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TOTAL CROSS SECTIONS AND THE RATIO OF THE REAL AND
IMAGINARY FORWARD SCATTERING AMPLITUDES AT ULTRA-HIGH ENERGIES

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1. FIT TO TOTAL CROSS SECTIONS

In the last few years the following facts have emerged from the total cross sections of hadrons against protons:

i) In the first total cross section measurements\(^1\) at Serpukhov (negative particles only), the cross sections were found to reach a constant plateau sooner than expected at that time; such expectations were based mainly on Regge pole models\(^2\).

ii) Subsequent measurements\(^3\) on positive particles showed the total cross section for K\(^+\)p to be rising, from about 17.3 mb at 15 GeV/c, to 18.2 mb at 55 GeV/c incident lab. momentum.

iii) ISR measurements\(^4\)\(^5\) show that the proton-proton total cross section is rising by about 10% in the ISR range of energies, \(\sqrt{s} = 23\) to \(\sqrt{s} = 53\) GeV. At this last energy it has reached a value of about 43 mb, to be compared to 38.4 \(\pm 0.3\) mb in the Serpukhov energy range, \(\sqrt{s} = 10\) GeV.

iv) The antiproton-proton total cross section is still decreasing appreciably even at the highest Serpukhov energy. It has reached there a value of 44 mb. Presumably this trend will be reversed at some higher energy and the \(\bar{p}p\) cross section will start to rise towards the pp cross section, if there is a common high energy limit as required by the Pomeranchuk theorem.

Thus two charged particle cross sections are known to be rising at high energies and a third one is expected to rise. Rising total cross sections are thus relatively common at high energies, presumably as a consequence of a universal phenomenon. In the following it will be assumed that for some not yet understood reason all total cross sections start to increase at sufficiently high energies.

Why is this phenomenon of increasing total cross sections manifesting itself at so much lower energies in the K\(^+\)p system than in the pp system? Presumably this has to be correlated with the fact that the pp cross section is much bigger than the K\(^+\)p cross section.
One is then led to the question whether along these lines a fit to the actual data is possible with an expression containing an economical set of parameters. The equation should have (i) a decreasing term for the low-energy cross sections, (ii) a constant term, equivalent to \( \pi R^2 \) with \( R \) being some constant radius of the order of 1 f, and (iii) a slowly increasing term, for instance a power of the logarithm of the energy, possibly the same for all particles.

Applying this procedure to the \( K^+ p \) and the \( p^+ p \) cases, one is led by trial and error to the following expression for a description of the high energy part (contribution (ii) and (iii)) of the total cross sections

\[
\sigma_h = \sigma_0 i \left[ 1 + C \left( \frac{E}{m_i} \right)^2 \right]
\]  

(1)

with three adjustable parameters, viz. \( C \) and two \( \sigma_0 i \), one for the \( p^+ p \) and one for the \( K^+ p \) system, respectively. \( E \) is the lab. energy of the incident particle and \( m_i \) its mass. Leaving the power of the logarithmic term and the mass \( m_i \) as free parameters, they are found in a typical fit to adjust themselves to \( 2.004 \pm 0.005 \) and \( 0.761 \pm 0.067 \) GeV, respectively. This latter quantity is just in between the kaon and the proton mass and fixing it to the mass of the incident particle does not deteriorate the goodness of the fit. Similarly the power is fixed at 2. This is also the maximum power with which the total cross section is allowed to rise according to Froissart\(^{12} \), but in this context it should be remarked that the Froissart bound seems to be two orders of magnitude higher than the rising term actually found.

In these trials, actual data above 8 GeV/c were fitted and the low energy contribution (i) has been described by four terms of the type \( \text{Constant} \times E^{-\alpha_i} \), two for the \( p^+ p \) and two for the \( K^+ p \) system. The Pomeranchuk theorem is obviously satisfied by this formulation.

This line of reasoning can be extended to the pion cross section and it is found that the following expression can be fitted to all existing \( \pi^- p, \pi^+ p, K^- p, K^+ p, \overline{p} p \) and \( pp \) total cross sections above 8 GeV/c:

\[
\sigma_{\text{tot}, i} = \sigma_{0 i} i E^{-\alpha_1 i} + \sigma_{2 i} E^{-\alpha_2 i} + \sigma_0 i \left[ 1 + C (\ln \gamma_i)^2 \right].
\]

(2)
In this formula \( i \) stands for the pion, kaon, or nucleon respectively. The + sign applies to the negative particle cross sections, the - sign to the positive ones. Thus actually \( \sigma_{1i} \) is determined by the even signature combination of the amplitudes, \( \sigma_{2i} \) by the odd signature combination. The energy \( E \) and the Lorentz factor \( \gamma_i \) refer to the lab. system.

The expression has been fitted to the existing data\(^3-11\) above 8 GeV/c, resulting in a \( \chi^2 \) of 163 for 175 data points and 16 adjustable variables. These variables are determined by the fitting procedure as follows:

\[
\begin{align*}
\sigma_{1\pi} &= 23.84 \pm 1.05 \text{ mb} & \alpha_{1\pi} &= 0.548 \pm 0.025 & \sigma_{0\pi} &= 17.05 \pm 0.10 \text{ mb} \\
\sigma_{2\pi} &= 2.12 \pm 0.24 \text{ mb} & \alpha_{2\pi} &= 0.332 \pm 0.039 & \sigma_{0K} &= 16.37 \pm 0.05 \text{ mb} \\
\sigma_{1K} &= 21.42 \pm 1.25 \text{ mb} & \alpha_{1K} &= 0.906 \pm 0.021 & \sigma_{0p} &= 30.60 \pm 0.22 \text{ mb} \\
\sigma_{2K} &= 9.18 \pm 0.69 \text{ mb} & \alpha_{2K} &= 0.544 \pm 0.024 & c &= 0.00654 \pm 0.00012 \\
\sigma_{1p} &= 42.67 \pm 0.56 \text{ mb} & \alpha_{1p} &= 0.450 \pm 0.009 & & \\
\sigma_{2p} &= 27.80 \pm 1.21 \text{ mb} & \alpha_{2p} &= 0.602 \pm 0.014 & & \\
\end{align*}
\]

The data below 8 GeV/c can be fitted qualitatively down to about 4 GeV/c incident lab. momentum. The fit is sketched in Fig. 1, where the heavy lines indicate the part fitted and the thinner lines are predictions.

One notices that the asymptotic cross sections do not satisfy the quark relation\(^13\) \[
\sigma_{0\pi} : \sigma_{0K} : \sigma_{0p} = 1 : 1 : \sqrt[3]{2}.
\]

The relation is not satisfied at fixed \( \gamma \) and not at fixed \( E \). Presumably this constraint could be satisfied at the cost of introducing three variables \( C \) instead of one. The powers \( \alpha_{1i} \) and \( \alpha_{2i} \) have values roughly consistent with those expected from Regge pole ideas\(^2\).
2. RATIO OF REAL AND IMAGINARY SCATTERING AMPLITUDES

Using the parametrization (2) as an input to dispersion relations the real parts of scattering amplitudes at \( t = 0 \) and high energies can be calculated. Thus one may investigate for instance whether an accurate measurement of real parts puts restrictions on the behaviour of total cross sections at even higher energies.

Dispersion relations as quoted by Söding\(^{14}\)) are used to compute the real parts \( D_+ \) and \( D_- \) for pp and \( \bar{p}p \) scattering amplitudes as a function of the laboratory energy \( E \). Using the optical theorem the dispersion relations with one subtraction may be put into the following form:

\[
D_\pm(E) = c' + \frac{\pi E}{8\pi^2} \int_0^\infty \frac{dE'}{\sqrt{E'^2 - m^2}} \left\{ \frac{\pm E' E - m^2}{E' + E} \sigma(E') + \frac{\pm E' E - m^2}{E' + E} \bar{\sigma}(E') \right\}
\]

(3)

where \( \sigma(E') \) and \( \bar{\sigma}(E') \) are the total cross sections for pp and \( \bar{p}p \) respectively.

In writing down Eq. (3), pole terms have been neglected, as well as the contribution from the unphysical region, since in this paper the interest is concentrated on energies above 50 GeV, where these contributions are small. The small subtraction constant \( c' \) is determined by low-energy data and has no influence on the high-energy behaviour. In the actual calculations the total cross sections were parametrized down to \( E_1 = 1.1 \) GeV and the lower limit of integration has been changed from \( m \) to this value without changing the result.

The region of integration in Eq. (3) can be split arbitrarily into two ranges: a low-energy part containing a pole at \( E' = E \), and a high-energy region, where the difference between pp and \( \bar{p}p \) cross sections is negligibly small, with the pole outside the limits of integration. Equation (3) can thus be re-written in the following form with an arbitrary energy cut \( E_0 > E \):

\[
D_\pm(E) = c' + \frac{\pi E}{8\pi^2} \int_{E_0}^{E} \frac{dE'}{E'^2 - E^2} E_0 \left( \sigma(E') + \bar{\sigma}(E') \right)
\]

\[
+ \frac{\pi E}{8\pi^2} \int_{E_1}^{\infty} \frac{dE'}{E'^2 - E^2} E_1 \left( \sigma(E') - \bar{\sigma}(E') \right)
\]

\[
+ \frac{\pi E}{8\pi^2} \int_{E_0}^{E} \frac{dE'}{E'^2 - E^2} E \left( \sigma(E') - \bar{\sigma}(E') \right)
\]

(3')
Deriving Eq. (3') from (3) the nucleon mass \( m \) has been neglected because of the high energies involved.

The dispersion relation (3') has been evaluated numerically using fits to experimental total cross section data and the parametrization of Eq. (2). The result of these calculations is shown in Fig. 2. The ratio \( \rho \) of the real to imaginary scattering amplitude for proton-proton scattering is in agreement with the existing high-energy data\(^{15,16}\). It crosses zero at a laboratory energy of about 300 GeV, goes through a very broad maximum of \( \rho \approx 12.5\% \) at an energy of about \( 10^6 \) GeV, which is not shown on the drawing and approaches zero at infinity from above. The behaviour of \( \rho \) for the antiproton-proton system is similar.

To investigate the sensitivity of \( \rho \) on the behaviour of total cross sections at high energies it was assumed that Eq. (2) holds only up to an energy \( E_\omega \), beyond which the pp total cross section remains constant, viz. \( \sigma_\omega \), with the pp cross section approaching that value according to a power law:

\[
\sigma(E) - \sigma_\omega = 1/E^{0.602}
\]

The result of these calculations is shown in Fig. 3 indicating that a measurement of \( \rho \) with a relative accuracy of \( \pm 10\% \) at the highest ISR energy of 31 + 31 GeV, corresponding to \( E = 2000 \) GeV, would provide information about the behaviour of the total cross section up to energies of about \( 10^4 \) GeV. The sensitivity of the result to the form of the equation chosen for the parametrization of the total cross section, has not been investigated.

3. DISCUSSION

Obviously the parametrization (2) of the total cross sections cannot have the pretension of being unique. Rather the parametrization aims at showing that a simple four-parameter expression, with a ln \( \gamma \) term present from threshold onwards, can give a good description of the presently known data on high energy total cross sections. That an additional twelve parameters are needed for a description of the low energy phenomena exhibits the complication of these phenomena. Whether the simplification suggested for higher energies actually is possible will require more experimental data.
The constant in front of the $(\log)^2$ term, $\sigma_0 C$, is about 0.2 mb
for the nucleon-nucleon case, 0.1 mb for the pion and kaon-nucleon case.
This is appreciably smaller than the value derived by Froissart and Martin\textsuperscript{12},
$4\pi/(2m_\pi^2) \approx 60$ mb, based on the lower limit of the mass exchanged in the
t-channel.

The qualitative behaviour of $\rho$ as a function of $E$ can be interpreted on the basis of Eq. (3') as follows.

As the difference between $\bar{\sigma}$ and $\bar{\sigma}$ is always negative, approaching
zero for $E \to \infty$, at infinite energies the following relations hold:

$$\rho_{pp} < \rho_{\bar{p} \bar{p}}$$
$$\lim_{E \to \infty} \rho_{pp}(E) = \rho_{\bar{p} \bar{p}}(E).$$

The energy $E_0$ in (3') can always be chosen in such a way that
the first integral becomes zero \textsuperscript{*).}  The second integral then always gives
a negative contribution at high energies for $pp$ and a positive one for $\bar{p}\bar{p}$.
The last integral is always positive and becomes the leading term at high
energies. The negative values of $\rho$ below 300 GeV are mainly determined
by the difference between proton-proton and antiproton-proton cross sections.
At high energies the asymptotic behaviour of the total cross section itself determines the shape of the $\rho$ curve which approaches zero from above
at infinity; the rate of approach is governed by the asymptotic energy
dependence of $\sigma_{\text{tot}}$.

Such a behaviour of $\rho$ - approaching zero from above - is a consequence of the rise of the total cross sections, as was demonstrated analytically by Khuri and Kinoshita\textsuperscript{17} from the analyticity and crossing properties of the scattering amplitudes.

One might wonder whether this behaviour of $\rho$ can be translated in
the elementary language of potential scattering, positive $\rho$ meaning attraction,
negative $\rho$ meaning repulsion. This has been tried by Goldberger\textsuperscript{18} years
ago for the negative $\rho$ values generally found in the 10 GeV region. The
idea does not seem fruitful to the present authors.

\textsuperscript{*}) If $\sigma(E') + \bar{\sigma}(E')$ were a constant, $E_0$ would in fact be given by
$E_0 = E^2/E_1$. 

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Though the actual calculation has been done for the pp and $\bar{p}p$ case, the result for the $\pi p$ and Kp system in the high energy limit will be identical.

Preprints by Kroll\textsuperscript{19)} and Bourrely and Fischer\textsuperscript{20)} have recently appeared similar in scope to the present paper, giving predictions for $\rho$ for indefinitely rising total cross sections.

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REFERENCES


FIGURE CAPTIONS

Figure 1  Total cross sections as derived from Eq. (2). The heavy lines indicate the part fitted to experimental data.

Figure 2  Ratio between real and imaginary parts of scattering amplitudes for pp and \( \bar{p} p \) scattering.

Figure 3  Ratio between real and imaginary parts of the scattering amplitudes for proton-proton scattering, assuming the total cross section to stop rising at an energy \( E_\infty \).
Fig. 1

Diagram showing the variation of $\sigma_{\text{TOT}}$ with $E_{\text{LAB}}$ for different particle interactions:
- $\bar{p}p$
- $pp$
- $\pi^- p$
- $\pi^+ p$
- $K^- p$
- $K^+ p$

The diagram includes a legend indicating 'Fitted' and 'Predicted'.