Closed-form decomposition of one-loop massive amplitudes

Ruth Britto,1 Bo Feng,2 and Pierpaolo Mastrolia3

1Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
2Center of Mathematical Science, Zhejiang University, Hangzhou, China
3Theory Division, CERN, CH-1211 Geneva 23, Switzerland

(Received 11 April 2008; published 28 July 2008)

We present formulas for the coefficients of 2-, 3-, 4-, and 5-point master integrals for one-loop massive amplitudes. The coefficients are derived from unitarity cuts in D dimensions. The input parameters can be read off from any unitarity-cut integrand, as assembled from tree-level expressions, after simple algebraic manipulations. The formulas presented here are suitable for analytical as well as numerical evaluation. Their validity is confirmed in two known cases of helicity amplitudes contributing to \( gg \to gg \) and \( gg \to gH \), where the masses of the Higgs and the fermion circulating in the loop are kept as free parameters.

I. INTRODUCTION

With the approaching kick-off of the CERN Large Hadron Collider, the calculation of one-loop multileg amplitudes has been under intense consideration [1], following a program of improvement of established techniques [2] and development of new methods (see [3] for an extensive review).

The unitarity method introduced in [4,5] is designed to compute any scattering amplitude by matching its unitarity cuts onto the corresponding cuts of its expansion in a basis of master integrals [6] with rational coefficients. Each of these coefficients can be determined quantitatively from prior knowledge of the master integrals and the singularity structure of the amplitude.

As the master integrals form a basis for amplitudes, so the unitarity cuts of master integrals have uniquely identifiable analytic properties, and can be used as a basis for the cuts of any amplitude. Therefore, the coefficients of the linear combination can be extracted systematically through the phase-space integration (instead of complete loop integration).

Recently, unitarity-based methods for one-loop amplitudes have been the subject of an intense investigation, through different implementations of the cut-constraints [7–25].

The holomorphic anomaly of unitarity cuts [7,26] simplifies the phase-space integration dramatically: cut-integrals can be done analytically by evaluating residues of a complex function in spinor variables [27], reducing the problem of so-called tensor reduction to one of algebraic manipulation.

Accordingly, in [12,14], a systematic method was introduced to evaluate any finite four-dimensional unitarity cut, yielding compact expressions for the coefficients of the master integrals. This method was successfully applied to the final parts of the cut-constructible part of the six-gluon amplitude in QCD. The same method, based on the spinor integration of the phase space, was later extended for the evaluation of generalized cuts in D dimensions [15–18], which is essential for the complete determination of any amplitude in dimensional regularization [28–30].

In this paper, we carry out the extension to the massive case of the analytic results presented in [19], stemming from an original study of compact formulas for the coefficients of the master integrals [17]. Following the same logic as in [19], we now present general formulas for the coefficients of the master integrals which can be evaluated without performing any integration. These formulas depend on input variables (indices, momenta, and associated spinors) that are specific to the initial cut-integrand, which is assembled from tree-level amplitudes. The value of a given coefficient is thus obtained simply by pattern matching, that is by specializing the value of the input variables to be inserted in the general formulas. The implementation of the general formulas into automatic tools is straightforward, as done for the current investigation with the program \( \text{S@M} \) [31].

In this paper, since the formulas for the coefficients are obtained via massive double cuts in D dimension, we do not present results for the coefficients of cut-free functions like tadpoles and bubbles with massless external momentum (which can be expressed in terms of tadpoles as well). The coefficients of such functions could be fixed either by imposing the expected UV- behavior of the amplitude, as described in [29], or computed with other techniques applicable in massive calculations [20–22,24,25].

The paper is organized as follows. In Sec. II, we describe the structure of the decomposition of one-loop amplitude in terms of master integrals. In Sec. III we explain the double-cut integration with spinor variables, which leads to the formulas of the coefficients of the master integrals, presented in Sec. IV. In Secs. V and VI, we apply our formulas to two examples of one-loop scattering amplitudes, respectively \( gH \to gg \) and \( gg \to gg \), where the Higgs mass and the mass of the internal fermion (in both cases) are kept as free parameters. In Sec. VII, we present both analytical and numerical methods to obtain, finally,
the explicit coefficients of the dimensionally shifted master integrals. In Appendix A, we record the translation between our basis of integrals and the ones used in the literature for the examples discussed in Secs. V and VI. In Appendix B, we present a proof of the decomposition into the dimensionally shifted basis, with rational coefficients independent of $\varepsilon$. In other words, we prove that the coefficients given by our algebraic expressions will be polynomial in our extra-dimensional variable $u$. As a by-product, we have produced equivalent and simpler algebraic functions for the evaluation of coefficients.

**II. DECOMPOSITION IN TERMS OF MASTER INTEGRALS**

We define the $n$-point scalar function with nonuniform masses as follows:

\[ I_n(M_1, M_2, m_1, \ldots, m_{n-2}) = \int \frac{d^{4-2\varepsilon} \rho}{(2\pi)^{4-2\varepsilon}} \frac{1}{(p^2 - M_1^2)((p - K)^2 - M_2^2) \prod_{j=1}^{n-2}((p - P_j)^2 - m_j^2)}, \tag{2.1} \]

Giele, Kunszt, and Melnikov [25] have given the decomposition of any one-loop amplitude in $D$ dimensions in terms of master integrals, represented pictorially in Fig. 1.

Here, with reference to [25]: (i) we have absorbed the residual $D$-dependence of the coefficients in the definition of the master integrals; (ii) for ease of notation, we have given as understood the sums on the partition of the $n$-points of the amplitude in the number of points corresponding to each master integral. Thus, the coefficients $e, d, c, b, a$ in Fig. 1 are independent of $D$. If, on both sides of the equation in Fig. 1, we apply the standard decomposition of the $D = 4 - 2\varepsilon$ dimensional loop variable $L$, in a four-dimensional component $\tilde{L}$, and its $(-2\varepsilon)$-dimensional orthogonal complement $\mu$,

\[ L = \tilde{L} + \mu, \tag{2.2} \]

then the integration measure becomes

\[ \int d^{4-2\varepsilon} L = \int d^{-2\varepsilon} \mu \int d^4 \tilde{L}, \tag{2.3} \]

\footnote{For ease of presentation, we are omitting the prefactor $i(-1)^{n+1}(4\pi)^{D/2}$ (which was included, for example, in [29]).}
The presence of $\pi(\mu^2)$ in the coefficient of the four-dimensional box is a unique signature of the pentagon. We conclude that the reconstruction of the four-dimensional kernel of any one-loop amplitude, given in Fig. 2, contains all the information for the complete reconstruction of the amplitude in $D$ dimensions, given in Fig. 1.

In the following pages, we present the general formulas of the coefficients of the box, $l_4^{(4)}$, triangle, $l_3^{(4)}$, and bubble, $l_2^{(4)}$, obtained from the double cut of the equation in Fig. 2, and represented in Fig. 3. Since the formulas for the coefficients are obtained via double cuts, we do not present the results for the coefficients of cut-free functions like tadpoles and bubbles with massless external momentum (which can be expressed in terms of tadpoles as well). Their coefficients could be fixed either by imposing the expected UV-behavior of the amplitude, as described in [29], or computed with alternative techniques [20–22,24,25].

III. THE DOUBLE-CUT PHASE-SPACE INTEGRATION

In this section, we review the $D$-dimensional unitarity method [15,18] as applied in cases with arbitrary masses [17]. Our goal is to describe the structure of the cut integrand, from which we will directly read off the coefficients from the formulas in the following section. The formulas will be the massive analogs of the ones in [19].

Recall the phase-space integration of a standard (double) cut in $D = 4 - 2\epsilon$ dimensions. We use the usual decomposition of the $D$-dimensional loop variable $L$, in a four-dimensional component $\ell$, and a transverse ($-2\epsilon$)-dimensional remnant $\mu$.

$$L = \ell + \mu. \quad (3.1)$$

The integration measure becomes

$$\int d^{4-2\epsilon}L = \int d^{2-2\epsilon}\ell \int d^4\ell = \frac{(4\pi)^{\epsilon}}{\Gamma(-\epsilon)} \int d\mu^2(\mu^2)^{-1-\epsilon} \int d^4\ell, \quad (3.2)$$

namely the composition of a four-dimensional integration and an integration over a ($-2\epsilon$)-dimensional masslike parameter. In order to write the four-dimensional part in terms of spinor variables associated to massless momentum, we proceed with the following change of variables:

$$\ell = \ell + zK, \quad \ell^2 = 0, \quad (3.3)$$

where $\ell$ is a massless momentum and $K$ is the momentum across the cut, fixed by the kinematics. Accordingly, the four-dimensional integral measure becomes

$$\int d^4\ell = \int dzd^4\ell \delta^+(\ell^2)(2\ell \cdot K). \quad (3.4)$$

The Lorentz-invariant phase space (LIPS) of a double cut in the $K^2$-channel is defined by the presence of two $\delta$-functions imposing the cut conditions:

$$\int d^{4-2\epsilon}\Phi = \int d^{4-2\epsilon}L \delta(L^2 - M_1^2) \delta((L - K)^2 - M_2^2). \quad (3.5)$$

Here $M_1$ and $M_2$ are the masses of the cut lines. By using the decomposition of the loop variable in Eq. (2.2), the four-dimensional integral can be separated, so that

$$\int d^{4-2\epsilon}\Phi = \frac{(4\pi)^{\epsilon}}{\Gamma(-\epsilon)} \int d\mu^2(\mu^2)^{-1-\epsilon} \int d^4\phi, \quad (3.6)$$

where the four-dimensional LIPS is

$$\int d^4\phi = \int d^4\ell \delta(\ell^2 - M_1^2 - \mu^2) \delta((\ell - K)^2 - M_2^2 - \mu^2). \quad (3.7)$$

The change of variables in Eq. (3.3), and the $z$-integration (trivialized by the presence of $\delta$’s), yield the four-dimensional LIPS to appear as
\[ \int d^4 \phi = \int d^4 \ell \delta^4(\ell^2) \delta((1-2z)K^2 - 2\ell \cdot K + M_1^2 - M_2^2), \] 
\[ \text{where} \]
\[ z = \frac{(K^2 + M_1^2 - M_2^2)}{2K^2}, \]
\[ \Delta[K, M_1, M_2] \equiv (K^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2K^2M_1^2 - 2K^2M_2^2 - 2M_1^2M_2^2. \]

We remark that the value of \( z \) in Eq. (3.9) is frozen to be the proper root \((K > 0)\) of the quadratic argument of \( \delta(z(1-z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2) \), coming from \( \delta(\ell^2 - M_1^2 - \mu^2) \). For later convenience, one can redefine the \( \mu^2 \)-integral measure as

\[ \int d\mu^2 \langle \mu^2 \rangle^{-1-\epsilon} = \left( \frac{\Delta[K, M_1, M_2]}{4K^2} \right)^{-\epsilon} \int_0^1 du \, u^{-1-\epsilon}. \]

where the relation between \( u \) and \( \mu^2 \) is given by

\[ u = \frac{4K^2\mu^2}{\Delta[K, M_1, M_2]}, \quad \mu^2 = \left( \frac{u\Delta[K, M_1, M_2]}{4K^2} \right). \]

We observe that the domain of \( u \), i.e., \( u \in [0, 1] \), follows from the kinematical constraints, as discussed in [17].

Finally, after the above rearrangement, the \( D \)-dimensional Lorentz-invariant phase space of a double cut in the \( K^2 \)-channel can be written in a suitable form,

\[ \int d^{4-2\epsilon} \Phi = \chi(\epsilon, K, M_1, M_2) \int_0^1 du \, u^{-1-\epsilon} \int d^4 \phi, \]

where

\[ \chi(\epsilon, K, M_1, M_2) = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \left( \frac{\Delta[K, M_1, M_2]}{4K^2} \right)^{-\epsilon}. \]

and where \( d^4 \phi \) was given in Eq. (3.8). By using the definition of \( u \) given in Eq. (3.11), we can write

\[ z = \frac{\alpha - \beta \sqrt{1 - u}}{2}, \]

where

\[ \alpha = \frac{K^2 + M_1^2 - M_2^2}{K^2}, \quad \beta = \frac{\sqrt{\Delta[K, M_1, M_2]}}{K^2}. \]

Notice that when \( M_1 = M_2 = 0 \) we have \( \alpha = \beta = 1 \), thus reproducing the massless case. A useful relation between \( z \) and \( u \) is the following:

\[ (1 - 2z) + \frac{M_1^2 - M_2^2}{K^2} = \beta \sqrt{1 - u}. \]

This relation will be used in Appendix B to prove that the coefficients given in this paper are polynomials in \( u \), or equivalently \( \mu^2 \). As discussed in the previous section, this feature is essential for the straightforward reconstruction of dimensionally shifted master integrals.

The main feature of a double-cut LIPS parametrized as in Eqs. (3.8) and (3.12), is that the kernel of the integration is represented by the four-dimensional integral. In fact, the \( u \)-integration (or equivalently, the \( \mu^2 \)-integration), is simply responsible for the rise of shifted-dimension master integrals. Thus, our interest in the extraction of the coefficients of the master integrals from a four-dimensional massive double cut, see Fig. 3, translates in focusing the discussion only on the \( \int d^4 \phi \).

The \( D \)-dimensional double cut of any one-loop amplitude is, in general form,

\[ \int d^{4-2\epsilon} \Phi A_L^{\text{tree}} \times A_R^{\text{tree}} = \chi(\epsilon, K, M_1, M_2) \int_0^1 du \, u^{-1-\epsilon} \int d^4 \phi A_L^{\text{tree}} \times A_R^{\text{tree}}. \]

where \( A_L^{\text{tree}} \) and \( A_R^{\text{tree}} \) are the two tree-level amplitudes on the left and right side of the cut. As discussed above, the kernel of the integration is represented by the four-dimensional part,

\[ \int d^4 \phi A_L^{\text{tree}} \times A_R^{\text{tree}}. \]

We proceed from the formula (3.8) by introducing spinor variables according to [26],

\[ \int d^4 \ell \delta(\ell^2) = \int (\ell d\ell)[\ell d\ell] \int t dt \]

and performing the integral over \( t \) trivially, with the second delta function. The general expression of the double-cut integral will then be
\[
\int d^4 \phi A_L^\text{tree} \times A_R^\text{tree} = \int d^4 \ell \delta^+(\ell^2)\delta((1 - 2z)K^2 - 2 \ell \cdot K + M_1^2 - M_2^2) \prod_{i} \langle \pi_j | b_j \rangle \left( (\ell - K_i)^2 - m_i^2 - \mu^2 \right) ^{\frac{1}{2}} \prod_{i} \langle a_j | \ell | b_j \rangle \left( (\ell - K_i)^2 - m_i^2 - \mu^2 \right) ^{\frac{1}{2}}
\]
\[
= \int d^4 \ell \delta^+(\ell^2)\delta((1 - 2z)K^2 - 2 \ell \cdot K + M_1^2 - M_2^2) \prod_{i} \langle \pi_j | \ell + zK | b_j \rangle \left( K_i^2 + M_1^2 - m_i^2 - 2(\ell + zK) \cdot K_i \right) ^{\frac{1}{2}} \prod_{i} \langle a_j | \ell | b_j \rangle \left( K_i^2 + M_1^2 - m_i^2 - 2(\ell + zK) \cdot K_i \right) ^{\frac{1}{2}}
\]
\[
= \int \langle \ell | \ell \rangle \int t dt \delta((1 - 2z)K^2 + t(\ell | K | \ell) + M_1^2 - M_2^2) \prod_{i} \langle \pi_j | K | b_j \rangle + t(\ell | P_j | \ell) \rangle \times \prod_{i} \langle a_j | K | b_j \rangle + t(\ell | K | \ell) \rangle \times \prod_{i} \langle \ell | K_i \rangle \left( K_i^2 + M_1^2 - m_i^2 - 2zK \cdot K_i + t(\ell | K_i | \ell) \right) \rangle.
\]

Here we have used \( \ell^2 = M_1^2 + \mu^2 \). Notice that \( \langle a_j | \ell | b_j \rangle = -2 \ell \cdot P \), with \( P = |a_j| |b_j| \).

After using the remaining delta function to perform the integral over \( t \), we have
\[
\int \langle \ell | \ell \rangle \int t dt \delta((1 - 2z)K^2 + t(\ell | K | \ell) + M_1^2 - M_2^2) \prod_{i} \langle \pi_j | K | b_j \rangle + t(\ell | P_j | \ell) \rangle \times \prod_{i} \langle a_j | K | b_j \rangle + t(\ell | K | \ell) \rangle \times \prod_{i} \langle \ell | K_i \rangle \left( K_i^2 + M_1^2 - m_i^2 - 2zK \cdot K_i + t(\ell | K_i | \ell) \right) \rangle.
\]

(2) Triangle: \( k = 1, n + k = 0 \).
\[
\int \langle \ell | \ell \rangle \int t dt \delta((1 - 2z) + \frac{M_1^2 - M_2^2}{K^2}) \times \frac{1}{(\ell | K | \ell) (\ell | Q | \ell)}
\]

(3) Box: \( k = 2, n + k = 0 \).
\[
\int \langle \ell | \ell \rangle \int t dt \delta((1 - 2z) + \frac{M_1^2 - M_2^2}{K^2}) \times \frac{(K^2)^{-1}}{(\ell | Q_1 | \ell) (\ell | Q_2 | \ell)}
\]

These formulas are the extension to the massive case of the corresponding ones given in [19]. We notice that the presence of the masses enters only the definitions of \( P_j, R_j, \) and \( Q_j \). Therefore the spinor integration performed in the massless case [19] is valid as well in this case.

The expression of the cut-integrand in Eq. (3.21), with its indices, \( n \) and \( k \), and its vectors \( P_j, R_j, \) and \( Q_j \) is the key to constructing the coefficients. In the next section we present general formulas for the coefficients of the master integrals (boxes, triangles, and bubbles), which depend on exactly these input parameters. Accordingly, given a specific amplitude (or integral), one can obtain its decomposition in terms of master integrals without any integration. Every coefficient is obtained from the general formulas simply by substituting the input parameters characterizing the specific amplitude. These parameters are obtained by pattern-matching onto the reference form in Eq. (3.21).

**IV. FORMULAS FOR THE COEFFICIENTS OF MASTER INTEGRALS**

The coefficients of master integrals are obtained by the procedure described in the previous section, which is a straightforward generalization of the massless case [19].

We list the results in this section. In fact, the expressions
take the same form as in the massless case; the mass 
dependence enters directly through the definitions (3.23) 
and (3.24), and through these formulas into the definitions 
(4.2) and (4.4).

A. Box coefficient

The formula for the coefficient of either of the box 
functions with external kinematics as shown in Fig. 4 is

\[
C[Q_s, Q_r, K] = \frac{(K^2)^{1+n}}{2} \left\{ \frac{\prod_{j=1}^{k+n} \langle P_{sr,1} | R_j | P_{sr,2} \rangle}{\prod_{j=1}^{k+n} \langle P_{sr,1} | Q_r | P_{sr,2} \rangle} \right\} \left( P_{sr,1} \leftrightarrow P_{sr,2} \right).
\]

where

\[
\Delta_{sr} = (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2,
\]

\[
P_{sr,1} = Q_s + \left( \frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r} \right) Q_r,
\]

\[
P_{sr,2} = Q_s + \left( \frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r} \right) Q_r.
\]

B. Triangle coefficient

The formula for the coefficient of the triangle function 
with external kinematics as shown in Fig. 5 is

\[
C[Q_s, K] = \frac{(K^2)^{1+n}}{2} \left\{ \frac{1}{(\sqrt{\Delta})^{n+1}(n+1)!} \langle P_{s,1} P_{s,2} \rangle^{n+1} \right\} \left( P_{s,1} \leftrightarrow P_{s,2} \right) - \sum_{\alpha=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{(-1)^n}{\alpha! \alpha!} \left( \frac{2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K.
\]

where

\[
\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2,
\]

\[
P_{s,1} = Q_s + \left( \frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K,
\]

\[
P_{s,2} = Q_s + \left( \frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K.
\]

Note that the triangle coefficient is present only when \( n \geq -1 \).

C. Bubble coefficient

The formula for the coefficient of the bubble function 
with the external momentum \( K \), shown in Fig. 6, is

\[
C[K] = (K^2)^{1+n} \sum_{q=0}^{\infty} \frac{(-1)^n}{q!} \frac{d^q}{ds^q} \left( \mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left( \mathcal{B}^{(r,a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r,a-q;2)}_{n,n-a}(s) \right) \right) \bigg|_{s=0}
\]

where
where \( \Delta_r, P_{r1}, P_{r2} \) are given by (4.4), and \( \eta, \bar{\eta} \) are arbitrary, generically chosen null vectors. Note that the bubble coefficient exists only when \( n \equiv 0 \).

V. EXAMPLE I: \( s_{12} \)-CHANNEL CUT OF \( A(1^+, 2^+, 3^+, H) \)

In this section as well as the next, we check our formulas by reconstructing some helicity amplitudes contributing to \( gH \rightarrow gg \) and \( gg \rightarrow gg \) at next-to-leading order (NLO) in QCD, both known in the literature [29,32]. We present our calculations in detail.

\[
A(1^+, 2^+, 3^+, H)_{s_{12} \text{-channel}} = -\frac{i}{(4\pi)^2} \frac{m^2}{v^2} \frac{stu}{(12)(3)(1)} \left\{ \frac{(t-u)}{2t(s-m_H^2)} I_3^{(2)} [4(m^2 + \mu^2) - s] - \frac{(s-m_H^2)}{2st} I_3^{(2)} [4(m^2 + \mu^2) - s] - \frac{1}{2} l_4 [4(m^2 + \mu^2) - s] \right\}.
\]

\[
(5.1)
\]

where \( s = s_{12}, t = s_{23}, u = m_H^2 - s - t \). One can thus read the following values for the coefficients:

\[
c_{12}^{4m} = -c_0^4 [4(m^2 + \mu^2) - s],
\]

\[
(5.2)
\]

\[
c_{12}[3H] = c_0 \frac{(s-m_H^2)}{2st} [4(m^2 + \mu^2) - s],
\]

\[
(5.3)
\]

\[
c_{12}[3H] = c_0 \frac{(t-u)}{2t(s-m_H^2)} [4(m^2 + \mu^2) - s],
\]

\[
(5.4)
\]

Our first example is the \( s_{12} \)-channel cut of \( A(1^+, 2^+, 3^+, H) \). This amplitude was first computed in [32]. Here, to facilitate comparison, we follow the setup of [30], where the amplitude was rederived using unitarity cuts. At one loop, every Feynman diagram has a massive quark circulating in the loop. The quark mass is denoted by \( m \).

The \( s \)-channel cut of \( A(1^+, 2^+, 3^+, H) \) admits a decomposition in terms of cuts of master integrals as shown in Fig. 7. Its expression, given in Eq. (4.20) of [30], reads

\[
c_{12}[3H] = 0
\]

\[
(5.5)
\]

with

\[
c_0 = -\frac{i}{(4\pi)^2} \frac{m^2}{v^2} \frac{stu}{(12)(3)(1)}.
\]

\[
(5.6)
\]

A. The reconstruction of the coefficients

We now show how to reconstruct the coefficients given above with our formulas from Sec. IV. We follow the

\[c_0^{12}[3H] = 0\]

\[c_0 = \frac{i}{(4\pi)^2} \frac{m^2}{v^2} \frac{stu}{(12)(3)(1)}\]

\[c_0 = \frac{i}{(4\pi)^2} \frac{m^2}{v^2} \frac{stu}{(12)(3)(1)}\]
By pattern-matching onto the reference form in Eq. (3.21), each integrand can be characterized by the parameters given in the following table.

| integrand                | n  | k  | $P_1 = |P_1|/|P_1|$ |
|--------------------------|----|----|---------------------|
| $N_1/D_2$                | -1 | 1  | -                   |
| $N_{2,1}/(D_2 D_4)$      | -2 | 2  | -                   |
| $N_{2,2}/(D_2 D_4)$      | -1 | 2  | $|1| [3]$            |
| $N_{2,3}/(D_2 D_4)$      | -1 | 2  | $k_1 [3] (1|k_4$    |

These data are the input values that we need in evaluating the formulas of the coefficients of the master integrals.

From this table we draw the following conclusions.

From (5.18), we use the definition (3.24) to construct

$$Q_1 = -(1-z)k_1 - z k_2,$$

$$Q_2 = (1-2z)k_4 + \left(1-z \frac{m_H^2}{K^2} - z\right) K.$$

Using (4.4), we also set up the following quantities useful for triangle coefficients:

$$\Delta_1 = (1-2z)^2 (K^2)^2,$$

$$P_{1,1} = (1-2z)k_2, \quad P_{1,2} = -(1-2z)k_1,$$

$$\Delta_2 = (1-2z)^2 (K^2 - m_H^2)^2,$$

$$P_{2,1} = -(1-2z)k_3,$$

$$P_{2,2} = -(1-2z)\frac{m_H^2}{K^2} k_3 + (1-2z)\left(1-\frac{m_H^2}{K^2}\right) k_4.$$

---

\[\text{Note that here we use “twistor” sign convention for the antiholomorphic spinor product, which is opposite of the “QCD” convention followed by [30] \footnote{x}_R = -|xy|_{BFM}.}\]
B. The box coefficient \( c_4^{1m} \)

The box coefficient \( c_4^{1m} \) takes contributions from \( N_{2,1}, N_{2,2}, N_{2,3} \):

\[
c_4^{1m} = c_{0,2} 8 m (m^2 + \mu^2) k_4 \cdot e_3^+ C[Q_1, Q_2, K]^{(2.1)} - c_{0,2} \frac{8 m (m^2 + \mu^2)}{\sqrt{2} (13)} C[Q_1, Q_2, K]^{(2.2)} + c_{0,2} \frac{\sqrt{2} m}{\langle 13 \rangle} C[Q_1, Q_2, K]^{(2.3)},
\]

where \( C[Q_r, Q_s, K] \), defined in Eq. (4.1), is

\[
C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2} \left( \frac{\prod_{j=1}^{k} (P_{sr,1}|R_j|P_{sr,2})}{\prod_{i=1}^{n} (P_{sr,1}|Q_i|P_{sr,2})} + \{P_{sr,1} \leftrightarrow P_{sr,2}\} \right).
\]

(i) \( C[Q_1, Q_2, K]^{(2.1)} \)

This term, corresponding to \( n = -2 \) is trivial, since \( k = 2 \) and \( N_{2,1} \) has no dependence on the loop variable,

\[
C[Q_1, Q_2, K]^{(2.1)} = 1.
\]

(ii) \( C[Q_1, Q_2, K]^{(2.2)} \) and \( C[Q_1, Q_2, K]^{(2.3)} \) both correspond to \( n = -1, k = 2 \). They differ only in the definition of \( P_1 \). Therefore we can compute them in parallel, and specialize later to the corresponding \( P_1 \). With \( n = -1, k = 2 \), the expression is

\[
C[Q_1, Q_2, K] = \frac{(K^2)^2}{2} \left( \frac{(P_{21,1}|R_1|P_{21,2})}{(P_{21,1}|K|P_{21,2})} + \{P_{21,1} \leftrightarrow P_{21,2}\} \right)
\]

\[
= -(1 - 2z) K^2 \left( \frac{(P_{21,1}|P_{1}|P_{21,2})}{(P_{21,1}|K|P_{21,2})} + \frac{(P_{21,2}|P_{1}|P_{21,1})}{(P_{21,2}|K|P_{21,1})} \right) + z(-2K \cdot P_1)
\]

\[
= -(1 - 2z) K^2 \left( \frac{(P_{21,1}|P_{1}|P_{21,2})(P_{21,2}|K|P_{21,1}) + (P_{21,2}|P_{1}|P_{21,1})(P_{21,1}|K|P_{21,2})}{(P_{21,1}|K|P_{21,2})(P_{21,2}|K|P_{21,1})} \right) + z(-2K \cdot P_1).
\]

(iii) The result for \( c_4^{1m} \)

The total coefficient of our box is

\[
c_4^{1m} = c_{0,2} 8 m (m^2 + \mu^2) k_4 \cdot e_3^+ - c_{0,2} \frac{8 m (m^2 + \mu^2)}{\sqrt{2} (13)} \frac{(1|2\rangle|3\rangle)}{2} + c_{0,2} \frac{\sqrt{2} m}{\langle 13 \rangle} \frac{(1|2\rangle|3\rangle)}{2} - m^2 \sqrt{2} s_{23} \frac{(1|2\rangle|3\rangle)}{2}
\]

\[
= -m^2 K^2 s_{23} \frac{4 (m^2 + \mu^2) - m^2 \sqrt{2} (1|2\rangle|3\rangle)}{2}. \tag{5.33}
\]
Multiplying by \(-i/(4\pi)^{2-\epsilon}\), to account for the difference in the definitions of master integrals, we confirm the result of [30].

C. The triangle coefficient \(c_{\{1\}[2][3H]}\)

The coefficient \(c_{\{1\}[2][3H]}\) gets contributions from \(N_1, N_{2,2}, \) and \(N_{2,3}\):

\[
c_{\{1\}[2][3H]} = c_{0,1} N_1 C[Q, K]^{(1)} - c_{0,2} \frac{8m(m^2 + \mu^2)}{\sqrt{2} \langle 13 \rangle} C[Q, K]^{(2,2)} + c_{0,2} \frac{\sqrt{2} m}{\sqrt{2} \langle 13 \rangle} C[Q, K]^{(2,3)},
\]

(5.34)

where the general triangle coefficient, given in Eq. (4.3), reads

\[
C[Q, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta})^{n+1}} \frac{1}{(n+1)!} \left( \prod_{i=1}^{n+1} (\sum_{j=1}^{n+1} \frac{1}{P_{s,i} P_{s,j}}) \right) \left( \prod_{i=1}^{n+1} \frac{P_{s,i}}{P_{s,j}} \right) \bigg|_{\tau = 0}.
\]

(5.35)

(i) \(C[Q, K]^{(1)}\)

We have already observed that the \(N_1\) term is trivial. Here is how that shows up in our formulas.

Read \(C[Q, K]\) for \(s = 1\) and \(k = 1\), \(n = -1\), and no \(R_j\).

The term inside the parentheses degenerates to 1.

\[
C[Q, K]^{(1)} = \frac{1}{2} (1 + 1) \bigg|_{\tau = 0} = 1.
\]

(5.36)

(ii) \(C[Q, K]^{(2,2)}\) and \(C[Q, K]^{(2,3)}\)

As we said already, \(N_{2,2}\) and \(N_{2,3}\) differ in the definition of \(P_1\). Therefore we start by manipulating the general formula, and only at the very end we specialize each contribution using the corresponding \(P_1\).

Since \(n = -1, k = 2\), there is no derivative at all, so we can set \(\tau = 0\) from the beginning:

\[
C[Q, K] = \frac{1}{2} \left( \frac{\langle P_{s,1} | R_1 Q_1 | P_{s,1} \rangle}{\prod_{i=1}^{n+1} (\sum_{j=1}^{n+1} \frac{1}{P_{s,i} P_{s,j}})} + \frac{\langle P_{s,2} | R_1 Q_1 | P_{s,2} \rangle}{\prod_{i=1}^{n+1} (\sum_{j=1}^{n+1} \frac{1}{P_{s,i} P_{s,j}})} \right).
\]

(5.37)

The one-mass triangle \((1)[2][3H]\) corresponds to the value \(s = 1\),

\[
C[Q, K] = -\frac{1}{2} \left( \frac{\langle 2 | P_1 | 1 \rangle}{\langle 2 | 1 \rangle} + \frac{\langle 1 | P_1 | 2 \rangle}{\langle 1 | 2 \rangle} \right).
\]

(5.38)

For \(|P_1\rangle = |1\rangle, |P_1\rangle = |3\rangle\), one gets

\[
C[Q, K]^{(2,2)} = -\frac{1}{2} \frac{\langle 1 | 2 | 3 \rangle}{s_{23}}.
\]

(5.39)

For \(|P_1\rangle = |k_4[3], |P_1\rangle = |k_4[1]\), one obtains

\[
C[Q, K]^{(2,3)} = -\frac{1}{2} \frac{\langle 2 | 4 | 3 | 1 | 4 | 1 \rangle}{\langle 2 | 4 | 1 \rangle} + \frac{\langle 1 | 4 | 3 | 1 | 4 | 2 \rangle}{\langle 1 | 4 | 2 \rangle} = \langle 1 | 2 | 3 \rangle \left( 1 - \frac{m_H^2}{2 s_{23}} \right).
\]

(5.40)

(iii) The result for \(c_{\{1\}[2][3H]}\)

The total coefficient of triangle \((1)[2][3H]\) is

\[
c_{\{1\}[2][3H]} = -\frac{m K^2 s_{23}}{(K^2 - m_H^2) (2) (3) (1)} \frac{m [4 (m^2 + \mu^2) - s_{12}]}{2 v (12)} - \frac{m [12]}{2 v (12)} \frac{\sqrt{2} m}{\sqrt{2} v (13)} \langle 1 | 2 | 3 \rangle \left( 1 - \frac{m_H^2}{2 s_{23}} \right) \frac{m [12]}{2 v (12)} \frac{\sqrt{2} m}{\sqrt{2} v (13)} \langle 1 | 2 | 3 \rangle \left( 1 - \frac{m_H^2}{2 s_{23}} \right) \frac{m [12]}{2 v (12)} \frac{\sqrt{2} m}{\sqrt{2} v (13)} \langle 1 | 2 | 3 \rangle \left( 1 - \frac{m_H^2}{2 s_{23}} \right)
\]

(5.41)

Multiplying by \(i/(4\pi)^{2-\epsilon}\), to account for the difference in the definitions of master integrals, we again confirm the result of [30].
D. The coefficient $c_{12[3]H}$

The coefficient $c_{12[3]H}$ gets contributions from $N_{2,2}$ and $N_{2,3}$, therefore it can be written as

$$c_{12[3]H} = -c_{0,2} \frac{8m(m^2 + \mu^2)}{\sqrt{2(13)}} C[Q_2, K]^{(2,2)}$$

$$+ c_{0,2} \frac{2m}{(13)} C[Q_2, K]^{(2,3)}. \tag{5.42}$$

(i) $C[Q_2, K]^{(2,2)}$ and $C[Q_2, K]^{(2,3)}$

The two-mass triangle $(12)[3]H$ corresponds to the value $s = 2$, and its coefficient can be obtained from Eq. (4.3),

$$C[Q_2, K] = \frac{1}{2} \left( \frac{3|P_1[K]|}{(3|1][2][3]} - \frac{3|KP_1[3]|}{[3][1][2][3]} \right). \tag{5.43}$$

By using $|P_1| = |1|$, $|P_1| = |3|$, one gets

$$C[Q_2, K]^{(2,2)} = \left( 1 - \frac{m_H^2}{K^2} \right) \frac{1}{2} \frac{|1][2][3]}{2s_{23}} \tag{5.44}$$

By using $|P_1| = k_4[3]$, $|P_1| = k_4[1]$, one obtains

$$C[Q_2, K]^{(2,3)} = \frac{1}{2} \left( \frac{3|4][3][1][4][K][3]}{3|1][2][3]} - \frac{3|K4][3][1][4][3]}{[3][1][2][3]} \right)$$

$$= \left( 1 - \frac{m_H^2}{K^2} \right) \frac{m_H^2}{2s_{23}} \frac{|1][2][3]}{2s_{23}}. \tag{5.45}$$

(ii) The result of $c_{12[3]H}$

The total coefficient of triangle $(12)[3]H$ is

$$c_{12[3]H} = -c_{0,2} \frac{8m(m^2 + \mu^2)}{\sqrt{2(13)}} \left( 1 - \frac{m_H^2}{K^2} \right) \frac{|1][2][3]}{2s_{23}}$$

$$+ c_{0,2} \frac{2m}{(13)} \left( 1 - \frac{m_H^2}{K^2} \right) \frac{m_H^2}{2s_{23}}$$

$$= \frac{m^2(K^2 - m_H^2)}{2\nu(12)(23)(31)} \left[ 4(m^2 + \mu^2) - m_H^2 \right]. \tag{5.46}$$

Multiplying by $i/(4\pi)^{2-k}$, to account for the difference in the definitions of master integrals, we again confirm the result of [30].

VI. EXAMPLE II: $s_{23}$-CHANNEL CUT OF $A(1^-, 2^-, 3^+, 4^+)$

Our second example features a nonvanishing bubble coefficient. We study the $t$-channel cut of the gluon amplitude $A(1^-, 2^-, 3^+, 4^+)$, with a massive quark circulating in the loop. (As usual, $t = s_{23}$.)

The $t$-channel cut of $A(1^-, 2^-, 3^+, 4^+)$ admits a decomposition in terms of cuts of master integrals as shown in Fig. 8, and its expression was given in Eq. (5.33) of [29]. After converting that expression into our basis of $D$-dimensional master integrals, as done in Appendix A, it reads

$$A_d^{\text{form}}(1^-, 2^-, 3^+, 4^+) \bigg|_{t-\text{cut}}$$

$$= \langle 12 \rangle^2[34]^2 \left( \frac{2}{3} I_2[s] + \frac{4}{3} I_3[m^2 + \mu^2] - \frac{3}{2} I_3[m^2 + \mu^2] \right)$$

$$+ 2t I_4[(m^2 + \mu)^2] - tI_4[(m^2 + \mu)^2] \bigg|_{t-\text{cut}}. \tag{6.1}$$

One reads the following values for the coefficients:

$$C_{4,0}^m = c_0 \left( \frac{2t}{s} (m^2 + \mu^2) - t \right)(m^2 + \mu^2), \tag{6.2}$$

$$C_{23[4]}[1] = 0, \tag{6.3}$$

$$C_{23[4]}[1] = 0, \tag{6.4}$$

$$C_{23[4]}[1] = c_0 \left( \frac{2}{3} + \frac{4}{3} t (m^2 + \mu^2) - \frac{2}{3} (m^2 + \mu^2) \right), \tag{6.5}$$

with

$$c_0 = \langle 12 \rangle^2[34]^2 \frac{2}{3}. \tag{6.6}$$

A. The reconstruction of the coefficients

We now apply our formulas of Sec. IV to construct the coefficients given above. We follow the definition of the integrand given by [29]. By sewing the tree-level amplitude $A_d^{\text{tree}}(-L_1, 2^-, 3^+, 4^+)$ and $A_d^{\text{tree}}(-L_2, 4^+, 1^-, 1^1)$

![FIG. 8. Double cut in the $s_{23}$-channel for $A(1^-, 2^-, 3^+, 4^+)$.

025031-11
given in Eq. (2.3) of [29], and using the Dirac equation for a massive fermion, it is shown that the four-dimensional integrand of the t-cut, $C_{23}$, can be written as:

$$C_{23} = -\frac{2N_1 + N_2}{D_1D_2}, \quad (6.7)$$

with

$$N_1 = \frac{1}{s_{23}}(1|\ell_1|4)(2|\ell_1|3), \quad (6.8)$$

$$N_2 = -\frac{1}{s_{23}}(12|34)(1|\ell_1|4)(2|\ell_1|3), \quad (6.9)$$

$$D_1 = (\ell_1 + k_2)^2 - \mu^2 - m^2, \quad (6.10)$$

$$D_2 = (\ell_1 - k_2)^2 - \mu^2 - m^2. \quad (6.11)$$

By pattern matching onto the reference form in Eq. (3.21), each integrand is characterized by the parameters given in the following table.

<table>
<thead>
<tr>
<th>integrand</th>
<th>$n$</th>
<th>$k$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$P_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_1/(D_1D_2)$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$N_2/(D_1D_2)$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

$$\text{(6.12)}$$

We define

$$K = k_2 + k_3; \quad K_1 = -k_1; \quad K_2 = k_2; \quad (6.13)$$

$$P_1 = P_3 = \lambda_1 \tilde{\lambda}_4, \quad P_2 = P_4 = \lambda_2 \tilde{\lambda}_3. \quad (6.14)$$

Moreover, since we have a quark of mass $m$ circulating in the loop, we take

$$M_1 = M_2 = m_j = m. \quad (6.15)$$

Then, by applying (3.9), we find

$$z(1 - z) = \frac{m^2 + \mu^2}{K^2}. \quad (6.16)$$

For analytic simplification, the following trace identity is helpful.

$$\langle P_1|R|P_2\rangle\langle P_2|S|P_1\rangle = \text{Tr}\left(\frac{1 - \gamma_5}{2} \bar{P} \gamma_5 P S\right) = \text{tr}_-. (P_1RP_2S).$$

For the $N_1$ term, $n = 2$. For the $N_2$ term, $n = 0$. Both terms give boxes, triangles and bubbles.

From the definitions (3.23) and (3.24), we have

$$Q_1 = (1 - z)k_1 + zk_4, \quad Q_2 = -(1 - z)k_2 - zk_3, \quad (6.17)$$

$$R_1 = R_3 = -(1 - 2z)\lambda_1 \tilde{\lambda}_4, \quad R_2 = R_4 = -(1 - 2z)\lambda_2 \tilde{\lambda}_3. \quad (6.18)$$

Further, the quantities defined in (4.2) and (4.4), become

$$\Delta_{12} = (1 - 2z)^4 s_{12}(K^2 + s_{12}) - (1 - 2z)^2 K^2 s_{12}, \quad (6.19)$$

$$\Delta_1 = (1 - 2z)^2 (K^2)^2, \quad (6.20)$$

$$P_{1,1} = -(1 - 2z)k_4, \quad P_{1,2} = (1 - 2z)k_1, \quad (6.21)$$

$$\Delta_2 = (1 - 2z)^2 (K^2)^2, \quad (6.22)$$

$$P_{2,1} = (1 - 2z)k_3, \quad P_{2,2} = -(1 - 2z)k_2. \quad (6.23)$$

**B. The box coefficient $c_j^{0m}$**

The box coefficient $c_j^{0m}$ receives contributions from both $N_1$ and $N_2$, and can be correspondingly decomposed as

$$c_j^{0m} = -\frac{2}{(K^2)^2} C[Q_1, Q_2, K]^{(1)}$$

$$+ \frac{12|34|}{K^2} C[Q_1, Q_2, K]^{(2)}. \quad (6.24)$$

We discuss the computation of $C[Q_1, Q_2, K]^{(1)}$ and $C[Q_1, Q_2, K]^{(2)}$ in detail, starting from the expression given in Eq. (4.1).

(i) $C[Q_1, Q_2, K]^{(1)}$

For the $N_1$ term, with $n = 2$, the expression is given by

$$C[Q_1, Q_2, K]^{(1)} = \frac{(K^2)^4}{2} \left(\frac{\langle P_{21,1}|R_1|P_{21,2}\rangle^2\langle P_{21,1}|R_2|P_{21,2}\rangle^2}{\langle P_{21,1}|K|P_{21,2}\rangle^4} + \frac{\langle P_{21,2}|R_1|P_{21,1}\rangle^2\langle P_{21,2}|R_2|P_{21,1}\rangle^2}{\langle P_{21,2}|K|P_{21,1}\rangle^4}\right). \quad (6.25)$$

Recall that here we use twistor sign convention for the antiholomorphic spinor product, which is the opposite of the QCD convention followed by [29] $\{\cdot\}^{\text{Bern-Morgan}} = -\{\cdot\}^{\text{BFM}}$. 

025031-12
In terms of vectors,
\[
\text{tr} \cdot (V_1 V_2 V_3 V_4) = \frac{1}{2}(2V_1 \cdot V_2)(2V_3 \cdot V_4) + (2V_1 \cdot V_4)(2V_2 \cdot V_3) - (2V_1 \cdot V_3)(2V_2 \cdot V_4) - 4i\epsilon_{\mu\nu\rho\sigma} V_1^\mu V_2^\nu V_3^\rho V_4^\sigma.
\]

The coefficient can then be expressed in terms of traces, and evaluated as follows:
\[
C[Q_1, Q_2, K]^{(1)} = \frac{(K^2)^2(\text{tr} \cdot (P_{21,1} K P_{21,1} R_1) \text{tr} \cdot (P_{21,2} K P_{21,2} R_2))^2 + (\text{tr} \cdot (P_{21,1} K P_{21,2} R_1) \text{tr} \cdot (P_{21,1} K P_{21,2} R_2))^2}{2(\text{tr} \cdot (P_{21,1} K P_{21,2} K))^4}
\]
\[
= \frac{(K^2)^4 z^2(1 - z)^2 [34]^2 (12)^2}{s_{12}^2}.
\]

(ii) \( C[Q_1, Q_2, K]^{(2)} \)
For the \( N_2 \) term with \( n = 0 \), the expression is given by
\[
C[Q_1, Q_2, K]^{(2)} = \frac{(K^2)^2}{2} \left( \frac{\langle P_{21,1} | R_1 | P_{21,2} \rangle \langle P_{21,1} | R_2 | P_{21,2} \rangle}{\langle P_{21,1} | P_{21,2} \rangle^2} + \{P_{21,1} \leftrightarrow P_{21,2}\} \right)
\]
\[
= \frac{(K^2)^2 z(1 - z) [34] (12)}{2(\text{tr} \cdot (P_{21,1} K P_{21,2} K))^2}.
\]

(iii) The result of \( c_4^{0m} \)
We add our two contributions together and replace \( z \) using (6.16). The total box coefficient is thus
\[
c_4^{0m} = -\frac{2}{(K^2)^2} \frac{(K^2)^4 z^2(1 - z)^2 [34]^2 (12)^2}{s_{12}^2} + \frac{\langle 12 | [34] \rangle (K^2)^2 z(1 - z) [34] (12)}{s_{12}}
\]
\[
= \frac{(m^2 + \mu^2) [34]^2 (12)^2}{s_{12}} \left( 1 - \frac{2(m^2 + \mu^2)}{s_{12}} \right).
\]

C. The triangle coefficients \( c[234|1] \) and \( c[2|341] \)
Both terms exhibit the symmetry of the amplitude, so our two triangles are not independent.
The triangle coefficients \( c[234|1] \) and \( c[2|341] \) receive contributions from both \( N_1 \) and \( N_2 \), and they can be correspondingly decomposed as
\[
c[234|1] = -\frac{2}{(K^2)^2} C[Q_1, K]^{(1)} + \frac{\langle 12 | [34] \rangle}{K^2} C[Q_1, K]^{(2)},
\]
\[
c[2|341] = -\frac{2}{(K^2)^2} C[Q_2, K]^{(1)} + \frac{\langle 12 | [34] \rangle}{K^2} C[Q_2, K]^{(2)}.
\]
We discuss in parallel, first the contribution due to \( N_1 \) to both coefficients, namely \( C[Q_1, K]^{(1)} \) and \( C[Q_2, K]^{(1)} \), and later the one due to \( N_2 \), namely \( C[Q_1, K]^{(2)} \) and \( C[Q_2, K]^{(2)} \), where the triangle coefficient was given in Eq. (4.3).

(i) \( C[Q_1, K]^{(1)} \) and \( C[Q_2, K]^{(1)} \)
Since the \( N_1 \) term with \( n = 2 \), the triangle coefficient expression is given by
\[
C[Q_1, K]^{(1)} = \frac{(K^2)^3}{2} \frac{1}{(\sqrt{\Delta t})^3} \frac{1}{3!} \frac{d^3}{d \tau^3} \left( \Pi_{i=1}^{4}(4 = \tau | R_i Q_1 | 4 = \tau) + \Pi_{i=1}^{4}(1 = \tau | R_i Q_1 | 1 = \tau) \right)_{\tau = 0} \\
= \frac{(1 - 2z)(1 - z)(K^2)^2}{2} \frac{1}{3!} \frac{d^3}{d \tau^3} \left( (4 | Q_1 | 4)^2 (4 = \tau, 2)^2 | Q_1 | 4 = \tau - 1) + \tau^2 (4 | Q_1 | 4)^2 (1 = \tau, 2)^2 | Q_1 | 1 = \tau - 1) \right)_{\tau = 0} \\
= \frac{(1 - z)^2(K^2)^2}{12} \frac{1}{d \tau^3} \left( ((4 - \tau (12))^2 ((1 - z)(2)) | Q_1 | 3 + \tau z(2)) \right)_{\tau = 0} = 0.
\]

A similar calculation shows that
\[
C[Q_2, K]^{(1)} = \frac{(K^2)^3}{2} \frac{1}{(\sqrt{\Delta t})^3} \frac{1}{3!} \frac{d^3}{d \tau^3} \left( \Pi_{i=1}^{4}(3 = \tau | R_i Q_2 | 3 = \tau) + \Pi_{i=1}^{4}(2 = \tau | R_i Q_2 | 2 = \tau) \right)_{\tau = 0} = 0,
\]

which can also be seen by the symmetry of the amplitude and the cut.

(ii) \( C[Q_1, K]^{(2)} \) and \( C[Q_2, K]^{(2)} \)

For the \( N_2 \) term with \( n = 0 \), the expression is simpler as
\[
C[Q_1, K]^{(2)} = -\frac{(1 - 2z)(1 - z)K^2}{2} \frac{d}{d \tau} \left( \frac{4(4 - \tau 12)[3 | Q_1 | 4 - \tau 1]}{(4 = \tau | Q_2 | 2) \left| Q_1 | 4 - \tau 1) + \tau^2(1 = \tau 4 2)[3 | Q_1 | 1 - \tau 4) \right| \right)_{\tau = 0} \\
= -\frac{(1 - 2z)(1 - z)K^2}{2} \frac{d}{d \tau} \left( \frac{(4(4 - \tau 12)[3 | Q_1 | 4 - \tau 1) - \tau(3 | Q_1 | 3) \right]}{(4 = \tau | Q_2 | 2) \left| Q_1 | 4 - \tau 1) \right]} = 0.
\]

A similar calculation shows
\[
C[Q_2, K]^{(2)} = \frac{(1 - 2z) d}{d \tau} \left( \frac{3 - \tau 2 1}[4] | Q_2 | 3 - \tau 2 | 3 | Q_2 | 3) \right) + \frac{\tau^2(2 - \tau 3 1)[4] | Q_2 | 2 - \tau 3 | 3 | Q_2 | 3) \right)}{(3 = \tau 2 1) | Q_2 | 3 - \tau 2 | 3 | Q_2 | 3) \right)}_{\tau = 0} = 0,
\]

which can also be seen by symmetry.

(iii) The results of \( C[23]^{[4]} \) and \( C[23]^{[4]} \)

Every term vanishes separately, so
\[
c^{[23]^{[4]}[1]} = 0, \quad c^{[23]^{[4]}[1]} = 0.
\]

The vanishing results for triangle coefficients is not obvious from the beginning. We suspect that there should be a more directly physical argument to see this point.

D. The bubble coefficient \( c^{[23]^{[4]}} \)

The bubble coefficient \( c^{[23]^{[4]}} \) receives contributions from both \( N_1 \) and \( N_2 \), and can be correspondingly decomposed as
\[
c^{[23]^{[4]}} = -\frac{2}{(K^2)^2} C[K]^{(1)} + \frac{12 [3 4]}{K^2} C[K]^{(2)}.
\]

There is one subtlety regarding the calculation of the bubble coefficient. The formulas involve an arbitrarily chosen, generic auxiliary null vector \( \eta \). If \( \eta \) coincides with one of the \( K_i \), we need to use a modified formula, given in Appendix B.3.1 of [19]. In this example, we illustrate both options. First, we show the result with a generic choice of \( \eta \); second, we use the formulas for the case \( \eta = K_1 \). Both are suitable for numerical evaluation, while the special choice of \( \eta \) may simplify the analytic expression. We will find that the two results agree with each other, as well as with [29].

I. Generic reference momentum \( \eta \)

Let us start with the formulas for generic \( \eta \), given in Eq. (4.5). There are two terms we need to calculate.

(i) \( C[K]^{(1)} \)

For the first term, with \( N_1 \) in the numerator, and \( n = 2 \), the coefficient is
\[
C[K]^{(1)} = (K^2)^3 \frac{2}{q = 0} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left( \mathcal{S}_{22-a}(s) + \frac{2}{r=1} \frac{2}{a=q} \left( \mathcal{S}_{22-a}(r = a, q = 1) - \mathcal{S}_{22-a}(r = a, q = 2) \right) \right)_{s = 0},
\]
where

\[
B_{2,2-a}^{(0)}(s) = \frac{d^2}{d\tau^2}\left[\frac{1}{[\eta\tilde{\eta}K]^2}\frac{(2\eta \cdot K)^{3-a}}{(3-a)(2K)^{3-a}} \frac{\prod_{j=1}^{3-a}(\langle\ell|R_{j}(K+s\eta)|\ell\rangle)}{\langle\ell|Q_p(K+s\eta)|\ell\rangle}\right]_{\tau=0},
\]

\[
B_{2,2-a}^{(r_1a-q;1)}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_a}} \frac{d^a-q}{d\tau^a-q}\left[\frac{1}{(3-a)} \frac{\langle P_{r_1} - \tau P_{r_2}\rangle|\eta|P_{r_2}\rangle[\eta^3-a]}{\langle P_{r_1} - \tau P_{r_2}\rangle[K|P_{r_2}\rangle[\eta^3-a]} \right]_{\tau=0},
\]

\[
B_{2,2-a}^{(r_2a-q;2)}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_a}} \frac{d^a-q}{d\tau^a-q}\left[\frac{1}{(3-a)} \frac{\langle P_{r_2} - \tau P_{r_1}\rangle|\eta|P_{r_2}\rangle[\eta^3-a]}{\langle P_{r_2} - \tau P_{r_1}\rangle[K|P_{r_2}\rangle[\eta^3-a]} \right]_{\tau=0}.
\]

After making some substitutions, and considering the summation ranges of \(a\) and \(q\), we get

\[
B_{2,2-a}^{(1a-q;1)}(s) = 0,
\]

\[
B_{2,2-a}^{(1a-q;2)}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_a}} \frac{d^a-q}{d\tau^a-q}\left[\frac{1}{(3-a)} \frac{\langle 1 - \tau 4|\eta|1 \rangle[\eta^3-a]}{(1 - \tau 4[K]|1 \rangle[\eta^3-a]} \right]_{\tau=0},
\]

\[
B_{2,2-a}^{(2a-q;1)}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_a}} \frac{d^a-q}{d\tau^a-q}\left[\frac{1}{(3-a)} \frac{\langle 3 - \tau 2|\eta|3 \rangle[\eta^3-a]}{(3 - \tau 2[K]|3 \rangle[\eta^3-a]} \right]_{\tau=0}.
\]

(ii) \(C[K]^{(2)}\)

For the second term \(N_2\) with \(n = 0\), the expression is much simpler:

\[
C[K]^{(2)} = K^2 \left( B_{0,0}^{(0)}(s) + \sum_{r=1}^{2} (B_{0,0}^{(r;1)}(s) - B_{0,0}^{(r;2)}(s)) \right)_{s=0},
\]

where

\[
B_{0,0}^{(0)}(s) = \left( \frac{2\eta \cdot K}{K^2} \frac{\langle \ell|R_{1}(K)|\ell\rangle\langle \ell|R_{2}(K)|\ell\rangle}{\langle \ell|Q_p(K)|\ell\rangle} \right)_{\ell=-[K]|\eta]} = \frac{1}{K^2} \left[ \frac{\eta|KR_1|\eta|KR_2|\eta}{|\eta|KQ_1|\eta|KQ_2|\eta} \right] - \frac{1}{K^2} \left[ \frac{\eta|4\eta|\eta}{|\eta|1|\eta|2} \right],
\]

\[
B_{0,0}^{(1;0)}(s) = \left( \frac{2\eta \cdot K}{K^2} \frac{\langle \ell|R_{1}(K)|\ell\rangle\langle \ell|R_{2}(K)|\ell\rangle}{\langle \ell|Q_p(K)|\ell\rangle} \right)_{\ell=-[K]|\eta]} = \frac{1}{K^2} \left[ \frac{\eta|4\eta|\eta}{|\eta|1|\eta|2} \right].
\]

025031-15
Since $k_3$, we can directly set $r = 2$:

\[
\mathcal{B}_{n,t}^{(2,0;1)}(s) = \frac{1}{(1 - 2z) - sz (1 - 2z)^{a+1}(K^2)^{a+r+2}a!/(32)^a} \frac{(-1)^a}{d^n(13)^{t+1}} \left. \frac{\ell}{(1 + 1)} \frac{\ell}{(1 + 1 - a)} \frac{\ell}{(1 + 1 - a)} \frac{\ell}{(1 + 1 - a)} \cdots \frac{\ell}{(1 + 1 - a)} \right|_{(t) - (3) - (2)}.
\]  

\[
\mathcal{B}_{n,t}^{(2,0;2)}(s) = \frac{1}{(1 - 2z) - sz (1 - 2z)^{a+1}(K^2)^{a+r+2}a!/(32)^a} \frac{(-1)^a}{d^n(13)^{t+1}} \left. \frac{\ell}{(1 + 1)} \frac{\ell}{(1 + 1 - a)} \frac{\ell}{(1 + 1 - a)} \frac{\ell}{(1 + 1 - a)} \cdots \frac{\ell}{(1 + 1 - a)} \right|_{(t) - (2) - (3)}.
\]
Now we proceed to evaluate.

(i) \( C[K]^{(2)} \)

For the \( N_2 \) term with \( n = 0 \), the evaluation is simple. In particular, there are no derivatives in \( s \), so we can set \( s = 0 \) directly.

\[
C[K]^{(2)} = K^2 (B_{0,0}^{(0)}(s) + B_{0,0}^{(2;0;1)}(s)) - B_{0,0}^{(2;0;2)}(s)|_{s=0}.
\]

(6.59)

Choosing \( \bar{\eta} = 3 \), we have

\[
B_{0,0}^{(0)}(s = 0) = \frac{1}{[1 \; 2][3 \; 4]},
\]

\[
B_{0,0}^{(2;0;1)}(s = 0) = \frac{1}{K^2} \frac{1}{[1 \; 2]^2},
\]

\[
B_{0,0}^{(2;0;2)}(s = 0) = 0
\]

so the total contribution comes to

\[
C[K]^{(2)} = \begin{bmatrix} 4 \; 3 \\ 1 \; 2 \end{bmatrix}.
\]

(6.61)

(ii) \( C[K]^{(1)} \)

For the \( N_1 \) term with \( n = 2 \), the calculation is a bit more involved.

\[
C[K]^{(1)} = (K^2)^3 \sum_{q=0}^{2} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left. \left( B_{2,2-q}^{(0)}(s) - B_{2,2-q}^{(2;2-q;2)}(s) \right) \right|_{s=0}.
\]

(6.62)

For the various terms, we have

\[
B_{2,2-q}^{(0)}(s) = \frac{1}{(3-q)!(3 \; 4)^2[1 \; 2]^2} \frac{d^3}{d^3s} \left. \left( \frac{(1-2z)^2(1+s)^2}{(1-2z) - sz} \right) \right|_{r=0}.
\]

(6.63)

\[
B_{2,2-a-q}^{(2;2-q;1)}(s) = \frac{(1+s)^2(1-2z)^{3-a+q}(-1)^{a-1}(3 \; 2)^2[1 \; 3]^{3-a}}{(1-2z) - sz} \frac{d^{a-q}}{d^{a-q}} \left. \left( K^2)^{4-q}(a-q)! \right) \right|_{r=0}.
\]

(6.64)

The term \( B_{2,2-a-q}^{(2;2-q;2)} \) vanishes after taking the derivatives with respect to \( s \). The reason is the following. Notice that

\[
B_{2,2-a-q}^{(2;2-a-q;2)}(s) = \frac{(1+s)^2(1-2z)^{3-a+q}(-1)^{a-q}(3 \; 2)^2[1 \; 2]^{3-a}+1}{(1-2z) - sz} \frac{d^{a-q}}{d^{a-q}} \left. \left( K^2)^{a-q+r+2}(a-q)! \right) \right|_{r=0}.
\]

(6.65)

We can see that the \( \tau \)-derivative vanishes unless \( a - q = 2 \), in which case we get

\[
B_{2,2}^{(2;2-a)}(s) = \frac{(1+s)^2(1-2z)(3 \; 2)^2[1 \; 2]^{3-a}+1}{(1-2z) - sz} \frac{d^{a-q}}{d^{a-q}} \left. \left( K^2)^{a-q+r+2}(a-q)! \right) \right|_{r=0}.
\]

However, the condition \( a - q = 2 \) implies \( a = 2, q = 0 \). Therefore we can set \( s = 0 \), and the expression vanishes:

\[
B_{2,2}^{(2;2)}(s) = 0.
\]

(6.67)

Now we collect the results of (6.63), (6.64), and (6.67). We take \( \bar{\eta} = k_3 \). Define

\[
C_1 = \frac{[1 \; 3][1 \; 3]}{(K^2)^2[1 \; 2]^2[3 \; 4]^2}.
\]

(6.68)

Let us begin with the terms with \( q = 0 \):
expression was given in Eq. (5.33) of [29], and reads
\[
B_{2;2}^{(0)}(s = 0) = \frac{K^2(1 - 2z)^2}{3(34)^2[12]^2},
\]
\[
B_{2;2}^{(2;0;1)}(s = 0) = C_1(-\frac{1}{4}(1 - 2z)^2[13]^2(s13)^2),
\]
\[
B_{2;1}^{(2;1;1)}(s = 0) = C_1(-\frac{1}{4}(1 - 2z)^2[13]^3(s13)^2 + \frac{1}{2}(3z - 2z + 3z^2)K^2[13](13)),
\]
\[
B_{2;0}^{(2;2;1;1)}(s = 0) = C_1((-3 + 6z - 3z^2)(K^2)^2 - (1 - 2z)^2[13]^2(s13)^2 + (6 - 18z + 12z^2)K^2[13](13)).
\]

For \( q = 1 \):
\[
- \frac{d}{ds} B_{2;0}^{(0)}(s) \bigg|_{s=0} = -\frac{(1 - 2z)^2}{2(34)^2[12]^2} \frac{[31](13) - (1 - 2z)K^2}{(1 - 2z)},
\]
\[
- \frac{d}{ds} B_{2;1}^{(2;0;1)}(s) \bigg|_{s=0} = C_1\left((1 - 2z)^2[13]^2(s13)^2 + \left(-1 + \frac{7}{2}z - 3z^2\right)K^2[13](13)\right).
\]
\[
- \frac{d}{ds} B_{2;0}^{(2;1;1)}(s) \bigg|_{s=0} = C_1((-4 - 10z + 6z^2)(K^2)^2 + 2(1 - 2z)^2[13]^2(s13)^2 + (-10 + 30z - 21z^2)K^2[13](13)).
\]

For \( q = 2 \):
\[
\frac{1}{2} \frac{d^2}{ds^2} B_{2;0}^{(0)}(s) \bigg|_{s=0} = \frac{1}{(34)^2[12]^2} z((1 - 2z)[31](13) - (1 - z)K^2),
\]
\[
\frac{1}{2} \frac{d^2}{ds^2} B_{2;0}^{(2;0;1)}(s) \bigg|_{s=0} = C_1((-1 + 2z - z^2)(K^2)^2 - (1 - 2z)^2[13]^2(s13)^2 + (4 - 14z + 12z^2)K^2[13](13)).
\]

All together, we get the following result for the \( N \) term:
\[
C[K]^{(1)} = \frac{(13)[13](K^2)^3(1 - 4z(1 - z))}{12(34)^3(13)^2} z((1 - 2z)[31](13) - (1 - z)K^2) + \frac{K^2[13](13)}{(43)^3[12]^2} \left(\frac{(13)[13](13)^2}{(6 - 2z(1 - z))}\left((1 - z)(13)^2 + \left(\frac{1}{2} - 3z(1 - z)\right)K^2[13](13)\right)\right).
\]

(iii) The result of \( c_{12}^{[34]} \)

Final bubble coefficient:
\[
c_{12}^{[34]} = -\frac{2}{(K^2)^2} \left(\frac{(13)[13](K^2)^3}{(34)^3[12]^2} \left(\frac{1}{2} - 4z(1 - z)\right) - \frac{5(K^2)^4}{3[12]^2(34)^2} z(1 - z) + \frac{(K^2)^4}{6[12]^2(34)^2} \right) + \frac{K^2[13](13)}{(43)^3[12]^2} \left(\frac{(13)[13](13)^2}{(6 - 2z(1 - z))}\left((1 - z)(13)^2 + \left(\frac{1}{2} - 3z(1 - z)\right)K^2[13](13)\right)\right) + \frac{(12)[34][43]}{K^2[13]} \left((13)[13](13)^2\right)
\]
\[
= \frac{(34)^2}{3x^2t^2} (-4(m^2 + \mu^2)s + 6(m^2 + \mu^2)t - 2st).
\]

where we used the definitions \( K^2 = t, \langle 13 \rangle[13] = -s - t \).

E. Comparison with the literature

The \( t \)-channel cut of \( A(1^-, 2^-, 3^+, 4^+) \) admits a decomposition in terms of cuts of master integrals as shown in Fig. 8. Its expression was given in Eq. (5.33) of [29], and reads
\[
A_{4}^{\text{fermion}}(1^-, 2^-, 3^+, 4^+) |_{t-\text{cut}} = -2A_{4}^{\text{scalar}}(1^-, 2^-, 3^+, 4^+) |_{t-\text{cut}} - \frac{1}{(4\pi)^2} \epsilon A_{4}^{\text{tree}}(tJ_4 - I_2(t)) |_{t-\text{cut}}
\]

with

025031-18
represent any coefficient of the master integral as
\[ A_4^{\text{scalar}}(1^-, 2^-, 3^+, 4^+) \bigg|_{t\to\text{cut}} = \frac{1}{(4\pi)^{2-\epsilon}} A_4^{\text{free}} \left( \frac{1}{t} I_2^{(1,3), D=6-2\epsilon} + \frac{1}{s} j_2^{(1,3)} - \frac{t}{s} K_4 \right) \bigg|_{t\to\text{cut}} \quad (6.75) \]

and
\[ A_4^{\text{fermion}}(1^-, 2^-, 3^+, 4^+) \bigg|_{t\to\text{cut}} = -2A_4^{\text{scalar}}(1^-, 2^-, 3^+, 4^+) \bigg|_{t\to\text{cut}} - \frac{1}{(4\pi)^{2-\epsilon}} A_4^{\text{free}}(tJ_4 - I_2(t)) \bigg|_{t\to\text{cut}} \quad (6.77) \]
\[ = -\frac{1}{(4\pi)^{2-\epsilon}} A_4^{\text{tree}} \left( \frac{2}{7} I_2^{(1,3), D=6-2\epsilon} + \frac{2}{s} j_2^{(1,3)} - \frac{2t}{s} K_4 + tI_4 - I_2(t) \right) \bigg|_{t\to\text{cut}} \quad (6.78) \]
\[ = -iA_4^{\text{free}} \left( \frac{2}{7} I_2^{\text{BFM}}[1] + \frac{1}{3} I_4 - \frac{2}{3} I_2^{\text{BFM}}(m^2 + \mu^2) + \frac{2}{s} I_2^{\text{BFM}}(m^2 + \mu^2) - \frac{2t}{s} I_4^{\text{BFM}}[(m^2 + \mu^2)^2] + tI_4^{\text{BFM}}(m^2 + \mu^2) - I_2^{\text{BFM}}[1] \right) \bigg|_{t\to\text{cut}} \quad (6.79) \]
\[ = \frac{(12)^2[34]^2}{st} \left( \frac{2}{3} I_2^{\text{BFM}}[1] + \frac{4}{3t} I_2^{\text{BFM}}(m^2 + \mu^2) - \frac{2}{s} I_2^{\text{BFM}}(m^2 + \mu^2) + \frac{2t}{s} I_4^{\text{BFM}}[(m^2 + \mu^2)^2] - tI_4^{\text{BFM}}(m^2 + \mu^2) \right) \bigg|_{t\to\text{cut}}. \quad (6.80) \]

We have reproduced every one of these coefficients, up to an overall minus sign in the amplitude.

VII. FROM POLYNOMIALS IN u TO FINAL COEFFICIENTS

As proven in Appendix B, the coefficients of 2-, 3-, and 4-point functions in four dimensions are polynomials in \( u \) (or equivalently \( \mu^2 \)), of known degree \( d \); for boxes, \( d = [(n+2)/2] \); for triangles, \( d = [(n+1)/2] \); for bubbles, \( d = [n/2] \); where \([x]\) denotes the greatest integer less than or equal to \( x \). Using this fact, we can generally represent any coefficient of the master integral as
\[ P_d(u) = \sum_{r=0}^{d} c_r u^r. \quad (7.1) \]
The coefficients \( c_r \) are in one-to-one correspondence to the coefficients of the \emph{shifted-dimension} master integrals (see Sec. II).

To compute the \( c_r \) analytically, one can proceed with the standard differentiations with respect to \( u \), at \( u = 0 \):
\[ c_r = \frac{1}{k!} \frac{d^k}{du^k} P_d(u) \bigg|_{u=0}. \quad (7.2) \]
When the differentiations are time consuming, or the analytic expression is not needed, one can switch to the following numerical procedure, and extract the \( c_r \), algebraically, by \emph{projections}.

(1) Generate the values \( P_{d,k} \) \((k = 0, \ldots, d - 1)\),
\[ P_{d,k} = P_d(u_k), \quad (7.3) \]
by evaluating \( P_d(u) \) at particular points:
\[ u_k = e^{-2\pi ik/d}. \quad (7.4) \]
(2) Using the orthogonality relations for plane waves, one can obtain the coefficient \( c_r \) simply by the following formula:
\[ c_r = \frac{1}{d} \sum_{k=0}^{d-1} P_{d,k} e^{2\pi i rk/d}. \quad (7.5) \]

ACKNOWLEDGMENTS

R.B. is supported by Stichting FOM. B.F. is supported by Zhejiang University, China.

APPENDIX A: CHANGE OF BASIS

To compare our results to the literature, we need to convert the master integrals used in \([29,30]\) to our canonical \((4 - 2\epsilon)\)-dimensional basis, \((2.1)\). For clarity, we now denote the basis used in this paper by \( I_n^{\text{BFM}} \), while the other integrals in this appendix are defined according to \([29,30]\). The first point is then that
\[ I_n = i(-1)^n(4\pi)^2(\epsilon_n)_{\text{BFM}}, \quad (A1) \]

We use the identities from Appendix A.4 of [29] to perform the conversion.

\[ J_4 = I_4[m^2 + \mu^2] = i(4\pi)^2 - \epsilon I_4^{\text{BFM}}[m^2 + \mu^2], \quad (A2) \]

\[ I_2(t) = i(4\pi)^2 - \epsilon I_2^{\text{BFM}}[1], \quad (A3) \]

\[ J_2^{(13,D=6-2\epsilon)} = \frac{t}{6}I_2(t) + \frac{1}{3}I_1 - \frac{2}{3}J_2^{(13)} \]

\[ = i(4\pi)^2 - \epsilon \left( \frac{t}{6}I_2^{\text{BFM}}[1] + \frac{1}{3}I_1 \right) - \frac{2}{3}I_2^{\text{BFM}}[m^2 + \mu^2]. \quad (A4) \]

\[ J_2^{(13)} = J_2^{(13)}[m^2 + \mu^2] = i(4\pi)^2 - \epsilon I_4^{\text{BFM}}[m^2 + \mu^2]. \quad (A5) \]

\[ K_4 = I_4[(m^2 + \mu^2)^2] = i(4\pi)^2 - \epsilon I_4^{\text{BFM}}[(m^2 + \mu^2)^2]. \quad (A6) \]

**APPENDIX B: THE \( u \)-DEPENDENCE OF THE COEFFICIENTS**

Here we analyze the \( u \)-dependence of the integral coefficients given by our formulas. First, we prove that they are polynomials in \( u \). Then, we present some alternate formulas where this polynomial dependence is more explicit. This material is a straightforward generalization of the analysis in the massless case [33], so we omit many of the details here.

To begin, we rewrite our vectors \( R_j, Q_j \) from (3.23) and (3.24) in the following way:

\[ R_j = (1 - 2z) + \frac{M_j^2 - M_j^2}{K^2} p_j + \beta_j K, \quad (B1) \]

\[ p_j = \left( p_j - \frac{P_j \cdot K}{K^2} K \right), \]

\[ \beta_j = - \left( \frac{P_j \cdot K}{K^2} \right) \left( 1 + \frac{M_j^2 - M_j^2}{K^2} \right), \quad (B2) \]

\[ Q_j = - (1 - 2z) + \frac{M_j^2 - M_j^2}{K^2} q_j + \alpha_j K. \quad (B3) \]

\[ q_j = \left( K_j - \frac{K_j \cdot K}{K^2} K \right). \quad (B4) \]

\[ \alpha_j = - \left( \frac{K_j \cdot K}{K^2} \right) \left( 1 + \frac{M_j^2 - M_j^2}{K^2} \right) + \frac{K_j^2 + M_j^2 - M_j^2}{K^2}. \quad (B5) \]

Notice that

\[ p_j \cdot K = 0, \quad q_j \cdot K = 0. \quad (B6) \]

Using (3.16), we have

\[ R_j(u) = - \beta(\sqrt{1 - u}) p_j + \beta_j K, \]

\[ Q_j(u) = - \beta(\sqrt{1 - u}) q_j + \alpha_j K. \quad (B7) \]

This is the same expression as in the massless case, except for the factor \( \beta \). The point is that now all \( u \)-dependence is in the factor \( \sqrt{1 - u} \), just as in the massless case. In fact, we should now consider the factor \( \beta \sqrt{1 - u} \) as our basic quantity. The proof that the integral coefficients are polynomials in \( u \) was performed by considering the (demonstrably finite) series expansion in \( \sqrt{1 - u} \), and showing that the odd powers drop out. Therefore, the same arguments now carry over to the series expansion in \( \beta \sqrt{1 - u} \).

1. **Triangle coefficients**

Let us begin with triangle coefficients. The null vectors \( P_{s,i} \) exhibit a simple dependence on \( u \). Specifically,

\[ P_{s,i}(u) = - \beta(\sqrt{1 - u}) P_{q,i}, \]

where

\[ P_{q,i} = \left( \sqrt{-\frac{q_j^2}{K^2}} \right) K, \quad (B8) \]

which is manifestly independent of \( u \). In defining the spinor components of \( P_{s,i} \), we can place the \( u \)-dependent factor inside the antiholomorphic spinor, i.e.,

\[ |P_{s,i}| = |P_{q,i}|, \quad |P_s| = - \beta(\sqrt{1 - u})|P_{q,i}|. \quad (B9) \]

Then, for the triangle coefficients, we have

\[ \begin{align*}
C[Q_s, K] &= \frac{(K^2)^{1+n}}{2} \frac{1}{(-\beta \sqrt{1 - u})^{n+1}} \frac{1}{(\sqrt{1 - 4q_j^2} K^2)^{n+1}} \frac{1}{(n + 1)!}(P_{q,1} P_{q,2})^{n+1} d\tau^{n+1} \\
&\times \left( \prod_{j=1}^{n} \left[ P_{q,1}(1 - \tau P_{q,2}) Q_{s}(1 - \tau P_{q,2}) - Q_{s}(1 - \tau P_{q,2}) P_{q,1}(1 - \tau P_{q,2}) + [P_{q,1} \equiv P_{q,2}] \right] \right)_{\tau=0}.
\end{align*} \quad (B10) \]

Further, we make use of some identities,
\[
\langle \ell | Q_s(u)Q_s(u) | \ell \rangle = \left( \ell | (Q_s - \frac{\alpha_s}{\alpha_s} Q_s(u))Q_s(u) | \ell \right) = -\beta \sqrt{1-u} \left( \ell | (q_i - \frac{\alpha_s}{\alpha_s} q_s)Q_s(u) | \ell \right) \langle \ell | R_s(u)Q_s(u) | \ell \rangle
\]
along with the definitions
\[
\tilde{q}_i = (q_i - \frac{\alpha_s}{\alpha_s} q_s), \quad \tilde{p}_i = (p_i - \frac{\beta_s}{\alpha_s} q_s).
\]

Our final form for the triangle coefficient is
\[
C[Q_s, K] = \frac{1}{2} \sqrt{\frac{K^2}{-4q_s^2}} \frac{1}{(n+1)!} \left( \frac{d^{n+1}}{d\tau^{n+1}} \left( \prod_{i=1}^{n+1} \langle p_{s_1} - \tau p_{s_2} | \tilde{q}_i Q_s(u) | p_{s_1} - \tau p_{s_2} \rangle \right) \right)_{\tau=0}
\]
where
\[
\sqrt{1-u}, \text{ in the following formulas:}
\]
\[
C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{1}{q!} \left( \frac{d^q}{d\tau^q} \left( B_{n,n-q}^{(0)}(s) \right) \right)_{\tau=0},
\]

2. Bubble coefficients

We follow the same procedure as with triangles, and make use of the same definitions (B8) and (B9). The \( u \)-dependence can be concentrated entirely within the vector \( Q_s(u) \), since we have made sure to choose the spinor components wisely in (B9), so that the holomorphic spinors are \( u \)-independent.

\[
B_{n,t}^{(0)}(s) = \frac{d^n}{d\tau^n} \frac{1}{n!} \left( \frac{d}{d\tau} \right)^{n+1} \left( \prod_{i=1}^{n+1} \langle \ell | R_s(u)(K-s\eta) | \ell \rangle \right)_{\tau=0},
\]

\[
B_{n,t}^{(c;h;1)}(s) = \frac{(-1)^{b+1}}{b!(\beta \sqrt{1-u})^{b+1} \sqrt{-4q_s^2} K^2} \frac{d^b}{d\tau^b} \left( \frac{1}{(t+1)} \left( \prod_{i=1}^{t+1} \langle p_{s_1} - \tau p_{s_2} | \eta p_{s_3} \rangle \right) \right)_{\tau=0},
\]

\[
B_{n,t}^{(c;h;2)}(s) = \frac{(-1)^{b+1}}{b!(\beta \sqrt{1-u})^{b+1} \sqrt{-4q_s^2} K^2} \frac{d^b}{d\tau^b} \left( \frac{1}{(t+1)} \left( \prod_{i=1}^{t+1} \langle p_{s_1} - \tau p_{s_2} | \eta p_{s_3} \rangle \right) \right)_{\tau=0},
\]

3. Box and pentagon coefficients

Although the formula (4.1) for box and pentagon coefficients looks simple, the \( u \)-dependence now gets complicated. We consider the separate cases \( k = 2, k = 3, \) and \( k \geq 4. \)

\textbf{a. The case} \( k = 2 \)

In this case, there is only one box, and no pentagons. The box coefficient is given by

\[
\langle \ell | R_j(u)Q_s(u) | \ell \rangle = \frac{1}{2} \left( \prod_{i=1}^{2} \left( \langle p_{s_1} - \tau p_{s_2} | \eta p_{s_3} \rangle \right) \right)_{\tau=0} + \left( \langle p_{s_1} - \tau p_{s_2} | \eta p_{s_3} \rangle \right)_{\tau=0},
\]

Given the vectors \( Q_s, Q_j, K \) that select a particular box, it is useful to construct a vector \( q^{(j,q,K)}_j \) that is orthogonal to all three, and independent of \( u \):
\[ (q_0)_{(q_i, q_j, K)}^{(q_0, q_i, K)} = \frac{1}{K^2} \epsilon_{\mu \nu \rho \xi} q_0^\mu q_i^\nu q_j^\rho K^\xi \]  
\[ = \frac{1}{K^2} \epsilon_{\mu \nu \rho \xi} K_i^\mu K_j^\nu K^\rho K^\xi. \]  
(B18)  
(B19)

As in the massless case, the \( u \)-dependence can be concentrated in a single factor, \( \alpha^{(q_0, q_i)}(u) \). If all input quantities are set to their values with \( u = 0 \), except for adjusting the definition of \( R_s(u) \) as follows:

\[ R_s(u) \rightarrow \tilde{R}_s(u) = \frac{P_s \cdot q_0^{(q_0, q_i, K)}}{(q_0^{(q_0, q_i, K)})^2} (\alpha^{(q_0, q_i)}(u) - 1) \times (-\beta q_0^{(q_0, q_i, K)}) + R_s(u = 0), \]  
(B20)

\[ \alpha^{(q_0, q_i)}(u) = \sqrt{\beta^2 (1 - u) + 4K^2(\alpha_s, \alpha_s, (2q_i, q_j) - \alpha_i^2 \alpha_j^2 \alpha_i^2 \alpha_j^2)}, \]  
(B21)

then the value of the box coefficient remains the same.

In summary, the box coefficient for \( k = 2 \) is given by

\[ C[K_i, K_j]_{k=2} = \frac{(K^2)^{2+n}}{2} \left( \prod_{i=1}^{n+2} \langle P_{j|i;1} \hat{R}_s(u) | P_{ji;2} \rangle \right) \left( \prod_{i=1}^{n+2} \langle P_{ji;1} | K | P_{ji;2} \rangle^{n+2} \right) + \{ P_{j|i;1} \leftrightarrow P_{ji;2} \}. \]  
(B22)

where

\[ \tilde{R}_s(u) = \frac{P_s \cdot q_0^{(q_0, q_i, K)}}{(q_0^{(q_0, q_i, K)})^2} (\alpha^{(q_0, q_i)}(u) - 1) (-\beta q_0^{(q_0, q_i, K)}) + R_s(u = 0) \]  
(B23)

and

\[ P_{ji;a} = P_{(q_i, q_j);a}(u = 0). \]  
(B24)

In evaluating (B23), it is useful to observe the following:

\[ (p_s \cdot q_0^{(q_0, q_i, K)}) = \frac{\epsilon(p_s, q_i, q_j, K)}{K^2} = \frac{\epsilon(p_s, K_i, K_j, K)}{K^2}. \]  
(B25)

This formula (B22) looks the same as in the massless case; the difference is the appearance of \( \beta \) in (B23), both explicitly and through the definition (B21) of \( \alpha^{(q_0, q_i)} \).

**b. The case \( k = 3 \)**

Here there is a pentagon, as well as three boxes. The differences from the massless case are all based in the definitions of \( R_s(u) \), \( Q_s(u) \): there is always a factor of \( \beta \) accompanying \( \sqrt{1 - u} \), and mass parameters enter into the definitions (B2) and (B5) of \( \beta \), \( \alpha \).

When we make these adjustments, we find that the pentagon coefficient takes the same form as in the massless case,

\[ C[S_i, S_j, S_k] = (K^2)^{3+n} \prod_{s=1}^{n+3} \beta_s^{(q_0, q_i, q_j, q_k)} \]  
(B26)

but the definition of \( \beta_s^{(q_0, q_i, q_j, q_k)} \) now includes mass parameters:

\[ \beta_s^{(q_0, q_i, q_j, q_k)} \equiv \beta_s^{(K_i, K_j, K_k, P_s)} \]

\[ = \frac{(K_i^2 + M_i^2 - m_i^2)\epsilon(K_i, K_j, K_k, K_k) + (K_j^2 + M_j^2 - m_j^2)\epsilon(K_j, K_k, K_k)}{K^2 \epsilon(K_i, K_j, K_k, K_k)} \]

\[ - \frac{(K_i^2 + M_i^2 - M_k^2)\epsilon(K_i, K_j, P_s, K_k) + (K_j^2 + M_j^2 - M_k^2)\epsilon(K_j, K_k, P_s)}{K^2 \epsilon(K_i, K_j, K_k, K_k)}. \]  
(B27)

The expression (B27) is symmetric in \( K_i, K_j, K_k \) (recall that \( M_2 \) is the mass associated with \( K \) in this context).

The box coefficients are given by

\[ C[S_i, S_j, S_k]_{k=3} = \frac{(K^2)^{2+n}}{2} \left( \prod_{i=1}^{n+2} \langle P_{j|i;1} \tilde{R}_s(u) | P_{ji;2} \rangle \right) \left( \prod_{i=1}^{n+2} \langle P_{j|i;1} | Q_s(u) | P_{ji;2} \rangle^{n+2} \right) - \sum_{s=1}^{n+3} \beta_s^{(q_0, q_i, q_j, q_k)} \left( \prod_{i=1}^{n+2} \langle P_{j|i;1} | K | P_{ji;2} \rangle^{n+2} \right) + \{ P_{j|i;1} \leftrightarrow P_{ji;2} \}. \]  
(B28)

The derivation of (B28) involved the result from the case \( k = 2 \). All mass dependence is already included in the definitions (B21) and (B23), along with the similarly defined vector \( Q_s(u) \):
\[ \tilde{R}_s(u) = \frac{p_s \cdot d_0^{(q_s, q_j, K)}}{(d_0^{(q_s, q_j, K)})^2} \left( (\alpha^{(q_s, q_j)})(u) - 1 \right) (\beta d_0^{(q_s, q_j, K)}) + R_s(u = 0), \]

(B29)

\[ \tilde{Q}_s(u) = \frac{q_t \cdot d_0^{(q_s, q_j, K)}}{(d_0^{(q_s, q_j, K)})^2} \left( (\alpha^{(q_s, q_j)})(u) - 1 \right) (\beta d_0^{(q_s, q_j, K)}) + Q_s(u = 0). \]

(B30)

c. The case \( k \geq 4 \)

In the derivation of the formulas, we introduce the following functions:

\[ \gamma_s^{(K_i, K_j, K_s, K_t)} = \frac{(K_i^2 + M_t^2 - m_t^2)\epsilon(K, K_j, K_s, K_t) + (K_j^2 + M_t^2 - m_t^2)\epsilon(K_i, K, K_s, K_t)}{K^2 \epsilon(K, K_j, K, K_i)} + \frac{(K_s^2 + M_t^2 - m_t^2)\epsilon(K_i, K_j, K, K_t) + (K_t^2 + M_t^2 - m_t^2)\epsilon(K_i, K_j, K_s, K)}{K^2 \epsilon(K, K_j, K, K_i)} - \frac{\epsilon(K_i, K_j, K_s, K_t)}{\epsilon(K_i, K_j, K, K_i)}. \]

(B31)

The numerator of \( \gamma_s^{(K_i, K_j, K_s, K_t)} \) is symmetric in \( K_i, K_j, K_s, K_t \); the denominator breaks this symmetry by singling out \( K_s \).

We find the following results:

The pentagon coefficients are given by

\[ C(Q_i, Q_j, Q_t) = (K^2)^{3+n} \frac{\prod_{i=1}^{n+k} d_0^{(q_s, q_j, q_t; p_i)}}{\prod_{i=1}^{n+k} d_0^{(q_s, q_j, q_t; p_i)}}. \]

(B32)

The box coefficients are given by

\[ C(Q_i, Q_j)_{k\geq 4} = \frac{(K^2)^{2+n}}{2} - \frac{\prod_{i=-1}^{k} \langle P_{ji:1} | \tilde{R}_s(u) | P_{ji:2} \rangle \prod_{i=-1}^{k} \langle P_{ji:1} | \tilde{Q}_s(u) | P_{ji:2} \rangle}{\prod_{i=-1}^{k} \langle P_{ji:1} | K | P_{ji:2} \rangle} \]

\[ - \frac{\prod_{i=-1}^{k} \langle P_{ji:1} | K | P_{ji:2} \rangle}{\prod_{i=-1}^{k} \langle P_{ji:1} | \tilde{Q}_s(u) | P_{ji:2} \rangle} \]

\[ + \langle P_{ji:1} | K | P_{ji:2} \rangle. \]

(B33)

Again, all the \( u \)-dependence is concentrated in \( \tilde{R}(u) \) and \( \tilde{Q}(u) \). The definitions of \( P_{ji:1}, \tilde{R}_s(u), \tilde{Q}_s(u), \beta_d^{(q_s, q_j; p_i)}, \) and \( \gamma_s^{(K_i, K_j, K_s, K_t)} \) are given in (B24), (B29), (B30), (B27), and (B31), respectively.


