Classical mechanics studies the motion of physical bodies at the macroscopic level. Newton and Galileo first laid its foundations in the 17th century. The essential physics of the mechanics of Newton and Galileo, known as Newtonian mechanics, is contained in the three laws of motion. Classical mechanics has since been reformulated in a few different forms: the Lagrange, the Hamilton, and the Hamilton-Jacobi formalisms. They are alternatives, but equivalent to the Newtonian mechanics. We will review Newtonian mechanics first and then Lagrangian dynamics.

A.1 Newtonian Mechanics

A.1.1 The Three Laws of Motion

The foundations of Newtonian mechanics are the three laws of motion that were postulated by Isaac Newton, resulting from a combination of experimental evidence and a great deal of intuition. The first law states:

A free particle continues in its state of rest or of uniform motion until an external force acts upon it.

Newton made the first law more precise by introducing the concepts of “quantity of motion” and “amount of matter” that we now call momentum, and mass of the particle, respectively. The momentum \( \vec{p} \) of a particle is related to its velocity \( \vec{v} \) and mass \( m \) by the relation

\[
\vec{p} = m \vec{v}
\]

The first law may now be expressed mathematically as

\[
\vec{p} = m \vec{v} = \text{constant}
\]

provided that there is no external force acting on the particle. Thus, the first law is a law of conservation of momentum.
The second law gives a specific way of determining how the motion of a particle is changed when a force acts on it. It may be stated as

*The time rate of change of momentum of the particle is equal to the external applied force:*

\[
\vec{F} = \frac{d\vec{p}}{dt}.
\]

The second law can also be written as, when mass \( m \) is constant, as

\[
\vec{F} = m\frac{d\vec{v}}{dt},
\]

which says that the force acting on a particle (of constant mass) is equal to the product of its acceleration and its mass, a familiar result from basic physics.

Mass that appears in Newton’s laws of motion is called the inertial mass of the particle, as it is indicative of the resistance offered by the particle to a change of its velocity.

The third law states that

*The force of action and reaction are equal in magnitude but opposite in direction.*

Physical laws are usually stated relative to some reference frame. Although reference frame can be chosen arbitrarily in an infinite number of ways, the description of motion in different frames will, in general, be different. There are frames of reference relative to which all bodies that do not interact with other bodies move uniformly in a straight line. Frames of reference satisfying this condition are called inertial frames of reference. It is evident that the three laws of motion apply in inertial frames. In fact, some scientists suggest that Newton singled out the first law just for the purpose of defining inertial frames of reference. But the true significance of Newton’s first law is that, in contrast to the view held by his science contemporaries or those before him, the state of a body at rest is equal to that of a body in uniform motion (with a constant velocity in a straight line).

Then how is it possible to determine whether or not a given frame of reference is an inertial frame? The answer is not quite as trivial as it might seem at first sight, for in order to eliminate all forces on a body, it would be necessary to isolate the body completely. In principle this is impossible because, unless the body were removed infinitely far away from all other matter, there are at least some gravitational forces acting on it. Therefore, in practice we merely specify an approximate inertial frame in accordance with the needs of the problem under investigation. For example, for elementary applications in the laboratory, a frame attached to Earth usually suffices. This frame is, of course, an approximate inertial frame, owing to the daily rotation of Earth on its axis and its revolution around the sun. It is also a basic assumption of classical mechanics that space is continuous and the geometry of space is Euclidean, and that time is absolute. The assumptions of absolute time and of the geometry of space have been modified by Einstein’s theory of relativity, which we will discuss in the following sections.
A.1.2 The Galilean Transformation

Any frame moving at constant velocity with respect to an inertial frame is also an inertial frame. To show this, let us consider two frames, S and S', which coincide at time \( t = t' = 0 \), as shown in Figure A.1. S' moves with a constant velocity \( \mathbf{V} \) relative to S. The corresponding axes of S and S' remain parallel throughout the motion. It is assumed that the same units of distance and time are adopted in both frames. An inspection of Figure A.1 gives

\[
\mathbf{r}' = \mathbf{r} - \mathbf{V} t \quad \text{and} \quad t' = t
\]

(A1-1)

The relations (A1-1) are called the Galilean transformations. The second relation expresses the universality, or absoluteness, of time. We shall see later that the Galilean transformations are only valid for velocities that are small compared with that of light.

Differentiating (A1-1) once, we obtain

\[
\dot{\mathbf{r}}' = \dot{\mathbf{r}} - \mathbf{V},
\]

(A1-2)

which says that if the velocity of a particle in frame S is a constant, then its velocity in frame S' is also a constant. Differentiating the velocity relation (A1-2) once with respect to time, we obtain

\[
\ddot{\mathbf{r}}' = \ddot{\mathbf{r}},
\]

(A1-3)

which indicates that the acceleration of the particle is also a Galilean invariant.

If frame S in Figure A.1 is inertial, so is S', since the linear equations of motion of free particles in frame S are transformed by (A1-1) into similar linear equations in S'. Any frame that moves uniformly (i.e., with constant velocity and without rotation) relative to any inertial frame is also itself an inertial frame. And there is an infinity of inertial frames, all connected by Galilean transformations.

A.1.3 Newtonian Relativity and Newton’s Absolute Space

The inertial mass \( m \) is also a Galilean invariant, so

\[
m \ddot{\mathbf{r}}' = m \ddot{\mathbf{r}},
\]
that is, the $m\ddot{a}$ term of Newton’s second law is a Galilean invariant. The applied force on the particle is also invariant under Galilean transformations, provided that it is velocity-independent. That is, if the applied force is velocity-independent, the form of Newton’s second law is preserved under Galilean transformations. Thus, not only Newton’s first law but also his second and third laws are valid in all inertial frames. This property of classical mechanics is often referred to as Newtonian (or Galilean) relativity.

Because of this relativity, the uniform motion of one inertial frame relative to another cannot be detected by internal mechanical experiments of Newton’s theory. According to the first law, a particle does not resist uniform motion, of whatever speed, but it does resist any change in its velocity, i.e., acceleration. Newton’s second law precisely expresses this, and mass $m$ is a measure of the particle’s inertia. Here we should ask, “Acceleration with respect to what?” One may give a simple answer: with respect to any one of the inertial frames. However, this answer is quite unsatisfactory. Why does nature single out such “preferred” frames as standards of acceleration? Inertial frames are unaccelerated and nonrotating, but relative to what? Newton found no answer and postulated instead the existence of an absolute space, which “exists in itself, as if it were a substance, with basic properties and quantities that are not dependent on its relationship to anything else whatsoever (i.e., the matter that is in this space).” Newton’s concept of an absolute space has never lacked critics. From Huyghens, Leibniz, and Bishop Berkeley to Ernst Mach and Einstein, these objections have been brought against absolute space:

(1) It is purely *ad hoc* and explains nothing.
(2) How are we to identify which inertial frame is at rest relative to absolute space?
(3) Newton’s absolute space is a physical entity; thus it acts on matter (it is the “seat of inertia” resisting acceleration in the absence of forces). But matter does not act on it. As Einstein said: “It conflicts with one’s scientific understanding to conceive of a thing (absolute space) that acts, but cannot be acted upon.”

Objection (3) is perhaps the most powerful one. It questions not only absolute space but also the set of all inertial frames.

Newton’s theory can do without absolute space. Space can be regarded as a concept necessary for the ordering of material objects. From this viewpoint space is just a generalization of the concept of place assigned to matter. Matter comes first and the concept of space is secondary. Empty space has no meaning. The ancient Greeks, led by the great philosopher Aristotle, thought about space in this fashion. If we take this view of space, then the space is not absolute. As we saw above there is an infinity of inertial frames, connected by Galilean transformations. One inertial frame of reference does not in any way differ from the others. A reference frame attached to Earth’s surface can be considered as an inertial frame, and a train moving with a constant velocity with respect to Earth is also an inertial frame; the laws of motion would hold inside the train. If a rubber ball on the train bounces straight up and down, it hits the floor twice on the same spot a certain time apart, say one second. However, to someone standing by the rails, the two bounces would take place a certain distance apart. Thus we cannot give an event an absolute position in space. The lack of absolute position means that space is not absolute.
A.1.4 Newton’s Law of Gravity

Historically, Newton began with his laws of motion and Kepler’s laws of planetary motion to arrive at his law of gravity. Kepler discovered three laws that planets obey as they move around the sun. These laws were based on Tycho Brahe’s observational data on Mars’ motion around the sun:

*The first law:* Each planet revolves around the sun in an elliptical orbit, with the sun at one focus of the ellipse.

*The second law:* The speed of a planet in its orbit varies in such a way that the radius connecting the planet and the sun sweeps over equal areas in equal time intervals. Figure A.2 illustrates the second law. Each planet moves fastest when it is nearest the sun, and slowest when it is farthest away.

*The third law:* The ratio between the square of a planet’s period of revolution $T$ and the cube of the major axis $a$ of its orbit has the same value for all the planets,

\[ T^2 / a^3 = K \]

where $K$ is a constant, the same for all the planets.

For the derivation of the law of gravity from these laws, see my text, *Classical Mechanics*, or any other textbook on classical mechanics. Here is a simplified version: As the orbits of the planets are very near to being circles, we assume for simplicity that the planets move in circular orbits around the sun. Now, a planet of mass $m$ circulating the sun at a radius $r$ with a velocity $v$ is acted upon by the centripetal force

\[ F_c = m v^2 / r. \]

The distance a planet travels in one revolution is the circumference of the circle, $2\pi r$. Thus we have

\[ 2\pi r = v T, \quad \text{or} \quad T = 2\pi r / v \]

where $T$ is the period of revolution; and Kepler’s third law becomes

\[ \frac{T^2}{r^3} = \frac{4\pi^2}{r v^2} = K,\]

or

\[ v^2 = \frac{4\pi^2}{K r}. \]

![Fig. A.2 Kepler’s second law.](image-url)
Substituting this into $F_c$ we obtain

$$F_c = 4\pi^2 m/Kr^2.$$  

The centripetal force $F_c$ is provided by the gravitational force of the sun exerted on the planet. Thus,

$$F_g = 4\pi^2 m/Kr^2.$$  

Now, Newton’s third law requires that the force a planet exerts on the sun be equal in magnitude to the force the sun exerts on the planet. This means that $F_g$ must be proportional to both $m$ and the mass of the sun $M$, and we may express the constant quantity $4\pi^2/K$ as $GM$. Accordingly, the gravitational force between the sun and any planet is

$$F_g = GMm/r^2.$$  

The quantity $G$ is the gravitational constant, and its value is determined experimentally to be $6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$.

This is a remarkable result, providing an astonishingly simple and unified basis for the description of the motion of all planets around the sun: it is the gravitational force of the sun that keeps the planets moving in their orbits. Newton went further and made a gigantic extrapolation from his law of gravity by claiming its universality. This extrapolation was based on the argument that the apple, the moon, the planets, and the sun were made of ordinary matter so that they were in no way special. A similar force might well be expected to act between any two masses $m_1$ and $m_2$ separated by a distance $r$ (Fig. A.3)

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}. \quad (A1-4)$$

According to (A1-4), gravitational forces act at a distance and are able to produce their effects through millions of miles of empty space. The law also tacitly implies that the gravitational influence propagates with infinite speed.

In the form of (A1-4) the law strictly applies only to point particles or objects that have spherical symmetry. If one or both of the objects has a certain extension, then we need to make additional assumptions before we can calculate the force. The most common assumption is that the gravitational force is linear. That is, the total gravitational force on a particle due to many other particles is the vector sum of all the individual forces. Thus, for example, if $m_1$ is a point particle of mass $m$ and $m_2$ is an extended object that has a continuous distribution of matter, then (A1-4) becomes

$$\vec{F} = -Gm \int_{V} \frac{\rho(\vec{r}')}{r^2} \, d\mathbf{v}' \quad (A1-4a)$$

where $\rho(\vec{r}')$ is the mass density and $d\mathbf{v}'$ is the element of volume at the position defined by the vector $\vec{r}'$, as shown in Figure A.4.

Fig. A.3 Gravitational force on mass $m_2$ due to mass $m_1$. 

\[ \text{Fig. A.3 Gravitational force on mass } m_2 \text{ due to mass } m_1. \]
A.1.5 Gravitational Mass and Inertial Mass

The mass of an object plays dual roles. The initial mass \( m_I \), which appears in Newton’s laws of motion, determines the acceleration of a body under the action of a given force. The mass entering in Newton’s law of gravity determines the gravitational forces between the object and other objects and is known as the object’s gravitational mass \( m_g \). The dual role played by mass has an astonishing consequence. If we drop a body near Earth’s surface, Newton’s second law of motion describes its motion

\[
F = m_I a
\]

where the force \( F \) acting on the object is its weight, the force of gravity of Earth

\[
F = G m_g M / r
\]

where \( m_g \) is the gravitational mass of the falling object, \( M \) is the mass of the earth, and \( r \) is the distance of the object from Earth’s center. Combining these two equations, and if \( m_I = m_g \), we then see that the acceleration of an object in any gravitational field is independent of its mass:

\[
a = G M / r.
\]

Hence, if only gravitational forces act, all bodies similarly projected pursue identical trajectories. Galileo and Newton knew this. Newton conducted experiments to test the equivalence of inertial and gravitational masses. Most recent experiments have found that the two masses are numerically equal to within a few parts in \( 10^{12} \). Einstein was so greatly intrigued by this that he attached a deep significance to it.

The assertion of the equivalence of the inertial mass and gravitational mass is known as the principle of equivalence. Pursuit of the consequences of the principle of equivalence led Einstein to formulate his Theory of General Relativity.
We now attempt to reformulate Newton’s theory of gravitation so that action at a distance is eliminated. This can be done very easily through the field concept. Here is the basic idea: consider any body—the sun, for example—in space. The presence of this body alters the properties of the space in its vicinity, setting up a gravitational field that stands ready to interact with any other masses brought into it. We now try to make these ideas quantitative.

The gravitational field vector $\vec{g}$ at any point in space is defined as the gravitational force per unit mass that would act on a particle located at that point, 

$$\vec{g} = \vec{F} / m.$$  (A1-5)

Obviously $\vec{g}$ has the dimension of acceleration. Near the surface of Earth, it is called the gravitation acceleration, and its magnitude is about 9.8 m/sec$^2$.

Now the gravitational force between a pair of masses $m$ and $M$ separated by distance $r$ is 

$$\vec{F} = -\frac{GMm}{r^2} \hat{r}$$

where $\hat{r}$ is a unit vector away from $M$. Therefore the intensity of the gravitational field at distance $r$ from $M$ is 

$$\vec{g} = -\frac{GM}{r^2} \hat{r}.$$  (A1-6)

If more than one body is present, the gravitational field is the vector sum of the individual fields produced by each body. For a body that consists of a continuous distribution of matter, (A1-6) becomes 

$$\vec{g} = \nabla \left( \int V G \rho(\vec{r}') \frac{\hat{r}}{r} dV' \right).$$  (A1-7)

Upon using the identity $\nabla \left( \frac{1}{r} \right) = -\left( \frac{1}{r^2} \right) \hat{r}$ we find that 

$$\vec{g} = G \int_{vol} \nabla \left( \frac{1}{r} \right) \rho(\vec{r}') dV'.$$

Since $\nabla$ does not operate on the variable $\vec{r}'$, it can be factored out of the integral and we have 

$$\vec{g} = \nabla \left( \int V \frac{G \rho(\vec{r}')}{r} dV' \right)$$  (A1-8)

which can be rewritten as the gradient of a scalar function $\Phi$

$$\vec{g} = -\nabla \Phi$$

with 

$$\Phi = -\int V \frac{G \rho(\vec{r}')}{r} dV'.$$  (A1-9)

The quantity $\Phi$ has the dimension of energy per unit mass and is called the gravitational potential.
A.1.7 Gravitational Field Equations

The gravitational potential $\Phi$ satisfies certain partial differential equations. To find out these equations, we consider a closed surface $S$ enclosing a mass $M$. The gravitational flux passing through the surface element $dS$ is given by the quantity $\mathbf{g} \cdot \hat{n} dS$, where $\hat{n}$ is the outward unit vector normal to $dS$. So the total gravitational flux through $S$ is then given by

$$\int_{S} \mathbf{g} \cdot \hat{n} dS = -GM \int_{S} \frac{\hat{r} \cdot \hat{n}}{r^2} dS. \quad (A1-10)$$

By definition $(\hat{r} \cdot \hat{n}/r^2) dS = \cos \theta \frac{dS}{r^2}$ is the element of solid angle $d\Omega$ subtended at $M$ by the element of surface $dS$ (Fig. A.5). And so (A1-10) becomes

$$\int_{S} \mathbf{g} \cdot \hat{n} dS = -GM \int d\Omega = -4\pi GM. \quad (A1-11)$$

If the surface $S$ encloses a number of masses $M_i$, we have

$$\int_{S} \mathbf{g} \cdot \hat{n} dS = -4\pi G \sum_{i} M_i. \quad (A1-12)$$

For a continuous distribution of mass within $S$, the sum becomes an integral

$$\int_{S} \mathbf{g} \cdot \hat{n} dS = -4\pi G \int_{V} \rho(\mathbf{r}) dV. \quad (A1-13)$$

Using Gauss’ divergence theorem, $\int_{S} \mathbf{g} \cdot \hat{n} dS = \int_{V} \nabla \cdot \mathbf{g} dV$, (A1-13) becomes

$$\int_{V} (\nabla \cdot \mathbf{g} + 4\pi G \rho) dV = 0.$$

---

Fig. A.5 The element of solid angle $d\Omega$. 
This equation holds true for any volume, and it can be true only if the integrand vanishes:

$$\nabla \cdot \vec{g} = -4\pi G \rho.$$  \hspace{1cm} (A1-14)

Since $\vec{g}$ is conservative, a second fundamental differential equation obeyed by $\vec{g}$ is

$$\nabla \times \vec{g} = 0.$$  \hspace{1cm} (A1-15)

Now, substituting $\vec{g} = -\nabla \Phi_1$ into (A1-14) we finally arrive at the result, which is the Poisson’s equation:

$$\nabla^2 \Phi_1(\vec{r}) = 4\pi G \rho(\vec{r}).$$  \hspace{1cm} (A1-16)

If $\rho = 0$, Poisson’s equation reduces to Laplace’s equation

$$\nabla^2 \Phi(\vec{r}) = 0.$$  \hspace{1cm} (A1-17)

Equations (A1-16) and (A1-17) constitute the field equations for the Newtonian theory of gravity.

A.2 Lagrangian Mechanics

As mentioned earlier, classical mechanics can be reformulated in a few different forms, such as the Lagrange, the Hamiltonian, and the Hamilton-Jacobi formalisms. Each is based on the ideas of work or energy, and each is expressed in terms of generalized coordinates. Any convenient set of parameters (quantities) that can be used to specify the state of the system can be taken as the generalized coordinates. Thus, the new generalized coordinates may be any quantities that can be observed to change with the motion of the system, and they need not be geometric quantities. They may be electric charges, for example, in certain circumstances. We shall write the generalized coordinates as $q_i, i = 1, 2, 3, \ldots n$, where $n$ is the number of degrees of freedom of the system.

In terms of these generalized coordinates, we can write the basic equations of motion in some form that is equally suitable for all coordinate systems, and they can be applied to a wide range of physical phenomena, particularly those involving fields with which Newton’s equations of motion are not usually associated.

A.2.1 Hamilton’s Principle

Joseph Louis Lagrange first developed the Lagrangian dynamics, and he selected d’Alembert’s principle as the starting point and obtained the equations of motion known as “Lagrange’s equations” from it. We will start with a variational principle, known as Hamilton’s principle, which may be stated as follows:

*Of all possible paths along which a dynamical system may move from one point to another within a specific time interval and consistent with any constraints, the*
actual path followed is that for which the time integral of the difference between the kinetic and potential energies has stationary value.

In terms of the calculus of variation, Hamilton’s principle becomes

$$\delta \int_{t_1}^{t_2} (T - V) dt = \delta \int_{t_1}^{t_2} L dt = 0 \quad (A1-18)$$

where $L$ is defined to be the difference between the kinetic and potential energies and is called the Lagrangian of the system. Energy is a scalar quantity, and so the Lagrangian is a scalar function. Hence the Lagrangian must be invariant with respect to coordinate transformations. We are therefore assured that no matter what generalized coordinates are chosen for the description of a system, the Lagrangian will have the same value for a given condition of the system. Although Lagrangian will be expressed by means of different functions, depending on the generalized coordinates used, the value of the Lagrangian is unique for a given condition. Therefore, we can write

$$L = L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t)$$

where $\dot{q}_i$ is the generalized velocity corresponding to $q_i$. Hamilton’s principle becomes

$$\delta I = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt \quad (A1-19)$$

where $q_i(t)$, and hence $\dot{q}_i(t)$, is to be varied subject to the end conditions: $\delta q_i(t) = \delta \dot{q}_i(t) = 0$. The symbol “$\delta$” refers to the variation in a quantity between two paths, and “$d$” as usual refers to a variation along a given path. If we label each possible path by a parameter, $\delta$ then stands as shorthand for the parametric procedure outlined below.

We label each possible path by a parameter $\alpha$ in the following way:

$$q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i(t) \quad (A1-20)$$

where $q_i(t, 0)$ is the actual dynamical path followed by the system (as yet unknown), and $\eta_i(t)$ is a completely arbitrary function of $t$, which has a continuous first derivative and subject to $\eta_i(t_1) = \eta_i(t_2) = 0$. In terms of the variation symbol, we can write

$$\delta q_i = \frac{\partial q_i}{\partial \alpha} d\alpha = \eta_i d\alpha. \quad (A1-21)$$

This corresponds to a virtual displacement in $q_i$ from the actual dynamical path to a neighboring varied path, as depicted schematically in Figure A.6.

The action integral $I$ is now a function of $\alpha$ only, for any given $\eta_i(t)$:

$$I(\alpha) = \int_{t_1}^{t_2} L(q_i(t, \alpha), \dot{q}_i(t, \alpha), t) dt \quad (A1-22)$$

Hence, the stationary values of $I(\alpha)$ occur when $\partial I / \partial \alpha = 0$. But by our choice of $q_i(t, 0)$, we know that this occurs when $\alpha = 0$, so that the necessary condition that
the action integral has a stationary value is $\partial I / \partial \alpha = 0$ when $\alpha = 0$. In terms of the variation symbol $\delta$, this necessary condition can be written as

$$\delta I = \left( \frac{\partial I}{\partial \alpha} \right)_{\alpha=0} d\alpha = 0 \quad (A1-23)$$

for arbitrary $\eta_i(t)$ and nonzero $\alpha$. The subscript $\alpha = 0$ means that we evaluate the derivative $\partial I / \partial \alpha$ at $\alpha = 0$.

**A.2.2 Lagrange’s Equations of Motion**

We now expand the integrand $L$ in (A1-22) in a Taylor’s series:

$$I(\alpha) = \int_{t_1}^{t_2} \left[ L(q_i(t, 0), \dot{q}_i(t, 0); t) + \alpha \eta_i(t) \frac{\partial L}{\partial q_i} + \alpha \dot{\eta}_i(t) \frac{\partial L}{\partial \dot{q}_i} + 0(\alpha^2) \right] dt \quad (A1-24)$$

Since the integration limits $t_1$ and $t_2$ are not dependent on $\alpha$, we can differentiate under the integral sign with respect to $\alpha$ and obtain

$$\frac{\partial I}{\partial \alpha} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} \eta_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i + 0(\alpha^2) \right] dt. \quad (A1-25)$$

Dropping terms in $\alpha^2$, $\alpha^3$, and integrating by parts the second term we obtain

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \dot{\eta}_i dt = \left. \left( \frac{\partial L}{\partial q_i} \eta_i \right) \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta_i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) dt \quad (A1-26)$$

The first term on the right is zero because $\eta_i(t_1) = \eta_i(t_2) = 0$. Substituting (A1-26) into (A1-25) we obtain

$$\left( \frac{\partial I}{\partial \alpha} \right)_{\alpha=0} d\alpha = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} \eta_i - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right]_{\alpha=0} \eta_i(t) d\alpha dt = 0.$$
To obtain the stationary condition in terms of the variation symbol $\delta$ we multiply both sides by $d\alpha$, resulting in
\[
\left( \frac{\partial I}{\partial \alpha} \right)_{\alpha=0} = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right]_{\alpha=0} \eta_i(t) dt
\]
or, in terms of the variation symbol $\delta$:
\[
\delta I = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right]_{\alpha=0} \delta q_i(t) dt = 0. \tag{A1-27}
\]
Since $\delta q_i(t)$ is arbitrary except when $\delta q_i(t_1) = \delta q_i(t_2) = 0$, it follows that a necessary condition for $\delta I = 0$ is that the square bracket vanishes, yielding Lagrange’s equations of motion:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, 3, \ldots n \tag{A1-28}
\]
This is also a sufficient condition for a stationary value of the action integral $I$. This is from the fact that (A1-28) implies that the integral in (A1-27) vanishes and result in the variation $\delta I$ being zero.

A.3 Problems

A.1. Under the influence of a force field a particle of mass $m$ moves in the $xy$-plane and its position vector is given by $\vec{r} = a \cos \omega t \hat{i} + b \sin \omega t \hat{j}$, where $a$, $b$, and $\omega$ are positive constants and $a > b$. Show that
(a) the particle moves in an ellipse;
(b) the force acting on the particle is always directed toward the origin; and
(c) $\vec{r} \times \vec{p} = mab\omega \hat{k}$, and $\vec{r} \cdot \vec{p} = \frac{1}{2} m (b^2 - a^2) \sin 2\omega t$
where $\vec{p}$ is the momentum of the particle.

A.2. Two astronauts of masses $M_A$ and $M_B$, initially at rest in free space, pull on either end of a rope. The maximum force with which astronaut A can pull, $F_A$, is larger than the maximum force with which astronaut B can pull, $F_B$. Find their motion if each pulls on the rope as hard as possible.

A.3. Show that the electromagnetic wave equation
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0
\]
is not invariant under the Galilean transformations.

A.4. (a) Find the force of attraction of a thin spherical shell of radius $a$ on a particle $P$ of mass $m$ at a distance $r > a$ from its center.
(b) Prove that the force of attraction is the same as if all the mass of the spherical shell were concentrated at its center.

A.5. Derive the result of Problem A.4a by first finding the potential due to the mass distribution.

A.6. A simple pendulum of mass $m$ and length $b$ oscillates in a plane about its equilibrium position. If $\theta$ is the angular displacement of the pendulum from equilibrium ($\theta = 0$), (a) write the Lagrangian of the system in terms of $\theta$ and $b$, and (b) use the Lagrange’s equation to show that

$$\ddot{\theta} + \frac{g}{b} \sin \theta = 0.$$ 

Reference

Appendix B
The Special Theory of Relativity

B.1 The Origins of Special Relativity

Before Einstein, the concept of space and time were those described by Galileo and Newton. Time was assumed to have an absolute or universal nature, in the sense that any two inertial observers who have synchronized their clocks will always agree on the time of any event (any happening that can be given a space and time coordinates). The Galilean transformations assert that any one inertial frame is as good as another one describing the laws of classical mechanics.

However, physicists of the 19th century were not able to grant the same freedom to electromagnetic theory, which did not seem to obey Galilean transformations. For example, the electromagnetism of Maxwell predicts that the speed of light is a constant, independent of the motion of the source and the observer. Now, a source at rest in an inertial frame $S$ emits a light wave, which travels out as a spherical wave at a constant speed ($3 \times 10^5$ km/sec). But observed in a frame $S'$ moving uniformly with respect to $S$ will see, according to Galilean transformations, that the light wave is no longer spherical and the speed of light is also different, so Maxwell’s equations are not invariant under Galilean transformations. Therefore, for electromagnetic phenomena, inertial frames of reference are not equivalent.

Does this suggest that Maxwell’s equations are wrong and need to be modified to obey the principle of Newtonian relativity? Or does this suggest that the existences of a preferred frame of reference in which Maxwell’s equations are valid? The idea of a preferred frame of reference is foreign to classical mechanics. So a number of theories were proposed to resolve this conflict.

Today we know that Maxwell’s equations are correct and have the same form in all inertial reference frames. There is some transformation other than the Galilean transformation that makes both electromagnetic and mechanical equations transform in an invariant way. But this proposal was not accepted without resistance. Owing to the works of Young and Fresnel, light was viewed as a mechanical wave, analogous to transverse waves on a string. Thus, its propagation required a physical medium. This medium was called ether and was required to have very strong restoring forces so that it could propagate light at such a great speed. But at the same
time the medium offers little resistance to the planets, as they suffered no observable reduction in speed even though they traveled through it year after year. It was necessary to demonstrate the existence of the ether so that this paradox might be resolved.

Since light can travel through space, it was assumed that ether must fill all of space and the speed of light must be measured with respect to the stationary ether.

### B.2 The Michelson-Morley Experiment

If ether does exist, it should be possible to detect some variation of the speed of light as emitted by some terrestrial source. As Earth travels through space at 30 km/s in an almost circular orbit around the sun, it is bound to have some relative velocity with respect to ether. If this relative velocity is added to that of the light emitted from the source, then light emitted simultaneously in two perpendicular directions should be traveling at different speeds, corresponding to the two relative velocities of the light with respect to the ether.

In 1887 Michelson set out to detect this velocity variation in the propagation of light. His ingenious way of doing this depends on the phenomenon of interference of light to determine whether the time taken for light to pass over two equal paths at right angles was different or not. Figure B.1 shows schematically the interferometer that Michelson used, which is essentially comprised of a light source S, a half-silvered glass plate A, and two mirrors B and C, all mounted on a rigid base. The
two mirrors are placed at equal distances \( L \) from plate \( A \). Light from \( S \) enters \( A \) and splits into two beams. One goes to mirror \( B \), which reflects it back; the other beam goes to mirror \( C \), also to be reflected back. On arriving back at \( A \), the two reflected beams are recombined as two superimposed beams, \( D \) and \( F \), as indicated. If the time taken for light to travel from \( A \) to \( B \) and back equals the time from \( A \) to \( C \) and back, the two beams \( D \) and \( F \) will be in phase and will reinforce each other. But if the two times differ slightly, the two beams will be slightly out of phase and interference will result. We now calculate the two times to see whether they are the same or not. We first calculate the time required for the light to go from \( A \) to \( B \) and back. If the line \( AB \) is parallel to Earth’s motion in its orbit, and if Earth is moving at speed \( u \) and the speed of light in the ether is \( c \), the time is

\[
t_1 = \frac{L_{AB}}{c-u} + \frac{L_{AB}}{c+u} = \frac{2L_{AB}}{c[1-(u/c)^2]} \approx \frac{2L_{AB}}{c} \left(1 + \frac{u^2}{c^2}\right)
\]

where \((c-u)\) is the upstream speed of light with respect to the apparatus, and \((c+u)\) is the downstream speed.

Our next calculation is of the time \( t_2 \) for the light to go from \( A \) to \( C \). We note that while light goes from \( A \) to \( C \), the mirror \( C \) moves to the right relative to the ether through a distance \( d = ut_2 \) to the position \( C' \); at the same time the light travels a distance \( ct_2 \) along \( AC' \). For this right triangle we have

\[
(ct_2)^2 = L_{AC}^2 + (ut_2)^2
\]

from which we obtain

\[
t_2 = \frac{L_{AC}}{\sqrt{c^2-u^2}}.
\]

Similarly, while the light is returning to the half-silvered plate, the plate moves to the right to the position \( B' \). The total path length for the return trip is the same, as can be seen from the symmetry of Figure B.1. Therefore if the return time is also the same, the total time for light to go from \( A \) to \( C \) and back is then \( 2t_2 \), which we denote by \( t_3 \):

\[
t_3 = \frac{2L_{AC}}{\sqrt{c^2-u^2}} = \frac{2L_{AC}}{c\sqrt{1-(u/c)^2}} \approx \frac{2L_{AC}}{c} \left(1 + \frac{u^2}{2c^2}\right).
\]

In (A2-1) and (A2-2) the first factors are the same and represent the time that would be taken if the apparatus were at rest relative to the ether. The second factors represent the modifications in the times caused by the motion of the apparatus. Now the time difference \( \Delta t \) is

\[
\Delta t = t_3 - t_1 = \frac{2(L_{AC} - L_{AB})}{c} + \frac{L_{AC} \beta^2 - 2L_{AB} \beta^2}{c} \quad (A2-3)
\]

where \( \beta = u/c \).

It is most likely that we cannot make \( L_{AB} = L_{AC} = L \) exactly. In that case we can rotate the apparatus 90 degrees, so that \( AC \) is in the line of motion and \( AB \) is
perpendicular to the motion. Small differences in length become unimportant. Now we have

$$\Delta t' = t'_3 - t'_1 = \frac{2(L_{AB} - L_{AC})}{c} + \frac{2L_{AB}}{c} \beta^2 - \frac{L_{AC}}{c} \beta^2. \quad (A2-4)$$

Thus,

$$\Delta t' - \Delta t = \frac{(L_{AB} + L_{AC})}{c} \beta^2. \quad (A2-5)$$

This difference yields a shift in the interference pattern across the crosshairs of the viewing telescope. If the optical path difference between the beams changes by one \(\lambda\) (wave-length), for example, there will be a shift of one fringe. If \(\delta\) represents the number of fringes moving past the crosshairs as the pattern shifts, then

$$\delta = \frac{c(\Delta t' - \Delta t)}{\lambda} = \frac{L_{AB} + L_{AC}}{\lambda} \beta^2 = \frac{\beta^2}{\lambda / (L_{AB} + L_{AC})}. \quad (A2-6)$$

In the Michelson-Morley experiment of 1887, the effective length \(L\) was 11 m; sodium light of \(\lambda = 5.9 \times 10^{-5}\) cm was used. The orbit speed of Earth is \(3 \times 10^4\) m/s, so \(\beta = 10^{-4}\). From (A2-6) the expected shift would be about \(4/10\) of a fringe

$$\delta = \frac{22m \times (10^{-4})^2}{5.9 \times 10^{-5}} = 0.37. \quad (A2-7)$$

The Michelson-Morley interferometer can detect a shift of 0.005 fringes. However, no fringe shift in the interference pattern was observed. So no effect at all due to Earth’s motion through the ether was found. This null result was very puzzling and most disturbing at the time. It was suggested, including by Michelson, that the ether might be dragged along by Earth, eliminating or reducing the ether wind in the laboratory. This is hard to square with the picture of the ether as an all-pervasive, frictionless medium. The ether’s status as an absolute reference frame was also gone forever. Many attempts to save the ether failed (see Resnick and Halliday 1985). We just mention one here, namely the contraction hypothesis.

George F. Fitzgerald pointed out in 1892 that a contraction of bodies along the direction of their motion through the ether by a factor \((1 - u^2/c^2)^{1/2}\) would give the null result. Because (A2-1) must be multiplied by the contraction factor \((1 - u^2/c^2)^{1/2}\), then (A2-2) reduces to zero. The magnitude of this time difference is completely unaffected by rotation of the apparatus through 90°.

Lorentz obtained a contraction of this sort in his theory of electrons. He found that the field equations of electron theory remain unchanged if a contraction by the factor \((1 - u^2/c^2)^{1/2}\) takes places, provided also that a new measure of time is used in a uniformly moving system. The outcome of the Lorentz theory is that an observer will observe the same phenomena, no matter whether the person is at rest in the ether or moving with velocity. Thus, different observers are equally unable to tell whether they are at rest or moving in the ether. This means that for optical phenomena, just as for mechanics, ether is unobservable.
Poincaré offered another line of approach to the problem. He suggested that the result of the Michelson-Morley experiment was a manifestation of a general principle that absolute motion cannot be detected by laboratory experiments of any kind, and the laws of nature must be the same in all inertial reference frames.

B.3 The Postulates of the Special Theory of Relativity

Einstein realized the full implications of the Michelson-Morley experiment, the Lorentz theory, and Poincaré's principle of relativity. Instead of trying to patch up the accumulating difficulties and contradictions connected with the notion of ether, Einstein rejected the ether idea as unnecessary or unsuitable for the description of the physical laws. Along with the exit of ether, gone also was the notion of absolute motion through space. The Michelson-Morley experiment proved unequivocally that no such special frame of reference exists. All frames of reference in uniform relative motion are equivalent, for mechanical motions and also for electromagnetic phenomena. Einstein further extended this as a fundamental postulate, now known as the principle of relativity. Furthermore, he argued that the speed of light, $c$, predicted by electromagnetic theory must be a universal constant, the same for all observers. He took an epoch-making step in 1905 and developed the Special Theory of Relativity from these two basic postulates (assumptions), which are rephrased as follows:

1. The laws of physics are the same in all inertial frames. No preferred inertial frame exists (the principle of relativity).
2. The velocity of light in free space is the same in all inertial frames and is independent of the motion of the emitting body (the principle of the constancy of the velocity of light).

According to Einstein, sometime in 1896, after he entered the Zurich Polytechnic Institute to begin his education as a physicist, he asked himself the question of what would happen if he could catch up to a light ray—that is, move at the speed of light. Maxwell’s theory says that light is a wave of electric and magnetic fields moving through space. But if you could catch up to a light wave, then the light wave would not be moving relative to you but instead be standing still. The light wave would then be a standing wave of electric and magnetic fields, which is not allowed if Maxwell’s theory is right. So, he reasoned, there must be something wrong with the assumption that you can catch a light wave the same way as you can catch a water wave. This idea was the seed from which the fundamental postulate of the constancy of the speed of light and the Special Theory of Relativity grew nine years later.

All the seemingly very strange results of special relativity came from the special nature of the speed of light. Once we understand this, everything else in relativity makes sense. So let us take a brief look at the special nature of the speed of light. The speed of light is very great, 186,000 mi/s or $3 \times 10^5$ km/sec. But the bizarre fact of the speed of light is that it is independent of the motion of the observer or
the source emitting the light. Michelson hoped to determine the absolute speed of Earth through ether by measuring the time differences required for light to travel across equal distances that are at right angles to each other. What did he observe? No difference in travel times for the two perpendicular light beams. It was as if Earth were absolutely stationary. The conclusion is that the speed of light does not depend on the motion of the object. This bizarre nature is not something that would be expected from common sense. The same common sense once told us that it was nonsense to think that Earth was round. So common sense is not always right!

How do we know that the speed of light is independent of the motion of the light source? There are many binary star systems in our galaxy, in which two stars revolve around a common center of mass. If the speed of light depended on the motion of the source, then the light emitted by the two stars in a binary system would have different speeds as they moved toward Earth, as shown in Figure B.2. The orbit is roughly edge-on to our line of sight, and its orbital speed about the center of mass of the system. If the distance to the binary system were right, we would receive light from the star at position A at the same time as the light sent to us at a slower speed and at an earlier time, when the star was at position B. Thus, under some circumstances we could be seeing the same star in a binary system at many different places in its orbit at once, and there would be multiple images or spread out images. But, in fact, we always see binary stars moving in a well-behaved elliptical orbit about each other. Thus, the motion of its source (the emitter) does not affect the speed of light.

Einstein’s two postulates radically revised our concepts of space and time. Newtonian mechanics abolished the notion of absolute space. Now absolute space is abolished in its Maxwellian role as the ether, the carrier of electromagnetic waves. Time is also not absolute any more either, since all inertial observers agree on how fast light travels but not on how far light travels. Time has lost its universal nature. In fact, we shall see later examples of moving clocks that run slow. This is known as time dilation.

**Fig. B.2** The nonexistence of light intensity variation from a binary star proves that the speed of light is independent of the motion of the light source.
B.4 The Lorentz Transformations

Since the Galilean transformations are inconsistent with Einstein’s postulate of the constancy of the speed of light, we must modify it in such a way that the new transformation will incorporate Einstein’s two postulates and make both mechanical and electromagnetic equations transforming in an invariant way. To this end we consider two inertial frames $S$ and $S’$. Let the corresponding axes of $S$ and $S’$ frames be parallel, with frame $S’$ moving at a constant velocity $u$ relative to $S$ along the $x_1$-axis. The apparatus for measurement of distances and times in the two frames are assumed identical, and the clocks are adjusted to read zero at the moment the two origins coincide. Figure B.3 represents the viewpoint of observers in $S$.

Suppose that an event occurred in frame $S$ at the coordinates $(x, y, z, t)$ and is observed at $(x’, y’, z’, t’)$ in frame $S’$. Because of the homogeneity of space and time, we expect the transformation relations between the coordinates $(x, y, z, t)$ and $(x’, y’, z’, t’)$ to be linear, for otherwise there would not be a simple one-to-one relation between events in $S$ and $S’$ frames. For instance, a nonlinear transformation would predict acceleration in one system even if the velocity were constant in the other, obviously an unacceptable property for a transformation between inertial systems.

Let us consider the transverse dimensions first. Since the relative motion of the coordinate systems occurs only along the $x$-axis, we expect the linear relations are of the forms $y’ = k_1 y$ and $z’ = k_2 z$. The symmetry requires that $y = k_1 y’$ and $z = k_2 z’$. These can both be true only if $k_1 = 1$ and $k_2 = 1$. Therefore, for the transverse direction we have

$$y’ = y, \quad z’ = z. \quad \text{(A2-8)}$$

These relations are the same as in Galilean transformations.

Along the longitudinal dimension, the relation between $x$ and $x’$ must depend on the time, so let’s consider the most general linear relation

$$x’ = ax + bt. \quad \text{(A2-9)}$$

![Fig. B.3 Relative motion of two inertial frames of reference.](image-url)
Now, the origin $O'$, where $x' = 0$, corresponds to $x = ut$. Substituting these into (A2-9), we have

$$0 = aut + bt$$

from which we obtain

$$b = -au$$

and (A2-9) simplifies to

$$x' = a(x - ut).$$  \hspace{1cm} (A2-10)

By symmetry, we also have

$$x = a(x' + ut').$$  \hspace{1cm} (A2-11)

Now we apply Einstein’s second postulate of the constancy of the speed of light. If a pulse of light is sent out from the origin $O$ of frame $S$ at $t = 0$, its position along the $x$-axis later is given by $x = ct$, and its position along $x'$-axis is $x' = ct'$. Putting these in (A2-10) and (A2-11), we obtain

$$ct' = a(c - u)t \quad \text{and} \quad ct = a(c + u)t'.$$

From these we obtain

$$\frac{t}{t'} = \frac{c}{a(c - u)} \quad \text{and} \quad \frac{t}{t'} = \frac{a(c + u)}{c}.$$

Therefore,

$$\frac{c}{a(c - u)} = \frac{a(c + u)}{c}.$$

Solving for $a$

$$a = \frac{1}{\sqrt{1 - (u/c)^2}},$$

then

$$b = -au = -\frac{u}{\sqrt{1 - (u/c)^2}}.$$

Substituting these in (A2-10) and (A2-11) gives

$$x' = \frac{x - ut}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (A2-12a)

and

$$x = \frac{x' + ut'}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (A2-12b)

where $\beta = u/c$. Eliminating either $x$ or $x'$ from (A2-12a) and (A2-12b), we obtain

$$t' = \frac{t - ux/c^2}{\sqrt{1 - \beta^2}}$$  \hspace{1cm} (A2-12c)
and
\[ t = \frac{t' + ux'/c^2}{\sqrt{1 - \beta^2}} \]  
(A2-12d)

Combining all of these results, we obtain the Lorentz transformations

\[
\begin{align*}
    x' &= \gamma (x - ut) \quad x = \gamma (x' + ut) \\
    y' &= y \quad y = y' \\
    z' &= z \quad z = z' \\
    t' &= \gamma (t - ux/c^2) \quad t = \gamma (t' + ux/c^2)
\end{align*}
\]  
(A2-13)

where
\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = u/c \]  
(A2-14)

is the Lorentz factor.

If \( \beta << 1 \), then \( \gamma \approx 1 \), and (A2-13) reduces to the Galilean transformations. That is, the Galilean transformations are a first approximation to the Lorentz transformations for \( \beta << 1 \).

When the velocity, \( \vec{u} \), of \( S' \) relative to \( S \) is in some arbitrary direction, (A2-13) can be given a more general form in terms of the components of \( \vec{r} \) and \( \vec{r}' \) perpendicular and parallel to \( \vec{u} \).

\[
\begin{align*}
    \vec{r}'_\parallel &= \gamma (\vec{r}_\parallel - \vec{u}t) \quad \vec{r}_\parallel = \gamma (\vec{r}'_\parallel + \vec{u}t) \\
    \vec{r}'_\perp &= \vec{r}_\perp \quad \vec{r}_\perp = \vec{r}'_\perp \\
    t' &= \gamma (t - \vec{u} \cdot \vec{r}/c^2) \quad t = \gamma (t' + \vec{u} \cdot \vec{r}/c^2)
\end{align*}
\]  
(A2-15)

The Lorentz transformations are valid for all types of physical phenomena at all speeds. As a consequence of this all physical laws must be invariant under a Lorentz transformation.

The Lorentz transformations that are based on Einstein’s postulates contain a new philosophy of space and time measurements. We now examine the various properties of these new transformations. In the following discussion, we still use Figure B.3.

### B.4.1 Relativity of Simultaneity and Causality

Two events that happen at the same time but not necessarily at the same place are called simultaneous. Now consider two events in \( S' \) that occur at \( (x'_1, t'_1) \) and \( (x'_2, t'_2) \); they would appear in frame \( S \) at \( (x_1, t_1) \) and \( (x_2, t_2) \). The Lorentz transformations give

\[ t_2 - t_1 = \gamma \left[ (t'_2 - t'_1) + \frac{u(x'_2 - x'_1)}{c^2} \right]. \]  
(A2-16)

Now, it is easy to observe that if the two events take place simultaneously in \( S' \) (so \( t'_2 - t'_1 = 0 \)), they do not occur simultaneously in the \( S \) frame, for there is a finite time lapse.
\[ \Delta t = t_2 - t_1 = \gamma \frac{u(x'_2 - x'_1)}{c^2} \neq 0. \]

Thus, two spatially separated events that are simultaneous in \( S' \) would not be simultaneous in \( S \). In other words, the simultaneity of spatially separated events is not an absolute property, as it was assumed to be in Newtonian mechanics. Moreover, depending on the sign of \((x'_2 - x'_1)\) the time interval \( \Delta t \) can be positive or negative, that is, in the frame \( S \) the “first” event in \( S' \) can take place earlier or later than the “second” one. The exception is the case when two events occur coincidentally in \( S' \); then they also occur at the same place and at the same time in frame \( S \).

If the order of events in frame \( S \) is not reversed in time, then \( \Delta t = t_2 - t_1 > 0 \), which implies that

\[
(t'_2 - t'_1) + \frac{u(x'_2 - x'_1)}{c^2} > 0
\]

or

\[
\frac{x'_2 - x'_1}{t'_2 - t'_1} < \frac{c^2}{u}
\]

which will be true as long as

\[
\frac{x'_2 - x'_1}{t'_2 - t'_1} < c.
\]

Thus the order of events will remain unchanged if no signal can be transmitted with a speed greater than \( c \), the speed of light.

### B.4.2 Time Dilation and Relativity of Co-locality

Two events that happen at the same place but not necessarily at the same time are called co-local. Now consider two co-local events in \( S' \) taking place at \( t'_1 \) and \( t'_2 \) but at the same place. For simplicity consider this to be on the \( x' \)-axis so that \( y' = z' = 0 \). These two events would appear in frame \( S \) at \((x_1, t_1)\) and \((x_2, t_2)\). The Lorentz transformations give

\[
\Delta x = \frac{u \Delta t'}{\sqrt{1 - \beta^2}} = \gamma u \Delta t', \quad \Delta t = \frac{\Delta t'}{\sqrt{1 - \beta^2}} = \gamma \Delta t'
\]

where \( \beta = u/c, \Delta t' = t'_2 - t'_1 \), and so forth. It is easy to observe that:

(1) Two co-local events in \( S' \) do not occur at the same place in \( S \), and so \( t_1 \) and \( t_2 \) must be measured by spatially separated synchronized clocks. Einstein’s prescription for synchronizing two stationary separated clocks is to send a light signal from clock 1 at a time \( t_1 \) (measured by clock 1) and reflected back from clock 2 at a time \( t_2 \) (measured by clock 2). If the reflected light returns to clock 1 at a time \( t_3 \) (measured by clock 1), then clocks 1 and 2 are synchronous if \( t_2 - t_1 = t_3 - t_2 \); that is, if the time measured for light to go one way is equal to the time measured for light to go in the opposite direction.
The time interval between two co-local events in an inertial reference is measured by a single clock at a given point and it is called the *proper time* interval between the two events. In the second equation of (A2-17), \( \Delta t' = t'_2 - t'_1 \) is the proper time interval between the events in \( S' \). Since \( \gamma \geq 1 \), the time interval \( \Delta t = t_2 - t_1 \) in \( S \) is longer than \( \Delta t' \); this is called time dilation, often described by the statement that “moving clocks run slow.” This apparent asymmetry between \( S \) and \( S' \) in time is a result of the asymmetric nature of the time measurement.

Time dilation has been confirmed by experiments on the decay of pions. Pions have a mean lifetime of \( T_0 = 2.6 \times 10^{-8} \) sec when they are at rest. When they are in fast motion in a synchrotron, their lifetimes become larger according to

\[
T = \frac{T_0}{\sqrt{1 - \beta^2}}.
\]

Time dilation between observers in uniform relative motion is a very real thing. All processes, including atomic and biological processes, slow down in moving systems.

We often hear the twin paradox. Consider one twin gets on a spaceship and accelerates to 0.866c, so \( \gamma = 2 \). If this twin travels for one year, as measured by his clock, then heads back at the same speed, the moving twin will report that the trip required two years. But his Earth-bound twin would report that the time for the journey was four years. Can the twin on the spaceship argue that he was at rest, and it was the twin on earth who was moving? The answer is no. To make a transition from a rest frame to a moving frame and to turn around heading for home, there must be acceleration. The twin who feels acceleration can no longer claim that his frame is the rest frame. Thus the twin on the spaceship cannot argue that it was his Earth-bound twin moving, and there is no paradox.

### B.4.3 Length contraction

Consider a rod of length \( L_0 \) lying at rest along the \( x' \) axis in the \( S' \) frame: \( L_0 = x'_2 - x'_1 \). \( L_0 \) is the proper length of the rod measured in the rod’s rest frame \( S' \). Now the rod is moving lengthwise with velocity \( u \) relative to the S frame. An observer in the S frame makes a simultaneous measurement of the two ends of the rod. The Lorentz transformations give

\[
x'_1 = \gamma [x_1 - ut_1], \quad x'_2 = \gamma [x_2 - ut_2]
\]

from which we get

\[
x'_2 - x'_1 = \gamma [x_2 - x_1] - \gamma u(t_2 - t_1) = \gamma [x_2 - x_1],
\]

where we dropped the \((t_2 - t_1)\) term, because the measurement in S is made simultaneously. The above result often is rewritten as
where \( L(u) = x_2 - x_1 \). Thus the length of a body moving with velocity \( u \) relative to an observer is measured to be shorter by a factor of \( (1 - \beta^2)^{1/2} \) in the direction of motion relative to the observer.

Since all inertial frames are equally valid, if \( L_0 = \gamma L \), is the expression \( L = \gamma L_0 \) to be equally true? The answer is no, because the measurement was not carried out in the same way in the two reference frames. The positions of the two ends of the rod were marked simultaneously in the \( S \) frame, but they are not simultaneous in the \( S' \) frame. This difference gives the asymmetry of the result. As a general expression, \( \Delta x' = \gamma \Delta x \) is not true. The full expression relating distances in two frames of reference is \( \Delta x' = \gamma(\Delta x - u \Delta t) \), and the symmetrical inverse relation is \( \Delta x = \gamma(\Delta x' + u \Delta t') \). In the case that was considered earlier, \( \Delta t = 0 \), so \( \Delta x' = \gamma \Delta x \), but \( \Delta t' \neq 0 \), so \( \Delta x \neq \gamma \Delta x' \).

A body of proper volume \( V_0 \) can be divided into thin rods parallel to \( u \). Each one of these rods is reduced in length by a factor \( (1 - \beta^2)^{1/2} \) so that the volume of the moving body measured by an observer in \( S \) is \( V = (1 - \beta^2)^{1/2} V_0 \).

An interesting consequence of the length contraction is the visual apparent shape of a rapidly moving object. This was shown first by James Terrell in 1959 [Physics Review, 116(1959), 1041; and American Journal of Physics, 28 (1960) 607]. The act of seeing involves the simultaneous light reception from different parts of the object. In order for light from different parts of an object to reach the eye or a camera at the same time, light from different parts of the object must be emitted at different times, to compensate for the different distances the light must travel. Thus, taking a picture of a moving object or looking at it does not give a valid impression of its shape. Interestingly, the distortion that makes the Lorentz contraction seem to disappear instead makes an object seem to rotate by an angle \( \theta = \sin^{-1}(u/c) \), as long as the angle subtended by the object at the camera is small. If the object moves in another direction, or if the angle it subtends at the camera is not small, the apparent distortion becomes quite complex.

Figure B.4 shows a cube of side \( l \) moving with a uniform velocity \( u \) with respect to an observer some distance away; the side \( AB \) is perpendicular to the line of sight of the observer. In order for light from corners \( A \) and \( D \) to reach the observer at the same instant, light from \( D \), which must travel a distance \( l \) farther than from \( A \), must have been emitted when \( D \) was in position \( E \). The length \( DE \) is equal to \( (l/c)u = l\beta \). The length of the side \( AB \) is foreshortened by Lorentz contraction to \( l\sqrt{1 - \beta^2} \). The net result corresponds to the view the observer would have if the cube were rotated through an angle \( \sin^{-1}\beta \). The cube is not distorted; it undergoes an apparent rotation. Similarly, a moving sphere will not become an ellipsoid; it still appears as a sphere. Weisskopf (Physics Today 13, 9, 1960) gives an interesting discussion of apparent rotations at high velocity.

Length contraction opens the possibility of space travel. The nearest star, besides the sun, is Alpha Centauri, that is about 4.3 light-years away; light from Alpha Centauri takes 4.3 years to reach us. Even if a spaceship can travel at the speed of light, it would take 4.3 years to reach Alpha Centauri. This is certainly true from the
point of view of an observer on Earth. But from the point of view of the crew of the spaceship, the distance between Earth and Alpha Centauri is shortened by a factor \( \gamma = (1 - \beta^2)^{1/2} \), where \( \beta = v/c \) and \( v \) is the speed of the spaceship. If \( v \) is, say, 0.99\( c \), then \( \gamma = 0.14 \), and the distance appears to be only 14% of the value as seen from Earth. If the crew, therefore, deduces that light from Alpha Centauri takes only 0.14 \( \times \) 4.3 = 0.6 year to reach Earth and sees Alpha Centauri coming toward them at a speed of 0.99\( c \), they expect to get there in 0.60/0.99 = 0.606 years, without having to suffer a long tedious journey. But, in practice, the power requirements to launch a spaceship near the speed of light are prohibitive.

### B.4.4 Velocity Transformation

The new and more complicated transformation for velocities can be deduced easily from Lorentz transformations. By definition the components of velocity in S and \( S' \) frames are given by, respectively,

\[
\begin{align*}
    v_x &= \frac{dx}{dt} = \frac{x_2 - x_1}{t_2 - t_1}, \\
    v_x' &= \frac{dx'}{dt'} = \frac{x'_2 - x'_1}{t'_2 - t'_1}, \\
    v_y &= \frac{dy}{dt} = \frac{y_2 - y_1}{t_2 - t_1}, \\
    v_y' &= \frac{dy'}{dt'} = \frac{y'_2 - y'_1}{t'_2 - t'_1},
\end{align*}
\]

and so on.

Applying the Lorentz transformations to \( x_1 \) and \( x_2 \) and then taking the difference, we get

\[
dx = \frac{dx' + u dt'}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{u}{c}.
\]
Similarly,
\[ dx' = \frac{dx - udt}{\sqrt{1 - \beta^2}}. \]

Do the same for the time intervals \( dt \) and \( dt' \):
\[ dt = \frac{dt' + udx'/c^2}{\sqrt{1 - \beta^2}}, \quad dt' = \frac{dt - udx/c^2}{\sqrt{1 - \beta^2}}. \]

From these we obtain
\[ \frac{dx}{dt} = \frac{dx' + udt'}{dt' + udx'/c^2}. \]

Dividing both numerator and denominator of the right side by \( dt' \) yields the right transformation equation for the \( x \) component of the velocity:
\[ v_x = \frac{v_x' + u}{1 + uv_x'/c^2}. \quad (A2-19a) \]

Similarly, we can find the transverse components:
\[ v_y = \frac{v_y'\sqrt{1 - \beta^2}}{1 + uv_y'/c^2} = \frac{v_y'}{\gamma \left(1 + uv_x'/c^2\right)} \quad (A2-19b) \]
\[ v_z = \frac{v_z'\sqrt{1 - \beta^2}}{1 + uv_z'/c^2} = \frac{v_z'}{\gamma \left(1 + uv_x'/c^2\right)}. \quad (A2-19c) \]

In these formulas, \( \gamma = (1 - \beta^2)^{-1/2} \) as before. We note that the transverse velocity components depend on the \( x \)-component. For \( v << c \), we obtain the Galilean result \( v_x = v_x' + u \). Solving explicitly or merely switching the sign of \( u \) would yield \((v_x', v_y', v_z')\) in terms of \((v_x, v_y, v_z)\).

It follows from the velocity transformation formulas that the value of an angle is relative and changes in transition from one reference frame to another. For an object in the \( S \) frame moving in the \( xy \)-plane with velocity \( v \) that makes an angle \( \theta \) with the \( x \)-axis, we have
\[ \tan \theta = \frac{v_y}{v_x}, \quad v_x = v \cos \theta, \quad v_y = v \sin \theta. \]

In the \( S' \) frame, we have
\[ \tan \theta' = \frac{v_y'}{v_x'} = \frac{v \sin \theta}{\gamma (v \cos \theta - u)} \quad (A2-20) \]

where \( \gamma = 1/\sqrt{1 - \beta^2} \), and \( \beta = u/c \).

As an application, consider the case of starlight, that is, \( v = c \); then
\[ \tan \theta' = \frac{\sin \theta}{\gamma (\cos \theta - u/c)}. \]
Let $\theta = \pi/2, \theta' = \pi/2 - \phi$ (Fig. B.5); from this we obtain the star aberration formula, to see a star overhead tilt the telescope at angle $\phi$:

$$\tan \phi = \frac{-u/c}{\sqrt{1 - \beta^2}}, \quad \sin \phi = -\frac{u}{c}$$

### B.5 The Doppler Effect

The Doppler effect occurs for light as well as for sound. It is a shift in frequency due to the motion of the source or the observer. Knowledge of the motion of distant receding galaxies comes from studies of the Doppler shift of their spectral lines. The Doppler effect is also used for satellite tracking and radar speed traps. We examine the Doppler effect in light only.

Consider a source of light or radio waves moving with respect to an observer or a receiver, at a speed $u$ and at an angle $\theta$ with respect to the line between the source and the observer (Fig. B.6). The light source flashes with a period $\tau_0$ in its rest frame (the $S'$ frame in which the source is at rest). The corresponding frequency is $\nu_0 = 1/\tau_0$, and the wavelength is $\lambda_0 = c/\nu_0 = c\tau_0$.

While the source is going through one oscillation, the time that elapses in the rest frame of the observer (the $S$ frame) is $\tau = \gamma \tau_0$ because of time dilation, where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = u/c$. The emitted wave travels at speed $c$, and therefore its front moves a distance of $\gamma \tau_0 c$; the source moves toward the observer with a speed $u \cos \theta$, so a distance of $\gamma \tau_0 u \cos \theta$. Then the distance $D$ separates the fronts of the successive waves (the wavelength):

$$D = \gamma \tau_0 c - \gamma \tau_0 u \cos \theta,$$

i.e.,

$$\lambda = \gamma \tau_0 c - \gamma \tau_0 u \cos \theta = \gamma \tau_0 c[1 - (u/c) \cos \theta].$$
but \( c \tau_0 = \lambda_0 \), so we can rewrite the last expression as

\[
\lambda = \lambda_0 \frac{1 - \beta \cos \theta}{\sqrt{1 - \beta^2}}.
\]  

(A2-21)

In terms of frequency, this Doppler effect formula becomes

\[
\nu = \nu_0 \frac{\sqrt{1 - \beta^2}}{1 - \beta \cos \theta}.
\]  

(A2-22)

Here \( \nu \) is the frequency at the observer, and \( \theta \) is the angle measured in the rest frame of the observer. If the source is moving directly toward the observer, then \( \theta = 0 \) and \( \cos \theta = 1 \). (A2-22) reduces to

\[
\nu = \nu_0 \frac{\sqrt{1 - \beta^2}}{1 - \beta} = \nu_0 \sqrt{1 + \beta}.
\]  

(A2-23a)

For a source moving directly away from the observer, \( \cos \theta = -1 \), (A2-24) reduces to

\[
\nu = \nu_0 \frac{\sqrt{1 - \beta^2}}{1 + \beta} = \nu_0 \sqrt{1 - \beta}.
\]  

(A2-23b)

At \( \theta = \pi/2 \), i.e., the source moving at right angles to the direction of the observer (A2-24) and reduces to

\[
\nu = \nu_0 \sqrt{1 - \beta^2}.
\]  

(A2-23c)

This transverse Doppler effect is due to time dilation.

### B.6 Relativistic Space-Time and Minkowski Space

In our daily experience we are used to thinking of a world of three dimensions. Objects in space have length, breadth, and height. We tend to think of time as being independent of space. However, as we have just seen, there is no absolute standard for the measurement of time or of space; the relative motion of observers affects both kinds of measurement. Lorentz transformations treat \( x^i \) \((i = 1, 2, 3)\) and \( t \) as equivalent variables. In 1907 H. Minkowski proposed that the three dimensions of space and the dimension of time should be treated together as a fourth dimension of space-time. Minkowski remarked: “Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.” And he called the four dimensions of space-time
the world space, and the path of an individual particle in space-time a world line. The four-dimension relativistic space-time is often called the Minkowski space.

It is now a common practice to treat \( t \) as a zeroth or a fourth coordinate

\[
x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z
\]

(A2-24a)

or

\[
x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ix^0.
\]

(A2-24b)

By analogy with the three-dimensional case, the coordinates of an event \( (x^0, x^1, x^2, x^3) \) can be considered as the components of a four-dimensional radius vector, for short, a radius four-vector in Minkowski space. The square of the length of the radius four-vector is given by

\[
(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = -(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.
\]

It does not change under Lorentz transformations.

The Lorentz transformations now take on the form

\[
\begin{align*}
x'^1 &= \gamma (x^1 + i\beta x^4) & x'^0 &= \gamma (x^0 - \beta x^1) \\
x'^2 &= x^2 & x'^1 &= \gamma (-\beta x^0 + x^1) \\
x'^3 &= x^3 \\
x'^4 &= \gamma (-i\beta x^1 + x^4) & x'^3 &= x^3.
\end{align*}
\]

(A2-25)

In matrix form, we have

\[
\begin{pmatrix}
x'^1 \\
x'^2 \\
x'^3 \\
x'^4
\end{pmatrix} =
\begin{pmatrix}
\gamma & 0 & 0 & i\beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-i\beta & 0 & 0 & \gamma
\end{pmatrix}
\begin{pmatrix}
x^1 \\
x^2 \\
x^3 \\
x^4
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
x'^0 \\
x'^1 \\
x'^2 \\
x'^3
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{pmatrix}.
\]

(A2-26)

We will use Greek indices \( (\mu \text{ and } \nu, \text{ etc.}) \) to label four-dimensional variables and Latin indices \( (i \text{ and } j, \text{ etc.}) \) to label three-dimensional variables.

The Lorentz transformations can be distilled into a single equation

\[
x'^\mu = \sum_{\nu=1}^{4} L_{\nu}^{\mu} x^\nu = L_{\nu}^{\mu} x^\nu \quad \mu, \nu = 1, 2, 3, 4
\]

(A2-27)

where \( L_{\nu}^{\mu} \) is the Lorentz transformation matrix in (A2-26). The summation sign is eliminated in the last step by Einstein summation convention; the repeated indexes appearing once in the lower and once in the upper position are summed over. However, the indexes repeated in the lower part or upper part alone are not summed over.

If (A2-27) reminds you of the orthogonal rotations, it is no accident! The general Lorentz transformations can indeed be interpreted as an orthogonal rotation of axes in Minkowski space. The \( xt \)-submatrix of the Lorentz matrix in (A2-26) is

\[
\begin{pmatrix}
\gamma & i\beta \\
-i\beta & \gamma
\end{pmatrix};
\]
let us compare it with the $xy$-submatrix of the two-dimensional rotation about the $z$-axis
\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}.
\]
Upon identification of matrix elements $\cos \theta = \gamma$, $\sin \theta = i \beta \gamma$, we see that the rotation angle $\theta$ (for the rotation in the $xt$-plane) is purely imaginary. Some books prefer to use a real angle of rotation $\phi$, defining $\phi = -i \theta$. Then note that
\[
\begin{align*}
\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{-\phi} + e^{\phi}}{2} = \cosh \phi \\
\sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{i[e^{\phi} - e^{-\phi}]}{2} = \sinh \phi
\end{align*}
\]
and the submatrix becomes
\[
\begin{pmatrix}
\cosh \phi & i \sinh \phi \\
-i \sinh \phi & \cosh \phi
\end{pmatrix}.
\]
We should note that the mathematical form of Minkowski space looks exactly like a Euclidean space; however, it is not physically so because of its complex nature as compared to the real nature of the Euclidean space.

### B.6.1 Interval $ds^2$ as an Invariant

We are always interested in an invariant quantity that is unaffected by different choices of coordinate systems. We will see that intervals are Lorentz invariants. Let us consider again the two frames $S$ and $S'$ in Figure B.3, moving relative to each other with constant velocity. The wave front of light that is emitted at the origin of frame $S$ when $t = 0$ is given by
\[
\begin{align*}
-\frac{c^2 t_2^2 - x_2^2 - y_2^2 - z_2^2}{(x_0^2 - x_1^2 - (x_2^2 - (x_3^2))} = 0. \quad (A2-28a)
\end{align*}
\]
The wave front of the light will give a cone around the $t$ axis. This is called the light cone (Figure B.7). The same wave front will have different coordinates
\[
\begin{align*}
-\frac{c^2 t_2^2 - x_2^2 - y_2^2 - z_2^2}{(x_0^2 - x_1^2 - (x_2^2 - (x_3^2))} = 0. \quad (A2-28b)
\end{align*}
\]
For any two events, such as sending out and receiving a light signal, the quantity $s_{12}$, where
\[
s_{12} = \left[ (x_2^0 - x_1^0)^2 - (x_2^1 - x_1^1)^2 - (x_2^2 - x_1^2)^2 - (x_2^3 - x_1^3)^2 \right]^{1/2} \quad (A2-29)
\]
is called the interval between the two events. (A2-28a) and (A2-28b) indicate that if the interval between two events is zero in one coordinate frame, it is also zero in all other frames.
The interval $ds$ between two events that are infinitesimally close to each other is

$$
ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

$$= - \left[ (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right]$$  \hspace{1cm} (A2-30)

From the formal point of view, $ds^2$ can be regarded as the square of the distance between two world points in Minkowski space. We may rewrite $ds^2$ in a more general form

$$
ds^2 = \sum_{\mu, \nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \hspace{1cm} (A2-30a)$$

where

$$g_{00} = 1, \ g_{11} = g_{22} = g_{33} = -1; \ g_{\mu\nu} = 0 \text{ if } \mu \neq \nu.$$  \hspace{1cm} (A2-30b)

The reader should be aware that the sign for the $g$’s is not standard. Others may define $-s_{12}$ as the interval between two events; if so, then $g_{00} = -1 \ g_{11} = 1$ etc.

If $ds = 0$ in frame $S$, then $ds' = 0$ in frame $S'$. Furthermore, $ds$ and $ds'$ are infinitesimal of the same order. It follows that $ds$ and $ds'$ must be proportional to each other

$$
ds^2 = ads'^2 \hspace{1cm} (A2-31)$$

where the proportionality constant $a$ may depend on the absolute value of the relative velocity of the two inertial frames. Owing to the homogeneity of space and time and the isotropy of space, the coefficient $a$ cannot depend on the coordinates or the
time or the direction of the relative velocity. Because of the complete equivalence of the two frames $S'$ and $S$, we also have

$$ds'^2 = ads^2.$$  \hspace{1cm} (A2-32)

Combining this with Equation A2-31, we find that

$$a^2 = 1 \quad \text{and} \quad a = \pm 1.$$  

It is natural to assume that the sign of the interval in all frames must be the same. Therefore the value of $a$ that is equal to $-1$ must be discarded. We thus arrive at the conclusion that

$$ds^2 = ds'^2.$$  \hspace{1cm} (A2-33)

From the equality of the infinitesimal intervals there follows the equality of finite intervals $s'^2 = s^2$, which can be expressed explicitly as

$$\sum_{\mu=0}^{3} (x'^{\mu})^2 = \sum_{\mu=0}^{3} (x^{\mu})^2.$$  \hspace{1cm} (A2-34)

This invariance of the interval between two events is the mathematical expression of the constancy of the velocity of light.

Equation (A2-34) is analogous to three-dimensional length-preserving orthogonal rotations, and indicates again that Lorentz transformations corresponding to a rotation in Minkowski space.

The invariance of the interval $ds^2$ is a very useful tool in many of its applications. The skillful use of this invariance often avoids an explicit Lorentz transformation. Some insight into the nature of the interval is gained by considering some special cases. First, we introduce the notations

$$t_{12} = t_2 - t_1, \quad d_{12} = (x_2^1 - x_1^1)^2 - (x_2^2 - x_1^2)^2 - (x_2^3 - x_1^3)^2.$$  

The interval between two events in frame $S$ now takes a simpler appearance:

$$s_{12}^2 = c^2 t_{12}^2 - d_{12}^2.$$  

If the two events occur at the same place in $S'$ frame, then $d_{12}' = 0$, and because of the invariance of the intervals, we have

$$s_{12}' = c^2 t_{12}'^2 - d_{12}'^2 = c^2 t_{12}^2 > 0,$$  \hspace{1cm} (A2-35)

and the interval is real. Real intervals are called timelike. The time interval between two events in $S'$ frame is

$$t_{12}' = \frac{s_{12}}{c} = \frac{\sqrt{c^2 t_{12}^2 - d_{12}^2}}{c}.$$
Every timelike interval that connects event 1 with another event lies within the light cones bounded by $x_1 = \pm ct$. All events that could have affected event 1 lie in the past light cone, and all events that event 1 is able to affect lie in the future light cone.

If the two events occur at one and the same time (simultaneously) in the $S'$ frame, then $t_{12}' = 0$, and we have

$$s_{12}'^2 = c^2 t_{12}'^2 - d_{12}'^2 = -d_{12}'^2 < 0,$$

and the interval is imaginary. Such an interval is called spacelike. There is no causal relationship between events 1 and 2. Every event that is connected with event 1 by a spacelike interval lies outside the light cone of event 1, and neither has interacted with event 1 in the past nor is capable of interacting with it in the future (Fig. B.8).

When two events can be connected with a light signal only, then

$$S_{12} = 0,$$

and such an interval is said to be lightlike. Events that can be connected with event 1 by lightlike intervals lie on the boundaries of the light cones.

The world line of a particle (the path of a particle in Minkowski space) must lie within its light cones. The division of intervals into spacelike, timelike, and lightlike intervals is, because of their invariance, an absolute concept. This means that the timelike, spacelike, or lightlike character of an interval is independent of the frame of reference.

### B.6.2 Four Vectors

By analogy with the three-dimensional case, the coordinates of an event $(x^0, x^1, x^2, x^3)$ can be considered as the components of a four-dimensional radius
vector, or, for short, a radius four-vector in a four-dimensional Minkowski space. The square of the length of the radius four-vector is given by
\[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\]
and it does not change under Lorentz transformations (or under any rotations of the four-dimensional coordinate system.)

In general, any set of four quantities \(A^\mu (\mu = 0, 1, 2, 3)\) that transforms like the components of the radius four-vector \(x^\mu\) under Lorentz transformations is called a contravariant four-vector:
\[
\begin{align*}
A^0 &= \gamma (A'^0 + \beta A'^1), \\
A^1 &= \gamma (A'^1 + \beta A'^0), \\
A^2 &= A'^2, \\
A^3 &= A'^3
\end{align*}
\]
(A2-38)
The square length of any four-vector is defined analogously to the radius four-vector,
\[
(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2.
\]
(A2-39)
The components of covariant four vectors \(A_\mu\) are related to contravariant vectors by the following equation:
\[
A_\mu = g_{\mu\nu} A^\nu
\]
(A2-40)
where \(g_{\mu\nu}\) is given by (A2-30b).

With the two types of four vectors, we can form the scalar product that is an invariant:
\[
\sum_{\mu=0}^{3} A_\mu A^\mu = A_\mu A^\mu.
\]
(A2-41)
The summation sign is eliminated in the last step by Einstein’s summation convention. The \(g_{\mu\nu}\) is a device to lower the indexes. Likewise, we can define \(g^{\mu\nu}\) to raise indexes. In the Cartesian coordinates used here
\[
g^{\mu\nu} = g_{\mu\nu}, \quad g_{\mu\nu} g^{\mu\sigma} = \delta^\sigma_\nu
\]
(A2-42)
where \(\delta^\nu_\mu\) is the kronecker delta symbol, \(\delta^\nu_\mu = 1\) if \(\mu = \nu\), and \(\delta^\nu_\mu = 0\) if \(\mu \neq \nu\).

We can define quantities \(A^{\mu\nu}\) or \(A_{\mu\nu}\) which, for each index, behave like a vector. Evidently such a quantity transforms like
\[
A'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} A^{\alpha\beta}
\]
(A2-43)
and is called a tensor of second rank or second-order tensor. A second-order tensor is said to be symmetric if \(A^{\mu\nu} = A^{\nu\mu}\), and antisymmetric if \(A^{\mu\nu} = -A^{\nu\mu}\). Tensors of higher rank are similarly defined:
\[
A'^{\mu\nu...\sigma} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \ldots \frac{\partial x'^\sigma}{\partial x^\lambda} A^{\alpha\beta...\lambda}.
\]
A partial differential operator behaves like a vector. This can be seen from its transformation equation. From the chain rule of differential calculus we get
\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial x'}{\partial x'}. \]

The coefficient can be read off from (A2-25),

\[ \frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial t'} + (-\beta \gamma c) \frac{\partial}{\partial x'} = \gamma \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right). \]  

(A2-44a)

Similarly

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial x'}{\partial x'} = \gamma \left( \frac{\partial}{\partial x'} - v c^2 \frac{\partial}{\partial t'} \right). \]  

(A2-44b)

For convenience, we will write \( \frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \) and so on.

Since the differentiation \( \partial_{\mu} = \partial/\partial x^{\mu} \) behaves as a vector, we can obtain new tensors by differentiating tensors, for example,

gradient: \( V_{\mu} = \partial_{\mu} \Phi \), curl: \( f_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \),

(A2-45a)

divergence: \( \alpha = \partial_{\mu} A^{\mu} \), d’Alembertian \( \lambda = \partial_{\mu} \partial^{\mu} \Phi \).

(A2-45b)

Under certain conditions, new tensors can also be formed by integration. To show this, consider the differentials

\[ d\Omega = dx^{0} dx^{1} dx^{2} dx^{3} = cdtdV, \quad dx^{\mu} = (dx^{0}, dx^{1}, dx^{2}, dx^{3}) \]

\[ dS_{\mu} = (dx^{2} dx^{3} dx^{0}, dx^{1} dx^{3} dx^{0}, dx^{1} dx^{2} dx^{0}, dx^{1} dx^{2} dx^{3}), \quad dS^{\mu\nu} = dx^{\mu} dx^{\nu}. \]

We can also construct the integral quantities, for example

\[ A = \int \Phi d\Omega \text{ scalar, } A = \int A_{\mu} dx^{\mu} \text{ scalar, } A^{\mu} = \int T^{\mu\nu} dS_{\nu}, \text{ vector } A = \int U_{\mu} dS_{\mu} \text{ scalar, etc.} \]

Among these expressions, the following are important:

- \( \int A_{\mu} dx^{\mu} \) line integration
- \( \int dx^{\mu} dx^{\nu} B_{\mu\nu} \) two-surface integration
- \( \int A^{\mu} dS_{\mu} \) three-surface integration
- \( \int \Phi d\Omega \) space-time integration.

There are theorems that enable us to transform four-dimensional integrals, analogous to the theorems of Gauss and Stokes in three-dimensional vector analysis. The integral over a closed hypersurface can be transformed into an integral over the four-volume contained within it by replacing the element of integration \( dS_{\mu} \) by the operator

\[ dS_{\mu} \rightarrow d\Omega \frac{\partial}{\partial x^{\nu}}. \]  

(A2-46)

For example, for the integral of a vector \( A^{\mu} \) we have

\[ \int A^{\mu} dS_{\mu} = \int \frac{\partial A^{\mu}}{\partial x^{\nu}} d\Omega. \]  

(A2-47)

This formula is the generalization of Gauss’ theorem. Thus, when \( \frac{\partial A^{\mu}}{\partial x^{\nu}} = 0 \), the result of integration is a true scalar and is independent of the choice of the three-surface.
B.6.3 Four-Velocity and Four-Acceleration

How do we define four vectors of velocity and acceleration? Obviously the set of the four quantities \( \frac{dx^\mu}{dt} \) doesn’t have the properties of a four-vector because \( dt \) is not an invariant. But the proper time \( d\tau \) is an invariant. Observers in different frames disagree about the time interval between events, because each is using his own time axis; all agree on the value of the time interval that would be observed in the frame moving with the particle. The components of the four-velocity are therefore are defined as

\[
u^\mu = \frac{dx^\mu}{d\tau}.
\]

(A2-48)

The second equation of (A2-17) relates the proper time \( d\tau \) (was \( dt' \) there) to the time \( dt \) read by a clock in frame S relative to which the object (S’ frame) moves at a constant \( u \):

\[
d\tau = dt\sqrt{1 - \beta^2}.
\]

We can rewrite \( u^\mu \) completely in terms of quantities observed in frame S as

\[
u^\mu = \frac{1}{\sqrt{1 - \beta^2}} \frac{dx^\mu}{dt} = \gamma \frac{dx^\mu}{dt}.
\]

(A2-49)

In terms of the ordinary velocity components \( v_1, v_2, v_3 \) we have

\[
u^\mu = (\gamma c, \gamma v_i), \; i = 1, 2, 3.
\]

(A2-50)

The length of four-velocity must be invariant, as shown by (A2-31)

\[
\sum_{\mu=0}^{3} (\nu^\mu)^2 = c^2.
\]

(A2-51)

Similarly, a four-acceleration is defined as

\[
\omega^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}.
\]

(A2-52)

Now differentiating (A2-51) with respect to \( \tau \), we obtain

\[
\omega^\mu u^\mu = 0
\]

(A2-53)

thus, the four-vectors of velocity and acceleration are mutually perpendicular.

B.6.4 Four-Momentum Vector

It is obvious that Newtonian dynamics cannot hold totally. How do we know what to retain and what to discard? This is found in the generalizations that grew from the
laws of motion but transcend it in their universality. These are the laws of conservation of momentum and energy. So we now generalize the definitions of momentum and energy so that in the absence of external forces the momentum and energy of a system of particles are conserved. In Newtonian mechanics the momentum \( \vec{p} \) of a particle is defined as \( m \vec{v} \), the product of particle’s inertial mass and its velocity. A plausible generalization of this definition is to use the four-velocity \( u^\mu \) and an invariant scalar \( m_0 \) that truly characterize the inertial mass of the particle and define the momentum four-vector (or four-momentum, for short) \( P^\mu \) as

\[
P^\mu = m_0 u^\mu.
\] (A2-54)

To ensure that the “mass” of the particle is truly a characteristic of the particle, it must be measured in the frame of reference in which the particle is at rest. Thus, the mass of the particle must be its proper mass. We customarily call this mass the rest mass of the particle and denote it by \( m_0 \). We can write \( P^\mu \) in terms of ordinary velocity \( v_i (i = 1, 2, 3) \)

\[
P^0 = \gamma m_0 c, \quad P^j = \gamma m_0 v_j, \quad j = 1, 2, 3 \quad (A2-55)
\]

where \( \gamma = (1 - \beta^2)^{-1/2} \). We see that as \( \beta = v/c \to 0 \), the spatial components of the four-momentum \( P^\mu \) reduce to \( m_0 v_j \), the components of the ordinary momentum. This indicates that (A2-48) appears to be a reasonable generalization.

Let us write the time component \( P^0 \) as

\[
P^0 = \frac{m_0 c}{\sqrt{1 - \beta^2}} = \frac{E}{c}. \quad (A2-56)
\]

Now, what is the meaning of the quantity \( E \)? For low velocities, the quantity \( E \) reduces to

\[
E = \frac{m_0 c^2}{\sqrt{1 - \beta^2}} \approx m_0 c^2 + \frac{1}{2} m_0 v^2.
\]

The second term on the right-hand side is the ordinary kinetic energy of the particle; the first term can be interpreted as the rest energy of the particle (it is an energy the particle has, even when it is at rest), which must contain all forms of internal energy of the object, including heat energy, internal potential energy of various kinds, or rotational energy if any. Hence we can call the quantity \( E \) the total energy of the particle (moving at speed \( v \)).

We now write the four-momentum as

\[
P^\mu = \left( \frac{E}{c}, P^j \right). \quad (A2-57)
\]

The length of the four-momentum must be invariant, just as the length of the velocity four-vector is invariant under Lorentz transformations. We can show this easily:

\[
\sum_\mu P^\mu P_\mu = \sum_\mu (m_0 u^\mu)(m_0 u_\mu) = m_0^2 c^2. \quad (A2-58)
\]
But (A2-57) gives
\[ \sum \mu P^\mu P^\mu = P^2 - \frac{E^2}{c^2}. \]

Combining this with (A2-58) we arrive at the relation
\[ P^2 - \frac{E^2}{c^2} = -m_0^2c^2 \quad \text{or} \quad E^2 - P^2c^2 = m_0^2c^4. \] (A2-59)

The total energy \( E \) and the momentum \( P^\mu \) of a moving body are different when measured with respect to different reference frames. But the combination \( P^2 - \frac{E^2}{c^2} \) has the same value for all frames of reference, namely \( m_0^2c^2 \). This relation is very useful. Another very useful relation is \( \vec{P} = \vec{v}(E/c^2) \). From (A2-56) we have \( \gamma m_0 = E/c^2 \); the second part of (A2-55) gives \( \vec{P} = m_0\vec{v}/\sqrt{1-\beta^2} \). Combining these two yields the very useful relation \( \vec{P} = \vec{v}(E/c^2) \).

The relativistic momentum, however, is not quite the familiar form found in general physics, because its spatial components contain the Lorentz factor \( \gamma \). We can bring it into the old sense, and the traditional practice was to introduce a “relativistic mass” \( m \):
\[ m = m_0\gamma = \frac{m_0}{\sqrt{1-\beta^2}}. \] (A2-60)

With this introduction of \( m \), \( P^j \) takes the old form: \( P^j = mv_j \). But some feel that the concept of relativistic mass often causes misunderstanding and vague interpretations of relativistic mechanics. So they prefer to include the factor \( \gamma \), with \( v_j \) forming the proper four-velocity component \( u_j \), and treating the mass as simply the invariant parameter \( m_0 \). For detail, see the article by Prof. Lev B Okun (The Concept of Mass, Physics Today, June 1989).

### B.6.5 The Conservation Laws of Energy and Momentum

It is now clear that the linear momentum and energy of a particle should not be regarded as different entities, but simply as two aspects of the same attributes of the particle, since they appear as separate components of the same four-vector \( P^\mu \), that transforms according to (A2-27):
\[ P'^\mu = L_{\nu\mu}P^\nu \]

where the Lorentz transformation matrix is given by (A2-26)
\[
(L_\nu^\mu) = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Thus,
\[ P'{}^0 = \gamma (P^0 - \beta P^1), \quad P'{}^1 = \gamma (-\beta P^0 + P^1), \quad P'{}^2 = P^2, \quad P'{}^3 = P^3. \] (A2-61)

We see that what appears as energy in one frame appears as momentum in another frame, and vice versa.

So far, we have not discussed explicitly the conservation laws. Since linear momentum and energy are not regarded as different entities but as two aspects of the same attributes of an object, it is no longer adequate to consider linear momentum and energy separately. A natural relativistic generalization of the conservation laws of momentum and energy would be the conservation of the four-momentum. Consequently, the conservation of energy becomes one part of the law of conservation of four-momentum. This is exactly what has been found to be correct experimentally, and, in addition, this generalized conservation law of four-momentum holds for a system of particles, even when the number of particles and their rest energies are different in the initial and final states. It should be emphasized that what we mean by energy \( E \) is the total energy of an object. It consists of rest energy that contains all forms of internal energy of the body and kinetic energy. The rest energies and kinetic energies need not be individually conserved, but their sum must be. For example, in an inelastic collision, kinetic energy may be converted into some form of internal energy or vice versa, accordingly the rest energy of the object may change.

Energy and momentum conservation go together in special relativity; we cannot have one without the other. This may seem a bit puzzling for the reader, for in classical mechanics the conservation laws of energy and momentum are on different footing. That is because energy and momentum are regarded as different entities. Moreover, classical mechanics does not talk about rest energy at all.

One of the consequences of the relativistic energy-momentum generalization is the possibility of “massless” particles, which possess momentum and energy but no rest mass. From the expression for the energy and momentum of a particle
\[ E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, \quad \vec{P} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \] (A2-62)

we can define a particle of zero rest mass possessing finite momentum and energy. To this purpose, we allow \( v \rightarrow c \) in some inertial system \( S \) and \( m_0 \rightarrow 0 \) in such a way that
\[ \frac{m_0}{\sqrt{1 - v^2/c^2}} = \chi \] (A2-63)
remains constant. Then (A2-62) takes the simple form
\[ E = \chi c^2, \quad \vec{P} = \chi c \hat{e} \]
where \( \hat{e} \) is a unit vector in the direction of motion of the particle. Eliminating \( \chi \) from the last two equations, we obtain
\[ E = P c, \] (A2-64)
which is consistent with (A2-59): \( E^2 - P^2 c^2 = m_0^2 c^4 \).
Now, as \((E/c, \vec{P})\) is a four-vector, \((\chi c, \chi \hat{c})\) is also a four-vector, the energy and momentum four-vector of a zero rest-mass particle in frame \(S\) and in any other inertial frame such as \(S'\). It can be shown that the transformation of the energy and momentum four-vector \((\chi c, \chi \hat{c})\) of a zero rest-mass particle is identical with that of a light wave, provided \(\chi\) is made proportional to the frequency \(\nu\). Thus if we associate a zero rest-mass particle with a light wave in one inertial frame, it holds in all other inertial frames. The ratio of the energy of the particle to the frequency has the dimensions of action (or angular momentum). This suggests that we can write this association by the following equations

\[
E = h\nu \quad \text{and} \quad P = \chi c = h\nu/c
\]

where \(h\) is Planck’s constant. This massless particle of light is called a photon in modern physics, introduced by Einstein in his paper on the photoelectric effect.

### B.6.6 Equivalence of Mass and Energy

The equivalence of mass and energy is the best-known relation Einstein gave in his special relativity in 1905:

\[
E = mc^2 \tag{A2-65}
\]

where \(E\) is the energy, \(m\) the mass, and \(c\) is the speed of light.

We can get this general idea of the equivalence of mass and energy from the consideration of electromagnetic theory. An electromagnetic field possesses energy \(E\) and momentum \(p\), and there is a simple relationship between \(E\) and \(p\):

\[
P = E/c.
\]

Thus, if an object emits light in one direction with momentum \(p\), in order to conserve momentum, the object itself must recoil with a momentum \(-p\). If we stick to the definition of momentum as \(p = mv\), we may associate a “mass” with a flash of light:

\[
m = \frac{p}{v} = \frac{p}{c} = \frac{E}{c^2}
\]

which leads to the famous formula

\[
E = mc^2.
\]

This mass is not merely a mathematical fiction. Let us consider a simple thought experiment, provided by Einstein some time ago. Imagine that an emitter and absorber of light is firmly attached to the ends of a box of mass \(M\) and length \(L\). The box is initially stationary, but is free to move. If the emitter sends a short light pulse of energy \(\Delta E\) and momentum \(\Delta E/c\) toward the right, the box will recoil toward the left by a small distance \(\Delta x\), with momentum \(p_x = -\Delta E/c\) and velocity \(v_x\), where \(v_x\) is given by
\[ v_x = \frac{p_x}{M} = -\Delta E/cM. \]

The light pulse reaches the right end of the box approximately in time \( \Delta t = L/c \) and is absorbed. The small recoil distance is then given by

\[ \Delta x = v_x \Delta t = -\Delta E L/Mc^2. \]

Now, the center of mass of the system cannot move by purely internal changes and there are no external forces. It must be that the transport of energy \( \Delta E \) from the left end of the box to the right end is accompanied by transport of mass \( \Delta m \), so the change in the position of the center of mass of the box (denoted by \( \delta x \)) vanishes. The condition for this is

\[ \delta x = 0 = \Delta m L + M \Delta x \]

from this we find

\[ \Delta m = -\frac{M}{L} \Delta x = \frac{M \Delta E L}{L M c^2} = \frac{\Delta E}{c^2}, \]

or

\[ \Delta E = \Delta m \cdot c^2. \]

We should not confuse the notions of equivalence and identity. The energy and mass are different physical characteristics of particles; “equivalence” only established their proportionality to each other. This is similar to the relation between the gravitational mass and inertial mass of a body; the two masses are indissolubly connected with each other and proportional to each other, but are at the same time different characteristics. The equivalence of mass and energy has been beautifully verified by experiments in which matter is annihilated and converted totally into energy. For example, when an electron and a positron, each with a rest mass \( m_0 \) come together, they disintegrate and two gamma rays emerge, each with the measured energy of \( m_0 c^2 \).
Based on Einstein’s mass-energy relation $E = mc^2$, we can show that the mass of a particle depends on its velocity. Let a force $F$ act on a particle of momentum $mv$. Then,

$$F dt = d(mv) \quad (A2-66)$$

If there is no loss of energy by radiation due to acceleration, then the amount of energy transferred in $dt$ is

$$dE = c^2 dm$$

This is put equal to the work done by the force $F$ to give

$$Fv dt = c^2 dm$$

Combining this with (A2-66), we have

$$v d(mv) = c^2 dm$$

Multiply this by $m$:

$$vmd(mv) = c^2 mdm,$$

integrating

$$(mv)^2 = c^2 m^2 + K.$$  

$K$ is a integration constant. Now $m = m_0$ as $v \rightarrow 0$, we find $K = -c^2 m_0^2$, and

$$m^2 v^2 = c^2 (m^2 - m_0^2).$$

The $m_0$ is known as the rest mass of the particle. Solving for $m$ we obtain (A2-60)

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}}.$$  

It is now easy to see that a material body cannot have a velocity greater than the velocity of light. If we try to accelerate the body, as its velocity approaches the velocity of light its mass becomes larger and larger as it becomes more difficult to accelerate it further. In fact, since the mass $m$ becomes infinite when $v = c$, we can never accelerate the body up to the speed of light.

As mentioned earlier, however, in the language of relativity theory and high-energy physics there is a trend to treat the mass as simply the invariant parameter $m_0$.

**B.7 Problems**

**B.1.** Observer O notes that two events are separated in space and time by $600 m$ and $8 \times 10^{-7} s$. How fast must Observer O’ be moving relative to O in order that the events be simultaneous to O’?
B.2. A meterstick makes an angle of 30° with respect to \( x' \)-axis of \( O' \). What must be the value of \( v \) if the meterstick makes an angle of 45° with respect to the \( x \)-axis of \( O \)?

B.3. Find the speed of a particle that has a kinetic energy equal to exactly twice its rest mass energy.

B.4. Find the law of transformation of the components of a symmetric four-tensor \( T^{\mu \nu} \) under Lorentz transformations.

B.5. A man on a station platform sees two trains approaching each other at the rate 7/5 \( c \), but the observer on one of the trains sees the other train approaching him with a velocity 35/37 \( c \). What are the velocities of the trains with respect to the station?

B.6. The equation for a spherical pulse of light starting from the origin at \( t = t' = 0 \) is

\[ c^2 dt^2 - x^2 - y^2 - z^2 = 0. \]

Show from the Lorentz transformations that \( O' \) will observe this same pulse as spherical, in accordance with Einstein’s postulate stating that the velocity of light is the same for all observers.

B.7. Referring to Figure B.4, frame \( S' \) moves with a velocity \( u \) relative to frame \( S \) along the \( x \) axis. A pair of oppositely charged plates is at rest in \( S' \) frame in a direction parallel to \( x' \) axis, and the field \( E \) between the plates is perpendicular to the plates (i.e., \( \perp \) to the \( x' \) axis) and has a value that depends on the charge density \( \sigma \) on the plates: \( E = \sigma/\varepsilon_0 \). Show that view from frame \( S \) in which the plates are now moving in the \( x \) direction with a velocity \( u \), the field \( E' \) is given by

\[ E' = (1 - u^2/c^2)^{-1/2} E. \]

B.8. Referring to the previous problem, now the plates are at rest in frame \( S' \) along the \( y' \) axis. Find the electric field in both frames.

B.9. A large metallic plate moves at a constant velocity \( \vec{v} \) perpendicular to a uniform magnetic field \( \vec{B} \). Find the surface charge density induced on the surface of the plate.

B.10. A point charge \( q \) moves at constant velocity \( \vec{v} \). Using the transformation formulas, find the magnetic field of this charge at a point whose radius vector is \( \vec{r} \).

**References**


Index

Absolute space 234
Accretion disk 102
Accelerating universe 156
Adiabatic expansion 127
Affine
  connections 28
  space 21
Antimatter
  mystery of 218
Area theorem 93
Baryons 165
Bending of a light beam 7
Big Bang 123
Birkhoff’s theorem 57
Blackbody radiation 125
Black hole,
  detection of 101
  entropy of 95
  event horizon 83
  ergosphere 91
  inside of 84
  formation of 86
  Kerr-Newman 89
  mini 86
  no-hair theorem 95
  quantum mechanics of 97
  static limit 91
  supermassive 104
  surface gravity 83
  thermodynamic s of 95
Black holes & particle physics 107
Black holes, physics of 81
Boson 164, 165
Christoffel symbols
  first kind 28
  second kind 28
  metric tensors and 28
  symmetry properties of 27
Closed model (of universe) 144
COBE 125, 127
Color force 170
Co-moving coordinate 135
Condition for flat space 36
Contraction of tensor indices 20
Contravariant derivative 26
Contravariant vector 19
Conservation of photon numbers 206
Conservation law
  baryon number 176
  lepton number 177
  of energy 270
  of momentum 270
Correspondence principle 45
Cosmic fluid 133
Cosmic background radiation 125
Cosmology 111
  the renaissance of 113
Cosmological constant /Lambda 157, 158
Cosmological principle 111
Cosmological redshift 124
Covariant derivative 25
Covariant differentation 23
Covariant vector 20
CP violation 219
Critical density 159
Curie temperature 179
Curvature 10
  negative 135
  positive 135
  scalar 140
Curvature (Riemann) tensor 32
Curved spacetime 8
Curvilinear coordinates 19
Cygnus X-1 104
D’Alembertian 47
Dark energy 158
Dark matter 148, 221
  cold 225
  hot 224
Deceleration parameter $q_0$ 151
Deflection of light rays
  in gravitational field 71
Density parameter 159
Distance interval 13
Doppler shift 120, 127, 259
Eddington-Finkelstein coordinates 85
Einstein-de Sitter model
Einstein’s equation
  a Heuristic derivation 46
Einstein’s law of gravitation 45
Einstein’s summation convention 19
Einstein tensor 34
Electroweak force 173
Energy density of vacuum 159
Energy extraction from black holes
  Penrose process 92
  from two coalescing black holes 94
Energy-momentum tensor 49, 51
Entropy 95, 152
Equation of geodesic deviation 37
Equivalence of
  gravitation and acceleration 6
Equivalence of mass and energy 272
Equivalence, principle 3
  strong 5
  weak 5
Era
  GUTs 213
  inflationary 213
  hadron 214
  lepton 215
  nuclear 216
Experimental tests of Einstein’s theory 65
Event horizon 83
False vacuum 180, 192, 194
Fermions 164
Feynman diagrams 165
Flat model (of universe) 143
Flatness problem 187
Four vector 265
  four-acceleration 268
  four-momentum 268
  four-velocity 268
  contravariant four vector 266
Four-momentum vector
Friedmann’s equations 139, 142
  solutions of 142
Frozen stars 84
Fundamental forces (interactions) 171
Fundamental observer 133
Galilean coordinates, 10, 26
Galilean transformation 233
Geodesic 29
  coordinate system 29
  curve 29
  deviation 37
  effect 91
  null 30
  Schwarzschild 60
  space-like 30
  stationary property of 30
  time-like 30
Glouon 170
Grand unification energy 182
Gravitational lens 75
Gravitational mass 4
Gravitational shift of spectral lines 8
Gravitational radiation 52
  from Binary pulsar PSR 1913+16) 77
Gravitational radius 58
Gravitational redshift 8
Graviton 53
  longitudinal 107
  mass 53
  spin of 53
  transverse 107
Gravity Probe B 91
GUT 182
H-R diagram 129
Hadrons 165
Hamilton’s principle 240
Hawking effect 97
Heisenberg uncertainty principle 97
Higgs particle (boson) 159, 165
  possible production of 176
Horizon distance 189
Horizon problem 188
Hubble’s law 120
Hubble constant 120
Hulse-Taylor pulsar (PSR 1913+16) 77
HUT (Hopkins Ultraviolet Telescope) 130
Inertial frames of reference 3
  local 6
  Lense-Thirring dragging of 91
Inertial mass 4
Inertial force 7
Inflationary theory (Alan Guth) 191
  The successes of 195
Intermediate vector bosons 165
Interval 262
  lightlike 265
  spacelike 265
solution of vacuum field equation 56
Shapiro experiment 75
Singularities
  coordinate 83
  intrinsic 86
  Naked 91
Space
  affine 21
  metric 21
Special theory of relativity 245
  postulates of 249
Spin 163
Spontaneous symmetry breaking 177, 178
Standard model of particle physics 163
Static curvature coordinates 57
Stefan-Boltzmann law 99
String theory 108
Supermassive black holes 104
Surface gravity of black hole 83
Symmetry properties of Christoffel symbols 27
Temperature
  of black hole 99
  of cosmic background radiation
Tensor
  contravariant 20
  covariant 20
  curvature 32
  fundamental 21
  four-momentum
Einstein 34
Energy-momentum 49, 51
metric 10, 21
mixed 20
rank (or order) of 20
Ricci 34
Riemann 33
skew-symmetric 22
symmetric 22
Threshold temperature 210
Time dilation 254
Time’s arrow 152
  thermodynamic 152
  cosmic 153
Time interval 13
Transition temperature 207
Turbulent viscosity 102
Unification of forces 180
Vacuum energy density 159
Vacuum field equation 46
Vacuum fluctuation 98, 170
Vacuum polarization 170
Vector
  contravariant 19
  covariant 20
  four
  four-momentum
Velocity transformation 257
WIMP 224