdots < x_N < x_{N+1} = l \) be a partition of the interval \([0, l]\) into subintervals \( I_e = [x_{e-1}, x_e] \) of length \( h^e = x_e - x_{e-1}, e = 1, 2, \ldots, N + 1 \). Let \( V_h \) be a set of functions \( b^e_i \) for \( i = 1, 2, 3, e = 1, 2, \ldots, N + 1 \) such that \( b^e_i \) is quadratic function on each interval \( I_e \), and continuous on \([0, l]\). Then it’s clear that \( V_h \subset H^1(0, l) \) (cf. [18]). The \( b^e_i \in V_h \) is given by

\[
\begin{align*}
b^e_1(x) &= \left( 1 - \frac{x - x_e}{h^e} \right) \left( 1 - \frac{2(x - x_e)}{h^e} \right), \\
b^e_2(x) &= \frac{4(x - x_e)}{h^e} \left( 1 - \frac{x - x_e}{h^e} \right), \\
b^e_3(x) &= -\frac{(x - x_e)}{h^e} \left( 1 - \frac{2(x - x_e)}{h^e} \right).
\end{align*}
\]

Its interpolation properties see [17]. The total approximate solution can be represented as

\[
\begin{align*}
\psi_h(t, x) &= \sum_{i=1}^{N} \psi_i^e(t, x) = \sum_{i=1}^{N} \sum_{j=1}^{3} \xi_i^e(t) b^e_j(x) \in V_h, \\
\phi_h(t, x) &= \sum_{i=1}^{N} \phi_i^e(t, x) = \sum_{i=1}^{N} \sum_{j=1}^{3} \zeta_i^e(t) b^e_j(x) \in V_h.
\end{align*}
\]

Thus by (2) to find \( \psi_i^e \) and \( \phi_i^e \) satisfy

\[
\begin{align*}
\sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e) &= e^2 \left( \sum_{i=1}^{3} \zeta_i^e(b_i^e, b_j^e)^2 \right) \left( \sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e)^2 \right) \\
&= -e\omega \left( \sum_{i=1}^{3} \zeta_i^e(b_i^e, b_j^e)^2 \right) \left( \sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e)^2 \right) \\
&\quad \left( \frac{h}{m} \sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e) + [m^2 - (\omega + e \sum_{i=1}^{3} \zeta_i^e(b_i^e, b_j^e)^2)] \sum_{i=1}^{3} \zeta_i^e(b_i^e, b_j^e) \right) \\
&= \left( \sum_{i=1}^{3} \zeta_i^e(b_i^e, b_j^e)^2 \right) \left( \sum_{i=1}^{3} (v, b_i^e) \right),
\end{align*}
\]

with \( \sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e) = \psi_0, \sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e) = \phi_0 \) and

\[
\sum_{i=1}^{3} \xi_i^e(b_i^e, b_j^e) = \phi_1.
\]

Set

\[
B^c = \{(b_i^e_j) \in (b_i^e_j)^{j=1,2,3}_{i=1,2,3} \in M_{3 \times 3}(R),
\]
\[
D^c = \{(a_i^e_j) \in (a_i^e_j)^{j=1,2,3}_{i=1,2,3} \in M_{3 \times 3}(R),
\]
\[
E^c = \{(e_i^e_j) \in (e_i^e_j)^{j=1,2,3}_{i=1,2,3} \in M_{3 \times 3}(R),
\]
\[
N^c(t) = (n_{11}^c n_{12}^c n_{13}^c) \in M_{3 \times 1}(R),
\]
\[
X^c(t) = (X_{i1}^c, X_{i2}^c, X_{i3}^c) \in M_{3 \times 1}(R),
\]
\[
Y^c(t) = (Y_{i1}^c, Y_{i2}^c, Y_{i3}^c) \in M_{3 \times 1}(R),
\]
Let’s introduce the following matrices and vectors.

\[
D = \begin{bmatrix}
  d_{11} & d_{12} & d_{13} \\
  d_{21} & d_{22} & d_{23} \\
  d_{31} & d_{32} & d_{33} + d_{41} \\
  \vdots & \vdots & \vdots \\
  d_{m1} & d_{m2} & d_{m3}
\end{bmatrix}
\]

Hence \( B \) has the same structure as \( D \) with \( b_{ij} \) instead of \( d_{ij} \).

\[
\Xi = \begin{bmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_{2N-1} \\
  \xi_{2N}
\end{bmatrix}, \quad \Sigma = \begin{bmatrix}
  \zeta_1 \\
  \zeta_2 \\
  \zeta_3 \\
  \vdots \\
  \zeta_{2N-1} \\
  \zeta_{2N}
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
  L_{11} & L_{12} & L_{13} \\
  L_{21} & L_{22} & L_{23} \\
  L_{31} & L_{32} & L_{33} \\
  \vdots & \vdots & \vdots \\
  L_{2N1} & L_{2N2} & L_{2N3}
\end{bmatrix}, \quad U = \begin{bmatrix}
  U_{1} \\
  U_{2} \\
  U_{3} \\
  \vdots \\
  U_{2N-1} \\
  U_{2N}
\end{bmatrix}
\]

Hence \( L_2 \) has the same structure as \( L_1 \) with \( L_{2j} \) instead of \( L_{1j} \) and \( L_{2j}^T \) instead of \( L_{1j}^T \) \( V \) has the same structure as \( U \) with \( V_j \) instead of \( U_j \) and \( V_j^T \) instead of \( U_j^T \). The matrix \( N_1, N_2 \) have the same configuration with \( L_1, L_2 \), just their components are composed of \( N_1^T, N_2^T \), respectively.

\[
X_0 = \begin{bmatrix}
  X_{01} \\
  X_{02} \\
  X_{03}(= X_{01}^T) \\
  \vdots \\
  X_{2N1}(= X_{2N2}^T)
\end{bmatrix}, \quad Y_0 = \begin{bmatrix}
  Y_{01} \\\n  Y_{02} \\\n  Y_{03}(= Y_{01}^T) \\\n  \vdots \\\n  Y_{2N1}(= Y_{2N2}^T)
\end{bmatrix}
\]

Hence \( X_0 \) has the same structure as \( Y_0 \) with \( X_{0j}^T \) instead of \( Y_{0j}^T \). Then by (6), the overall equation in the vector form can be expressed as

\[
\begin{align*}
D\Xi + L_1(\Xi, \Sigma) &= N_1 + U, \\
\frac{h}{m} D\Sigma + L_2(\Xi, \Sigma) &= N_2 + V, \\
\Xi(0) &= X_0, \quad \Sigma(0) = Y_0.
\end{align*}
\]

(7)

As in [17], applying Gauss-Legendre integrate method to the components of \( g_i \), divide the element interval \([x_i, x_{i+1}]\) into \( m = 6 \) points to obtain the abscissa \( p_1, p_2, \ldots, p_m \) on \([x_i, x_{i+1}]\) and weights \( r_1, r_2, \ldots, r_m \). Thus \( S_1, S_2 \) are approximated by the new function \( \tilde{S}_1, \tilde{S}_2 \), respectively. By introducing

\[
M = \begin{bmatrix}
  D & 0 \\
  0 & \frac{h}{m} D
\end{bmatrix}, \quad L_1(\Xi, \Sigma) = \begin{bmatrix}
  L_{11}(\Xi, \Sigma) \\
  L_{21}(\Xi, \Sigma)
\end{bmatrix}, \quad N = \begin{bmatrix}
  N_1 \\
  N_2
\end{bmatrix}, \quad \Sigma = \begin{bmatrix}
  \Sigma_1 \\
  \Sigma_2
\end{bmatrix}, \quad U = \begin{bmatrix}
  U \\
  V
\end{bmatrix}, \quad \Sigma(0) = \begin{bmatrix}
  \Sigma(0)_1 \\
  \Sigma(0)_2
\end{bmatrix}
\]

with initial \( \Sigma(0) = [X_0, Y_0] \). Since the inverse of \( M \) is exists, then (7) implies that

\[
\Sigma = M^{-1}(-L + N + U).
\]

(8)

with initial guess \( \Sigma(0) \). Therefore, the first order ODE (8) can be solved using the 4th order Runge-Kutta method (cf.[18]). Using \( \xi_i \) and \( \zeta_i \) (i = 1, 2, 3) to obtain the numerical solution on domain \([0, T] \times (0, I)\). The converge of the approach refer to [17].

B. Numerical control solution

Let \( u_k = (u_h, v_h) \) be the approximate control of \( u = (u, v) \). The formulation of finite element approximation is by minimizing the approximate cost functions,

\[
J_h = J(u_h) = \int_0^T (u_h(T) - z_h)^2 dx + \int_0^T (v_h(T) - z_h)^2 dx
\]

\[
+ \int_0^T (u_h(t) - u_h(t')) dt + \int_0^T (v_h(t) - v_h(t')) dt.
\]

(9)

**Theorem 3.1**: The existence theorem of optimal control in [14] implies that there exists at least a minimizer to the finite element problems (9).

Denote the Gâteaux derivative of \( J_h \) at any point \( u_h \) and \( \psi_h \) in \( V_h \) by \( J_h'(u_h, \psi_h) \). The Gâteaux derivative for solution \( \psi_h \) at any direction \( \phi_h \) in \( V_h \) denotes as \( \psi_h', \phi_h \) satisfying (2). The discrete adjoint system and necessary optimality condition (5) for \( u^* = (u_h^*, v_h^*) \) in \( U_{ad} \times U_{ad} \) can be obtained easily.

**Theorem 3.2**: Let \( \{u_k^*\} \) be a sequence of minimizer to finite element problems (9). Then each subsequence of \( \{u_k^*\} \) has a sub-subsequence convergence in \( L^2(0, L) \) minimizer of the continuous problems (9).

C. Computational procedure

Suppose \( u_k^* = (u_k^*, v_k^*) \) is available at iteration \( k \), \( u_k = \{u_k^*\}, k = 1, 2, \ldots \) are minimize sequence of \( \{u_k^*\} \) such that the cost function (9) achieve minimization.

**Step 1** For given \( z^d = (z_1^d, z_2^d) \) and \( u(0) \) in \( U_{ad} \), using construct approximate solution \( \psi_h(x, t), \phi_h(x, t) \) to solve the directly problems for state equation for \( \psi \).

Let \( P_k(t) = -\nabla J(u(0)); k = 0 \).

**Step 2** Compute the search step size \( \beta^k \) such that

\[
J_h(u_h^* + \beta^k u_k^*) = \min \{J_h(u_h^* + \beta u_k^*); \beta \geq 0\}.
\]
Given $0 < \xi < \frac{1}{2}$ and $0 < \tau < 1$. Let $\rho^0 = 1$, for
\[ m = 0, 1, 2, \cdots, \beta_0. \]
If
\[ J_h(u_h^k - \rho^m P^k) \leq J_h(u_h^k) - \xi \rho^m J'_h(u_h^k), \]
then $\beta^m = \rho^m$; else $\rho^{m+1} = \tau \rho^m$. 

Step 3. $u_h^{k+1} = u_h^k + \beta^k P^k$. The convergence of iteration
procedure in minimizing $J_h$ is guaranteed in [14].

Step 4. The stopping criterion $\varepsilon$ is a small specified number. If
$J_h(u_h^{k+1}) < \varepsilon$, then stop ($u_h^{k+1}$ is the solution).

Step 5. Compute the gradient of $J'_h(t)$ to obtain $J'_h(u_h^k)\varphi_h$
for all $\varphi_h \in V_h$.

Step 6. Compute the updated conjugate coefficients
\[ \gamma^k = \frac{\varepsilon_1 \int_0^T (J_h^k)' 2 dt}{\varepsilon_2 \int_0^T (J_h^{k-1})' 2 dt}, \quad \text{with} \quad \gamma^0 = 0. \]

Using $\varepsilon_1, \varepsilon_2$ to get proper $\gamma^k$.

Step 7. Compute the directions of descent $P^k(t) :=
\gamma^k J_h^k(t) + \gamma^k P^{k-1}(t), k := k + 1$; return to step
2, and so on.

D. Convergence of nonlinear algorithm

The convergence proof can be given as in [17], hence
$J_h(u_h)$ can be minimized by sequence $\{u_h^k\}$. Let $u_h^k$
denotes the solution of discrete problem such that $u_h^k \rightarrow u^*$.
It’s clearly that $u_h$ converges to $u^*$ in the order of $O(h)$ as $h \rightarrow 0$.

IV. LABORATORY EXPERIMENTS

Let $\Omega = (0, 1), t_0 = 0.0, T = 1.0, h = \frac{1}{23}$ and take $\varepsilon = 0.02$ in
Step 4 of part C in Section III. The desired state $z_3^1 =
e^{-\frac{8}{50} x} \sin(\omega x - 0.5)$ and $z_3^2 = 0.5 \sin(3\pi x)$. Take
initial functions $\psi(0) = e^{-\frac{8}{50} x} \sin(\omega x - 0.5), \phi(0) = \sin(3\pi x)$. Set the physics constants $m = 9.10983188 \times
10^{-31}, \omega = 9.10983188 \times 10^{-32}, p = 5, e = 1.60217091 \times
10^{-34}$ and $h = 1.05435715964207855 \times 10^{-34}$. Let $c_1 =
0.05, c_2 = 0.5 \times 10^{-4}$, and experiment control inputs as $c_1 u$ and $c_2 v$. The initial and desired states are shown in
Figures 1-2. The start controls functions $u_0(t) = 1 +
2 \sin(\frac{2}{7} t), v_0(t) = 1 + 2 \cos(\frac{2}{7} t)$, the desired controls
$u_T(t) = 1 + 0.0001 \sin(\frac{2}{7} t), v_T(t) = 1 + 0.0002 \cos(\frac{2}{7} t)$.
See Figures 3-4.
Remark 4.1: The Maxwell equation express the electromagnetic phenomena, and the KG equation changes its state from wave phenomena to electromagnetic phenomena in control iterations.

The vector plots of electromagnetic field for Maxwell equations in first and last iterations are shown in Figures 17-18.

Optimal control functions obtained as

\[ u^* = 0.999933 + 0.000199858\cos(5\pi t) \]
\[ + 0.00128123\sin(5\pi t); \]
\[ v^* = 0.95173 - 0.822073\cos(7\pi t) \]
\[ + 0.00021738\sin(7\pi t). \]

Optimal control graphics of two systems in Figures 19-20.

Controls functions iteration are listed in Figures 21-22.

Optimal cost function value calculated as:

\[ J(u^*) = 0.461625. \]

The cost functions is shown in Figure 23.

For \( u = (u, v), \) the cost iterations are calculated in below.

\[ J(1) = 2.00002; \quad J(2) = 1.97408; \quad J(3) = 1.85106; \]
\[ J(4) = 1.76527; \quad J(5) = 1.69754; \quad J(6) = 1.64626; \]
\[ J(7) = 1.60478; \quad J(8) = 1.58636; \quad J(9) = 1.53517; \]
\[ J(10) = 1.50289; \quad J(11) = 1.47076; \quad J(12) = 1.43814; \]
\[ J(13) = 1.40472; \quad J(14) = 1.37041; \quad J(15) = 1.33534; \]
\[ J(16) = 1.29977; \quad J(17) = 1.26400; \quad J(18) = 1.22832; \]
\[ J(19) = 1.19299; \quad J(20) = 1.15818; \quad J(21) = 1.12405; \]
\[ J(22) = 1.09067; \quad J(23) = 1.05811; \quad J(24) = 1.0284; \]
\[ J(25) = 0.963562; \quad J(26) = 0.965562; \quad J(27) = 0.936434; \]
\[ J(28) = 0.908152; \quad J(29) = 0.880701; \quad J(30) = 0.854062; \]
\[ J(31) = 0.828218; \quad J(32) = 0.803147; \quad J(33) = 0.778828; \]
\[ J(34) = 0.755241; \quad J(35) = 0.732365; \quad J(36) = 0.71018; \]
\[ J(37) = 0.688665; \quad J(38) = 0.66780; \quad J(39) = 0.647567; \]
\[ J(40) = 0.627946; \quad J(41) = 0.608919; \quad J(42) = 0.590469; \]
\[ J(43) = 0.572577; \quad J(44) = 0.555227; \quad J(45) = 0.538402; \]
\[ J(46) = 0.522088; \quad J(47) = 0.506267; \quad J(48) = 0.490926; \]
\[ J(49) = 0.47605; \quad J(50) = 0.461625. \]
Let $\eta_i$ represents the left hand of necessary condition (5) at $i$th iteration, then

$$\eta_1 = -3.1372, \quad \eta_2 = -3.15292, \quad \eta_3 = -2.8756, \quad \eta_4 = -2.55266, \quad \eta_5 = -2.19919, \quad \eta_6 = -1.84398, \quad \eta_7 = -1.5099, \quad \eta_8 = -1.18069, \quad \eta_9 = -0.86215, \quad \eta_{10} = -0.556352, \quad \eta_{11} = -0.267066, \quad \eta_{12} = -0.252251, \quad \eta_{13} = 0.237728, \quad \eta_{14} = 0.438674, \quad \eta_{15} = 0.595879, \quad \eta_{16} = 0.703969, \quad \eta_{17} = 0.759697, \quad \eta_{18} = 0.76237, \quad \eta_{19} = 0.713267, \quad \eta_{20} = 0.615796, \quad \eta_{21} = 0.47492, \quad \eta_{22} = 0.296857, \quad \eta_{23} = 0.88758, \quad \eta_{24} = -0.14159, \quad \eta_{25} = -0.386042, \quad \eta_{26} = -0.63636, \quad \eta_{27} = -0.884463, \quad \eta_{28} = -1.12266, \quad \eta_{29} = -1.34385, \quad \eta_{30} = -1.51473, \quad \eta_{31} = -1.71093, \quad \eta_{32} = -1.84715, \quad \eta_{33} = -1.97425, \quad \eta_{34} = -2.00928, \quad \eta_{35} = -2.03254, \quad \eta_{36} = -2.01753, \quad \eta_{37} = -1.96589, \quad \eta_{38} = -1.88033, \quad \eta_{39} = -1.7645, \quad \eta_{40} = -1.62288, \quad \eta_{41} = -1.46056, \quad \eta_{42} = -1.28312, \quad \eta_{43} = -1.09643, \quad \eta_{44} = -0.906454, \quad \eta_{45} = -0.719063, \quad \eta_{46} = -0.53988, \quad \eta_{47} = -0.374105, \quad \eta_{48} = -0.226369, \quad \eta_{49} = -0.100616, \quad \eta_{50} = 0.014254.$$

Necessary optimality condition is checked numerically.

V. CONCLUSIONS AND FUTURE WORKS

This work investigated the control problem for Klein-Gordon-Maxwell equations. Both theoretic and numerical study are considered completely. Experiment demonstration interpret that the approach is efficiently and can be applied to widely nonlinear control systems. For example, the Klein-Gordon-Schrodinger system (cf. [19]) and diffusion neural network system (cf. [20],[21]).

Several key problems will be proposed in the future research. (i) The laboratory realization for the control problems. (ii) Large scale computations. (iii) Different fields interdisciplinary research. (iv) Research topics concerning with control processing. (v) Other controls (e.g. initial control) application. (vi) Two and three spatial dimensions control problems. All of these will be hopeful directions in the future.

VI. ACKNOWLEDGMENTS

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REFERENCES