LINEAR COUPLING IN STORAGE RINGS

WITH RADIATING PARTICLES

by

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Abstract

This report shows how it is possible to extend the existing linear coupling theory to radiating particles. This analytical treatment gives the possibility to predict the equilibrium emittances in the presence of coupling and of vertical momentum dispersion. The application to electron storage rings with a separated function structure is treated and numerical values are given for the LEP machine.

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FIGURE
1. INTRODUCTION

In electron or positron rings of given magnetic structure, the transverse
emittances are governed by synchrotron radiations and by the coupling of the two
transverse motions. On the one side, there is a theory for calculating the
amplitude variations of radiating particles, on the other side, a theory of
the coupling has been elaborated for non radiating particles. These two
theories may be combined, as is shown in this report, in order to have an analyt-
cal description of linear coupling in the presence of synchrotron radiation.
This then gives the possibility to investigate what the respective importance of
these two mechanisms is and to give analytical expressions for the equilibrium
emittances.

The question of controlling the transverse emittances is indeed very
important for electron or positron rings. Some conditions are always prescribed
on the emittances in order to optimise the luminosity. On the one hand, the
emittance ratio $g$ should be adjusted at the design energy to equalize the tune
shifts in the two planes, i.e.

$$g = \frac{E_z}{E_x} = \frac{\beta_z^*}{\beta_x^*} = \frac{\sigma_z^*}{\sigma_x^*},$$

(1)

the luminosity then being given by the standard expression

$$L = \frac{f \, N^2}{4\pi \, k_b \, \sigma_x^* \, \sigma_y^*},$$

(2)

On the other hand, when the number of particles $N$ is either limited by the avai-
able RF power at energies above design energy, or restricted in a physics run
by the decay rate due to beam-beam bremsstrahlung, the emittance ratio $g$ should
be adjusted such that the vertical tune shift is at the limit. In this case,
the luminosity is given by

$$L = \frac{f \, \Delta Q_z^*}{2 \, r_e \, \beta_z^*} \, N_Y \, (1 + \frac{\sigma_z^*}{\sigma_x^*}),$$

(3)

and is proportional to $N_Y$ rather than $N^{2\gamma-1}$ (1). Hence, the reduction of
luminosity will follow $N$ rather than $N^\gamma$. The first numerical calculations for
LEP have shown that a gain in luminosity of a factor two could be achieved
at 5 GeV above the nominal energy, that the average luminosity in a physics run
might be some 5% higher and that the optimum time for a run could be 35% longer.

These results assume that g could be varied in a large range of values, with a lower limit as close as possible to zero. (In Ref. 3 the lower limit was taken equal to 0.01.) The present theory, which gives a complete picture of the effects on the emittances of the coupling and of the vertical momentum compaction, can be used for checking these assumptions.

2. AMPLITUDE VARIATION IN THE PRESENCE OF RADIATION

In this section, the main results of Pivinski's theory1) are summarized. Let us introduce a vector \( \mathbf{Y}(\theta) \) such as,

\[
\mathbf{Y}(\theta) = (x, p_x, z, p_z),
\]

where \( x, z, p_x, p_z \) are the transverse coordinates and their conjugate momenta. Let us then assume that the components of \( \mathbf{Y} \) may be written in the following form:

\[
\mathbf{Y}_i(\theta) = \sum_k w_{ik}(\theta) A_k,
\]

with

\[
w_{i2} = \bar{w}_{i1}, \quad w_{i4} = \bar{w}_{i3}.
\]

Consequently, the so-called amplitudes \( A_k \) are given by the relations

\[
A_k = \frac{W(\mathbf{Y}_1, \bar{w}_{ik})}{W(w_{ik}, \bar{w}_{ik})},
\]

provided that the quadratic function \( W(w_{ik}, \bar{w}_{ik}) \) is zero for \( w_{ij} \neq \bar{w}_{ik} \).

This function being defined by

\[
W(w_{ij}, \bar{w}_{ik}) = w_{1k} w_{2j} - w_{2j} w_{1k} + w_{3k} w_{4j} - w_{4j} w_{3k}.
\]

It is necessary to know the variation of \( A_k \) due to quantum fluctuations and to energy gain in the cavities, neglecting the diffusion in the residual gas. Owing to the relation (5), any variation of \( A_k \) can be expressed as follows:

\[
\delta A_k^2 = A_k \delta A_k + A_k \delta A_k + |\delta A_k|^2 = \frac{1}{W(w_{ik}, \bar{w}_{ik})} \left\{ A_k W(\delta Y_i, \bar{w}_{ik}) - A_k W(\delta Y_i, \bar{w}_{ik}) + W(\delta Y_i, \bar{w}_{ik}) W(\delta Y_i, \bar{w}_{ik}) \right\}.
\]
Considering firstly the amplitude variation due to photon emission, it is well known\(^4\) that only the equilibrium orbit of the emitting particle is changed by a quantity which is proportional to the photon energy \(e\) and to the momentum compaction vector \(D[D = (D_x, D_x', D_z, D_z')]\).

Hence,
\[
\delta Y = \frac{e D}{E_s},
\]
where \(E_s\) is the nominal energy of the particle. Introducing the variation \(\delta Y\) (8) in the expression (7) and averaging on time gives the amplitude variation due to quantum fluctuations,
\[
\langle \delta |A_k|^2 \rangle = - \frac{P_Y \Delta t 2i |A_k|^2}{E_s W(w_{ik}, \bar{w}_{ik})} < \text{Im} \{(C_{\bar{x}w_{ik}} + C_{\bar{z}w_{ik}})(D_{xw_{ik}} - D'_{xw_{ik}} + + D_{zw_{ik}} - D'_{zw_{ik}})\} > - \frac{Q_E \Delta t}{E_s^2 W^2(w_{ik}, \bar{w}_{ik})} < |D_{xw_{ik}} - D'_{xw_{ik}} + + D_{zw_{ik}} - D'_{zw_{ik}}|^2.\]

\(P_Y\) stands for the rate of loss of energy by radiation, whereas \(Q_E\) represents the mean value over one revolution of the product of the mean square quantum energy with the mean emission rate. They may be written\(^4\)
\[
P_Y = \frac{2}{3} \frac{r e c}{(mc^2)^3} \frac{E^4}{\rho^2},
\]
\[
Q_E = \frac{1}{\rho^3} \frac{55}{16\sqrt{3}} h c^3 \frac{1}{\gamma} \frac{1}{1/\rho^3}.\]

The definition of the coefficients \(C_y\) (where \(B^2\) stands for \(B_x^2 + B_z^2\)) is
\[
C_x = \frac{1}{p_x} + \frac{2}{B^2} \left( \frac{3B_z}{3x} - B_x \frac{2B_z}{3z} \right),
\]
\[
C_z = \frac{1}{p_z} + \frac{2}{B^2} \left( \frac{3B_x}{3x} + B_z \frac{2B_x}{3z} \right).
\]

Considering secondly the amplitude variation due to longitudinal acceleration in a cavity, the change of the position vector \(Y\) is given by
\[
\delta x = \delta z = 0, \quad \delta p_x = - \frac{\delta E}{E_s} p_x, \quad \delta p_z = - \frac{\delta E}{E_s} p_z,
\]
which means that the average of the amplitude variation (7) becomes

\[ <\delta |A_k|^2> = - \frac{P_Y \Delta t}{E_s} |A_k|^2. \]  

(13)

The total variation of the amplitudes \( A_k \) can be deduced from the two relations (9) and (13), and may be written as follows:

\[ \frac{d |A_k|^2}{dt} = - 2\alpha_k |A_k|^2 + \gamma_k, \]  

(14)

where the damping coefficients \( \alpha_k \) are

\[ \alpha_k = \frac{1}{2} \frac{P_Y}{E_s} J_k, \]

\[ J_k = 1 + \frac{\text{Im} \left\{ (C \ddot{w}_{1k} + C \ddot{w}_{2k}) (D \ddot{w}_{1k} - D' \ddot{w}_{1k} + D \ddot{w}_{2k} - D' \ddot{w}_{2k}) \right\}}{\text{Im}(\omega_{1k} \ddot{w}_{2k} + \ddot{w}_{1k} \ddot{w}_{2k})} ; \]

(15)

and the radiation excitation coefficients are

\[ \gamma_k = \frac{Q_e}{4 \pi^2} \frac{\left< \frac{|D \ddot{w}_{1k} - D' \ddot{w}_{1k} + D \ddot{w}_{2k} - D' \ddot{w}_{2k}|^2}{\text{Im}(\omega_{1k} \ddot{w}_{2k} + \ddot{w}_{1k} \ddot{w}_{2k})} \right>}{\text{Im}(\omega_{1k} \ddot{w}_{2k} + \ddot{w}_{1k} \ddot{w}_{2k})}. \]

(16)

Stationary conditions for the amplitude evolution are obtained when the derivatives (14) are zero. Hence the equilibrium amplitudes become

\[ |A_k|^2 = \frac{\gamma_k}{2\alpha_k}. \]  

(17)

Equations (5), (15), (16) and (17) give the equilibrium characteristics of the transverse motions. It is now interesting to relate them to the commonly used emittances. Such relations come directly from Eq. (5)

\[ E_y = \frac{\sigma_y}{\beta_y} = \frac{2}{\beta_y} \sum_{k=1,3} |A_k|^2 |w_{1k}|^2, \]

(18)

when using the properties of the functions \( w_{1k} \).

Finally, the parameter which is commonly used for electron machines is the ratio of the vertical emittance to the horizontal one, i.e.

\[ g = \frac{E_z}{E_x} = \frac{\beta_x}{\beta_z} \sum_{k=1,3} |A_k|^2 |w_{1k}|^2 / |w_{3k}|^2. \]

(19)
3. **LINEAR COUPLING AND SYNCHROTRON RADIATION**

As mentioned in the introduction, the theory of linear coupling in a three-dimensional magnetic field already exists\textsuperscript{2}. In this section it will be shown how the results of this theory can be used without major changes for predicting the effects of linear coupling in the presence of synchrotron radiation.

Let us first show that the canonical variables (4), which are used in Ref. 2, have the right form (5). This indeed comes directly from the equations (1.4.2) in Ref. 2, where \( a_1 \) and \( a_2 \) are linear combinations of oscillating components. These equations also indicate that the following relations are satisfied:

\[
\begin{align*}
\omega_{ik} (\theta + 2\pi) &= \omega_{ik} (\theta) e^{i2\pi \lambda_{ik}}, \\
\lambda_{i1} &= -\lambda_{i2}, \quad \lambda_{i3} = -\lambda_{i4}, \\
\omega_{i1} &= \omega_{i2}, \quad \omega_{i3} = \omega_{i4}.
\end{align*}
\]

The relations (1.4.4) in Ref. 2 can then be used for demonstrating that the quantity \( W \) defined after Eq (6) is an invariant of the motion. It then follows from this and from Eqs(20) that

\[
W(\omega_{ik} (\theta), \omega_{ij} (\theta)) = W(\omega_{ik} (\theta + 2\pi), \omega_{ij} (\theta + 2\pi)) =
\]

\[
e^{i2\pi (\lambda_{ik} + \lambda_{ij})} W(\omega_{ik} (\theta), \omega_{ij} (\theta)).
\]

Equating the two extreme quantities in Eqs(21) and taking into account the relations (20) for \( \lambda_{ik} \) and \( \omega_{ik} \) proves that the following is true

\[
W(\omega_{ik}, \omega_{ij}) = 0, \quad \text{for} \quad \omega_{ij} \neq \omega_{ik}.
\]

It is now clear that the solution of the perturbed motion which is stated in Ref. 2 satisfies the conditions assumed at the beginning of Section 2.

Following exactly the analysis made in the absence of radiation\textsuperscript{2}, it may be assumed that only the low frequency part of the Hamiltonian gives the important variations of the amplitudes. For this assumption it is necessary to define which specific resonance will be looked at. Since the working point of a storage ring is usually chosen close to a difference resonance, the following case will be considered from now on

\[
Q_x - Q_z = p.
\]
The explicit solution for this resonance is given by the equations (1.5.17) together with Eq (1.4.2) of Ref. 2. From these results, it is possible to deduce the eigenfunctions \( w_{ik}(\theta) \) associated with an isolated difference resonance

\[
\begin{align*}
\omega_1 &= \frac{k}{\omega} \sqrt{\frac{\beta X}{2R}} \exp \left[ i \left( \mu_x + \omega \theta \right) \right], \\
\omega_2 &= \frac{k}{\omega} \sqrt{\frac{R}{2\beta X}} \left( \frac{\beta Z}{2R} + i \right) \exp \left[ i \left( \mu_x + \omega \theta \right) \right], \\
\omega_3 &= \sqrt{\frac{R}{2\beta Z}} \exp \left[ i \left( \mu_z - \omega \theta \right) \right], \\
\omega_4 &= \sqrt{\frac{R}{2\beta Z}} \left( \frac{\beta Z}{2R} + i \right) \exp \left[ i \left( \mu_z - \omega \theta \right) \right], \\
\omega_5 &= \frac{k}{\omega} \sqrt{\frac{\beta X}{2R}} \exp \left[ i \left( \mu_x + \omega \theta \right) \right], \\
\omega_6 &= \frac{k}{\omega} \sqrt{\frac{R}{2\beta X}} \left( \frac{\beta Z}{2R} + i \right) \exp \left[ i \left( \mu_x + \omega \theta \right) \right], \\
\omega_7 &= \sqrt{\frac{R}{2\beta Z}} \exp \left[ i \left( \mu_z - \omega \theta \right) \right], \\
\omega_8 &= \sqrt{\frac{R}{2\beta Z}} \left( \frac{\beta Z}{2R} + i \right) \exp \left[ i \left( \mu_z - \omega \theta \right) \right].
\end{align*}
\]

(24)

where \( \beta' \) means the derivative of \( \beta \) w.r.t. \( \theta \). The coefficients \( \omega_\pm \) and \( \kappa \) are given in Ref. 2, i.e.

\[
\omega_\pm = -\frac{\Delta}{2} \pm \sqrt{\left( \frac{\Delta}{2} \right)^2 + |\kappa|^2} \quad \text{with} \quad \Delta = Q_x - Q_z - p,
\]

\[
\kappa = \frac{R}{8\pi B \rho} \int_0^{2\pi} \sqrt{\beta X \beta Z} \left[ \left( \frac{-3B}{\beta_x} - \frac{3B}{\beta_z} \right) + \frac{B \theta}{2R} \left( -\frac{\beta_x'}{\beta_x} - \frac{\beta_z'}{\beta_z} \right) - iB \left( \frac{1}{\beta_x'} + \frac{1}{\beta_z'} \right) \right] \exp \left\{ i \left( \mu_x - \mu_z - (Q_x - Q_z - p) \theta \right) \right\} d\theta.
\]

Note that the amplitudes \( A_k \) (k=1,2,3,4) associated with the functions \( w_{ik} \) (24) are nothing but the coefficients \( A_+, A_-, A_+ \) and \( A_- \) of Ref. 2.

In order to have the equilibrium amplitudes (17) for coupled motions in the presence of radiation, it is necessary to calculate the damping and radiation excitation coefficients by following Section 2 and using the relations (24).
By virtue of (18), only the coefficients having the indices 1 and 3 are relevant. Since combined function structures do not appear very suitable for large storage rings, let us consider a separated function machine here. In this case, the "normalized" damping coefficients $J_k$ ($k = 1, 3$) are very close to one, since only the dipoles where $\rho$ is finite but $\partial B_z/\partial y$ is zero, contribute to the second term of $J_k$ (15) and since this contribution, proportional to $<D>\partial y$ is then negligible. Hence,

$$\alpha_3 = \frac{1}{2} \frac{P}{E_s}.$$

(26)

The next task consists of calculating $\gamma_1$ and $\gamma_3$ by introducing Eqs (24) into Eq (16). These two coefficients may be rewritten

$$\gamma_1 = <\frac{E_s}{4E_s^2} \sqrt{\beta_x}} >,$$

(27)

where $H_k$ and $F_k$ are given by comparing (27) with (16). The detailed algebrae give for these coefficients

$$F_3 = \frac{1}{4} (1 + \frac{|\kappa|^2}{\omega_x^2}),$$

$$H_3 = \frac{|\kappa|^2}{\omega_x^2} \frac{R}{2\beta_x} \left[ D_x^2 + \frac{1}{R^2} \left( \frac{\beta_x^2}{R} - \frac{1}{2} \frac{\beta_x^2}{R} D_x \right)^2 \right] + \frac{R}{2\beta_z} \left[ D_z^2 + \frac{1}{R^2} \left( \frac{\beta_z^2}{R} - \frac{1}{2} \frac{\beta_z^2}{R} D_z \right)^2 \right]

+ \frac{R}{\omega_x} \sqrt{\frac{\beta_x^2}{R}} \Re \left\{ \frac{1}{\kappa} \left( \frac{\beta_x^2}{R} + \frac{i}{\omega_x} \right) \exp \left\{ i(\mu_x - \mu_z - \omega \Delta) \right\} \right\} D_x \bar{D}_x

- \frac{1}{\omega_x} \sqrt{\frac{\beta_x^2}{R}} \Re \left\{\frac{\beta_x^2}{R} + \frac{i}{\omega_x} \exp \left\{ i(\mu_x - \mu_z - \omega \Delta) \right\} \right\} D_x^2$$

$$+ \frac{1}{\omega_x} \sqrt{\frac{\beta_x^2}{R}} \Re \left\{\frac{\beta_x^2}{R} \exp \left\{ i(\mu_x - \mu_z - \omega \Delta) \right\} \right\} D_x D_x^2.$$

(28)

When the coupling coefficient $\kappa$ vanishes, the following quantities remain finite:

$$\frac{H_3}{F_3} = 0$$

$$\frac{|\kappa|^2}{\omega_x^2} \frac{H_3}{F_3} = \frac{2R}{\beta_x} \left[ D_x^2 + \frac{1}{R^2} \left( \frac{\beta_x^2}{R} \frac{1}{R^2} \right)^2 \right],$$

(29)
which is equivalent to say that in the absence of coupling the equilibrium emittances are defined by the momentum compaction functions. If on top of this, the vertical compaction is identical to zero, Eqs (29) show that the vertical emittance tends to zero, as is well known. When the vertical compaction vanishes, the quantities $F$ and $H$ are

$$F_3 = \frac{1}{4} \left(1 + \frac{|\kappa|^2}{\omega^2}\right),$$

$$H_3 = \frac{|\kappa|^2 R}{2 \beta} \left[D^2 + \frac{1}{R^2} \left(\beta \frac{\partial D'}{\partial x} - \frac{1}{2} \frac{\partial^2 D}{\partial x^2}\right)^2\right],$$

which is equivalent to say that the ratio of the two transverse emittances depends upon the quantity $|\kappa|/|\omega^2|$ (if $|\kappa|/|\omega^2| = 1$ the two emittances will be equal).

The relations (28) to (30) clearly indicate which terms contributing to the transverse emittances are due either to coupling resonance or to unwanted vertical dispersion.

Putting together the relations (17) to (19) and (26) to (28) and introducing in the Eqs (18) and (19) the expressions (24) of the eigenfunctions $\psi_{ik}$, gives simultaneously the horizontal emittance $E_x$ and the transverse emittance ratio $g$ in the presence of linear coupling and synchrotron radiation.

$$E_x = \frac{Q_E}{4 RE \rho} \left[\frac{|\kappa|^2}{\omega^2} \left\langle \frac{H_1}{F_1} + \frac{|\kappa|^2}{\omega^2} \left\langle \frac{H_3}{F_3}\right.\right\rangle\right],$$

$$g = \frac{E_z}{E_x} = \frac{|\kappa|^2}{\omega^2} \left\langle \frac{H_1}{F_1} + \frac{|\kappa|^2}{\omega^2} \left\langle \frac{H_3}{F_3}\right.\right\rangle,$$

where $F_k$ and $H_k$ are given in Eqs (28), and $\omega^2$ and $\kappa$ in Eqs (25).

4. APPLICATION TO ELECTRON STORAGE RINGS

The present theory of linear coupling in the presence of radiation indicates how the transverse emittances depend on the coupling coefficient $\kappa$ and on the transverse momentum compaction functions $D_x$ (6), while with similar conditions in a proton machine, the emittances only depend on $\kappa^2$.

The analytical expressions (31) give the possibility of calculating the transverse emittances for radiating particles at the equilibrium and show that the vertical emittance may only vanish if the coupling and the vertical momentum compaction are compensated.
Linear coupling comes from possible solenoids in the interaction regions, from the quadrupole tilts and from vertical orbit distortions in the sextupoles for chromaticity correction. Vertical momentum compaction arises from the two last causes only, since the horizontal momentum compaction vanishes in the interaction regions (assuming a machine with corrected chromaticity and neglecting the residual effects of the local chromaticity). Seeing that the solenoids are usually compensated locally\(^6\) where \(D_x = 0\) and that the effects of tilted quadrupoles are an order of magnitude smaller than the sextupole effects, it is interesting to apply the preceding theory to the sole perturbations due to the sextupoles, assuming a reasonable vertical closed orbit distortion.

It is easy to dissociate the effects of the vertical compaction \(D_z\) from the linear coupling \(k\) by calculating the emittance ratio \(g\) in the extreme conditions where one of these parameters is zero. In this case, the relations (29) and (30) apply and it is sufficient to calculate the average of the momentum compaction invariants \(I_y\)

\[
I_y = \frac{1}{\beta_y} \left[ D_y^2 + \frac{1}{R^2} \left( \beta_y D_y - \frac{1}{2} \beta_y' D_y' \right)^2 \right],
\]

(32)

Let us first consider the horizontal invariant \(I_x\). The horizontal momentum compaction is defined by the cell structure and the average of \(I_x\) has to be taken over the part of the circumference where \(\rho\) is finite, i.e. inside the bending magnets. Given \(D_x, D_x', \beta_x, \beta_x'\) at the entrance of a magnet, the average value of \(I_x\) may be written \(^7\)

\[
\langle I_x \rangle = \frac{1}{\gamma_x} \left\{ D_x^2 - \frac{\beta_x'}{R^2} D_x D_x' + \frac{\beta_x}{R^2} D_x'^2 \right. \\
- \left. (2\gamma_x D_x - \frac{\beta_x'}{R^2} D_x') \frac{\rho}{\phi} - \sin \phi + \left( 2 - \frac{\beta_x}{R} D_x' - \frac{\beta_x'}{R} D_x \right) \frac{1 - \cos \phi}{\phi} \right\}
\]

(33)

where \(\rho\) is the constant bending radius and \(\phi\) the bending angle of the magnet.

The definition of \(\gamma_x\) is simply

\[
\gamma_x = \frac{1}{\beta_x} \left( 1 + \frac{\beta_x^2}{4R^2} \right).
\]
Let us then find an analytical expression for the average vertical invariant $I_z$. This average may be approximated by

$$<I_z> = \frac{1}{2\pi} \int_0^{2\pi} (\gamma_\frac{D^2}{z} - \frac{\beta}{z} \frac{D^1}{z} + \frac{\beta}{z} \frac{D^{1^2}}{z}) \, d\theta,$$  \hspace{1cm} (34)

where the definition of $\gamma_z$ is similar to the one given above for $\gamma_x$.

In order to simplify the calculations, let us replace the exact betatron amplitude $\bar{\beta}_Z(\theta)$ by a constant equal to its average $\bar{\beta}_z$. In this case Eq (34) becomes

$$<I_z> = \frac{1}{2\pi} \int_0^{2\pi} \frac{D^2}{\bar{\beta}_z} + \frac{\beta}{R^2} \frac{D^{1^2}}{z} \, d\theta.$$ \hspace{1cm} (35)

If the vertical momentum compaction is analysed in Fourier's series

$$D_z = \sum_p D_p \cos (p\theta + \delta_p),$$ \hspace{1cm} (36)

the average vertical invariant (35) may be written

$$<I_z> = \sum_p \frac{D^2_p}{\bar{\beta}_z^2} \left[ 1 + \left( \frac{\bar{\beta}_z}{R} \right)^2 \right].$$ \hspace{1cm} (37)

Since we want to have the $D_z$ which arises from vertical orbit distortions in the sextupoles, it may be calculated as a closed orbit by using the formula 2.4 of Ref. 8 in which $G_z$ is replaced by $z_0^2 \frac{\partial^2 B_z}{\partial x^2}$ ($z_0$ being the vertical orbit deviation in the sextupoles). $D_z$ may then be analysed in Fourier's series, as a closed orbit (formula 2.2 of Ref. 8), and the harmonic coefficients $D_p$ can be written

$$D_p = \frac{Q^2_z}{Q^2_z - p^2} \frac{\beta_z^2}{\pi} \int_0^{2\pi} \frac{\partial^2 B_z}{\partial x^2} \frac{z_0}{|B_p|} D_x \cos (p\theta + \delta_p) \, d\theta.$$ \hspace{1cm} (38)

If we consider that the sextupoles are short and well localized and that the vertical orbit deviations $z_0$ in the sextupoles are randomly distributed, the integral (38) becomes

$$D_p = \frac{\ell \bar{\beta}_z^2}{2\pi R |B_p|} \frac{Q^2_z}{Q^2_z - p^2} \sqrt{\sum_{\text{syst,}} \left( \frac{\partial^2 B_z}{\partial x^2} \right)^2 x z_0^2},$$ \hspace{1cm} (39)

where $\ell$ is the sextupole length and $z_0^2$ the vertical orbit standard deviation.
Introducing the coefficients (39) in the Eq (37) gives the average vertical invariant due to localised sextupoles

\[
\langle I_x \rangle = \frac{\xi^2}{8\pi^2} \frac{\bar{\beta}_e^3}{R^2} \frac{z^2}{\beta_0^2} \sum_{\text{sext.}} \frac{\partial^2 \bar{z}}{\partial x^2} J_x \sum_p \frac{Q_z^2}{Q_x^2 - p^2} \left[ 1 + \left( \frac{p^2}{R^2} \right)^2 \right]. \tag{40}
\]

When the closed orbit is not corrected, Eq. (40) gives the dominant contribution to \( \langle I_x \rangle \). For a well corrected orbit, however, the residual effects of the local chromaticity may become comparable to the sextupole effects and \( \langle I_x \rangle \) may then be \( \sim \sqrt{2} \) times larger than given by Eq. (40).

With the expressions (33) and (40), the horizontal emittance and the emittance ratio (31) can be calculated in the conditions where either \( \kappa \) or \( D_z \) is zero.

i) \( \kappa \equiv 0, \ D_z \equiv 0 \),

\[
E_{x,0} = \frac{Q_e}{2E_p} \frac{I_x}{\langle I_x \rangle},
\]

\[
\frac{g}{\langle I_x \rangle} = \frac{\langle I_z \rangle}{\langle I_x \rangle}. \tag{41}
\]

ii) \( \kappa \neq 0, \ D_z \equiv 0 \),

\[
E_x = E_{x,0} \frac{2(|\kappa|_\Delta)^2 + 1}{4(|\kappa|_\Delta)^2 + 1},
\]

\[
g = \frac{(|\kappa|_\Delta)^2}{(|\kappa|_\Delta)^2 + 1}. \tag{42}
\]

where \( \Delta = Q_x - Q_z - p \) is the distance from the resonance and \( E_{x,0} \) is the horizontal emittance (41) in the absence of coupling.

In the very special case where both \( \kappa \) and \( D_z \) vanish, the expressions (41) and (42) give the same results, i.e. the emittance \( E_x \) is equal to \( E_{x,0} \) and the emittance ratio \( g \) goes to zero, as expected. Other extreme conditions are reached when \( D_z \equiv 0 \) and \( \kappa \) is much larger than the distance \( \Delta \) from the coupling resonance. In this case, the emittance \( E_x \) tends to \( E_{x,0}/2 \) and the ratio \( g \) becomes equal to 1, i.e. the two emittances become equal to half the horizontal emittance without coupling. The complete variation of \( E_x \) and \( g \) with the ratio \( \kappa/\Delta \) in the absence of vertical dispersion is given in Fig. 1.
In general, both $\kappa$ and $D_z$ are different from zero and the exact expressions (28) and (31) have to be used. Nevertheless, assuming that the emittance ratio is adjusted by linear coupling, the vertical dispersion being maintained constant, the second relation (41) gives the lowest attainable value of $g$ while the second relation (42) gives the order of magnitude of $\kappa/\Delta$ which is necessary to obtain a larger value of $g$. If the lowest value of $g$ (41) due to the residual vertical momentum compaction after closed orbit correction is small enough, the luminosity may be optimised by the sole adjustment of the coefficient $\kappa$. In the opposite case, the vertical dispersion has to be corrected at first.

5. **NUMERICAL APPLICATION TO LEP**

In the LEP-version $^9$, the tune values are $Q_x = 66.208$ and $Q_z = 74.272$, so that the predominant coupling resonance (23) is

$$Q_x - Q_z = -8,$$

(43)

with $\Delta = Q_x - Q_z - p = -0.064$. These conditions agree with the assumptions made, since $\Delta$ is small for this difference resonance while it is equal to 0.48 for the next, closest resonance.

Let us now calculate for this machine what is the lowest value of $g$ due to the residual vertical momentum compaction after closed orbit correction. Following chapter 3, let us calculate $<I_x>$ and $<I_z>$. The average horizontal invariant is given by Eq (33) and using the LEP parameters of a normal cell $^9$ gives

$$<I_x> = 2.4 \times 10^{-2}.$$  

(44)

The horizontal emittance in the absence of coupling (41) then becomes at 70 GeV/c

$$E_{x,0} = \frac{55}{32} \frac{hc\gamma^2}{\sqrt{s} mc^2} <I_x> = 7.37 \times 10^{-8} \text{ rad m},$$

(45)

by virtue of the Eqs (10). This value agrees within 8% with the value given by the PETROS program $^{10}$. The vertical invariant is given by Eq (40) and using the LEP parameters associated with the chromaticity correction gives

$$<I_z> = 24.2 <z_o^2>.$$  

(46)
If it is assumed that the variance of the corrected closed orbit is of the order of $6 \times 10^{-6}$ m$^2$ and if we introduce the factor $\sqrt{2}$ mentioned in the paragraph following Eq. 40, the vertical invariant becomes

$$<I_z> = 2.05 \times 10^{-4},$$

which gives the emittance ratio (41)

$$g = 0.85 \%.$$  

Taking into account the uncertainty of this result due to approximations, it is possible to conclude that a careful compensation of the most important coupling resonance should permit to reduce the emittance ratio up to $\sim 1 \%$, once the vertical orbit is corrected.

This lower boundary (48) for $g$ is well below the optimum value at the design energy, i.e. 6.25 %. Hence, it seems possible to vary $g$ in a large range of values by controlling the linear coupling $\kappa$, in order to optimize the luminosity above the nominal energy.

This possible control of $\kappa$ should maintain constant the vertical momentum compaction $D_z$ and this implies the use of skew quadrupoles sitting in the straight sections where $D_x$ is zero. The skew quadrupoles, which are foreseen for the compensation of possible solenoids, are obviously suitable for this purpose, since they are effectively in such regions and they can easily produce orthogonal vectors $\kappa$ in the complex plane. These quadrupoles may be foreseen powerful enough to achieve $\kappa = 5\Delta$, which corresponds closely to a fully coupled beam ($g = 0.98$) and the necessary skew fields may advantageously be distributed between the interaction regions, which are all in phase for the coupling resonance (43). Hence, the optimisation of the LEP luminosity above the nominal energy seems to be achievable with a limited number (32 in the design of Ref. 6) of skew quadrupoles, which have to be present in any case for the solenoid compensation.
REFERENCES


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11. B. Zotter, Effects of perturbations on LEP, LEP note, LEP-70/100.
Figure 1: Horizontal emittance and emittance ratio as functions of $\kappa/\Delta$ with vanishing vertical dispersion.