POTENTIAL AND FIELD CREATED BY A RECTANGULAR BEAM INSIDE AN INFINITE CYLINDRICAL VACUUM CHAMBER OF CIRCULAR CROSS-SECTION

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ABSTRACT

Starting from the isolated rectangular beam, the theory of the rectangular beam surrounded by a metallic vacuum chamber of infinite conductivity is developed. Special considerations is given to the image field (i.e. to the reaction of the vacuum chamber on the beam), particularly to the components of the image field along the symmetry axes. Various other practical cases are examined. Finally the displaced beam is investigated and a simple formula is given for the image field created by a displaced rectangular beam at the centre of the vacuum chamber.
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1. INTRODUCTION

Among the models one may consider for representing a beam of particles circulating inside a vacuum chamber, the beam of rectangular cross-section and the beam of elliptic cross-section come probably closest to reality. Although the potential and field of an isolated rectangular beam has been considered in the literature [1], the problem of the potential and field of such a beam surrounded by a metallic vacuum chamber does not seem to have been solved as yet, and the behaviour of an elliptical beam inside a cylindrical vacuum chamber of circular cross-section is even less known.

An attempt is made in this paper to derive closed analytical formulae for the case of a rectangular beam surrounded by a cylindrical vacuum chamber of perfect conductivity and kept at a constant potential. A direct attack on this problem by trying to solve Laplace's and Poisson's equations by the usual techniques of separation turns out to be unsuccessful because the beam envelope and the vacuum chamber have different geometrical shapes. If one tries to formulate the problem in polar coordinates, the boundary conditions at the vacuum chamber are extremely simple but the boundary conditions at the beam envelope become untractable; the reverse is true for a solution in rectangular coordinates. In the case under consideration it seems appropriate to put to work an outstanding feature of a hollow cylindrical conductor of circular cross-section, namely the possibility to define a simple and unique electric (or geometric) image and base the calculations on it. In principle the image technique can also be applied to a vacuum chamber of rectangular shape but the multiply infinite number of images one has to deal with makes the procedure complicated and there is no way of obtaining a solution is closed form, a series expansion of the solution is the best one can do. Finally in the case of a vacuum chamber of elliptic cross-section, the image concept loses its significance and other techniques must be looked for to solve the problem.
2. POTENTIAL PRODUCED BY A UNIFORM LINE CHARGE INSIDE A CYLINDRICAL TUBE

Consider an infinite line charge of uniform linear density \( \lambda \) (coulombs/m). If the line charge is isolated in space, it produces a potential

\[
\phi = -\frac{\lambda}{2\pi \varepsilon_0} \ln D_0 + C
\]

where \( D_0 \) is the distance of the field point (or observation point) from the line charge. The problem being two-dimensional we identify the line charge with the "source point". The constant \( C \) cannot be taken to be zero because the charge extends to infinity.

If the line charge is placed inside a hollow cylindrical conductor, the potential in the hollow is given by

\[
\phi = -\frac{\lambda}{2\pi \varepsilon_0} \ln D_0 + \frac{\lambda}{2\pi \varepsilon_0} \ln D_{01} + C'
\]

where \( D_0 \) is as before the distance of the field point \( P \) from the source point \( S \), and \( D_{01} \) the distance of the field point from the image of the source (Fig. 1). If we call \( r_0 \) the distance of the source point from the origin of the system of coordinates, and \( a \) the radius of the vacuum chamber, the image point (or inverse point) of the source is determined by

\[
OS \cdot OI_0 = a^2
\]

i.e.

\[
OI_0 = \frac{a^2}{r_0}
\]

We assume that the vacuum chamber is at zero potential. The constant \( C' \) in Eq (2) is then given by

\[
C' = \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{r_0}{a}
\]

and the potential produced by the line charge at an arbitrary point inside the vacuum chamber can be written

\[
\phi = \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{r_0 D_{01}}{a D_0}
\]
Except for the factor $\Lambda$ this is actually the Green function for the interior of the circular cylinder $^2$.

3. GENERAL FORMULÆ FOR THE POTENTIAL AND FIELD PRODUCED BY A BEAM OF RECTANGULAR SECTION COASTING COAXIALLY INSIDE A VACUUM CHAMBER OF CIRCULAR CROSS-SECTION

We assume now that the line charge has an infinitesimal cross-section $dS$ and put

$$d\lambda = \rho dS$$

(7)

where $\rho$ is the volume density (in coulomb/m$^3$). The elementary potential is then given by

$$d\Phi = \frac{\rho}{2\pi\varepsilon_0} \ln \frac{r_0^2}{a_0^2} dS$$

(8)

and we obtain finally for the potential inside the vacuum chamber

$$\Phi = \frac{1}{2\pi\varepsilon_0} \int \rho \ln \frac{r_0^2}{a_0^2} dS$$

(9)

In Eq (9) the integrations are meant to be carried out over the populated area (i.e. $\rho \neq 0$) inside the vacuum chamber.

We now consider a rectangular beam of dimensions $2g$ and $2p$ (where $g$ stands for "grand" and $p$ for "petit") coasting coaxially inside the vacuum chamber (Fig.2) and use cartesian coordinates. Let $x,y$ be the coordinates of the field point and $x_0,y_0$ those of the source point. The coordinates of the image point (i.e. the image of the source point in the vacuum chamber) are then given by

$$x_{01} = \frac{a^2}{x_0^2 + y_0^2} x_0, \quad y_{01} = \frac{a^2}{x_0^2 + y_0^2} y_0$$

(10)

and we have (Fig.1)

$$r_0^2 = x_0^2 + y_0^2$$

(11)
\[ d_0^2 = (x - x_0)^2 + (y - y_0)^2 \]  
\[ d_{0i}^2 = (x - x_{0i})^2 + (y - y_{0i})^2 \]  

Substituting these expressions in Eq (9) we find

\[
\phi(x,y) = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \ln \left( \frac{(x^2+y^2)(x_0^2+y_0^2)-2a^2(xx_0+yy_0)+a^4}{a^2 [(x-x_0)^2 + (y-y_0)^2]} \right) \, dx_0 \, dy_0
\]  

For any charge distribution \( \rho(x,y) \) and at any point inside the vacuum chamber the potential can therefore be calculated, at least numerically, by means of this formula to which we associate the field components

\[
E_x(x,y) = \frac{1}{2\pi \varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \left[ \frac{a^2 x_0 - x(x_0^2+y_0^2)}{(x^2+y^2)(x_0^2+y_0^2)-2a^2(xx_0+yy_0)+a^4} \right. \\
+ \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} \right] \, dx_0 \, dy_0
\]  

\[
E_y(x,y) = \frac{1}{2\pi \varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \left[ \frac{a^2 y_0 - y(x_0^2+y_0^2)}{(x^2+y^2)(x_0^2+y_0^2)-2a^2(xx_0+yy_0)+a^4} \right. \\
+ \frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} \right] \, dx_0 \, dy_0
\]  

Whatever the charge distribution, important parameters in these calculations will undoubtedly be the potential and the field components at the centre of the beam. From the preceding formulae we obtain for these quantities the simple expressions

\[
\phi(0,0) = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \ln \frac{a^2}{x_0^2+y_0^2} \, dx_0 \, dy_0
\]
\[ E_x(o,0) = \frac{1}{2\pi\varepsilon_o} \int_{-g}^{g} \int_{-p}^{p} \rho(x_o,y_o)(\frac{1}{a^2} - \frac{1}{x_o^2 y_o^2}) x_o \, dx_o \, dy_o \] (18)

\[ E_y(o,0) = \frac{1}{2\pi\varepsilon_o} \int_{-g}^{g} \int_{-p}^{p} \rho(x_o,y_o)(\frac{1}{a^2} - \frac{1}{x_o^2 + y_o^2}) y_o \, dx_o \, dy_o \] (19)

For any even distribution in x and y, the field components will be zero at the centre of the beam and the maximum potential in the beam will be given by Eq (17).

Another parameter which can easily be expressed by means of quadratures is the electric field at the intersections of the coordinate axes with the vacuum chamber. We have from Eqs (15) and (16)

\[ E_x(a,o) = \frac{1}{2\pi\varepsilon_o a} \int_{-g}^{g} \int_{-p}^{p} \rho(x_o,y_o) \frac{a^2 - (x_o^2 + y_o^2)}{(a-x_o)^2 + y_o^2} \, dx_o \, dy_o, \quad E_y(a,o) = 0 \] (20)

\[ E_y(o,a) = \frac{1}{2\pi\varepsilon_o a} \int_{-g}^{g} \int_{-p}^{p} \rho(x_o,y_o) \frac{a^2 - (x_o^2 + y_o^2)}{x_o^2 + (a-y_o)^2} \, dx_o \, dy_o, \quad E_x(o,a) = 0 \] (21)

4. ISOLATED BEAM AND IMAGE BEAM

In our approach we have actually separated the isolated or free beam from the virtual or image beam. Putting

\[ \Phi(x,y) = \Phi_f(x,y) + \Phi_i(x,y) \] (22)

\[ E_x = E_{fx}(x,y) + E_{ix}(x,y) \] (23)

\[ E_y = E_{fy}(x,y) + E_{iy}(x,y) \] (24)

we have from Eq (14)

\[ \Phi_f(x,y) = \frac{1}{4\pi\varepsilon_o} \int_{-g}^{g} \int_{-p}^{p} \rho(x_o,y_o) \ln \left\{ \frac{(x-x_o)^2 + (y-y_o)^2}{2} \right\} \, dx_o \, dy_o \] (25)
for the potential of the beam in the absence of the vacuum chamber, and

\[ \phi_i(x,y) = \frac{1}{4\pi\varepsilon_0} \int_{-p}^{+p} \int_{-g}^{+g} \frac{\rho(x_o,y_o)ln\left(\frac{(x^2+y^2)(x^2+y^2)-2a^2(xx_0yy_0)+a^4}{a^2}\right)}{(x-x_0)^2+(y-y_0)^2} \, dx_0 \, dy_0 \]  

(26)

for the potential produced by the image beam, or in more physical terms, the potential created by the charges which the actual beam induces by influence in the wall of the vacuum chamber.

It should be noted that considered separately, \( \phi_f \) and \( \phi_i \) are determined only to within arbitrary constants [cf. Eqs (1) and (2)], this explains why the logarithms are applied to dimensional quantities. It is only in the expression \( \phi_f + \phi_i \) that the logarithms will be applied to non-dimensional quantities, for in this case the constant is determined by the choice we have made for the potential of the vacuum chamber.

For the field components we find from Eqs (15) and

\[ E_{fx}(x,y) = \frac{1}{2\pi\varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \frac{\rho(x_0,y_0)}{(x-x_0)^2+(y-y_0)^2} \, dx_0 \, dy_0 \]  

(27)

\[ E_{fy}(x,y) = \frac{1}{2\pi\varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \frac{\rho(x_0,y_0)}{(x-x_0)^2+(y-y_0)^2} \, dx_0 \, dy_0 \]  

(28)

\[ E_{ix}(x,y) = \frac{1}{2\pi\varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \frac{\rho(x_0,y_0)}{(x^2+y^2)(x^2+y^2)-2a^2(xx_0+yy_0)+a^4} \, dx_0 \, dy_0 \]  

(29)

\[ E_{iy}(x,y) = \frac{1}{2\pi\varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \frac{\rho(x_0,y_0)}{(x^2+y^2)(x^2+y^2)-2a^2(xx_0+yy_0)+a^4} \, dx_0 \, dy_0 \]  

(30)

Contrary to the separated potentials, the field components are uniquely determined.

Fig. 3 shows how the action of the vacuum chamber can be replaced by the virtual beam. The contour of the virtual beam has
been obtained by inverting the four sides of the rectangle with respect to the circle representing the vacuum chamber. The contour of the virtual beam is therefore described by the four circles

\[ g(x^2 + y^2) = a_x^2 = 0 \]  
\[ p(x^2 + y^2) = a_y^2 = 0 \]

(31)
(32)

going all through the origin and intersecting orthogonally at the points

\[ x = \pm \frac{a_x^2}{g+p} g, \quad y = \pm \frac{a_y^2}{g+p} p \]

(33)

We now turn to a more specific consideration of the case where the density is constant, i.e.,

\[ \rho(x,y) = \rho \]

(34)

All the integrations in the preceding formulae can then be carried out effectively.

5. THE ISOLATED BEAM OF UNIFORM DENSITY

We will examine separately the field in the region situated outside the beam and the field in the beam itself; we know that in the first case the field must satisfy Laplace's equation whereas in the second case it should obey Poisson's equation.

5.1. Potential and field outside the beam

The potential is given by

\[ \phi_f(x,y) = -\frac{\rho}{4\pi\varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \ln \left[ (x-x_0)^2 + (y-y_0)^2 \right] dx_0 dy_0 \]

(35)

and the integrations have to be carried out for the case where at least one of the relations

\[ |x| > g, \quad |y| > p \]

(36)
is satisfied. This problem has actually been treated in the literature.\footnote{1}

We write the result here in a symmetric form, suitable for quick derivation of particular results and for developing the extensions which will follow. Using the two general formulae

\[
\begin{align*}
\ln(u^2+c^2) dz &= u \ln(u^2+c^2) - 2u + 2c \arctan \frac{u}{c} \\
\arctan \frac{c}{u} dz &= \frac{1}{2} \left[ \ln(u^2+c^2) \arctan \frac{c}{u} + cu \right]
\end{align*}
\] (37)

(which can readily be checked by calculating the derivatives), we obtain upon performing the integrations in Eq (35)

\[
\phi_{\text{ext}}(x,y) = \frac{\psi_0}{4\pi \rho_0} \left\{ 12 \frac{g \rho}{D} ight. \\
+ (x+g)(y-p) \ln[(x+g)^2+(y-p)^2] - (x+g)(y+p) \ln[(x+g)^2+(y+p)^2] \\
- (x+g)^2(\arctan \frac{y-p}{x+g} - \arctan \frac{y-p}{x-g}) + (x+g)^2(\arctan \frac{y-p}{x-g} - \arctan \frac{y+p}{x-g}) \\
- (y+p)^2(\arctan \frac{x+g}{y+p} - \arctan \frac{x-g}{y+p}) + (y+p)^2(\arctan \frac{x+g}{y+p} - \arctan \frac{x-g}{y+p}) \left. \right\}
\] (39)

As mentioned, this potential is determined to within an arbitrary constant which could be chosen so as to make the arguments of the logarithms nondimensional. These arguments as well as the angles involved in Eq (39) can easily be given a geometrical meaning (Fig. 4).

Putting

\[
\begin{align*}
r_{g,p}^2 &= (x-g)^2+(y-p)^2 \\
r_{-g,-p}^2 &= (x-g)^2+(y+p)^2 \\
r_{g,-p}^2 &= (x-g)^2+(y-p)^2 \\
r_{-g,p}^2 &= (x-g)^2+(y+p)^2
\end{align*}
\] (40)

and

\[
\begin{align*}
t_{g,p} &= \frac{y-p}{x+g} \\
t_{-g,p} &= \frac{y-p}{x+g} \\
t_{g,-p} &= \frac{y+p}{x-g} \\
t_{-g,-p} &= \frac{y+p}{x+g}
\end{align*}
\] (41)
we notice that the arguments of the logarithms are the squares of the
distances of the field point from the four vertices \((g, p), (-g, p),
(g, -p), (-g, -p)\) of the rectangular beam while the angles are those deter-
mined at the vertices by the \(x\)-axis on one hand and the straight lines
going from the vertices to the field point on the other hand (Fig. 4). In
order to achieve complete symmetry between \(x\) and \(y\) we have accepted some
redundancy in Eq (39) by introducing

\[
\begin{align*}
tg_{g, p}^\beta &= \frac{x-g}{y-p}, & tg_{-g, p}^\beta &= \frac{x+g}{y-p} \\
tg_{g, -p}^\beta &= \frac{x-g}{y+p}, & tg_{-g, -p}^\beta &= \frac{x+g}{y+p}
\end{align*}
\] (42)

The potential defined by Eq (38) is invariant with
respect to the simultaneous permutations \(x \leftrightarrow y, g \leftrightarrow p\). It is also even
with respect to \(x\) and \(y\), i.e.

\[
\phi(-x, y) = \phi(x, y) = \phi(x, -y)
\] (43)

Obviously this relation holds not only for \(\phi^{ext}\) but for all potentials
produced by a coaxial symmetric beam having a symmetric density distri-
bution. All our considerations can therefore be confined to the first
quadrant, i.e., \(x \geq 0, y \geq 0\).

A careful sign convention is essential in consid-
ering the angles defined by Eqs (41) and (42). Our convention is as
follows: the \(\alpha\)'s are obtained by starting from the positive \(x\)-axis or a
parallel to it and going over, via the shortest possible angle, to the
positive \(VP\)-axis where \(V\) is the vertex and \(P\) the field point. The \(\beta\)'s are
obtained by starting from the positive \(VP\)-axis and going over, via the
shortest possible angle, to the positive \(y\)-axis or a parallel to it. The
\(\alpha\)'s are positive if the rotation is counterclockwise and negative if the
rotation is clockwise; the same applies to the \(\beta\)'s.

As we restrict ourselves to the first quadrant,
they are only three possibilities for the field point, namely

\[
\begin{align*}
x &\geq g, & y &\leq p \\
x &\geq g, & y &\geq p \\
x &\leq g, & y &\geq p
\end{align*}
\] (44)
Fig. 4 shows the three possibilities with the corresponding angles. In the three cases we have, according to our sign convention,

\[
\begin{align*}
\arctg \frac{y-p}{x-g} + \arctg \frac{x-g}{y-p} &= \frac{\pi}{2}, \\
\arctg \frac{y-p}{x+g} + \arctg \frac{x+g}{y-p} &= \frac{\pi}{2}
\end{align*}
\]

(45)

\[
\begin{align*}
\arctg \frac{y+p}{x-g} + \arctg \frac{x-g}{y+p} &= \frac{\pi}{2}, \\
\arctg \frac{y+p}{x+g} + \arctg \frac{x+g}{y+p} &= \frac{\pi}{2}
\end{align*}
\]

We now turn to the field components. From Eq (39) we find for these components

\[
E_{fx}^{ext}(x,y) = \frac{\rho}{4\pi\varepsilon_0} \left[ (y+p) \ln \frac{(x+g)^2+(y+p)^2}{(x-g)^2+(y+p)^2} + (y-p) \ln \frac{(x-g)^2+(y-p)^2}{(x+g)^2+(y-p)^2} \\
+ 2(x+g)(\arctg \frac{y+p}{x+g} - \arctg \frac{y+p}{x-g}) - 2(x-g)(\arctg \frac{y+p}{x+g} - \arctg \frac{y-p}{x-g}) \right]
\]

(46)

and

\[
E_{fy}^{ext}(x,y) = \frac{\rho}{4\pi\varepsilon_0} \left[ (x+g) \ln \frac{(x+g)^2+(y+p)^2}{(x+g)^2+(y-p)^2} + (x-g) \ln \frac{(x-g)^2+(y+p)^2}{(x-g)^2+(y-p)^2} \\
+ 2(y+p)(\arctg \frac{x+g}{y+p} - \arctg \frac{x-g}{y+p}) - 2(y-p)(\arctg \frac{x+g}{y+p} - \arctg \frac{x-g}{y-p}) \right]
\]

(47)

From these expressions we can calculate

\[
\frac{\partial E_{fx}^{ext}}{\partial x} = \frac{\rho}{2\pi\varepsilon_0} \left[ (\arctg \frac{y+p}{x+g} - \arctg \frac{y-p}{x+g}) - (\arctg \frac{y+p}{x-g} - \arctg \frac{y-p}{x-g}) \right]
\]

(48)

\[
\frac{\partial E_{fy}^{ext}}{\partial y} = \frac{\rho}{2\pi\varepsilon_0} \left[ (\arctg \frac{x+g}{y+p} - \arctg \frac{x-g}{y+p}) - (\arctg \frac{x+g}{y-p} - \arctg \frac{x-g}{y-p}) \right]
\]

(49)

and using Eqs (45), we can check that the field is Laplacian, i.e.

\[
\frac{\partial E_{fx}^{ext}}{\partial x} + \frac{\partial E_{fy}^{ext}}{\partial y} = 0
\]

(50)
5.2. Potential and field inside the beam

We now consider the region

\[ |x| < g \quad , \quad |y| < p \]  
\[ (51) \]

but, as already mentioned, for symmetry reasons we may restrict ourselves to the first quadrant. Performing the integrations in the same way as before, we find

\[ \Phi^\text{int}(x,y) = \frac{\rho}{4\pi\varepsilon_0} \left( 12 gp - (g+y)(p+y)\ln[(g+x)^2+(p+y)^2] - (g-x)(p+y)\ln[(g-x)^2+(p+y)^2] \right) 
- (g+x)(p-y)\ln[(g+x)^2+(p-y)^2] - (g-x)(p-y)\ln[(g-x)^2+(p-y)^2] 
- (g+x)^2(\arctan\frac{p+y}{g+x} + \arctan\frac{p-y}{g-x}) - (g-x)^2(\arctan\frac{p+y}{g-x} + \arctan\frac{p-y}{g-x}) 
- (p+y)^2(\arctan\frac{g+x}{p+y} + \arctan\frac{g-x}{p-y}) - (p-y)^2(\arctan\frac{g+x}{p-y} + \arctan\frac{g-x}{p-y}) \right) \]
\[ (52) \]

Formally this expression is the same as Eq (39) for the potential outside the isolated beam. However the four quantities \( g+x, g-x, p+y, p-y \) are now all positive and all the angles involved have values between zero and \( \pi/2 \). There is no need for a particular sign convention and the equations

\[ \arctan\frac{p+y}{g+x} + \arctan\frac{g+x}{p+y} = \frac{\pi}{2} , \quad \arctan\frac{p+y}{g-x} + \arctan\frac{g-x}{p+y} = \frac{\pi}{2} \]
\[ (53) \]

\[ \arctan\frac{p-y}{g+x} + \arctan\frac{g+x}{p-y} = \frac{\pi}{2} , \quad \arctan\frac{p-y}{g-x} + \arctan\frac{g-x}{p-y} = \frac{\pi}{2} \]

hold under all circumstances.

Taking the derivatives of the potential we find for the field components

\[ E^\text{int}_x(x,y) = \frac{\rho}{4\pi\varepsilon_0} \left[ (p+y)\ln\frac{(g+x)^2+(p+y)^2}{(g-x)^2+(p+y)^2} + (p-y)\ln\frac{(g+x)^2+(p-y)^2}{(g-x)^2+(p-y)^2} \right] 
+ 2(g+x)(\arctan\frac{p+y}{g+x} + \arctan\frac{p-y}{g-x}) - 2(g-x)(\arctan\frac{p+y}{g-x} + \arctan\frac{p-y}{g-x}) \]
\[ (54) \]
and

\[
E_{f}^{x\int}(x,y) = \frac{\rho}{4\pi\varepsilon_{0}} \left[ (g+x) \ln \frac{(g+x)^{2} + (p+y)^{2}}{(g-x)^{2} + (p-y)^{2}} + (g-x) \ln \frac{(g-x)^{2} + (p+y)^{2}}{(g-x)^{2} + (p-y)^{2}} + 2(p+y)(\arctan \frac{g+x}{p+y} + \arctan \frac{g-x}{p-y}) - 2(p-y)(\arctan \frac{g+x}{p+y} + \arctan \frac{g-x}{p-y}) \right]
\]

(55)

From these expressions we calculate

\[
\frac{\partial E_{f}^{x\int}}{\partial x} = \frac{\rho}{2\pi\varepsilon_{0}} \left( \arctan \frac{p+y}{g+x} + \arctan \frac{p-y}{g-x} + \arctan \frac{p+y}{g-x} + \arctan \frac{p-y}{g+x} \right)
\]

(56)

\[
\frac{\partial E_{f}^{y\int}}{\partial y} = \frac{\rho}{2\pi\varepsilon_{0}} \left( \arctan \frac{g+x}{p+y} + \arctan \frac{g-x}{p+y} + \arctan \frac{g+x}{p-y} + \arctan \frac{g-x}{p-y} \right)
\]

and making use of Eqs (53) we check that the field is Poissonian, i.e.

\[
\frac{\partial E_{f}^{x\int}}{\partial x} + \frac{\partial E_{f}^{y\int}}{\partial y} = \frac{\rho}{\varepsilon_{0}}
\]

(57)

To summarize, the potential produced by an isolated rectangular beam is given by Eq (39) or by Eq (52) which are formally the same. The field components are given by Eqs (46) and (47) or by Eqs (54) and (55) which are again formally the same. It is only the angular constellation, or in other words the contributions from the \(\arctan\)‘s, which ensure the Laplacian nature of the field outside the beam and its Poissonian behaviour inside the beam.

5.3. Particular cases

5.3.1. Electric field along the symmetry axes

On the x-axis the y-component of the electric field is zero. For the x-component we have either from Eq (46) or from Eq (54)

\[
E_{f}^{x}(x,0) = \frac{\rho}{\pi\varepsilon_{0}} \left[ p \ln \frac{\sqrt{(x+g)^{2} + p^{2}}}{\sqrt{(x-g)^{2} + p^{2}}} + (x+g) \arctan \frac{p}{x+g} - (x-g) \arctan \frac{p}{x-g} \right]
\]

(58)
To this we associate the slope

\[
\frac{\partial E_{f_x}(x, o)}{\partial x} = \frac{\rho}{\pi \varepsilon_o} \left( \arctg \frac{p}{x+g} - \arctg \frac{p}{x-g} \right) \quad (59)
\]

For \( x \) varying from zero to \( g \), the derivative stays positive. It undergoes a jump discontinuity at \( x = g \), changing suddenly its value by \( \rho/\varepsilon_o \) [this corresponds to the transition (inside the beam) \& (outside the beam)]; it remains then negative up to infinity. Consequently, when \( x \) varies from zero to \( g \), \( E_{f_x}(x, o) \) increases from zero to its peak value \( (E_{f_x})_{\text{peak}} \); it then decreases to zero as \( x \) goes to infinity. We have

\[
(E_{f_x})_{\text{peak}} = E_{f_x}(g, o) = \frac{\rho}{\pi \varepsilon_o} \left( p \ln \frac{\sqrt{2x^2 + p^2}}{p} + 2g \arctg \frac{p}{2g} \right) \quad (60)
\]

Similarly, on the \( y \)-axis the \( x \)-component of the electric field is zero and for the \( y \)-component we find from Eq (47) or from Eq (55)

\[
E_{f_y}(o, y) = \frac{\rho}{\pi \varepsilon_o} \left[ g \ln \frac{\sqrt{(y+p)^2 + g^2}}{\sqrt{(y-p)^2 + g^2}} + (y+p)\arctg \frac{g}{y+p} - (y-p)\arctg \frac{g}{y-p} \right] \quad (61)
\]

Here again the slope

\[
\frac{\partial E_{f_y}(o, y)}{\partial y} = \frac{\rho}{\pi \varepsilon_o} \left( \arctg \frac{g}{y+p} - \arctg \frac{g}{y-p} \right) \quad (62)
\]

stays positive for \( y \) varying from zero to \( p \). A jump discontinuity occurs at \( y = p \), and the slope remains negative thereafter. Consequently, when \( y \) varies from zero to \( p \), \( E_{f_y}(o, y) \) increases from zero to its peak value \( (E_{f_y})_{\text{peak}} \); it then decreases to zero as \( y \) goes to infinity. We have

\[
(E_{f_y})_{\text{peak}} = E_{f_y}(o, p) = \frac{\rho}{\pi \varepsilon_o} \left( g \ln \frac{\sqrt{2p^2 + 4g^2}}{g} + 2p \arctg \frac{g}{2p} \right) \quad (63)
\]
5.3.2. Electric field at the vertices of the beam envelope

Putting \( x = g, y = p \) in Eqs (46) and (47) or in Eqs (54) and (55) we find

\[
E_{fx}(g,p) = \frac{p}{\pi \varepsilon_0} \left( p \ln \frac{\sqrt{g^2 + p^2}}{g} + g \arctan \frac{p}{g} \right) \tag{64}
\]

\[
E_{fy}(g,p) = \frac{p}{\pi \varepsilon_0} \left( g \ln \frac{\sqrt{g^2 + p^2}}{p} + p \arctan \frac{g}{p} \right) \tag{55}
\]

5.3.3. The square beam

Putting

\[ g = p = l \tag{66} \]

we have

i) at the intersections of the coordinate axes with the beam envelope

\[
(E_f)_{\text{max}} = \frac{\sigma l}{2\pi \varepsilon_0} \left( \ln 5 + 4 \arctan \frac{1}{2} \right) \tag{67}
\]

ii) at the vertices of the beam envelope

\[
E_{fx}(x,l) = E_{fy}(y,l) = \frac{\sigma l}{2\pi \varepsilon_0} \left( \ln 2 + \frac{\pi}{2} \right) \tag{68}
\]

6. THE VIRTUAL BEAM; IMAGE EFFECTS

We now proceed to the investigation of the effect caused by the virtual beam which translates the reaction of the vacuum chamber.

6.1. Potential and field due to the virtual beam

The density distribution being assumed constant, we obtain from Eq (26) for the potential produced by the virtual beam
\[ \phi_i(x,y) = \frac{p}{4\pi \varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \ln \left( \frac{(x^2+y^2)(x_o^2+y_o^2)-2a^2(xx_o+yy_o)+a^4}{a^2} \right) \, dx \, dy \]  

(89)

We now remark that this equation can be written either in the form

\[ \phi_i(x,y) = \frac{p}{4\pi \varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \ln \left( \frac{x_o^2+y_o^2}{a^2} \right) \left[ \left( x - \frac{a^2}{x^2+y^2} x_o \right)^2 + \left( y - \frac{a^2}{x^2+y^2} y_o \right)^2 \right] \, dx \, dy \]  

(70)

involving the coordinates

\[ x_o = \frac{a^2}{x^2+y^2} x_o, \quad y_o = \frac{a^2}{x^2+y^2} y_o \]  

(71)

of the image of the source point, or in the form

\[ \phi_i(x,y) = \frac{p}{4\pi \varepsilon_0} \int_{-g}^{+g} \int_{-p}^{+p} \ln \left( \frac{x^2+y^2}{a^2} \right) \left[ \left( -\frac{a^2}{x^2+y^2} x - x_o \right)^2 + \left( -\frac{a^2}{x^2+y^2} y - y_o \right)^2 \right] \, dx \, dy \]  

(72)

involving the coordinates

\[ x_i = \frac{a^2}{x^2+y^2} x, \quad y_i = \frac{a^2}{x^2+y^2} y \]  

(73)

of the image of the field point. Geometrically, the equality

\[ \left( x_o^2+y_o^2 \right) \left[ \left( x - \frac{a^2}{x^2+y^2} x_o \right)^2 + \left( y - \frac{a^2}{x^2+y^2} y_o \right)^2 \right] \]  

\[ = \left( x^2+y^2 \right) \left[ \left( \frac{a^2}{x^2+y^2} x - x_o \right)^2 + \left( -\frac{a^2}{x^2+y^2} y - y_o \right)^2 \right] \]  

(74)

follows from the similarity of the triangles involving the origin of the system of coordinates, the source point, the field point, and their images. We have (Fig. 5)
\[
\frac{O_{i}}{O_{f}} = \frac{s^{2}/r_{0}}{s^{2}/r} = \frac{r}{r_{0}}
\]  

(75)

The triangles \(O_{i}, P\) and \(OIS\) are therefore similar and there results

\[
r_{0} O_{oi} = r D_{1}
\]

(76)

which is nothing else than Eq (74).

Determination of the potential requires integration over the source space \(x_{0}, y_{0}\). It is obviously more convenient under these conditions to use Eq (72) than Eq (70). In fact we can write Eq (72) in the form

\[
\phi(x, y) = \frac{\sigma p}{4 \pi \varepsilon_{0}} \ln \frac{x^{2}+y^{2}}{a^{2}} + \int_{-p}^{+p} \int_{-p}^{+p} \ln \left[ (x-x_{0})^{2}+(y-y_{0})^{2} \right] dx_{0} dy_{0}
\]

(77)

or in other terms, if we take into account Eq (35)

\[
\phi(x, y) = \frac{\sigma p}{\pi \varepsilon_{0}} \ln \frac{x^{2}+y^{2}}{a^{2}} - \phi(x_{1}, y_{1})
\]

(78)

This equation can be given the form of a "reciprocity relation", viz.

\[
\phi(x, y) + \phi(x_{1}, y_{1}) = \frac{\sigma p}{\pi \varepsilon_{0}} \ln \frac{x^{2}+y^{2}}{a^{2}} - \frac{\sigma p}{\pi \varepsilon_{0}} \ln \frac{a^{2}}{x_{1}^{2}+y_{1}^{2}}
\]

(79)

or

\[
\phi(x, y) + \phi(x_{1}, y_{1}) = \frac{\sigma p}{2 \pi \varepsilon_{0}} \ln \frac{x^{2}+y^{2}}{x_{1}^{2}+y_{1}^{2}}
\]

(80)

The preceding considerations show that the investigation of the image field does not require further integrations; we merely use the results we have obtained before. The point \(x_{1}, y_{1}\) being always outside the beam, we rather use Eq (39) and replace in this equation \(x\) and \(y\) by \(x_{1}\) and \(y_{1}\), where \(x_{1}\) and \(y_{1}\) are given by Eq (73). We then apply Eq (78) to find for the image potential (or the potential produced by the charges in the wall of the vacuum chamber)
\[
\phi_1(x,y) = \frac{\rho}{4\pi \epsilon_0} \left\{ -12 \frac{g}{p} - 4 \frac{g}{p} \ln \frac{x_1^2 + y_1^2}{a^2} \right\} 
+ (x_1 + g)(y_1 + p) \ln [(x_1 + g)^2 + (y_1 + p)^2] - (x_1 + g)(y_1 - p) \ln [(x_1 + g)^2 + (y_1 - p)^2] 
+ (x_1 - g)(y_1 - p) \ln [(x_1 - g)^2 + (y_1 - p)^2] - (x_1 - g)(y_1 + p) \ln [(x_1 - g)^2 + (y_1 + p)^2] 
+ (x_1 + g)^2 \left( \arctan \frac{y_1 + p}{x_1 + g} - \arctan \frac{y_1 - p}{x_1 + g} \right) 
- (x_1 - g)^2 \left( \arctan \frac{y_1 + p}{x_1 - g} - \arctan \frac{y_1 - p}{x_1 - g} \right) 
+ (y_1 + p)^2 \left( \arctan \frac{x_1 + g}{y_1 + p} - \arctan \frac{x_1 - g}{y_1 + p} \right) 
- (y_1 - p)^2 \left( \arctan \frac{x_1 + g}{y_1 - p} - \arctan \frac{x_1 - g}{y_1 - p} \right) 
\]
\[\text{(81)}\]

This relation holds throughout the region \(x^2 + y^2 \leq a^2\), or alternatively, \(x_1^2 + y_1^2 \geq a^2\). Let us recall at this point that, as in the case of the free (or isolated beam), the image potential is only determined to within an arbitrary constant. It is only when one associates \(\phi_f\) and \(\phi_1\) to calculate the potential of the actual beam in the presence of the vacuum chamber that this constant is eliminated (for a given value of the potential of the vacuum chamber).

To calculate the field components we start from Eq (78). Using the relations
\[
\frac{\partial x_1}{\partial x} = \frac{y_1^2 - x_1^2}{a^2}, \quad \frac{\partial y_1}{\partial y} = \frac{x_1^2 - y_1^2}{a^2} \quad \text{(82)}
\]
\[
\frac{\partial x_1}{\partial y} = \frac{\partial y_1}{\partial x} = -\frac{2x_1y_1}{a^2} \quad \text{(83)}
\]
we find upon differentiation
\[
E_{ix}(x,y) = \frac{1}{a} \left[ -\frac{2g}{\pi \epsilon_0} x_1 + (x_1^2 - y_1^2) E_{fx}(x_1,y_1) + 2x_1 y_1 E_{fy}(x_1,y_1) \right] 
\quad \text{(84)}
\]
\[
E_{iy}(x,y) = \frac{1}{a} \left[ -\frac{2g}{\pi \epsilon_0} y_1 + (y_1^2 - x_1^2) E_{fy}(x_1,y_1) + 2x_1 y_1 E_{fx}(x_1,y_1) \right] 
\quad \text{(85)}
\]
where \(E_{fx}(x_1,y_1)\) and \(E_{fy}(x_1,y_1)\) are simply obtained from Eqs (46) and (47) by changing \(x,y\) into \(x_1,y_1\). Substituting these expressions in Eqs (84) and (85) we finally have for the components of the image field

$$E_{ix}(x,y) = \frac{p}{4\pi\varepsilon_0 a^2} \left\{ -8gpx_1 + \left[ \begin{array}{c} \frac{(x_1 + p)^2}{(x_1 - g)^2 + (y_1 - p)^2} \right] \ln \left[ \frac{(x_1 + g)^2}{(x_1 - g)^2 + (y_1 - p)^2} \right] \\
+ 2(x_1 + g) \left( \arctg \frac{y_1 + p}{x_1 + g} - \arctg \frac{y_1 - p}{x_1 + g} \right) - 2(x_1 - g) \left( \arctg \frac{y_1 + p}{x_1 - g} - \arctg \frac{y_1 - p}{x_1 - g} \right) \right\} x_1^2 y_1$$

$$E_{iy}(x,y) = \frac{p}{4\pi\varepsilon_0 a^2} \left\{ -8gpy_1 + \left[ \begin{array}{c} \frac{(x_1 + g)^2}{(x_1 - g)^2 + (y_1 - p)^2} \right] \ln \left[ \frac{(x_1 + g)^2}{(x_1 - g)^2 + (y_1 - p)^2} \right] \\
+ 2(y_1 + p) \left( \arctg \frac{x_1 + g}{y_1 + p} - \arctg \frac{x_1 - g}{y_1 + p} \right) - 2(y_1 - p) \left( \arctg \frac{x_1 + g}{y_1 - p} - \arctg \frac{x_1 - g}{y_1 - p} \right) \right\} y_1^2 x_1$$

and

We have verified previously [Eq (50)] that $E^\text{ext}$ is a Laplacian field. It is thus easy to check, using Eqs (82) and (83), that $E_1$ is equally a Laplacian field.

The expressions giving the field components of the virtual beam are considerably more complicated that those of the isolated beam. This is due to at least three circumstances:

1) The complicated boundary of the virtual beam (Fig. 3)

2) The reversals of the field direction (with respect to $x$ and $y$)
when going round the boundary of the virtual beam.

iii) The highly non-linear charge density in the virtual beam (see Appendix).

6.2. Particular cases

Although the expressions we have derived above for the field components of the image beam are relatively complicated in the general case, they become manageable in simpler cases of physical interest.

6.2.1. Image field components along the symmetry axes

In any application of the present theory to beam dynamics, the important parameters are likely to be the field components of the virtual beam along the symmetry axes. We shall therefore make an attempt to investigate in more detail these parameters.

On the x-axis the y-component of the image field is zero, just as for the isolated beam. For the x-component we find from Eq (66)

\[ E_{ix}(x,0) = \frac{1}{\pi \epsilon_0 x^3} \left[ \frac{2e}{a^2} x^2 + px \ln \sqrt{\frac{(a^2+gx)^2 + p^2 x^2}{(a^2-gx)^2 + p^2 x^2}} \right. \]

\[ + \frac{(a^2+gx)}{a^2+gx} \tan^{-1} \frac{px}{a^2+gx} - \left. \frac{(a^2-gx)}{a^2-gx} \tan^{-1} \frac{px}{a^2-gx} \right] \tag{68} \]

Similarly, the x-component of the image field is zero on the y-axis and we find from Eq (87) for the y-component

\[ E_{iy}(0,y) = \frac{1}{\pi \epsilon_0 y^3} \left[ \frac{2e}{a^2} y^2 + gy \ln \sqrt{\frac{(a^2+py)^2 + g^2 y^2}{(a^2-py)^2 + g^2 y^2}} \right. \]

\[ + \frac{(a^2+py)}{a^2+py} \tan^{-1} \frac{gy}{a^2+py} - \left. \frac{(a^2-py)}{a^2-py} \tan^{-1} \frac{gy}{a^2-py} \right] \tag{69} \]

The behaviour of these components is quite different from the corresponding components of the isolated beam. To see this at
Least qualitatively we perform a series expansion of the two components in the neighbourhood of the origin and find

$$E_{ix}(x,0) = \frac{2\rho g p}{3\pi \sigma_0 a^4} \left[ (g^2 - p^2)x + \frac{6}{5a} (g^2 - 3p^2)(3g^2 - p^2)x^3 + \ldots \right] \quad (90)$$

and

$$E_{iy}(0,y) = \frac{2\rho g p}{3\pi \sigma_0 a^4} \left[ (p^2 - g^2)y + \frac{6}{5a} (g^2 - 3p^2)(3g^2 - p^2)y^3 + \ldots \right] \quad (91)$$

Assuming $g > p$, we notice that in the neighbourhood of the origin the image field along the $x$-axis is directed away from the origin whereas the image field along the $y$-axis is directed towards the origin. Both fields increase in absolute value at the same rate. These properties are of course consistent with the Laplacian nature of the image field. Further away from the origin the behaviour of the image field depends, if the considered range for $x$ or $y$ is not too large, only on the sign of the quantity $g^2 - 3p^2$. If $g > \sqrt{3}p$ the component $E_{ix}(x,0)$ increases more rapidly than linear whereas the component $E_{iy}(0,y)$ decreases less rapidly than linear (Fig. 6a). Possibly $E_{iy}(0,y)$ could go through a minimum (Fig. 6b) or even reverse its sign (Fig. 6c). On the other hand, if $g < \sqrt{3}p$ (but always keeping our assumption $g > p$), $E_{iy}(0,y)$ decreases more rapidly than linear whereas $E_{ix}(x,0)$ increases less rapidly than linear (Fig. 6d). Possibly $E_{ix}(x,0)$ could go through a maximum (Fig. 6e) or even reverse its sign (Fig. 6f). If $g = \sqrt{3}p$ the two components $E_{ix}(x,0)$ and $E_{iy}(0,y)$ are essentially linear (and opposite in sign) in the neighbourhood of the origin.

The curves 6a-6f are only applicable to the neighbourhood of the origin. To obtain more specific information about a possible maximum of $E_{ix}(x,0)$ or $E_{iy}(0,y)$ over the entire range $0 < x \leq a$ resp. $0 < y \leq a$, we consider the derivatives

$$\frac{3E_{ix}(x,0)}{2x} = \frac{2\rho g p}{\pi \sigma_0 a^4} \left[ \frac{6p}{a^2} x - px \ln \frac{(a+gx)^2 + p^2}{(a-gx)^2 + p^2} \right]$$

$$- \left( \frac{3a^2}{2} + gx \right) \arctg \frac{px}{a^2 + gx} + \left( \frac{3a^2}{2} - gx \right) \arctg \frac{px}{a^2 - gx}$$
and

\[
\frac{aE_{iy}(0,y)}{ay} = \frac{2ae^2}{\pi e_y} \left[ \frac{gp}{a} y^2 - g_y \ln \frac{(a^2 + py)^2 + g_y^2}{(a^2 - py)^2 + g_y^2} \right. \\
\left. - \left( \frac{3a^2}{2} + py \right) \arctg \frac{g_y}{a^2 + py} - \left( \frac{3a^2}{2} - py \right) \arctg \frac{g_y}{a^2 - py} \right]
\] (93)

and ask for the solutions of the transcendental equations

\[
\frac{gp}{a^2} x^2 \left( \frac{3a^2}{2} - gx \right) \arctg \frac{px}{a^2 - gx} = px \ln \frac{(a^2 + gx)^2 + p^2 x^2}{(a^2 - gx)^2 + p^2 x^2} + \left( \frac{3a^2}{2} - gx \right) \arctg \frac{px}{a^2 + gx}
\] (94)

resp.

\[
\frac{gp}{a^2} y^2 \left( \frac{3a^2}{2} - py \right) \arctg \frac{gy}{a^2 - py} = gy \ln \frac{(a^2 + py)^2 + g^2 y^2}{(a^2 - py)^2 + g^2 y^2} + \left( \frac{3a^2}{2} - py \right) \arctg \frac{gy}{a^2 + py}
\] (95)

in the interval \(0 < x < a\) resp. \(0 < y < a\). On the other hand, to obtain more definite information about a possible sign reversal of \(E_{ix}(x,a)\) or \(E_{iy}(0,y)\) we look for the solutions of the transcendental equations

\[
px \ln \frac{(a^2 + gx)^2 + p^2 x^2}{(a^2 - gx)^2 + p^2 x^2} + (a^2 + gx) \arctg \frac{px}{a^2 + gx} = 2 \frac{gp}{a^2} x^2 \left( a^2 - gx \right) \arctg \frac{px}{a^2 - gx}
\] (96)

resp.

\[
gy \ln \frac{(a^2 + py)^2 + g^2 y^2}{(a^2 - py)^2 + g^2 y^2} + (a^2 + py) \arctg \frac{gy}{a^2 + py} = 2 \frac{gp}{a^2} y^2 \left( a^2 - py \right) \arctg \frac{gy}{a^2 - py}
\] (97)

in the same intervals.

Considering first the situation in the \(x\)-direction, we remark that Eqs (94) and (96) which contain in principle four parameters, viz. \(g, p, a,\) and \(x\) can actually be transformed into two-parameter relations by introducing the dimensionless quantities
\[ \frac{\delta y}{a^2} = G, \quad \frac{\delta x}{a^2} = P \]  

(98)

The position of the maximum of \( E_{ix}(x,0) \) is then given by the solution of the equation

\[ P \ln \frac{(1 + G)^2 + P^2}{(1 - G)^2 + P^2} + \left( \frac{3}{2} + G \right) \arctan \frac{P}{1 + G} = 2G^2 + \left( \frac{3}{2} - G \right) \arctan \frac{P}{1 - G} \]  

(99)

whereas the sign reversal of \( E_{ix}(x,0) \) is determined by the solution of the equation

\[ P \ln \frac{(1 + G)^2 + P^2}{(1 - G)^2 + P^2} + \left( 1 + G \right) \arctan \frac{P}{1 + G} = 2GP + (1 - G) \arctan \frac{P}{1 - G} \]  

(100)

Eqs (99) and (100) are "universal" in the sense that they contain only numerical coefficients. They both apply to the domain

\[ G^2 + P^2 \leq 1 \]  

(101)

To investigate the situation in the y-direction we put

\[ \frac{\delta y}{a^2} = G, \quad \frac{\delta y}{a^2} = P \]  

(102)

Eqs (95) and (97) determining the position of the maximum of \( E_{iy}(0,y) \) and its possible sign reversal become then respectively

\[ G \ln \frac{(1 + P)^2 + G^2}{(1 - P)^2 + G^2} + \left( \frac{3}{2} + P \right) \arctan \frac{G}{1 + P} = 2GP + \left( \frac{3}{2} - P \right) \arctan \frac{G}{1 - P} \]  

(103)

and

\[ G \ln \frac{(1 + P)^2 + G^2}{(1 - P)^2 + G^2} + \left( 1 + P \right) \arctan \frac{G}{1 + P} = 2GP + (1 - P) \arctan \frac{G}{1 - P} \]  

(104)

Actually these equations can be obtained from Eqs (99) and (100) by a simple interchange of \( G \) and \( P \).

The result of this calculation is summarized in Fig. 7 where the solutions of Eqs (99), (100), (103), and (104), obtained by numerical procedures, have been plotted. The upper curves, represen-
ting the solutions of Eqs (103) and (104), are symmetric with respect to the lower curves, representing the solutions of Eqs (99) and (100), the axis of symmetry being the bisector G = P. Because of this symmetry the situation in both directions, x and y, can be represented in one diagram. The bisector itself corresponds to a square beam where $E_{ix}(x,0)$ is formally identical to $E_{iy}(0,y)$.

The four curves of Fig. 7 divide the G-P plane in five regions but if we restrict ourselves to $g > p$ there are only three regions to consider. For $p/g = \text{const} = \nu$ and x variable the representative point in the G-P diagram describes a straight line $P = G\nu$ going through the origin. If this line stays entirely in region I we have $\partial E_{ix}(x,0)/\partial x > 0$ for all values of x and there is no maximum for $E_{ix}(x,0)$. If the straight line goes through region I and II, there will be a maximum for $E_{ix}(x,0)$ for sufficiently large values of x, but no sign reversal. Finally, if the straight line goes through region I, II, and III, there will be a maximum for $E_{ix}(x,0)$ and also, for larger values of x, a sign reversal of $E_{ix}(x,0)$. In the three regions considered we have $\partial E_{iy}(0,y)/\partial y < 0$ and $E_{iy}(0,y) < 0$. Consequently for $g > p$ (which is our assumption) $E_{iy}(0,y)$ cannot go through a minimum or reverse its sign, it can only decrease steadily as y increases; this rules out the curves 6b and 6c.

According to Fig. 7, the straight line will be entirely in region I for $g/p > 1.89$, partially in region I and II for $1.21 < g/p < 1.89$, and partially in region I, II, and III for $g/p < 1.21$. We can therefore distinguish the following possibilities:

(a) $g/p > 1.89$. The behaviour of the field components is given in Fig. 6a which now applies to the whole range of x and y.

(b) $\sqrt{3} < g/p < 1.89$. The component $E_{ix}(x,0)$ has now a maximum; the situation is represented in Fig. 6g.

(c) $1.21 < g/p < \sqrt{3}$. Fig. 6e shows the situation.

(d) $1 < g/p < 1.21$. The behaviour of the field components is given in Fig. 6f.

In the preceding discussion we have always assumed $g > p$. However, a similar discussion could be carried out for the case $g < p$ by considering the upper half of the G-P diagram.
Finally, we remark that both $E_{ix}(x,0)$ and $E_{iy}(0,y)$ go to zero for $a \to \infty$, in accordance with the properties of image field components.

6.2.2. Image field at the intersections of the symmetry axes with the beam envelope

From Eqs (88) and (89) we have

$$E_{ix}(g,\theta) = \frac{\rho a^2}{\pi \varepsilon_0 g^2} \left[ -\frac{2g}{a^2} \ln \sqrt{\frac{(a^2+g^2)^2 + 2g \cdot 2}{(a^2+g^2)^2 + 2g \cdot 2}} ight] + \left( a^2 + g^2 \right) \arctan \frac{g}{a^2+g^2} - \left( a^2 - g^2 \right) \arctan \frac{g}{a^2-g^2}$$

and

$$E_{iy}(g,\theta) = \frac{\rho a^2}{\pi \varepsilon_0 g^2} \left[ -\frac{2g}{a^2} \ln \sqrt{\frac{(a^2+p^2)^2 + 2g \cdot 2}{(a^2+p^2)^2 + 2g \cdot 2}} ight] + \left( a^2 + p^2 \right) \arctan \frac{g}{a^2+p^2} - \left( a^2 - p^2 \right) \arctan \frac{g}{a^2-p^2}$$

6.2.3. Image field at the intersections of the symmetry axes with the vacuum chamber

Again, from Eqs (88) and (89) we find

$$E_{ix}(a,0) = \frac{\rho a}{\pi \varepsilon_0} \left[ -\frac{2g}{a} \arctan \frac{g}{(a-g)^2} + (a+g) \arctan \frac{g}{a+g} - (a-g) \arctan \frac{g}{a-g} \right]$$

and

$$E_{iy}(a,0) = \frac{\rho a}{\pi \varepsilon_0} \left[ -\frac{2g}{a} \arctan \frac{g}{(a+p)^2} + (a+p) \arctan \frac{g}{a+p} - (a-p) \arctan \frac{g}{a-p} \right]$$

6.2.4. Image field along the bisectors of the beam

In the preceding expressions we have been able to simplify considerably the formulae giving the components of the virtual
field by taking \( y = 0 \) in Eq (86) and \( x = 0 \) in Eq (87). Another substantial simplification occurs along the geometrical bisectors of the beam, viz. for \( x^2 = y^2 \).

For \( x = y = u \) \( \text{(109)} \)

we have from Eqs (73)

\[
\begin{align*}
\frac{x_1}{a} &= \frac{y_1}{a} = \frac{a^2}{2u} \\
\text{(110)}
\end{align*}
\]

Replacing this in Eqs (86) and (87) and using again the non-dimensional quantities

\[
\frac{su}{a^2} = G, \quad \frac{du}{a^2} = P \text{ (111)}
\]

we obtain

\[
E_{ix}(u,u) = \frac{\rho a^4}{8\pi u^3} \left[ -8Gp^2 (1+2G) \ln \frac{(1+2G)^2 + (1+2P)^2}{(1+2G)^2 + (1-2P)^2} + (1-2G) \ln \frac{(1-2G)^2 + (1+2P)^2}{(1-2G)^2 + (1-2P)^2} \right. \\
\left. + (1+2P) (\arctan \frac{1+2G}{1+2P} - \arctan \frac{1-2G}{1-2P}) - (1-2P) (\arctan \frac{1+2G}{1-2P} - \arctan \frac{1-2G}{1+2P}) \right] \text{ (112)}
\]

and

\[
E_{iy}(u,u) = \frac{\rho a^4}{8\pi u^3} \left[ -8Gp^2 (1+2P) \ln \frac{(1+2G)^2 + (1+2P)^2}{(1-2G)^2 + (1+2P)^2} + (1-2P) \ln \frac{(1-2G)^2 + (1-2P)^2}{(1+2G)^2 + (1-2P)^2} \right. \\
\left. + (1+2G) (\arctan \frac{1+2P}{1+2G} - \arctan \frac{1-2P}{1+2G}) - (1-2G) (\arctan \frac{1+2P}{1-2G} - \arctan \frac{1-2P}{1-2G}) \right] \text{ (113)}
\]

where

\[
G^2 + P^2 \leq 1 \text{ (114)}
\]

The discussion of the behaviour of these components could proceed along the lines indicated above for \( E_{ix}(x,0) \) and \( E_{iy}(c,y) \).

**6.2.5 Image field at the vertices of the beam envelope**

To get a feeling just how complicated the virtual field components can be in the general case, we calculate these compo-
ments at the vertices of the beam envelope. Putting
\[ \frac{a^2}{g^2 + p^2} = \lambda \]  
we find from Eqs (86) and (87) for the components of the virtual field at the vertices of the beam envelope

\[
E_{ix}(g,p) = \frac{\rho}{4\pi \varepsilon_0 a^2} \left\{ -8 \lambda g^2 p \\
+ \lambda^2 \left[ p(\lambda+1) \ln \frac{(\lambda+1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda+1)^2 p^2} + p(\lambda-1) \ln \frac{(\lambda-1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda-1)^2 p^2} \\
+ 2g(\lambda+1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda-1}{\lambda+1} \right) - 2g(\lambda-1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) \right] (g^2 - p^2) \\
+ 2\lambda^2 \left[ g(\lambda+1) \ln \frac{(\lambda+1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda+1)^2 p^2} + g(\lambda-1) \ln \frac{(\lambda-1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda-1)^2 p^2} \\
+ 2p(\lambda+1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) - 2p(\lambda-1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) \right] p^2 \right\}
\]

and

\[
E_{iy}(g,p) = \frac{\rho}{4\pi \varepsilon_0 a^2} \left\{ -8 \lambda gp^2 \\
+ \lambda^2 \left[ g(\lambda+1) \ln \frac{(\lambda+1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda-1)^2 p^2} + g(\lambda-1) \ln \frac{(\lambda-1)^2(g^2 + p^2)}{(\lambda-1)^2 g^2 + (\lambda+1)^2 p^2} \\
+ 2p(\lambda+1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda-1}{\lambda+1} \right) - 2p(\lambda-1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) \right] (g^2 - p^2) \\
+ 2\lambda^2 \left[ p(\lambda+1) \ln \frac{(\lambda+1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda+1)^2 p^2} + p(\lambda-1) \ln \frac{(\lambda-1)^2(g^2 + p^2)}{(\lambda+1)^2 g^2 + (\lambda-1)^2 p^2} \\
+ 2g(\lambda+1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) - 2g(\lambda-1) \left( \arctan \frac{g}{p} - \arctan \frac{\lambda+1}{\lambda-1} \right) \right] p^2 \right\}
\]
These expressions undergo a substantial simplification in the case of the inscribed beam which we now consider.

**6.2.6. The inscribed beam**

The inscribed beam is defined by

\[ a^2 = g^2 + p^2 \]  \hspace{1cm} (118)

or

\[ \lambda = 1 \]  \hspace{1cm} (119)

Using this, Eqs (118) and (117) for the field components at the vertices become respectively

\[
E_{ix}(g,p) = \frac{\mu}{\pi \varepsilon_0 a^2} \left[ -2g^{-2} \cdot \left( p \ln \frac{a}{p} + g \arctg \frac{g}{p} \right) \left( g^2 - p^2 \right) + 2g(p \ln \frac{a}{p} + g \arctg \frac{g}{p}) \right]
\]  \hspace{1cm} (120)

\[
E_{iy}(g,p) = -\frac{\mu}{\pi \varepsilon_0 a^2} \left[ -2g^{-2} \cdot \left( g \ln \frac{a}{g} + g \arctg \frac{g}{p} \right) (p^2 - g^2) + 2g(p \ln \frac{a}{p} + g \arctg \frac{g}{p}) \right]
\]  \hspace{1cm} (121)

The change of sign of the second term when going over from \( E_{ix}(g,p) \) to \( E_{iy}(g,p) \) should be noticed.

**6.2.7. The square beam**

We now consider the case where

\[ g = p = \lambda \]  \hspace{1cm} (67)

The image field component on the x-axis has then the same functional form as the image field component on the y-axis, and putting

\[
\left( \frac{E_x}{a^2}, \frac{E_y}{a^2} \right) \right) = z \]  \hspace{1cm} (122)
we find from Eqs (88) and (89) the "universal" equation

\[ E_{iz}(z) = \frac{\rho \kappa^3}{\pi \varepsilon_0 a^2 z^3} \left[ -2z^2 + z \ln \frac{(1+z)^2 + z^2}{(1-z)^2 + z^2} + (1+z)\arctg \frac{z}{1+z} - (1-z)\arctg \frac{z}{1-z} \right] \]  

(124)

where

\[ 0 < z < \frac{1}{\sqrt{2}} \]  

(125)

Using the series expansions (90) and (91) or expanding directly Eq (124) we find that in the neighbourhood of the origin

\[ E_{iz}(z) = -\frac{16\rho \kappa^3}{5\pi \varepsilon_0 a^2} z^3 + \ldots \]  

(126)

This means that the curve \( E_{iz}(z) \) starts out from the origin with zero slope and zero curvature. The derivative

\[ \frac{\partial E_{iz}(z)}{\partial z} = \frac{2\rho \kappa^4}{\pi \varepsilon_0 a^4 z^4} \left[ z^2 - z \ln \frac{(1+z)^2 + z^2}{(1-z)^2 + z^2} - (\frac{3}{2}z)\arctg \frac{z}{1+z} + (\frac{3}{2}z)\arctg \frac{z}{1-z} \right] \]  

(127)

is negative throughout the range \( 0 < z < \frac{1}{\sqrt{2}} \) so that \( E_{iz}(z) \) stays negative and decreases (increases in absolute value) when \( z \) varies from zero to \( 1/\sqrt{2} \). Consequently the image field of a square beam in a circular vacuum chamber is everywhere focusing along the directions Ox and Oy. Fig. 8 gives the normalized value \( -\pi \varepsilon_0 a^2 E_{iz}(z)/\rho k^3 \) as a function of \( z \).

Particular values of \( E_{iz}(z) \) can easily be obtained. For instance, at the intersections of the symmetry axes with the beam envelope we have \( z = \frac{a^2}{2a^2} \) whereas at the intersections of the symmetry axes with the vacuum chamber we have \( z = \frac{a}{2} \). For an inscribed square beam we have in the former case \( z = \frac{1}{2} \) and in the latter case \( z = 1/\sqrt{2} \).

Along the bisectors of a square beam we have from Eqs (86) and (67)
\[ E_{ix}(u,u) = E_{iy}(u,u) = \frac{\rho \xi^3}{8\pi \varepsilon_0 a^2 z^3} \left( -8z^2 \ln \frac{1-4z^2}{1+4z^2} + 2z \ln \frac{1+2z}{1-2z} + 2z \arctg \frac{4z}{1-4z^2} \right) \]  
\text{(128)}

where
\[ z = \frac{kx}{a^2} = \frac{ky}{a^2} = \frac{ku}{a^2} \]  
\text{(129)}

and
\[ z \leq \frac{1}{2} \]  
\text{(130)}

Putting \( E_1(u) = E_{ix}(u,u) = E_{iy}(u,u) \) we notice that \( E_1(u) \) is positive everywhere. Consequently the image field of a square beam is defocusing everywhere along the directions of the beam bisectors. Fig. 9 gives the normalized value \( \pi_1 a^3 E_0(u)/\rho \xi^3 \) as a function of \( z \).

At the vertices of a square beam the field components of the image beam are according to Eqs (116) and (117)

\[ E_{ix}(\xi,\xi) = E_{iy}(\xi,\xi) = \frac{\rho \xi^3}{4\pi \varepsilon_0} \left[ -\frac{4}{\lambda} + (\lambda+1) \ln \frac{(\lambda+1)^2}{\lambda^2+1} + (\lambda-1) \ln \frac{(\lambda-1)^2}{\lambda^2+1} + 2 \arctg \frac{2\lambda}{\lambda^2-1} \right] \]  
\text{(131)}

where
\[ \lambda = \frac{a^2}{2 \xi} \]  
\text{(132)}

Putting \( u = \xi \) in Eq (128) we also have

\[ E_{ix}(\xi,\xi) = E_{iy}(\xi,\xi) = \frac{\rho \xi^3}{8\pi \varepsilon_0} \left( -8 \frac{2}{a^2} + \frac{a^2}{a^2} \ln \frac{a^4-4\xi^4}{a^4} + 2 \ln \frac{a^2+2\xi^2}{a^2-2\xi^2} + 2 \arctg \frac{4a^2\xi^2}{a^4-4\xi^4} \right) \]  
\text{(133)}

Eqs (131) and (133) are equivalent.

Finally if the square is inscribed we have \( \lambda = 1 \) and Eq (131) becomes

\[ E_{ix}(\xi,\xi) = E_{iy}(\xi,\xi) = \frac{\rho \xi}{2\pi \varepsilon_0} \left( \ln 2 + \frac{\pi}{2} - 2 \right) \]  
\text{(134)}
7. THE ACTUAL BEAM

We have practically solved the problem of the rectangular beam coating coaxially inside a cylindrical vacuum chamber of circular cross-section and infinite conductivity. What remains to be done is to obtain the final formulae for the potential and the field components produced by the actual beam; to that effect we simply add the corresponding expressions derived for the isolated beam and for the virtual beam.

7.1. Potential produced by the actual beam

To write the formula for the potential we use the notations introduced in Eq (40) to which we add the generalisations

\[
(r^i_{g,p})^2 = (x_i - g)^2 + (y_i - p)^2 \quad \quad (r^i_{-g,p})^2 = (x_i + g)^2 + (y_i - p)^2
\]

\[
(r^i_{g,-p})^2 = (x_i - g)^2 + (y_i + p)^2 \quad \quad (r^i_{-g,-p})^2 = (x_i + g)^2 + (y_i + p)^2
\]

with \( x_i \) and \( y_i \) given by Eq (73). The addition of Eq (39) and Eq (81) yields then

\[
\Phi(x,y) = \frac{e}{4\pi \varepsilon_0} \left\{ 4 \ln \frac{x^2 + y^2}{a^2} + 2 \ln \frac{\sqrt{r^i_{g,p} r^i_{-g,p} r^i_{g,-p} r^i_{-g,-p}}}{r^i_{g,p} r^i_{-g,p} r^i_{g,-p} r^i_{-g,-p}} \right. \\
+ 2xy \left[ \ln \frac{r^i_{-g,p} r^i_{g,-p}}{r^i_{g,p} r^i_{-g,p}} - \frac{a^2}{2} \ln \frac{r^i_{g,p} r^i_{-g,p}}{r^i_{g,-p} r^i_{-g,-p}} \right] \\
+ 2px \left[ \ln \frac{r^i_{g,p} r^i_{g,-p}}{r^i_{-g,p} r^i_{-g,-p}} - \frac{a^2}{2} \ln \frac{r^i_{g,p} r^i_{g,-p}}{r^i_{-g,p} r^i_{-g,-p}} \right] \\
+ 2gy \left[ \ln \frac{r^i_{g,p} r^i_{g,-p}}{r^i_{-g,p} r^i_{-g,-p}} - \frac{a^2}{2} \ln \frac{r^i_{g,p} r^i_{g,-p}}{r^i_{-g,p} r^i_{-g,-p}} \right] \right\}
\]

(136)
\[ + (x_1 + g)^2 \left( \arctg \frac{y_1 + p}{x_1 + g} - \arctg \frac{y_1 - p}{x_1 + g} \right) - (x + g)^2 \left( \arctg \frac{y + p}{x + g} - \arctg \frac{y - p}{x + g} \right) \]

\[ -(x_1 - g)^2 \left( \arctg \frac{y_1 + p}{x_1 - g} - \arctg \frac{y_1 - p}{x_1 - g} \right) + (x - g)^2 \left( \arctg \frac{y + p}{x - g} - \arctg \frac{y - p}{x - g} \right) \]

\[ + (y_1 + p)^2 \left( \arctg \frac{x_1 + g}{y_1 + p} - \arctg \frac{x_1 - g}{y_1 + p} \right) - (y + p)^2 \left( \arctg \frac{x + g}{y + p} - \arctg \frac{x - g}{y + p} \right) \]

\[ -(y_1 - p)^2 \left( \arctg \frac{x_1 + g}{y_1 - p} - \arctg \frac{x_1 - g}{y_1 - p} \right) + (y - p)^2 \left( \arctg \frac{x + g}{y - p} - \arctg \frac{x - g}{y - p} \right) \]

This formula gives the potential everywhere inside the vacuum chamber, i.e., inside or outside the beam envelope. The arbitrary constant involved in Eqs (39) and (81) has been eliminated by the choice we have made for the potential of the vacuum chamber, and the logarithms act therefore only on non-dimensional quantities.

The Poissonian or Laplacian nature of the potential given by Eq (136) results from its construction. On the other hand, it is easily checked that \( \phi(x, y) \) satisfies the given boundary condition. Indeed, on the circle \( x^2 + y^2 = a^2 \) we have \( x_1 = x, y_1 = y, r_1 = r \) and Eq (136) yields immediately \( \phi(x, y) = 0 \) in this case. As it satisfies Laplace's resp. Poisson's equation and as it takes on the prescribed value on the boundary, the potential given by Eq (136) is the unique solution of the problem; there is no way of putting it in a simpler form. Although we have been able to obtain a closed form for this potential, a mere glance at Eq (136) shows how difficult it would have been to derive it by the usual technique of separation of Laplace's resp. Poisson's equation.

### 7.2. Field components of the actual beam

From Eq (46) and (86) we have
\[
E_x(x,y) = \frac{\rho}{2\pi \varepsilon_0} \left[ \left( y + p \right) \ln \frac{r - g - p}{r - g, p} + \left( y - p \right) \ln \frac{r - g - p}{r - g, p} \right]
\]

\[
+ (x + g) \left( \arctan \frac{y + p}{x + g} - \arctan \frac{y - p}{x + g} \right) - (x - g) \left( \arctan \frac{y + p}{x - g} - \arctan \frac{y - p}{x - g} \right)
\]

\[
+ \frac{\rho}{2\pi \varepsilon_0} a^2 \left[ -4g p x_1 y_1 \left\{ \left( y_1 + p \right) \ln \frac{r - g - p}{r - g, p} + \left( y_1 - p \right) \ln \frac{r - g - p}{r - g, p} \right\} \right]
\]

\[
E_y(x,y) = \frac{\rho}{2\pi \varepsilon_0} \left[ \left( x + g \right) \ln \frac{r - g - p}{r - g, p} + \left( x - g \right) \ln \frac{r - g - p}{r - g, p} \right]
\]

\[
+ (y + p) \left( \arctan \frac{x + g}{y + p} - \arctan \frac{x - g}{y + p} \right) - (y - p) \left( \arctan \frac{x + g}{y - p} - \arctan \frac{x - g}{y - p} \right)
\]

\[
+ \frac{\rho}{2\pi \varepsilon_0} a^2 \left[ -4g p x_1 y_1 \left\{ \left( y_1 + p \right) \ln \frac{r - g - p}{r - g, p} + \left( y_1 - p \right) \ln \frac{r - g - p}{r - g, p} \right\} \right]
\]

whereas Eqs (47) and (67) yield

\[
E_x(x,y) = \frac{\rho}{2\pi \varepsilon_0} \left[ \left( y + p \right) \ln \frac{r - g - p}{r - g, p} + \left( y - p \right) \ln \frac{r - g - p}{r - g, p} \right]
\]

\[
+ (y + p) \left( \arctan \frac{x + g}{y + p} - \arctan \frac{x - g}{y + p} \right) - (y - p) \left( \arctan \frac{x + g}{y - p} - \arctan \frac{x - g}{y - p} \right)
\]

\[
+ \frac{\rho}{2\pi \varepsilon_0} a^2 \left[ -4g p x_1 y_1 \left\{ \left( y_1 + p \right) \ln \frac{r - g - p}{r - g, p} + \left( y_1 - p \right) \ln \frac{r - g - p}{r - g, p} \right\} \right]
\]

\[
+ (y + p) \left( \arctan \frac{x + g}{y + p} - \arctan \frac{x - g}{y + p} \right) - (y - p) \left( \arctan \frac{x + g}{y - p} - \arctan \frac{x - g}{y - p} \right)
\]
These equations are again applicable to any point inside the vacuum chamber, be it inside or outside the beam.

7.3. Particular cases

We first consider the potential at the centre of the beam; we did not investigate this parameter for the isolated beam or the virtual beam because of the undetermined constants. In all the other cases we then consider, the pertinent parameters are obtained by superposition of the corresponding quantities applying to the isolated and to the virtual beam.

7.3.1. Potential at the centre of the beam

From Eq (136) we find for the potential at the centre of the vacuum chamber which is at the same time the maximum potential in the beam

\[
\phi_{\text{Max}} = \frac{\rho}{\pi \varepsilon_0} \left[ g p (3 + \ln \frac{a^2}{g^2+p^2}) - (g^2 \arctan \frac{p}{g} + p^2 \arctan \frac{g}{p}) \right] \tag{139}
\]

This formula can also be found from Eq (17) by taking \( \rho = \text{const}, \) viz.

\[
\phi(0,0) = \frac{\rho}{4\pi \varepsilon_0} \int_{-p}^{+p} \int_{-g}^{+g} \ln \frac{a^2}{x^2+y^2} \, dx \, dy \tag{140}
\]

and carrying out the integrations.

For an inscribed beam where \( a^2 = g^2+p^2 \), Eq (139) becomes

\[
\phi_{\text{Max}} = \frac{\rho}{\pi \varepsilon_0} \left[ 3gp - (g^2 \arctan \frac{p}{g} + p^2 \arctan \frac{g}{p}) \right] \tag{141}
\]

whereas for a square beam where \( g = p = \lambda \) we have

\[
\phi_{\text{Max}} = \frac{\rho \lambda^2}{\pi \varepsilon_0} \left( 3 - 2\ln 2 - \frac{\pi}{2} + 2\ln \frac{a}{\lambda} \right) \tag{142}
\]

Finally if the square beam is inscribed, i.e. \( a = \lambda \sqrt{2} \), this becomes
\[ \phi_{\text{max}} = \frac{\rho z^2}{\pi \varepsilon_0} \left( 3 - \frac{\pi}{2} \right) \]

7.3.2. Electric field along the symmetry axes

From Eqs (58) and (88) we have

\[
E_x(x,0) = \frac{\rho}{\pi \varepsilon_0} \left[ \ln \frac{\sqrt{(x+g)^2 + p^2}}{(x-g)^2 + p^2} + (x+g) \text{arc} \tan \frac{p}{x+g} - (x-g) \text{arc} \tan \frac{p}{x-g} \right]
\]

\[+ \frac{p a^2}{\pi \varepsilon_0 x^3} \left[ p x \ln \frac{\sqrt{(a^2+gx)^2 + p^2 x^2}}{(a^2-gx)^2 + p^2 x^2} + (a^2+gx) \text{arc} \tan \frac{p x}{a^2+gx} - (a^2-gx) \text{arc} \tan \frac{p x}{a^2-gx} \right] \]

whereas Eqs (61) and (89) yield

\[
E_y(0,y) = \frac{\rho}{\pi \varepsilon_0} \left[ \ln \frac{\sqrt{(y+p)^2 + g^2}}{(y-p)^2 + g^2} + (y+p) \text{arc} \tan \frac{g}{y+p} - (y-p) \text{arc} \tan \frac{g}{y-p} \right]
\]

\[+ \frac{p a^2}{\pi \varepsilon_0 y^3} \left[ g y \ln \frac{\sqrt{(a^2+py)^2 + g^2 y^2}}{(a^2-py)^2 + g^2 y^2} + (a^2+py) \text{arc} \tan \frac{g y}{a^2+py} - (a^2-py) \text{arc} \tan \frac{g y}{a^2-py} \right] \]

7.3.3. Electric field at the intersections of the symmetry axes with the beam envelope

Putting \( x = g \) resp. \( y = p \) in the preceding equations we obtain

\[
E_x(g,0) = \frac{\rho}{\pi \varepsilon_0} \left( \ln \frac{4 g^2}{p^2} + 2g \text{arc} \tan \frac{p}{2g} - 2p \right)
\]

\[+ \frac{p a^2}{\pi \varepsilon_0 g^3} \left[ g p \ln \frac{(a^2+g^2)^2 + g^2 p^2}{(a^2-g^2)^2 + g^2 p^2} + (a^2+g^2) \text{arc} \tan \frac{g p}{a^2+g^2} - (a^2-g^2) \text{arc} \tan \frac{g p}{a^2-g^2} \right] \]
and

\[ E_y(o,p) = \frac{\rho}{\pi\varepsilon_0} \left( g \sin \frac{\sqrt{2^2 + 4p^2}}{g} + 2p \arctg \frac{g}{2p} - 2g \right) \]

\[ + \frac{oa^2}{\pi\varepsilon_0 p^3} \left[ g p \sin \sqrt{\frac{(a^2 + p^2)^2 + 2^2}{2^2 + 2^2}} \arctg \frac{g}{a^2 + p^2} - \frac{(a^2 - p^2)^2 + 2^2}{a^2 - p^2} \arctg \frac{g}{a^2 - p^2} \right] \]  

(147)

7.3.4. Electric field at the intersections of the symmetry axes with the vacuum chamber

Puting \( x = a \) resp. \( y = a \) in Eqs (144) and (145) we find

\[ E_x(a,0) = \frac{2p}{\pi\varepsilon_0} \left[ \frac{g p}{a} + g \sin \sqrt{\frac{(a + g)^2 + p^2}{(a - g)^2 + p^2}} + (a + g) \arctg \frac{p}{a + g} - (a - g) \arctg \frac{p}{a - g} \right] \]

(148)

and

\[ E_y(0,a) = \frac{2p}{\pi\varepsilon_0} \left[ \frac{g p}{a} + g \sin \sqrt{\frac{(a + p)^2 + g^2}{(a - p)^2 + g^2}} + (a + p) \arctg \frac{g}{a + p} - (a - p) \arctg \frac{g}{a - p} \right] \]

(149)

7.3.5. The square beam

Eq (139) gives the potential at the centre of a square beam. Three other field parameters may be of interest.

i) Electric field at the intersections of the coordinate axes with the beam envelope

Putting \( g = p = 1 \) in Eqs (146) and (147) we find

\[ E_x(1,0) = E_y(0,1) = \frac{\sqrt{2}}{\pi\varepsilon_0} \left( \frac{1}{2} \ln 5 + 2 \arctg \frac{1}{2} - 2 \right) \]

(150)

\[ + \frac{oa^2}{\pi\varepsilon_0} \left[ \ln \sqrt{\frac{(a^2 + 1)^2 + 1^4}{(a^2 - 1)^2 + 1^4}} + (a^2 + 1) \arctg \frac{1}{a^2 + 1} - (a^2 - 1) \arctg \frac{1}{a^2 - 1} \right] \]
For an inscribed beam \((a = \ell \sqrt{2})\) this becomes

\[
E_x(\ell, \ell) = E_y(\ell, \ell) = \frac{2\varepsilon_0}{\pi \ell} \left( \frac{3}{4} \ln 5 + \arctg \frac{1}{2} + 3 \arctg \frac{1}{3} - \frac{\pi}{4} - 1 \right) \tag{151}
\]

\[\text{ii) Electric field at the intersections of the coordinate axes with the vacuum chamber.}\]

Putting \(a = p = \ell\) in Eqs (148) and (149) we find

\[
E_x(a, a) = E_y(a, a) = \frac{2\varepsilon_0}{\pi \ell} \left[ - \frac{\ell}{\ell} + \ln \frac{(\ell + \ell)^2 + \ell^2}{(\ell - \ell)^2 + \ell^2} + \left( \frac{\ell}{\ell} - 1 \right) \arctg \frac{\ell}{\ell} - \left( \frac{\ell}{\ell} - 1 \right) \arctg \frac{\ell}{\ell - \ell} \right] \tag{152}
\]

For an inscribed beam this becomes

\[
E_x(a, a) = E_y(a, a) = \frac{2\varepsilon_0}{\pi \ell} \left[ - \sqrt{2} - \ln \sqrt{3 + 2\sqrt{2}} + (\sqrt{2} + 1) \arctg \frac{1}{\sqrt{2} + 1} - (\sqrt{2} - 1) \arctg \frac{1}{\sqrt{2} - 1} \right] \tag{153}
\]

\[\text{iii) Electric field at the vertices of a square beam}\]

From Eqs (68) and (133) we have

\[
E_x(\ell, \ell) = E_y(\ell, \ell) = \frac{\varepsilon_0}{2\pi \ell} \left( \ell \ln 2 + \frac{\pi}{2} \right) + \frac{\rho \ell^2}{6\pi \varepsilon_0} \left( -8 \frac{\ell^2}{a^2} + \frac{\ell^2}{\ell^2} \ln \frac{a - 4\ell^2}{a + 4\ell^2} + 2 \ln \frac{a^2 + 2\ell^2}{a^2 - 2\ell^2} + 2 \arctg \frac{4\ell^2}{a - 4\ell^2} \right) \tag{154}
\]

For an inscribed beam this becomes

\[
E_x(\ell, \ell) = E_y(\ell, \ell) = \frac{\varepsilon_0}{\pi \ell} \left( \ell \ln 2 + \frac{\pi}{2} - 1 \right) \tag{155}
\]
8. THE DISPLACED BEAM

In our preceding calculations we have assumed that the beam is coasting coaxially inside the vacuum chamber. The extension to a non-coaxial beam is however straightforward. Let \( d_x \) and \( d_y \) be the displacement of the beam centre with respect to the origin of the coordinate system. We consider a beam of dimensions \( 2g \) and \( 2p \) as before, displaced parallel to the coordinate axes (Fig. 10).

9.1. The displaced beam of arbitrary density distribution

The only modification which has to be made in the three basic equations giving the potential and the field components of the beam [Eqs (14), (15) and (16)] amounts to changing the integration limits. We now have

\[
\phi(x,y) = \frac{1}{4\pi \epsilon_0} \int_{d_x-g}^{d_x+g} \int_{d_y-p}^{d_y+p} \rho(x_0,y_0) \ln \frac{(x^2+y^2)(x_0^2+y_0^2) - 2a^2(xx_0 + yy_0) + a^4}{a^2[(x-x_0)^2 + (y-y_0)^2]} \, dx_0 \, dy_0
\]

(156)

for the potential, and

\[
E_x(x,y) = \frac{1}{2\pi \epsilon_0} \int_{d_x-g}^{d_x+g} \int_{d_y-p}^{d_y+p} \rho(x_0,y_0) \left[ \frac{a^2 x_0 - x(x_0^2 + y_0^2)}{(x^2 + y^2)(x_0^2 + y_0^2) - 2a^2(xx_0 + yy_0) + a^4} \right. \\
+ \left. \frac{x-x_0}{(x-x_0)^2 + (y-y_0)^2} \right] \, dx_0 \, dy_0
\]

(157)

\[
E_y(x,y) = \frac{1}{2\pi \epsilon_0} \int_{d_x-g}^{d_x+g} \int_{d_y-p}^{d_y+p} \rho(x_0,y_0) \left[ \frac{a^2 y_0 - y(x_0^2 + y_0^2)}{(x^2 + y^2)(x_0^2 + y_0^2) - 2a^2(xx_0 + yy_0) + a^4} \right. \\
+ \left. \frac{y-y_0}{(x-x_0)^2 + (y-y_0)^2} \right] \, dx_0 \, dy_0
\]

(158)

for the field components. However the potential at the centre of the vacuum chamber
\[ \phi(\alpha, \beta) = \frac{1}{4\pi\epsilon_0} \int_{x-g}^{x+p} \int_{y-g}^{y+p} \rho(x, y) \ln \frac{a^2}{x^2 + y^2} \, dx \, dy \]  

is no more a maximum for the potential and the field components at the centre of the vacuum chamber

\[ E_x(\alpha, \beta) = \frac{1}{2\pi\epsilon_0} \int_{x-g}^{x+p} \int_{y-g}^{y+p} \rho(x, y)(\frac{1}{a^2} - \frac{1}{x^2 + y^2}) \, dx \, dy \]  

\[ E_y(\alpha, \beta) = \frac{1}{2\pi\epsilon_0} \int_{x-g}^{x+p} \int_{y-g}^{y+p} \rho(x, y)(\frac{1}{a^2} - \frac{1}{x^2 + y^2}) \, dx \, dy \]

are no more zero even for a symmetric distribution in \( x \) and \( y \).

8.2. The displaced beam of uniform density

Considering first the isolated beam it is obvious that a displaced beam corresponds to a simple change of coordinates. The only thing one has to do in the three basic relations (39), (46), (47) or (52), (54), (55) is to make the substitutions

\[ x \rightarrow x - d_x \]  
\[ y \rightarrow y - d_y \]  

(162)

However potential and field components are no more invariant under the simultaneous permutations \( x \leftrightarrow y, \, g \leftrightarrow p \) and are no more even with respect to \( x \) and \( y \).

Considering next the virtual beam, we notice that the reciprocity relation (79) still holds, if we write it in the form

\[ f(x, y) = \frac{\rho \rho'}{\pi\epsilon_0} \ln \frac{x^2 + y^2}{a^2} - f(x_1 - d_x, y_1 - d_y) \]  

(163)
where \( \Phi_{f}(x,y) \) is the potential of the undisplaced isolated beam. The potential of the virtual beam can thus be determined and superposition of the two potentials (isolated beam and image beam) gives, as before, the potential created by the actual beam inside the vacuum chamber.

As an application of the preceding theory we calculate the potential and the field components of a displaced beam at the centre of the vacuum chamber. Carrying out the calculations we find for the potential

\[
\Phi(o,0) = \frac{e}{4\pi\epsilon_0} \left\{ \begin{array}{l}
12 \text{ gp} \\
+ g p \ln \frac{a^6}{[(d_x + g)^2 + (d_y + p)^2] [(d_x - g)^2 + (d_y - p)^2] [(d_x + g)^2 + (d_y + p)^2] [(d_x - g)^2 + (d_y - p)^2]}
\end{array} \right.
\]

\[+ d_x d_y \ln \frac{[(d_x + g)^2 + (d_y - p)^2] [(d_x - g)^2 + (d_y - p)^2]}{[(d_x + g)^2 + (d_y + p)^2] [(d_x - g)^2 + (d_y + p)^2]} \]

\[+ g d_y \ln \frac{[(d_x + g)^2 + (d_y - p)^2] [(d_x - g)^2 + (d_y - p)^2]}{[(d_x + g)^2 + (d_y + p)^2] [(d_x - g)^2 + (d_y + p)^2]} \]

\[+ p d_x \ln \frac{[(d_x - g)^2 + (d_y + p)^2] [(d_x - g)^2 + (d_y - p)^2]}{[(d_x + g)^2 + (d_y - p)^2] [(d_x + g)^2 + (d_y + p)^2]} \]

\[-(d_x + g)^2 (\arctg \frac{y + p}{x + g} - \arctg \frac{y - p}{x + g}) + (d_x - g)^2 (\arctg \frac{y + p}{x - g} - \arctg \frac{y - p}{x - g}) \]

\[-(d_y + p)^2 (\arctg \frac{y + p}{x + p} - \arctg \frac{y - p}{x + p}) + (d_y - p)^2 (\arctg \frac{y + p}{x - p} - \arctg \frac{y - p}{x - p}) \}

Clearly, this expression is considerably more complicated than \( \Phi(o,0) \) for a coaxial beam given by Eq (139).
For the field components we find

\[
\begin{align*}
E_x(o,o) &= \frac{2\epsilon_0 \rho a_2}{\pi \epsilon_0} d_x - \frac{\rho}{4\pi \epsilon_0} \left[ \ln \frac{(d_y+p)^2 + (d_x+p)^2}{(d_y-g)^2 + (d_x+g)^2} \right. \\
&\quad + 2(d_x+g)(\arctg \frac{d_y+p}{d_x+g} - \arctg \frac{d_y-p}{d_x+g}) - 2(d_x-g)(\arctg \frac{d_y+p}{d_x-g} - \arctg \frac{d_y-p}{d_x-g}) \bigg] \\
\end{align*}
\]  
(165)

and

\[
\begin{align*}
E_y(o,o) &= \frac{2\epsilon_0 \rho a_2}{\pi \epsilon_0} d_y - \frac{\rho}{4\pi \epsilon_0} \left[ \ln \frac{(d_x+g)^2 + (d_y+p)^2}{(d_x-g)^2 + (d_y-p)^2} \right. \\
&\quad + 2(d_y+g)(\arctg \frac{d_x+p}{d_y+g} - \arctg \frac{d_x-g}{d_y+g}) - 2(d_y-g)(\arctg \frac{d_x+p}{d_y-g} - \arctg \frac{d_x-g}{d_y-g}) \bigg] \\
\end{align*}
\]  
(166)

For a coaxial beam these expressions reduce to zero.

The first term in Eq (165) resp. (166) is easily identified as being due to the virtual beam whereas the quantities in brackets represent the contributions of the isolated beam. We can therefore write in the case of a displaced beam

\[
\begin{align*}
E_x(o,o) &= \frac{2\epsilon_0 \rho a_2}{\pi \epsilon_0} d_x \\
E_y(o,o) &= \frac{2\epsilon_0 \rho a_2}{\pi \epsilon_0} d_y \\
\end{align*}
\]  
(167)

(168)

The image field at the origin is proportional to the displacement and to the ratio (area of the beam cross-section)/(area of the vacuum chamber cross-section).

Even in the case of a uniform distribution, the problem of the maximum potential produced by a displaced beam is best treated numerically using Eq (158). A program can easily be written to calculate the double integral for a set of points inside the beam (there
is no need to consider in this case the space situated between the beam envelope and the vacuum chamber as the maximum cannot be there) and to refine progressively the mesh in the neighbourhood of the first detected maximum.

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APPENDIX

Charge Density in the Virtual Beam

Let \( x_1, y_1 \) be the coordinates of a point inside the virtual beam. We assume that the Poisson equation for the virtual beam can be written in the form

\[
\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} = -\frac{\rho_1(x_1, y_1)}{\varepsilon_0}
\]  

(A1)

and attempt to establish a relation between the density \( \rho_1(x_1, y_1) \) in the virtual beam and the density \( \rho(x, y) \) in the real beam. To that effect we start from the Poisson equation for the real beam

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\rho(x, y)}{\varepsilon_0}
\]  

(A2)

and operate a transformation from the real space \( x, y \) to the image space \( x_1, y_1 \). The transformation formulae are

\[
x_1 = \frac{a}{x^2 + y^2} x \quad , \quad y_1 = \frac{a}{x^2 + y^2} y
\]  

(A3)

Expressed in the image space coordinates the Laplacian becomes

\[
\text{Lap} \phi = \frac{\partial^2 \phi}{\partial x_1^2} \left[ \frac{a x_1}{a x} \right]^2 \left( \frac{\partial x_1}{\partial x} \right)^2 + \left( \frac{\partial x_1}{\partial y} \right)^2 + \frac{\partial^2 \phi}{\partial y_1^2} \left[ \frac{a y_1}{a y} \right]^2 \left( \frac{\partial y_1}{\partial x} \right)^2 + \left( \frac{\partial y_1}{\partial y} \right)^2
\]  

(A4)

\[+2 \frac{\partial^2 \phi}{\partial x_1 \partial y_1} \left( \frac{a x_1}{a x} \frac{a y_1}{a y} + \frac{\partial x_1}{\partial x} \frac{\partial y_1}{\partial x} + \frac{\partial x_1}{\partial y} \frac{\partial y_1}{\partial y} \right) + \frac{\partial \phi}{\partial x_1} \left( \frac{\partial^2 x_1}{\partial x^2} + \frac{\partial^2 y_1}{\partial x^2} \right) + \frac{\partial \phi}{\partial y_1} \left( \frac{\partial^2 y_1}{\partial y^2} + \frac{\partial^2 x_1}{\partial y^2} \right)
\]

From the transformation formulae (A3) we can calculate

\[
\frac{\partial x_1}{\partial x} = a^2 \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad \frac{\partial y_1}{\partial y} = a^2 \frac{x^2 - y^2}{(x^2 + y^2)^2}
\]
\[
\frac{3x_1}{\partial y} = \frac{3y_1}{\partial x} = -2 \frac{a^2 xy}{(x^2 + y^2)^2} \quad (A5)
\]

\[
\frac{3^2x_1}{\partial x^2} = 2a^2 x \frac{x^2 - 3y^2}{(x^2 + y^2)^3}, \quad \frac{3^2x_1}{\partial y^2} = 2a^2 x \frac{3y^2 - x^2}{(x^2 + y^2)^3}
\]

\[
\frac{3^2y_1}{\partial y^2} = 2a^2 y \frac{y^2 - 3x^2}{(x^2 + y^2)^3}, \quad \frac{3^2y_1}{\partial x^2} = 2a^2 y \frac{3x^2 - y^2}{(x^2 + y^2)^3}
\]

Introducing these quantities in Eq (A4) we get

\[
\text{Lap} \Phi = \left( \frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial y_1^2} \right) \frac{a^4}{(x^2 + y^2)^2} \quad (A6)
\]

so that Eq (A2) becomes

\[
\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial y_1^2} = -\frac{(x^2 + y^2)^2}{a^4} \frac{\rho(x,y)}{\epsilon_0} \quad (A7)
\]

Eq (A7) is therefore justified if we take

\[
\rho_{1}(x_1, y_1) = \frac{(x^2 + y^2)^2}{a^4} \rho(x, y) \quad (A8)
\]

In the neighbourhood of the image point \(x_1, y_1\), transform of the real point \(x, y\), the density is given by Eq (A8). There is no angular dependence in the density transformation because the factor in front of \(\rho(x, y)\) is simply \((r/a)^4\) where \(r\) is the distance of the real point from the origin.

Eq (A8) can also be written in the form

\[
\rho_{1}(x_1, y_1) = \frac{a^4}{(x_1^2 + y_1^2)^2} \rho(x, y)
\]
Even if $\rho$ is uniform $\rho_1$ is highly non-linear. In neighbourhood of the origin $(r/a)^4$ or $(a/r)^4$ is very small whereas for $r = a = r^s$ the scaling factor is equal to one. The origin itself maps to infinity where the dilution is complete.
DEFINITION OF ANGLES AND DISTANCES

Fig. 4
\[ OS = r_o \quad OP = r \]
\[ OI_o = \frac{a^2}{r_o} \quad OI = \frac{a^2}{r} \]

Fig. 5
Fig. 6  COMPONENTS OF THE IMAGE FIELD ALONG THE SYMMETRY AXES
\[-\pi \epsilon_0 \sigma^2 \epsilon_{12}(z)/\rho t^2\]
Fig. 9
Fig. 10  DISPLACED BEAM