ISR PERFORMANCE REPORT

ISR-MA/PJB/rh of 21st January 1974

Runs, 390, 400 and 403

Please exchange pages 1/2, 3/4, 11/12, 17/18, 21/Figs. 1 and 2 and Figures 3 to 8 of the above report against the ones attached. As a result of these changes, the final solenoid is expected to be able to work with a Q-separation of 0.02 instead of 0.023. Some corrections have been made in Appendix C. More photographs of coupling have been included in Figure 2 and two points in Figure 5 have been corrected.

P.J. Bryant
G. Kantardjian
Preliminary investigations of the perturbations caused
by a solenoid inclined to the beam axis

Conclusions

Tests have been carried out with a solenoid inclined at 7.4° to the beam axis. Three configurations were tested: i) with end plates; ii) open-ended and iii) one end open and the other with an end plate. The results and conclusions are summarized below:

i) Closed orbit

This presented no problems.

ii) Q-shifts

No significant Q-shifts were added by the test solenoid when mounted with two end plates or when left without end plates, but for the unbalanced case of only one end plate, the Q-values were strongly perturbed by the coupling and the true Q-shifts are not known.

iii) Coupling

The theory for coupling in skew fields and axial fields is given in Appendix C. A dimensionless coupling coefficient has been defined and expressions for the interchanged energy are given.

The coupling in ring 2 under normal conditions was found to be equivalent to a systematic tilt in all magnet units of 0.06 mrad. The difference due to the solenoid open-ended or with both end plates is not easily measured but is estimated to be approximately 25% of the existing coupling. This additional coupling principally arises from the axial field. The end effects for the balanced configurations are self-compensating to a very high degree as expected. The coupling was compensated with the skew quadrupoles, but the correction was not valid across the whole aperture. However,
the levels of coupling involved for the test solenoid were not sufficient to prevent stacking or to deteriorate medium intensity stacks (6 - 8 A) made on the FP line. Experiments at lower energies would be interesting.

The unbalanced configuration with only one end plate excited very strong coupling as expected. This arose mainly from the end effects, which are equivalent to an integrated skew gradient of 0.72 T or to powering the Q1 skew quadrupole series by 10.5%. It was still possible to make a stack, although not a good one. One new effect of the coupling was the appearance of both the horizontal and vertical Schottky scans on the vertical scan.

iv) **Excitation of non-linear resonances**

This was not directly studied but there is some evidence that the open-ended solenoid caused some excitation. This will have to be the subject of a further series of measurements.

v) **Extrapolation to the detector solenoid**

**Effect of axial field.** For 25 GeV/c operation at full field, the coupling coefficient is estimated as $4.35 \times 10^{-3}$.

**Effect of end plates.** The self-compensation of the end plates will be less efficient as their separation is far greater than for the test solenoid. Assuming there is a 10% residual effect, the coupling coefficient will be $2.4 \times 10^{-3}$ at 25 GeV/c for full field.

Assuming the residual coupling in the machine ($3.57 \times 10^{-3}$) arises from tilts, then the total coupling would be $7.4 \times 10^{-3}$ (i.e. linearly adding quadrupolar effect and taking the square root of the sum of the squares when adding quadrupolar and axial field effects). Table 6 summarizes the emittance interchanges for various cases assuming the FP working line.

Table 6 is also interesting with respect to the predicted 5.6% emittance transfer for the normal machine. Several people have measured an emittance blow-up during the first few turns after injection and this could be a partial explanation. Taking typical emittances measured in the transfer lines, $\varepsilon_H = 1.2 \pi$ and $\varepsilon_V = 0.4 \pi \text{mm.mrad}$, this effect increases $\varepsilon_V$ from $0.4 \pi$ to $0.47 \pi$. However, for the detector solenoid this would be more serious, i.e. $0.4 \pi$ to $0.61 \pi$. Compensation with the skew quadrupoles is a
TABLE 6

Expected coupling for detector solenoid compared to present conditions and the test solenoid
(assuming Δ = 0.01 i.e. FP line)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Coupling coeff. C x 10^{-3}</th>
<th>Period of interchange of energies T x 10^{-3} sec.</th>
<th>Fraction of horizontal emittance coupled into vertical plane*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present machine</td>
<td>3.57 (measured)</td>
<td>0.325</td>
<td>5.6 %</td>
</tr>
<tr>
<td>Plus test solenoid with end plates</td>
<td>4.34 (measured)</td>
<td>0.315</td>
<td>8.0 %</td>
</tr>
<tr>
<td>Plus new detector solenoid</td>
<td>7.4 (calculated)</td>
<td>0.253</td>
<td>17.7 %</td>
</tr>
</tbody>
</table>

* The peak fraction, e, is calculated from equation (20.c), Appendix C.
The emittance interchange is assumed to be equal to the average energy interchanged, i.e. e/2.

possibility, but one sure method would be to retreat from the diagonal. In order to reduce the emittance interchange to the present levels, the Q-separation would have to be increased from 0.01 to 0.02 as shown in Figure 11. It should be understood that it would still be possible to work in much of the shaded region, but increased emittance interchange would occur.

A consequence of the above would be that high intensity stacks would have to be made across the 5th order resonances in order to get a sufficiently large Q-spread. The resonance excitation in the slots then becomes important and very little is known about this at present.

FIGURE 11
Band in which emittance interchange exceeds 5%
Introduction

Following a proposal\(^1\) to build a particle detector using a solenoid around an intersection region in the ISR, a test solenoid was constructed and installed in SS416 by G. Kantardjian. This test solenoid is the first axial field device to be mounted in the ISR.

Table 1 compares the beam parameters and principal characteristics corresponding to the proposed solenoid detector at an intersection and the test solenoid in SS416.

### TABLE 1

Comparison of principal parameters for the proposed solenoid detector and the test solenoid

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Solenoid Detector</th>
<th>Test Solenoid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial field B(_s)</td>
<td>1.5 T</td>
<td>1.46 T</td>
</tr>
<tr>
<td>Overall length L</td>
<td>2.8 m</td>
<td>0.465 m</td>
</tr>
<tr>
<td>Thickness of end plates (\ell)</td>
<td>0.6 m</td>
<td>0.080 m (removable)</td>
</tr>
<tr>
<td>Angle w.r.t. beam (\theta)</td>
<td>7.4°</td>
<td>7.4°(variable)</td>
</tr>
<tr>
<td>(\beta_H) (on 15°)</td>
<td>22.6</td>
<td>36.5</td>
</tr>
<tr>
<td>(\beta_V)</td>
<td>14.9</td>
<td>12.8</td>
</tr>
<tr>
<td>(\gamma_P)</td>
<td>2.30</td>
<td>2.33</td>
</tr>
</tbody>
</table>

The test solenoid is not a prototype for the detector solenoid and although the machine parameters and the axial field strength are well matched, there is almost an order of magnitude difference in the overall lengths and thicknesses of the end plates.

Perturbations arising from a solenoid

The perturbations caused by a solenoid can be considered as arising from two sources:

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\(^1\) Proposal to ISRC by CERN-Columbia-Rockefeller Collaboration, CERN/ISRC 73-13
i) **The uniform central field region**

This causes coupling and closed orbit distortions if the solenoid axis is inclined to the beam.

ii) **The ends**

At each end, the axial flux diverges. The exact shape of these fields depends on the geometry, and conversely it is possible to partially control these fields by varying the geometry (e.g. end plates with slots). Depending on their shape, the end fields can cause appreciable closed orbit distortions, Q-shifts, coupling or excite non-linear resonances. The latter is probably the most serious.

**Outline of investigation**

The test solenoid was studied in three configurations:

i) **Fitted with end plates with horizontal slots** (Run 390)

This configuration is described in Appendix A. Its expected properties are: negligible Q-shifts; end plates mutually compensating for closed orbit, coupling and non-linearities; minimizes non-linearities by minimizing integral of transverse flux seen by beam; very uniform central field.

ii) **Open-ended solenoid** (Run 403)

This configuration is described in Appendix B. Its expected properties are: negligible Q-shifts; end fields compensate half of the orbit distortion arising from the central field and are mutually compensating for coupling; non-linearities are increased; central field is less uniform.

iii) **One end plate mounted, the other end open** (Run 400)

This configuration is also described in Appendix B. It is unbalanced and is expected to have the largest disruptive effect.

In each case, the closed orbit distortion was corrected, Q-shifts checked, coupling measured and stacks made. No detailed studies of non-linear resonance excitation were made as the available machine time was too limited.
Measurements and results

These tests were made on the 'FP' lines at 22 and 26 GeV/c, since the Q-separation is constant across the aperture and the line is close to the diagonal. By using one set of the Téwilliger quadrupoles, the whole working line can be displaced uniformly with respect to the diagonal and coupling measurements can then be made across the aperture under similar conditions.

i) Solenoid with end plates (Run 390 - 22 GeV/c) and without end plates (Run 403-- 22 GeV/c)

a) Closed orbit. In both of these cases, the vertical closed orbit was corrected by $H_{416}$. The correction scheme (Fig. 1) is not perfect, but the residual closed orbit distortion is quite tolerable (for the end plate case the overall distortion was actually reduced). The corrections are given by $(I_{H_{416}} = 0.387 I_{solenoid})$ with end plates and $(I_{H_{416}} = 0.136 I_{solenoid})$ without end plates. In both cases, the correction is independent of momentum. The closed orbit values are given in Table 2.

<table>
<thead>
<tr>
<th>Solenoid with end plates</th>
<th>Solenoid off</th>
<th>Solenoid on</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal pk-to-pk</td>
<td>9.1 mm</td>
<td>9.1 mm</td>
</tr>
<tr>
<td>vertical pk-to-pk</td>
<td>6.6 mm</td>
<td>5.5 mm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solenoid without end plates</th>
<th>Solenoid off</th>
<th>Solenoid on</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal pk-to-pk</td>
<td>4.3 mm</td>
<td>4.4 mm</td>
</tr>
<tr>
<td>vertical pk-to-pk</td>
<td>5.0 mm</td>
<td>7.6 mm</td>
</tr>
</tbody>
</table>

b) $Q$-values. No significant $Q$-shifts were attributable to the test solenoid in its two balanced configurations (see Table 3).

c) Coupling. Figure 2 shows a typical set of polaroid photographs demonstrating coupling. The amplitude of the traces represents the amplitude of the beam's coherent oscillation and the modulation period is the period of the energy interchange between the two planes. In the first case, the
### TABLE 3

Comparison of Q-values

<table>
<thead>
<tr>
<th>Radial position</th>
<th>with end plates</th>
<th>without end plates</th>
</tr>
</thead>
<tbody>
<tr>
<td>mm. α p av.</td>
<td>Q_H</td>
<td>Q_V</td>
</tr>
<tr>
<td>inj.</td>
<td>.598</td>
<td>.586</td>
</tr>
<tr>
<td>-20</td>
<td>.618</td>
<td>.607</td>
</tr>
<tr>
<td>0</td>
<td>.637</td>
<td>.627</td>
</tr>
<tr>
<td>+20</td>
<td>.656</td>
<td>.644</td>
</tr>
</tbody>
</table>

Coupling is incomplete and only part of the total energy is interchanged. After some experimentation, it was decided to standardize on the following conditions for coupling measurements.

**Coherent oscillation** : $Q_H$ kicker with normal gain

**Observation** : $\beta_v$-low filter

**Oscilloscope** : timebase 0.1 msec/cm, gain 5 V/cm

(Note: by kicking in one plane and observing in the other, it is easier to determine the period of energy interchange.)

Figure 3 shows the period of energy interchange ($T$) as a function of the position across the chamber for the solenoid off and at 100% both with and without end plates. There is very little difference between the three cases. The measurements on central orbit are summarized in Table 4 and by using equation (19.c) of Appendix C, the coupling coefficients have been calculated. The axial field alone would be expected to give $c = 0.8 \times 10^{-3}$ and, perhaps by coincidence, the difference between the normal ISR machine and the normal machine plus the solenoid at 100% with end plates is $0.77 \times 10^{-3}$. For the case without end plates, the overall coupling was reduced. Three explanations exist:

A) the measurements are spread over two runs separated by 2 weeks and conditions changed,

B) the open ends have an appreciable effect but it happens to correct the overall effect, or

C) the measurements were not sufficiently accurate.
### TABLE 4

**Coupling coefficient measurements on central orbit**

<table>
<thead>
<tr>
<th>Conditions on central orbit</th>
<th>Period T msec.</th>
<th>Coupling coefficient $c \times 10^{-3}$</th>
<th>Effect of solenoid $c_s \times 10^{-3}$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal ISR-2</td>
<td>.325</td>
<td>3.57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solenoid at 100 %:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>with end plates</td>
<td>.315</td>
<td>4.34</td>
<td>+0.77</td>
<td></td>
</tr>
<tr>
<td>without end plates</td>
<td>.330</td>
<td>3.15</td>
<td>-0.42</td>
<td></td>
</tr>
</tbody>
</table>

The residual coupling in the ISR must arise from random tilts of the main magnet units. The coupling coefficient from Table 4, $c = 3.57 \times 10^{-3}$ is equivalent to an average skew gradient of $1.0 \times 10^{-3}$ T/m around the whole ISR. Since only half the circumference is occupied by magnet units, this would correspond to a systematic tilt of 0.059 mrad.

Thus, the test solenoid has an effect equivalent to about 25% of the residual coupling already existing in ISR-2 and the end plates appear to be almost perfectly self-compensating, but this is not surprising for a solenoid only 46.5 cm long.

Figure 4 shows $T$ as a function of the Q-separation ($\Delta$) for beams on central orbit for the solenoid powered at 100% with end plates. As can be seen, these points lie exactly on the curve for $c = 4.3 \times 10^{-3}$ (Table 4) using equation (19.c), Appendix C. The other curve shown is the limiting curve for $c \rightarrow 0$. $\Delta$ was varied by means of the TD2 quadrupoles (the perturbation of the Q-values with coupling is dealt with in Appendix D).

An attempt was made in Run 390 to compensate the coupling with the Q1 series of skew quadrupoles (with the solenoid at 100% with both end plates). Figure 5 shows the variations of $T$ and of the amplitude of the interchanged energy on central orbit for different settings of Q1 (since $\Delta$ = constant and the signal levels were low, the filter output was a reliable measure of the interchanged energy). Figure 6 shows the same parameters across the chamber at the optimum setting (i.e., Q1 = -1.75%).
No improvement was obtained by varying the Q2 quadrupoles and these were left at the standard value of -0.3% in the working line file. It is interesting to note that for Q1 = -1.75%, 0.12 T integrated gradient is added to the ISR and for the coupling coefficient of $c = 4.3 \times 10^{-3}$ (Table 4), 0.2 T would be expected to exist. As can be seen, T changes very little during optimization, but this is not surprising as the limit curve in Figure 4 is very close to the measured curve. The variation in the coupled energy is far stronger, but it is not possible to say with certainty that the coupling is being reduced. It may be that the normal modes are being rotated locally. The theory in Appendix C is not sufficiently sophisticated to show that the normal modes can be rotated by varying degrees according to their azimuthal position. However, this is believed to occur. Figure 6 shows that the compensation is not valid across the whole aperture. This could prove to be a more serious problem as the only cure is to retreat from the diagonal, which reduces the amount of resonance-free space for stacks. Doubts as to the efficiency of the compensation were also raised since:

A) unaccountably, the damping magnet started to anti-damp and blow-up the injected pulses radially, and

B) when pulses were accelerated across the aperture, they "rocked" when watched on the Vosicki monitor (the radial blow-up made this more visible). This could indicate changes in the normal modes across the aperture.

d) Stacking

- Run 390 with end plates. Despite the above strange effects, seen after optimizing the coupling, a normal stack of 7.8 A was made on 22FP with a near zero decay rate (Fig. 7).

- Run 403 without end plates. In this run, a 7.3 A stack was made and then the solenoid was progressively powered. The beam apparently did not suffer ($dI/dt = 3$ ppm/min. throughout). The residual coupling was not optimized in this run.
e) **Excitation of non-linear resonances**

No direct measurements were made, but in Run 403 with the open-ended solenoid a 7.0 A stack was steered radially inside the solenoid. A +10 mm bump had little effect but a -10 mm bump caused the decay rate to jump from 10 ppm/min. to 200 ppm/min. On removing the bump, some beam was lost and considerable radial blow-up was found. Once re-scraped, the beam's dI/dt returned to normal. It is strange that the beam loss occurred when removing the -10 mm bump and not on applying it. This may indicate some resonance excitation, which was in any case expected to be more apparent with the open-ended solenoid.

**ii) Solenoid with one end plate and the other end open (Run 400 - 26 GeV/c)**

This unbalanced configuration of the solenoid proved to be disastrous from an operational point of view. The closed orbit was satisfactorily corrected \( I_{H416} Z = 0.284 I_{solenoid} Z \), but the Q-values were strongly perturbed (see Table 5 and Figure 8 where values are plotted).

**TABLE 5**

Comparison of Q-values for the unbalanced solenoid

<table>
<thead>
<tr>
<th>Radial position ( \alpha_p ) av.</th>
<th>Solenoid off</th>
<th>Solenoid at 100% with one end plate only</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Q_H )</td>
<td>( Q_V )</td>
</tr>
<tr>
<td>inj. -42.7</td>
<td>.595</td>
<td>.580</td>
</tr>
<tr>
<td>-21</td>
<td>.615</td>
<td>.604</td>
</tr>
<tr>
<td>0</td>
<td>.634</td>
<td>.627</td>
</tr>
<tr>
<td>+20</td>
<td>.655</td>
<td>.645</td>
</tr>
</tbody>
</table>

Standard Q-shifts were applied using the PFW's and for the solenoid at 100%, the measured Q-shifts varied according to the Q-separation. Thus, the apparent Q-shift is at least partly due to coupling perturbing the Q-meter (see Appendix D).
Nevertheless, it was found to be possible to stack without compensating the coupling (compensation was not tried because of lack of time). The stack started at 6.8 A with a decay rate fluctuating between 500 and 200 ppm/min. After some scraping, the beam settled at 6.6 A with 100 ppm/min. The Schottky scans showed it to be resting across the 3rd order resonances and to have a very large hole corresponding to $3Q_H = 26$ (see Figure 8). The Vosicki monitor showed singly injected pulses to be rotated (Figure 9) and perhaps the strangest thing of all was the Schottky scans of the stack showing horizontal and vertical images on the same picture (Figure 10), which is again attributed to a rotation of the normal modes of vibration.

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G. Kantardjian
APPENDIX A

Solenoid with end plates

$u, v, z$ co-ordinate system of solenoid

Beam co-ordinate system is rotated by $7.4^\circ$ w.r.t. solenoid system about $z$-axis

i) Coupling

Providing the entry and exit slots in the end plates are wide compared to their height, the fields will only have vertical and axial components.

Using Gauss' Theorem inside slot, $B_z = \frac{B_v}{2\ell} z$ ($\ell =$ thickness of end plate) (1.a)

and $\frac{dB_z}{dz} = \frac{B_v}{2\ell}$ (2.a)

Since, div $B = 0$ and $\frac{dB}{du} = B_u = 0$ (away from ends of slot)

$\frac{dB_z}{dz} = -\frac{dB_v}{dv}$ (3.a)

Since the beam crosses the slot at $7.4^\circ$ to the solenoid's axis,

$\frac{dB_x}{dx} = \frac{dB_v}{dv} \sin 7.4^\circ$ and $\frac{dB_s}{ds} = \frac{dB_v}{dv} \cos 7.4^\circ$ (4.a)
Thus, to the beam a slot is effectively a skew quadrupole whose vertical gradient is strong and whose radial gradient is weak. For the present test solenoid, using (2.a), (3.a) and (4.e),

\[ \int \frac{dB_z}{dz} \, ds = 0.72 \, T \quad \int \frac{dB_x}{dx} \, ds = -0.09 \, T. \]

These values will be virtually unchanged for the final detector solenoid. The theory in Appendix C does not strictly apply to devices where \( \frac{dB_x}{dx} \neq -\frac{dB_z}{dz} \). Assuming that a single slot is approximately equivalent to a conventional skew quadrupole with its gradient equal to the vertical gradient in the slot, then the effect would be similar to powering all the Q1 skew quadrupoles with 10.5% of their maximum current. However, it can be seen that the two slots represent quadrupoles of opposite signs and since they are close together, they tend to compensate each other. The test solenoid is extremely short and it is probably not far from the truth to say that the compensation is nearly perfect. However, the detector solenoid will be nearly 10 times longer.

ii) **Closed orbit**

If the beam passes through the slot vertically off-axis, there is a net horizontal kick, but the second end plate provides an equal and opposite kick and for all practical purposes the resulting orbit distortion can be neglected.

iii) **Non-linearities**

Equation (1.a) assumes that the flux distributes itself in a uniform fashion. This may be far from the truth and it is from this source that non-linear terms arise. However, the slot has the advantage that the integral of the flux that diverges across the beam is kept very small and intuitively this means the non-linear terms will be small.
APPENDIX B

Open-ended solenoid and a solenoid with a single end plate

\( u, v, z \) co-ordinate system of solenoid
\( x, s, z \) " " " beam

The two co-ordinate systems are mutually rotated by 7.4° about the \( z \)-axis

i) Coupling

In the case of an open-ended solenoid, the field is rotationally symmetric and Busch's Theorem applies. This theorem shows that the beam receives an azimuthal kick when crossing an end field region,

\[
\Delta \left( \frac{d\theta}{dt} \right) = \frac{\omega_c}{2} \tag{1.b}
\]

where:

\( \theta \) is the azimuthal co-ordinate in the solenoid system
\( \omega_c \) is the cyclotron frequency for the beam in the solenoid's field.

However, it receives exactly the opposite kick upon leaving the solenoid. These kicks add and subtract from the azimuthal kick given by the body of the solenoid. Thus, from the point of view of coupling, the open ends are self-compensating. This is shown in the diagram below.
Alternatively, the reasoning of Appendix A can be used to give by Gauss' Theorem,

$$B \propto r, \quad \frac{dB}{dr} = \text{const.}; \quad B_\theta = \frac{dB}{d\theta} = 0.$$  

Thus, the incoming and outgoing beams see a horizontal skew gradient but these gradients are of opposite sign and compensate. For this reason, the coupling situation will be much the same as for the end plates.

ii) **Closed orbit**

Although the end field kicks are azimuthally compensating in the solenoid co-ordinate system, they give a net vertical kick to the beam. Fortunately, this net kick partially compensates the vertical kick arising in the body of the solenoid. Thus, the overall closed orbit distortion is less than for the solenoid with end plates.

iii) **Non-linearities**

When crossing the fringe fields at the end of the solenoid, the integral of the transverse flux is proportional to the beam's distance from the solenoid's axis. In the case of a slot, this integral is much smaller, since the distance from the centre of the slot is then the important factor. For the test solenoid, the beam enters and exits ~2 cm off-axis, whereas in the longer detector solenoid, this will be ~13 cm. Intuitively the non-linearities are proportional to the integral
of the transverse flux. Thus, for an $h_{\text{eff}} \approx 3$ mm, the small test solenoid is estimated to be 7 times worse without end plates than with end plates and the final detector solenoid to 43 times worse.

iv) Combination of one open end and one end plate

The end effects appear principally as a vertical skew gradient in the slot and a horizontal skew gradient in the open end. The integrated strengths will be approximately equal and of opposite sign. Both for the test solenoid and final detector solenoid, this will be about equivalent to a skew quadrupole with 0.72 T integrated gradient which is approximately equivalent to powering the Q1 skew quadrupoles by 10.5 %. In practice, the skew quadrupoles are powered by typically 1 % of their strength. This configuration would, therefore, be expected to cause very strong coupling.
APPENDIX C

Betatron coupling in axial and skew quadrupolar fields

General theory

Using a sinusoidal approximation for the uncoupled betatron oscillations and then adding the coupling terms, we have:

\[ x'' + \left( \frac{Q_x}{R} \right)^2 x' = \left( \frac{1}{B_p} \right) \left( -B_s z' + \frac{\partial B}{\partial z} z \right) \] (1.c)

\[ z'' + \left( \frac{Q_z}{R} \right)^2 z' = \left( \frac{1}{B_p} \right) \left( B_s x' - \frac{\partial B}{\partial x} x \right) \] (2.c)

where:

- \( Q_x \) and \( Q_z \) are the horizontal and vertical tunes
- \( R = \) (machine circumference /2\( \pi \))
- \( (B_p) = \) magnetic rigidity
- \( B_s = \) axial field
- \( \frac{\partial B}{\partial x}, \frac{\partial B}{\partial z} \) are skew gradients

'\( \cdot \)' indicates differentiation w.r.t. s. and the co-ordinate system is shown opposite.

Putting,

\[ \beta_x = \frac{Q_x}{R} \; \beta_z = \frac{Q_z}{R} \; b = \frac{B_s}{B_p} \]

\[ k = \frac{\frac{\partial B}{\partial x}}{B_p} = -\frac{\frac{\partial B}{\partial z}}{B_p} \quad \text{(i.e. a skew quadrupole)} \]

\[ x'' + \beta_x^2 x' = -b z' - k z \] (3.c)

\[ z'' + \beta_z^2 z' = b x' - k x \] (4.c)

By trying solutions of the form \( x = X(s) e^{i\beta_x s} \) and \( z = Z(s) e^{i\beta_z s} \) and neglecting \( x'' \) and \( z'' \) terms on the basis that \( X \) and \( Z \) will be slowly varying, two coupled first order differential equations are obtained.

Since \( \beta_x \) and \( \beta_z \) differ by typically only a few parts in a thousand, we can put \( \beta = \beta_x = \beta_z \) everywhere except for difference expressions where we write \( \delta = (\beta_x - \beta_z) \).
We have,
\[
X' = \frac{1}{b^2 - 4b^2} \left[ b (k - i\beta) X - 2i\beta e^{-i\delta s} (k + i\beta) Z \right] \quad (5.c)
\]
\[
Z' = \frac{1}{b^2 - 4b^2} \left[ -2i\beta e^{i\delta s} (k - i\beta) X - b (k + i\beta) Z \right] \quad (6.c)
\]
For skew quadrupolar fields only (i.e. \(b = 0\)), the solutions to (5.c.)
and (6.c.) are:
\[
X = C e^{\frac{i}{2} (\eta - \delta)s} - D \frac{\beta}{k} (\eta - \delta) e^{-\frac{i}{2} (\eta + \delta)s} \quad (7.c)
\]
\[
Z = D e^{-\frac{i}{2} (\eta - \delta)s} + C \frac{\beta}{k} (\eta - \delta) e^{\frac{i}{2} (\eta + \delta)s} \quad (8.c)
\]
where:
\[
\eta = + \sqrt{\delta^2 + \frac{k^2}{\beta^2}} \quad (9.c)
\]
For axial fields only (i.e. \(k = 0\)), the solutions to (5.c.) and (6.c.) are:
\[
X = C e^{\frac{i}{2} (\eta - \delta - \phi)s} + \frac{D (b^2 - 4\beta^2)}{4\beta^2 b} (\eta - \delta) e^{-\frac{i}{2} (\eta + \delta + \phi)s + \frac{\pi}{2}} \quad (10.c)
\]
\[
Z = D e^{-\frac{i}{2} (\eta - \delta + \phi)s} + \frac{C (b^2 - 4\beta^2)}{4\beta^2 b} (\eta - \delta) e^{\frac{i}{2} (\eta + \delta - \phi)s + \frac{\pi}{2}} \quad (11.c)
\]
where:
\[
\eta = + \sqrt{\delta^2 + \frac{16 \beta^4 b^2}{(b^2 - 4\beta^2)^2}} \quad (12.c)
\]
and
\[
\phi = \frac{2\beta b^2}{b^2 - 4\beta^2} \quad (13.c)
\]
The energy in the x and z planes is given by:
\[
E_x = |X|^2 = C^2 + D^2 d^2 (\eta - \delta)^2 + 2CDd (\eta - \delta) \sin(\eta s) \quad (14.c)
\]
\[
E_z = |Z|^2 = D^2 + C^2 d^2 (\eta - \delta)^2 - 2CDd (\eta - \delta) \sin(\eta s) \quad (15.c)
\]
(7.c) and (8.c) give \(\cos(\eta s)\) but there is no loss in generality
by using \(\sin(\eta s)\) corresponding to (10.c) and 11.c.

where:
\[ d = \frac{B}{k} \] for skew quadrupole coupling, equations (7.c) & (8.c) \hspace{1cm} (16.c)

or

\[ d = \left( \frac{b^2 - 4\beta^2}{4\beta^2 b} \right) \] for axial field coupling, eq. (10.c) & (11.c) \hspace{1cm} (17.c)

Thus, there is a sinusoidal energy exchange with a period, \( \lambda = \frac{2\pi}{\eta} \).

**Boundary conditions for present tests**

In the present tests, a beam was kicked horizontally and observed vertically. The subsequent coherent oscillations in the x and z planes are governed by the above equations. Since the coherent oscillations slowly de-bunch, the signals appear to be damped and this is clearly visible on the photographs although individually the protons continue indefinitely. When the beam is kicked, all of the coherent signal resides in the horizontal plane and \( E_x \) has its maximum value. At the same time, \( E_z \) is a minimum and is zero.

Thus,

\[ E_{xo} = C^2 + D^2d^2 (\eta - \delta)^2 + 2CDd (\eta - \delta) \]

\[ 0 = D^2 + C^2d^2 (\eta - \delta)^2 - 2CDd (\eta - \delta) \] \( (\eta s = \pi/2 \text{ at kick}) \)

the peak energy interchanged, \( E_T \), is:

\[ E_T = \frac{E_{xo}}{\eta^2} (\eta^2 - \delta^2) \] \hspace{1cm} (18.c)

where:

\[ \eta = + \sqrt{\delta^2 + \frac{1}{d^2}} \] and \( d \) is defined in eq. (16.c) & (17.c).

Complete energy interchange will occur when \( \delta = 0 \), i.e. on the diagonal \( Q_x = Q_z \).

---

The coupling and no coupling situations when the beam is kicked horizontally.
Calculation of coupling

The easiest parameter to measure is the period of the energy interchange. Providing the Q-values are known, this is sufficient to calculate the fraction of the energy interchanged. The latter being the more important for the ISR, since it determines the increase in the vertical emittance.

Thus, we have finally:

\[
\text{Period of interchange, } T = \frac{1}{f_{\text{rev}} \sqrt{\Delta^2 + C^2}} \tag{19.c}
\]

Peak energy interchanged as a fraction of energy in kick,

\[
e = \left( \frac{E_T}{E_{xo}} \right) = \frac{C^2}{(\Delta^2 + C^2)} \tag{20.c}
\]

where:

\[
\Delta = (Q_x - Q_z) \tag{21.c}
\]

\(f_{\text{rev}}\) is revolution frequency

"C" is defined as the coupling coefficient which is for:

skew fields, \( C = \left( \frac{R^2}{Q} \right) \left( \frac{1}{B_{p}} \right) \left( \frac{dB_x}{dx} \right) \) \tag{22.c}

axial fields, \( C = \left( \frac{R}{B_{p}} \right) B_8 \) \tag{23.c}

(Note: \( b^2 \) is neglected from (17.c) to give (23.c).

The theory applies to uniform skew or axial fields. Until the theory is extended, localized devices such as the solenoid, must be represented by the zero harmonic they excite. Since we are primarily interested in the zero order coupling resonance (i.e. the diagonal), this approximation appears reasonable.

Why the amplitude of the energy interchange is not easily measured

This was found to be unsatisfactory for the following reasons:

a) the filter (β-low Q-meter) response is strongly Q dependent;

b) the following amplifier saturated at very low signal levels.
APPENDIX D

Measurement of Q in the presence of coupling

In Appendix C it was shown that in the presence of weak coupling the oscillations in the horizontal or vertical planes would be the sum of an oscillation close to the unperturbed frequency and a slower oscillation. The Q-meter will respond to the former. The difference between the measured frequency and the unperturbed one increases as the diagonal is approached. The figure below illustrates this effect for two cases:

i) the unbalanced solenoid \( (c = 26.5 \times 10^{-3}) \)
ii) the solenoid with end plates \( (c = 4.3 \times 10^{-3}) \).

As can be seen, under normal conditions the effect is negligible for \( \Delta > 0.005 \). On the diagonal, the coupled oscillations interchange their frequencies. This has often been observed close to the diagonal, when the Q-meter suddenly gives the symmetric value on the other side.

![Diagram](image)

Perturbation of Q-values due to coupling
Figure 1 Vertical Orbit Compensation Scheme for Test Solenoid

Oscillations seen when kicking the beam horizontally (during set-up Run 380)

Horizontal plane
note that not all energy is transferred.

Vertical plane.

Coupling measured on central orbit with Δ = 0.009 (Run 390 - 22 GeV/c)

Solenoid off
T = 0.325 msec  C = 3.6 x 10^-3

Solenoid at 100% with end plates
T = 0.315 msec  C = 4.3 x 10^-3

Coupling measured on central orbit (Run 400 - 26 GeV/c)

Solenoid at 100% but with only one end plate.
(Oscilloscope gains are not standardized)

T = 0.11 msec  C = 2.7 x 10^-3

From Appendix B Section IV the estimated integrated skew gradient for this configuration is 0.72 T. At 26 GeV/c this gives  C = 22.5 x 10^-3

Figure 2 Typical Coupling Conditions
Figure 3. Period of Energy Interchange Across Chamber

Figure 4. Period of Energy Interchange with Q-Separation

Figure 5. Optimisation of Coupling

Figure 6. Variation Across Chamber
Figure 7. Stack of 78A made with solenoid at 100% with both end plates (decay rate ~2 ppm/min).

Figure 8. Working line measurements with solenoid at 100% with only one end plate and Schottky scan of a 6.66A stack.
Figure 9. Rotation of a single pulse on the injection orbit (solenoid at 100% with only one end plate)

Figure 10. Schottky scans with horizontal trace appearing on vertical scan (solenoid at 100% with only one end plate)