Tune and chromaticity diagnostics

R. J. Steinhagen
CERN, Geneva, Switzerland

Abstract
Tune and chromaticity, the oscillation frequency of the transverse motion and its dependence on the particle’s momentum, are important observables that are used to estimate, measure, and control beam stability in circular accelerators. They are also the first, low-order, non-trivial beam parameters that cannot be directly derived by a single beam instrument. This contribution summarizes diagnostics methods to estimate tune and chromaticity and aims to highlight less obvious key experimental aspects.

1 Introduction
The beam particle’s tune, defined as the frequency of the transverse oscillation, and chromaticity as its dependence on particle momentum, require precise control as they may drive self-amplified instabilities leading to an increase in beam size, chaotic particle motion, and therefore, particle loss. Furthermore, since tune and chromaticity can be measured with great precision and small systematic errors, they provide valuable diagnostics for a variety of higher-order beam physics processes such as measurement of beam optics, effects related to machine impedance, beam–beam, electron-cloud, and other collective effects.

The concept of tune and chromaticity in particle accelerators finds analogies in other physics domains such as acoustics (e.g., musical string vibration) or light (e.g., refraction and decomposition of light according to its wavelength in prisms and lenses). Similar to the refraction index $n$ that defines the bending angle of light on its wavelength (or equivalently its photon energy), in accelerators chromaticity describes the dependence of the tune on momentum. The range of tunes providing stable beam conditions is typically limited. Owing to the intrinsic momentum spread of every particle beam, large values of chromaticity may also create undesirable large tune spread.

In contrast to other beam parameters such as trajectory, orbit, beam current, and beam size, tune and chromaticity are the first non-trivial beam parameters that cannot be derived from a single individual measurement. They typically rely on a coherent beam excitation, followed by measurement of the driven oscillation, and post-processing, and in the case of chromaticity, an adjustment of the accelerator’s radio-frequency (RF) cavities.

2 Review of linear transverse particle motion
This section summarizes the basic principles and formalism required to describe longitudinal and transverse particle dynamics and its control in a circular accelerator. Further information can be found in Refs. [1–6]. In order to describe particle motion in storage rings, a local co-rotating coordinate system is defined, describing the perturbed particle motion in relation to the design particle’s trajectory, as illustrated in Fig. 1. By convention, the local axis $x$ usually points outwards and the axis $y$ points upwards, with respect to the closed trajectory. The local longitudinal axis $s$ is parallel to the propagation direction of the reference particle. This coordinate system implicitly assumes the periodicity of the local coordinates $x$ and $y$, the local bending radius $\rho$, and various multipole strengths $b_n(s)$:

$$x(s + C) = x(s), \quad y(s + C) = y(s), \quad \rho(s + C) = \rho(s), \quad \text{and} \quad b_n(s + C) = b_n(s),$$

(1)

with $C$ being the accelerator circumference. Unless otherwise stated, primed notation $x'$ is used to indicate a differentiation $x' := \partial x/\partial s$ with respect to the longitudinal coordinate $s$. Using this local
Fig. 1: Co-rotating local coordinate system. The circular design trajectory (red) and an exemplary perturbed trajectory (blue) are shown. The local bending radius $\rho(s)$ is a function of the path length $s$ inside the accelerator.

co-rotating coordinate system, the perturbed particle motion is described by Hill’s equation, which can be written, expanding only leading terms, as

$$
\begin{align*}
\ddot{x} + \left( k(s) + \frac{1}{\rho^2(s)} \right) \dot{x} &= \frac{1}{\rho(s)} \frac{\Delta p}{p} + f_x(s,t) \\
\ddot{y} - k(s) \cdot y &= 0 + f_y(s,t)
\end{align*}
$$

with $\rho(s)$ being the local bending radius of the main dipoles and

$$k(s) = \frac{q}{p} \cdot \frac{dB}{dx} = \frac{dB/dx}{(B\rho)} = -\frac{dB/dy}{(B\rho)},$$

the normalized local quadrupole strength, $\Delta p/p$ the particle momentum offset with respect to the reference particle, and $f_x/y(s,t)$ perturbation terms. The quadrupole strength $k(s)$ is positive for focusing quadrupoles and negative for defocusing ones. The multipole strengths are commonly normalized by the magnetic rigidity $(B\rho) = p/q$ to make the equations independent of the design particle’s absolute momentum $p$ and charge $q$. Positive $k(s)$ quadrupole strengths in one plane intrinsically correspond to negative strengths in the other plane but with the same magnitude.

2.1 Betatron motion

The homogeneous solution, that is, the momentum- and perturbation-independent part of Eq. (2), is commonly solved using Floquet’s theorem through the following oscillatory combined amplitude–phase-modulation ansatz

$$z_\beta(s) = \sqrt{\epsilon_\beta(s)} \cdot \cos(\Delta \mu z(s) + \phi_z),$$

with $z$ being either the $x$ or $y$ coordinate and $\epsilon_\beta$ and $\phi_z$ the initial particle conditions. Where required, the optical functions and parameters are marked with subscripts (e.g., $\beta_x$ and $\beta_y$) to distinguish between
planes. Both lattice parameters, the betatron function $\beta(s)$ and betatron phase $\mu(s)$, are always positive. The betatron phase advance $\Delta \mu(s)$ depends on the betatron function and is defined as

$$\Delta \mu(s) = \int_{s_0}^{s} \frac{1}{\beta(s')} \, ds', \quad (5)$$

with $s_0 \leq s$ being an arbitrary reference location in the ring and $C$ its circumference. Since $\beta(s)$ is positive, $\Delta \mu$ is positive and monotonic increasing, according to the definition in Eq. (5).

The momentum-wise, transverse sorting of charged particles and dipolar field imperfections are intrinsic properties of all magnets in an accelerator. One distinguishes two momentum effects — the momentum spread of the beam distribution that defines the beam width and the systematic average beam momentum shift. Taking into account these particular solutions, the general solution can be written as

$$z(s) = z_{co}(s) + D(s) \cdot \frac{\Delta p}{p} + z_\beta(s), \quad (6)$$

with $z_{co}(s)$ the closed-orbit, which, on the timescale of tune oscillation, can be considered as static offset, $\Delta p/p$ being the individual particle’s momentum offset, $D(s)$ the periodic, so-called dispersion function, and $z_\beta(s)$ the betatron oscillations given by Eq. (4). The dispersion function, describing the systematic shift of the transverse equilibrium beam position due to a systematic average beam momentum shift, is usually defined for a normalized momentum error of $\Delta p/p = 1$. For most accelerators, the dispersion function has values in the range of a few centimetres up to two metres. Owing to the absence of strong bending magnets in the vertical plane, the vertical dispersion function is usually much smaller than in the horizontal plane.

Generally, with the exception of a few special cases, the betatron function $\beta(s)$, phase advance $\mu(s)$, and dispersion function $D(s)$ cannot be solved analytically but rely on numeric estimates with tools such as MAD or others described in Refs. [7–11]. The ensemble of the above lattice functions are referred to as Twiss parameters, after R. Q. Twiss, who first introduced them to describe particle motions in proton synchrotrons [12].

### 2.2 Tune and perturbation sources

As a measurable observable in storage rings, one defines the number of transverse oscillations a particle describes per turn as tune $Q$ of the machine\(^1\), which is related to the total phase advance of the machine via

$$Q := \frac{1}{2\pi} \oint_C \mu(s) \, ds, \quad (7)$$

with $C$ being the circumference of the machine. Using Eq. (4), the tune is thus determined by the number of transverse oscillations that the particles describe during one revolution, as illustrated in Fig. 2.

The tune is further split into an integer $Q_{\text{int}}$ and non-integer (fractional) part $q_{\text{frac}}$:

$$Q = Q_{\text{int}} + q_{\text{frac}}. \quad (8)$$

A direct measurement of the full integer and fraction of the tune is usually performed using injection and single-turn trajectories using all available beam position monitors in the ring, as indicated in Fig. 2. A tune is given either by counting the oscillation periods or, more precisely, by performing a harmonic analysis along the longitudinal coordinate $s$. Besides the described method, most other diagnostics methods measure only the fractional part of the tune $q_{\text{frac}}$. In order to avoid resonances due to magnetic imperfections and, consequently, amplitude growth of the particles’ oscillations inside the machine, the tune is usually a non-integer and ideally an irrational non-fractional number. Additionally, the horizontal

---

\(^1\)Americans often denote it by $\nu$. 
and vertical tunes usually differ in order to avoid coupling resonances between planes. One defines \( \Delta = Q_x - Q_y - p \) as the fractional unperturbed tune-split, with \( p \) an integer. In the Large Hadron Collider (LHC) at CERN, the tune-split is, for example, \( \Delta = 64.28 - 59.31 - 5 = -0.03 \) during injection and \( \Delta = 64.31 - 59.32 - 5 = -0.01 \) during collision. As shown in Ref. [6], it is further required to avoid the following resonance condition

\[
m \cdot Q_x + n \cdot Q_y = p
\]  

with \( m, n, \) and \( p \) being arbitrary integers. The order of the resonance is given by \( |m| + |n| \). Figure 3(a) shows a tune diagram with resonance lines plotted up to the twelfth order. As a rule of thumb, lower order resonances, unless compensated separately, usually have faster instability growth times than higher order ones. It is visible that (including all up to twelfth order resonances) many tune working points are excluded and that the largest resonance-free spaces are usually close to lower order resonances which are to be avoided. Nevertheless, many lepton accelerators have their tune working points close to half- or third-order resonances, as the intrinsic electron synchrotron-radiation damping time is typically much less than the growth time of higher than fourth-order resonances. However, for hadron accelerators such as the LHC or SpS the negligible radiation damping makes it necessary to avoid resonances up to the twelfth order as indicated in Fig. 3(b). The solid surfaces show the effective estimated tune width and the hatched surfaces show the tune width including the tolerance margins for control. Figure 4(a) shows the expected uncompensated tune drifts in the LHC based on offline magnet calibration measurements that have been performed on a subset of LHC magnets. Compared to the injection and collision tune tolerance of 0.003 and 0.001 respectively, it is visible that these perturbations exceed the required tolerance by orders of magnitude. Since some of these perturbations are known from the magnet test bench measurements, it is expected that they may be partially compensated. In any case, to reach the final tolerances, beam-based measurements and automatic control of the tune are needed.

Some of the tune perturbation sources are (sorted according to relative magnitude):

- magnetic field errors of quadrupole magnets due to magnet calibration errors, decay and snap-back of persistent fields in superconducting magnets, power-converter ripple, saturation of the magnet yoke;
- momentum shifts due to, for example, a systematic mismatch between reference momenta of main dipole and main quadrupole magnets, mismatch of the injection energy with respect to the matched storage ring energy;
- feed-down effects due to off-centre orbits in higher-order magnets (e.g., horizontal offsets in lattice sextupoles) creating magnetic quadrupole components; and
Fig. 3: Full tune diagram up to the twelfth order and zoomed to the LHC tune working point. First and second order resonance lines are plotted red, third are plotted blue, and the higher orders with brightening shades of grey. The LHC injection (green, \(q_x = 0.28, q_y = 0.31\)) and collision tune working points (blue, \(q_x = 0.31, q_y = 0.32\)) are indicated. The solid surfaces show the effective estimated tune width and the hatched surfaces show the tune width including the tolerance margins. The red dotted circles describe the maximum allowed tune tolerance during particle injection.

- higher-order collective effects such as beam optics, machine impedance, beam–beam, and electron-cloud.

With the exception of collective effects, most tune perturbations are caused by quadrupolar field errors \(\Delta k(s)\) that, in a thin-lens approximation, create tune shifts described by Ref. [1]:

\[
\Delta Q = \frac{1}{4\pi} \int \beta(s) k(s) \, ds \Delta p.
\]  

(10)

It is visible that quadrupole errors at locations with large betatron function values are more critical, particularly at the final focus quadrupole magnets, as these have both large quadrupole gradients and large values of the betatron function.

Another source for tune shifts arises if the beam momentum does not match the reference momentum for which the magnet was designed. In this case the focusing strength depends on the particle momentum, as illustrated in Fig. 5. Assuming small momentum errors, \(p\) can be replaced by \(p \rightarrow p_0 + \Delta p\), using the Maclaurin series for \(\frac{1}{p} = 1 - x + x^2 + \text{higher order terms (h.o.)}\) and inserting in Eq. (3) can be reduced to a similar form as Eq. (10):

\[
k(s) = \frac{q}{p} \frac{\partial B}{\partial z}
\]

\[
= k_0(s) - k_0(s) \cdot \frac{\Delta p}{p} + k_0(s) \cdot \left( \frac{\Delta p}{p} \right)^2 + \text{h.o.}.
\]  

(11)

It is visible that the off-momentum quadrupole strength can be decomposed into a series starting with the nominal unperturbed field \(k_0(s) \frac{\Delta p}{p}\) and a series of momentum-shift-dependent strengths \((-1)^n k_0(s) \frac{\Delta p}{p}\). Inserting the sources of Eq. (11) into Eq. (10) yields

\[
\Delta Q = \frac{1}{4\pi} \int_C \beta(s) k(s) \, ds \cdot \frac{\Delta p}{p} + \text{h.o.}.
\]  

(12)
Fig. 4: Expected uncompensated LHC tune and chromaticity drift and corresponding drift velocities. For comparison, the nominal requirements are \( \Delta Q < 0.001 \) and \( Q' = 2 \pm 1 \) which are orders of magnitude tighter than these perturbations. Parts of these perturbations will be partially compensated via feed-forward systems.

\[ f_x(s) = \kappa(s) \cdot y \]
\[ f_y(s) = \kappa(s) \cdot x \]  

(13)

and skew-quadrupole gradient

\[ \kappa(s) = \frac{q}{2p} \left\{ \frac{\partial B}{\partial x} - \frac{\partial B}{\partial y} \right\}. \]  

(14)

It is visible that with the driving term in Eq. (13), the horizontal motion becomes dependent on the vertical motion and vice versa due to the \( y' \) (\( x' \)) term in the horizontal (vertical) driving term. This coupling is commonly referred to as betatron coupling and is described by the lattice-dependent parameter \( C^- \) via

\[ C^- = \left| C^- \right| e^{i\xi} = \frac{1}{2\pi} \int \sqrt{\frac{\partial f_x}{\partial y} \kappa(s)} e^{i(\mu_x - \mu_y - 2\pi\Delta)} ds, \]  

(15)

with \( \xi \) being the angle of the eigensystem rotation, and \( \Delta = q_x - q_y \) the positive definite distance of the unperturbed tunes (which are the tunes in the absence of coupling). The coupling has a fundamental impact on the control of the accelerator. The importance of betatron coupling lies in the fact that it may compromise the control of a plane-dependent, transverse beam parameter by rotating the initially orthogonal horizontal and vertical tune eigensystems with respect to the lattice magnet reference coordinate system. For practical purposes, the tune eigenmodes are rotated by 45 degrees with respect to

Fig. 5: Schematic quadrupole focusing dependence on the particle momentum
the reference horizontal plane if $|C^-| \gg \Delta$. In this case, a horizontal dipole kick would be seen with the same r.m.s. oscillation amplitude in both planes, which would render impossible any orbit, tune, or chromaticity control.

The new observable oscillation eigenmode frequencies $Q_1$ and $Q_2$ are given by

$$Q_{1,2} = \frac{1}{2} \left( q_x + q_y \pm \sqrt{\Delta^2 + |C^-|^2} \right).$$

(16)

Here the convention is that for small betatron-coupling ($|C^-| \approx 0$), eigenmode $Q_1$ relates to the horizontal tune, and eigenmode $Q_2$ to the vertical tune. It is visible that the minimum distance of these eigenmodes is equal to $|C^-|$, which provides a first coupling measurement estimate, as discussed in the following section. A thorough treatment and recommended reading concerning the exact betatron-coupling formalism can be found in Ref. [6].

2.3 Chromaticity and perturbation sources

The dependence of the tune on chromaticity, visible in Eq. (12), cannot only be applied to a single particle, but also to a particle population forming a beam. In this case, the momentum spread relates to the tune spread of a beam. This dependence is commonly referred to as chromaticity, which in its general form can be written as

$$Q_n^{(n)} := \frac{\partial^{(n)} Q}{\partial \delta} \delta \quad \text{with} \quad \delta := \frac{\Delta p}{p},$$

(17)

with $n$ being the order of chromaticity. For a given, specific order of chromaticity, the index $(n)$ is usually replaced with a prime notation. For instance, the first order of chromaticity is denoted as $Q'$, the second order is denoted as $Q''$, the third order is denoted as $Q'''$, and so on. Sometimes, especially for smaller accelerators, the chromaticity is normalized to the machine tune $Q$ and denoted by $\xi = Q/Q$. Inserting the tune perturbation of Eq. (12) into Eq. (17) yields the first-order definition of the so-called natural chromaticity $Q_n^{(n)}$ which is inherent to all strong-focusing accelerators:

$$Q_n^{(n)} := \frac{\Delta Q}{\Delta p/p} = -\frac{1}{4\pi} \oint k(s)\beta(s)ds.$$

(18)

For most strong-focusing, circular accelerators, quadrupole strengths are predominantly negative and the betatron function is by definition positive, hence the natural chromaticity is typically negative. As is visible in Eq. (18), the natural chromaticity depends on the size (number of quadrupole magnets) and focusing strength of the machine. The relative beam momentum spread $\Delta p/p$ is usually of the order of $10^{-3}$, which, combined with the usually large natural chromaticity, would lead to a relative tune spread $|\Delta Q/Q|$ of the order of $10^{-2}$. This rather large tune spread would inevitably force some particles to oscillate on some high-order resonances, eventually leading to growth of amplitude and particle loss. In addition, since negative chromaticities lead to the so-called head–tail instability for machines operating above transition [13], this chromaticity needs to be compensated for and is usually set at around zero and slightly positive. In the LHC, a working point for the chromaticity is between 2 and 10 and is required to be controlled on the level of ±1.

The compensation is typically performed at locations in the ring where the particles are sorted according to their momenta, which are essentially locations with non-vanishing values of the dispersion function $D(s)$. The compensation itself is established with sextupole magnets. Their driving term can be expressed as

$$f_x(s) = \frac{1}{2}m(s) \cdot (x^2 - y^2)$$

$$f_y(s) = +m(s) \cdot xy$$

(19)

with the sextupole strength

$$m(s) = \frac{q}{p} \frac{\partial^2 B}{\partial x^2}.$$

(20)
In order to illustrate the sextupole function, it is useful to develop Eq. (20) around the individual particle trajectory with momentum shift $\Delta p/p$ and non-momentum-related betatron oscillation amplitude $x_\beta(s)$, by replacing the unperturbed $x$ coordinate with $x \rightarrow D(s) \cdot \frac{\Delta p}{p} + x_\beta(s)$ which yields

$$f_x(s) = \frac{1}{2} m(s) \cdot \left[ \left( D(s) \cdot \frac{\Delta p}{p} + x_\beta(s) \right)^2 \right]$$

$$= \frac{1}{2} m(s) \cdot \left[ \left( D(s) \cdot \frac{\Delta p}{p} \right)^2 + 2 \left( D(s) \cdot \frac{\Delta p}{p} \right) \cdot x_\beta(s) + x_\beta^2(s) \right]$$

$$= m(s) \left( D(s) \cdot \frac{\Delta p}{p} \right) \cdot x_\beta(s) + \frac{1}{2} m(s) \cdot \left( D(s) \cdot \frac{\Delta p}{p} \right)^2 + \frac{1}{2} m(s) \cdot x_\beta^2(s).$$

Comparing with Eq. (2), one can see [Eq. (23)] that the first term has the same form as a momentum-perturbed quadrupole lens. Using this dependence and matching

$$m(s) \cdot D(s) \approx k(s)$$

with $k(s)$ being the nominal lattice quadrupole strength, yields the first-order compensation of the natural chromaticity. Combining the first term of Eq. (23) and Eq. (18), the total chromaticity including the sextupole magnets is given by

$$Q' = \pm \frac{1}{4\pi} \oint \beta(s) \left[ k(s) + m(s)D(s) \right] ds.$$  

The sign depends on the plane: ‘−’ for the horizontal and ‘+’ for the vertical plane as quadrupoles ($k(s)$) and sextupoles ($m(s)$) are focusing in one plane and at the same time defocusing in the other.

It is worth while to also discuss the additional expansion terms in Eq. (23) quantitatively, as these are often (hastily) discarded while describing the compensation of the first-order chromaticity, and not re-included for the discussion of higher-order effects:

Firstly, comparing the second term in Eq. (23) with the chromaticity definition (17), it is visible that sextupoles are the first type of non-linear magnets that not only compensate first-order chromaticity but also intrinsically introduce second-, third- and other higher-order chromaticities. Please note that, for better readability, the higher-order expansion terms of the form $\left( \frac{\Delta p}{p} \right)^n$ of the sextupole strength $k(s)$ have been omitted in Eq. (23).

Secondly, in most accelerators, the dispersion function $D(s)$ typically has values of the order of a metre, the momentum perturbation of the order of $10^{-4}$ and, the betatron oscillation of the order $10^{-4}$ m. Thus the momentum-related oscillations $D(s) \frac{\Delta p}{p}$ and sextupole focusing are about the same order as the pure betatron oscillations and amplitude detuning. Assuming that the natural chromaticity is constant and the effective chromaticity primarily affected by sextupolar terms, the decoherence time-constant introduced by the third term $+\frac{1}{2} m(s) \cdot x_\beta^2(s)$ in Eq. (23) is linearly dependent on sextupole strength. Since, visible in Eq. (25), the first-order chromaticity itself is linearly dependent on sextupole strength, it is possible to use the sextupole-driven decoherence of the beam to estimate the chromaticity, as discussed in Section 4.6.

2.4 Landau detuning

The individual particle motion in an accelerator is usually inaccessible and any detector normally only measures the centre-of-mass beam motion, as described in Eq. (26), for an ensemble of $N$ particles, each describing an individual trajectory $x_i(n)$ for a given turn $n$:

$$\langle x(n) \rangle = \frac{1}{N} \sum_{i=0}^{N} x_i(n).$$  

(26)
Provided that \( N \) is large and the beam is not excited, this signal usually vanishes, even if all particles have the same oscillation frequency, because of the incoherent sum caused by the random phases of the individual particle oscillation. However, if excited coherently, the particle oscillation would prevail infinitely.

In real life, however, the bunch population is nearly always affected by a small finite tune spread that causes Landau detuning, also historically referred to as ‘Landau damping’.

The damping of the centre-of-mass motion can be explained by the fact that some particles of the bunch distribution oscillate faster or more slowly than the reference particle and consequently dephase, causing an incoherent superimposition to the visibly decaying centre-of-mass motion. Figure 6 shows the step response of some individual particles and the decay of the bunch’s centre-of-mass motion due to Landau damping as a function of turns in the machine. The increasing decoherence of the individual particle trajectories and corresponding decay of the centre-of-mass motion is visible.

Please note that the term Landau damping to describe the above effect is an actual misnomer, especially for hadron (such as proton, ion, etc.) accelerators that are practically unaffected by synchrotron radiation losses, as there is no actual energy loss. While the observed centre-of-mass motion decays, the sum of the individual particle energies is preserved. Nevertheless this term is still widely used in the accelerator physics field.

Using beam representation with a reduced number of particles may cause the inherent small statistical r.m.s. fluctuations (commonly referred to as Schottky noise) to appear as a coherent particle motion. Most numeric simulations thus require a significant number of particles (tens to hundreds of thousands or more) to suppress this spurious common mode artefact and to adequately represent the signal. Another analytic approach to circumvent this problem is to introduce an effective homogeneous bunch-charge distribution \( f(\omega) \) with \( \int_{-\infty}^{+\infty} f(\omega) \, d\omega := 1 \). Using this distribution, the average measured transverse

\[ \text{turns} \quad 0 \quad 500 \quad 1000 \quad 1500 \quad 2000 \]
\[ \text{amplitude [a.u.]} \quad -1 \quad -0.5 \quad 0 \quad 0.5 \quad 1 \]

Fig. 6: Landau detuning: some individual particle motions (thin, coloured lines) and the corresponding centre-of-mass motion (black, dashed line) are indicated. The plotted decoherence assumes a Cauchy–Lorentz distributed tune frequency spread.
bunch motion can be described by

$$\langle x(n) \rangle = X_0 \int_{-\infty}^{+\infty} f(\omega) \cos(\omega t) \, d\omega.$$  \hfill (27)

The tune width is usually very small compared to the tune resonance frequency $\omega_0/\omega$, and one can introduce a coordinate transform and move the integration range according to $\omega' = \omega_0 - \omega$. Using the trigonometric identity $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$, Eq. (27) can be reduced to

$$\langle x(n) \rangle = X_0 \int_{-\infty}^{+\infty} f(\omega') \cos \left( \omega' t + \omega_0 t \right) \, d\omega'$$  \hfill (28)

$$= X_0 \int_{-\infty}^{+\infty} f(\omega') \left[ \cos(\omega' t) \cos(\omega_0 t) - \sin(\omega' t) \sin(\omega_0 t) \right] \, d\omega'$$  \hfill (29)

$$= X_0 \left[ \cos(\omega_0 t) \int_{-\infty}^{+\infty} f(\omega') \cos(\omega' t) \, d\omega' - \sin(\omega_0 t) \int_{-\infty}^{+\infty} f(\omega') \sin(\omega' t) \, d\omega' \right]$$

It is visible that the second term in the square brackets is asymmetrical with respect to the centre frequency and that hence the integral from minus to plus infinity vanishes. Equation (29) thus reduces to

$$\langle x(n) \rangle = X_0 \cos(\omega_0 t) \int_{-\infty}^{+\infty} f(\omega') \cos(\omega' t) \, d\omega' = X_0 \cdot E(t) \cdot \cos(\omega_0 t)$$  \hfill (30)

with $X_0$ being the initial particle oscillation amplitude and $\cos(\omega_0 t)$ the harmonic oscillation. The envelope function $E(t)$ depends on the assumed tune frequency distribution of the particle ensemble. For example, using the cosine transform, one can show that the corresponding impulse response signal for a Gaussian and Cauchy–Lorentz distributed tune spread can be written as

$$\langle x(n) \rangle = X_0 \cdot e^{-\frac{1}{2} \sigma^2 \cdot n^2} \cdot \cos(\omega_0 \cdot n) ,$$  \hfill (31)

$$\langle x(n) \rangle = X_0 \cdot \pi e^{-\Gamma \cdot n} \cdot \cos(\omega_0 \cdot n) ,$$  \hfill (32)

with $\sigma$ being the r.m.s. tune spread, and $\Gamma$ being the full width at half maximum (FWHM) of the distribution, respectively. Both distribution widths are normalized by the revolution period, and are thus without any further dimension. It is visible in Eqs. (31) and (32) that the form of the decay differs significantly between the assumed cases, which can be used to distinguish between these and provide an estimate of the initial particle distribution. Assuming that the tune spread (or resonance width) is caused by chromaticity and the inherent beam momentum spread (which is certainly valid for unbunched beams), the measured decay constants can be used either to estimate the beam momentum spread if the chromaticity is known, or to provide an estimate of the chromaticity if the momentum spread $\Delta p/p$ is known:

$$Q' \approx \frac{\sigma}{\Delta p/p} \text{ resp. } Q' \approx \frac{\Gamma}{\Delta p/p} .$$  \hfill (33)

The definition of momentum spread differs for Gaussian (r.m.s. definition) and Cauchy–Lorentz (FWHM definition) distributions, and must thus be adjusted accordingly, if these different conventions are mixed (e.g., Cauchy–Lorentz tune width distribution and r.m.s. momentum spread). Landau detuning is also the driving mechanism that explains the effective emittance increase after transverse kicks of the beam, as schematically illustrated in Fig. 7. The corresponding relative beam size growth $\Delta \sigma/\sigma$ for a single kick with absolute beam shift of $\delta a$ and r.m.s. beam width $\sigma$ can be approximated to

$$\frac{\Delta \sigma}{\sigma} \propto \frac{1}{2} \left( \frac{\delta a}{\sigma} \right)^2 .$$  \hfill (34)

A more detailed treatment of Landau detuning can be found in Ref. [14].
3 Tune diagnostics

For the purpose of signal analysis, one can describe the effective collective oscillation of the bunch as a simple harmonic oscillator that can be described by

$$\ddot{x} + 2\Gamma \omega_{\beta_0} \cdot \dot{x} + \omega_{\beta_0}^2 \cdot x = \omega_{\beta_0}^2 \cdot e(n)$$  \hspace{1cm} (35)$$

with $x$ being the transverse coordinate, $\omega_{\beta_0}$ the unperturbed tune of the bunch, and $\Gamma$ the damping factor (e.g., due to Landau detuning). Here, the dot notation indicates differentiation with respect to the turn number $n$ (e.g., $\dot{x} = \frac{dx}{dn}$). It is important to point out that, while Eq. (2) is the exact description of the particle motion, Eq. (35) is merely a simplified, qualitative model of the bunch motion. One of the obvious flaws of this representation is that the damping in Eq. (35) is dissipative causing a real energy loss, while the real Landau detuning affects only the decoherence of the visible signal but not the particles’ energy. Nevertheless, for the application of tune measurements, this simplification is sufficient.

Applying the Laplace transform to Eq. (35) yields

$$s^2 \cdot X(s) + 2\Gamma \omega_{\beta_0} s \cdot X(s) + \omega_{\beta_0}^2 \cdot X(s) = \omega_{\beta_0}^2 \cdot E(s)$$  \hspace{1cm} (36)$$

and

$$X(s) = \frac{\omega_{\beta_0}^2}{s^2 + 2\Gamma \omega_{\beta_0} + \omega_{\beta_0}^2} \cdot E(s) = G(s)$$  \hspace{1cm} (37)$$

which illustrates that the beam-transfer-function (BTF) $G(s)$ can be analysed like any other control system, by exciting the system with a known perturbation source $E(s)$ while observing its response signal $X(s)$. Please note that exciter and beam pickup usually contribute to an additional non-beam-related amplitude and phase response that may to be taken into account.
The measurement of the betatron tune can be grouped into three fundamental diagnostic methods:

1. Methods involving beam excitations over a finite-time interval and observation of the ‘impulse’ or so-called chirp response using beam position monitors (BPM). The typical achievable tune resolutions are of the order of \( \Delta Q \approx 10^{-3} - 10^{-4} \), limited by the duration of the beam oscillation.

2. Passive monitoring of residual beam oscillations using Schottky monitors or diode-detection-based Base-Band-Q (BBQ) detection techniques. The typical achievable tune resolutions are of the order of \( \Delta Q \approx 10^{-3} - 10^{-4} \).

3. Active techniques such as the phase-locked-loop (PLL) systems that resonantly excite the beam on or close to the tune resonance. The typical tune resolutions can be as low as \( \Delta Q \approx 10^{-6} \).

The choice of which method to use usually depends on:

- the available aperture and/or maximum allowed beam excursions that respect constraints due to machine protection and emittance blow-up minimization. Typical excitation amplitudes are of the order of two or three r.m.s. beam widths \( \sigma \), with the LHC having probably the tightest constraints of less than 25 \( \mu \)m absolute or \( \frac{1}{30} \sigma \) relative to the beam size at the collimators;
- whether excitations can be applied to the beam at all (clearly favouring passive methods); and
- the required tune measurement rate such as continuous tracking, which would favour the use of active techniques.

### 3.1 Classic finite-time turn-by-turn based methods

The most common tune diagnostic method relies on an on-turn excitation and consecutive measurement of the beam response over a finite number of turns \( N \), spaced typically by the revolution sampling period \( T_{rev} \). Using the Laplace transform, the ‘kick-type’ excitation can be modelled as

\[
\mathcal{L} \{ f(n) = \delta(n) \} = 1 ,
\]

which indicates the kick spectrum’s independence of \( s \), thus the excitation energies are evenly distributed over all frequencies. Replacing \( s \to i\omega \) in Eq. (37), the absolute or magnitude spectrum of the beam response \( |G(\omega)| \) due to a kick-type excitation can be written as

\[
|G(\omega)| := \left| \frac{X(s)}{E(s)} \right| = \frac{\omega_0}{\sqrt{\left( \omega - \omega_0 \right)^2 + (2\Gamma \omega_0)^2}} ,
\]

which is also known as ‘Cauchy–Lorentz’ or ‘Breit–Wigner’ distribution. It is visible that the Cauchy–Lorentz response has a maximum at the tune frequency \( \omega_0 \) and a FWHM equal to the damping factor \( \Gamma \). Figures 8(a) and 8(b) show a beam response to a kick and corresponding magnitude spectrum. The magnitude spectrum is commonly obtained using the discrete Fourier transform (DFT) of the turn-by-turn oscillation signal \( x(n) \) which, for a total of \( N \) turns, is given by

\[
\hat{x}(k) = \sum_{n=0}^{N-1} x(n)e^{-i2\pi k \frac{n}{N}} \quad k = 0, \ldots, N - 1
\]

for a normalized discrete frequency \( k \). A first-order estimate of the fractional tune is then given by

\[
q_{frac} \approx \frac{k_0}{N} ,
\]
Fig. 8: Beam response to a single-turn kick excitation and corresponding magnitude spectrum. The signal-to-noise ratio of a factor of ten and the corresponding tune peak are visible.

with $k_0$ being the index of the highest bin of the tune peak in the magnitude spectrum $|x(k)|$, and $N$ the total number of samples used for the DFT computation. While the numerical complexity of the standard DFT is $O(n^2)$, it can be improved using the so-called Fast Fourier Transform (FFT) algorithm with $O(n \log(n))$ [15].

With the choice of the discrete Fourier transform, using equality of noise power in time-domain and frequency-domain, it can be shown that the width of the binning and thus frequency resolution $\Delta Q_{res}$, and noise floor $\sigma_f$ of the magnitude spectrum is limited to

$$\Delta Q_{res} \approx \frac{1}{N},$$  \hspace{1cm} (42)  

$$\sigma_f = \sqrt{\frac{2}{N}}\sigma_t, \hspace{1cm} (43)$$

with $\sigma_t$ being the r.m.s. turn-by-turn noise in the time domain (unit: [m]). The factor of ‘two’ in Eq. (43) derives from the fact that the power is folded below the Nyquist frequency $0.5f_{rev}$. While for most real-world machine applications, the given tune estimate in Eq. (41) is often sufficient, improved frequency estimates for special tune diagnostic applications are discussed below in Section 3.1.1. The tune and the tune spectrum itself are often plotted as a function of time to provide additional diagnostics for non-tune beam signals also, as shown for exemplary beam spectrum evolution at CERN’s PS Booster in Figs. 9(a) and 9(b). The large low-frequency amplitude relates to longitudinal oscillations during the acceleration ramp.

An alternative to the kick excitation is a frequency sweep, also called ‘chirp’-type excitation in analogy to the sound birds make, generated using a sinusoidal-type excitation with time-varying frequency

$$e(n) = \sin(\omega_0 \cdot n + \frac{\Delta \omega}{N} \cdot n^2), \hspace{1cm} (44)$$

with $\omega_0$ the initial chirp frequency, $n$ the turn number, and $\Delta \omega$ the frequency range (in units of $f_{rev}$) that is covered within the specified number of turns $N$. Note that the frequency-sweep method, especially when used in the context of beam transfer function measurements, is also known as ‘BTF scan’, sometimes with the notion that the frequency increase is slower compared to the sweeps performed to diagnose the spectral tune peak only. Otherwise it is identical to the chirp-type excitation. The advantage of a chirp-type excitation is twofold:
1. From the point of view of accuracy, both excitation types are identical, as the functional dependence of emittance blow-up over a given time interval $\Delta t$ scales proportionally to $(\delta a / \sigma)^2 N$ [Eq. (34)], and the signal-to-noise ratio scales as $\sqrt{N \delta a}$ [Eq. (43)]. Thus (to first order) using twice the kick excitation amplitude ($2 \delta a$) gives the equivalent signal-to-noise ratio as chirping on the tune, for an elongated period of $4N$. However, the peak power that the given kicker magnet has to supply in the latter case is significantly smaller. Compared to the kick-type excitation where the kicker has to supply the full power within a turn, the chirp exciter can distribute the total required power over many turns, which simplifies its design.

2. For an excitation range of $[0, 0.5 f_{\text{rev}}]$, kick- and chirp-type excitations give practically the same beam magnitude response and differ mainly in the corresponding phase response. However, in the case of a chirp, the excitation can be voluntarily narrowed to the frequency range where the tune resonance is expected, so as to not unnecessarily excite other non-tune resonances such as the synchrotron tune, and to further improve the resolution. The kick-type resolution can essentially only be improved by increasing the kick amplitude which is ultimately limited by the available maximum kicker power. However, the chirp-type resolution can be increased to the desired accuracy by simply extending the measurement time over which the chirp is applied, or by narrowing its total frequency sweep range while maintaining a constant excitation power.

### 3.1.1 Improved tune frequency estimation

There are multiple techniques to estimate the beam response spectra and recover the tune frequency. In order to discuss tune frequency estimation, it is useful to revise some of the aspects of the Fourier transform analysis, which, in the continuous time domain for an arbitrary signal $a(t)$ and given choice of normalization, can be written as

$$
\mathcal{F}\{a(t)\} = \hat{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a(t) e^{i\omega t} \, dt.
$$

(45)

Owing to the given integration limits, the transform is a priori defined for infinite lasting signals only. From an experimental point of view, this assumption and a direct transform of $a(t)$ is quite "unrealistic", as most beam responses decay after a given time, below the instrument noise floor $\sigma_t$, due to Landau detuning. This can be related to an effective finite acquisition length $T = N \cdot T_{\text{rev}}$. Ignoring the sampling for the sake of argument, the finite length can be addressed by either changing the integration limits to $\pm NT_{\text{rev}}$ which would complicate the expressions of other Fourier transform properties, or by introducing
a so-called apodization function which sets the signal $a(t)$ to zero outside the valid integration range. Therefore, the real-world signal $a(t)$ can be replaced with $a_0(t) \text{rect} \left( \frac{t}{NT_{\text{rev}}} \right)$ and the rectangular function defined as

$$\text{rect}(x) = \begin{cases} 
0 & \text{if } |x| > \frac{1}{2} \\
\frac{1}{2} & \text{if } |x| = \frac{1}{2} \\
1 & \text{if } |x| < \frac{1}{2}.
\end{cases}$$ (46)

The corresponding Fourier version is

$$\hat{\text{rect}}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sin(\pi x)}{\pi x} = \frac{1}{\sqrt{2\pi}} \cdot \text{sinc}(\omega) .$$ (47)

In its normalized form, the sinc($x$) function has its zero-crossing at non-zero integers. Using the definition in Eq. (46), Eq. (45) leads to

$$\hat{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-NT_{\text{rev}}/2}^{NT_{\text{rev}}/2} a(t) e^{i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} a_0(t) \text{rect} \left( \frac{2t}{NT_{\text{rev}}} \right) e^{i\omega t} \, dt .$$ (49)

Using the fundamental Fourier space theorem which states that the convolution of two functions in time-domain corresponds to a multiplication in Fourier-space and vice versa, it is visible that any finite beam response spectrum $\hat{a}_0(\omega)$ is always intrinsically convoluted with the sinc($x$) function as defined in Eq. (47). A harmonic signal $x(n)$ with an exact frequency $k_0/N$ that is recorded over $N$ turns would yield a spectral tune peak with two side-lobes at $k-2$, $k+2$, $k-3$, and so on, referred to as spectral leakage. Spectral leakage is commonly misconceived to be an ‘artefact of the discrete Fourier transform’, and ‘can be improved by applying additional apodization function’. In this context, these functions are also referred to as ‘window’ functions. The most frequently used window functions are the Hamming window

$$w(n) = 0.5386 - 0.46164 \cos \left( \frac{2\pi n}{N-1} \right)$$ (50)

and the Hann window

$$w(n) = 0.5 \left[ 1 - \cos \left( \frac{2\pi n}{N-1} \right) \right] .$$ (51)

However, the initial leakage is actually due to the finite length of the signal, and any other additional convolution merely widens the spectral tune peak distribution thus reducing the resolution of the spectrum. Taking the sampling of the beam response signal into account $x[n]$, the Fourier series given in Eq. (49) transforms to the discrete-time Fourier transform (DTFT) which is another special version of the Fourier transform

$$\hat{x}(\omega) = \sum_{n=0}^{N} x[n] e^{-i\omega n} ,$$ (52)

with $n$ the index of the individual sample. Sometimes, especially in older accelerator physics articles, the DTFT is also referred to as harmonic analysis. The main difference between DTFT and DFT is that while the DFT assumes that the signal is periodic and thus develops the signal using equally spaced frequencies $\omega = 2\pi \frac{k}{N}$, the DTFT develops the signal at any arbitrary continuous frequency. Figure 10(a) illustrates the difference between the DFT (blue dots) and DTFT (solid lines) using the frequency magnitude spectrum of a perfect finite-length sinusoidal oscillation. The frequency has been chosen to be exactly in between two DFT bins maximizing the representation error of the DFT. The DFT frequency spacing can be improved by zero-padding of the input, in other words, extending the data set with a given
Fig. 10: Sinc representation error. The DTFT spectrum (solid) line and DFT representation with regular (blue dots) and zero-padding (orange dots) are shown. The red dashed curve shows a Gaussian interpolation of the central peak using the highest and neighbouring bins of the non zero-padded spectrum. The interpolation error $\Delta A(l)$ is visible, with $l$ defining the additional $(l - 1) \cdot N$ sampled zeros.

number of additional zeros, typically, in multiples of $N$. An exemplary zero-padding doubling the number of samples is shown in Fig. 10(a). At the cost of an additional numeric complexity, higher spectral sampling improves the representation error defined as the difference between the DTFT and DFT-based peak computation, as shown in Fig. 10(b). Using the Taylor series of $\text{sinc}(x)$, the first non-constant, non-vanishing order is $x^2$. Thus by increasing the spacing of the test frequencies $\omega$, which is equivalent to increasing the total number of samples used for the computation of the DFT, it is visible that the relative worst-case error $\Delta A(l)$ scales as

$$dA = \Delta A(l) \propto \frac{1}{(lN)^2}$$

as also visible in Fig. 10(b). While the differences between DFT and DTFT spectrum representation decrease with increasing zero-padding, excessive zero-padding is inefficient in terms of the time required for the numerical computation, compared to the improvement of the estimate of a single tune value. The ultimate resolution $\Delta f_{\text{res}}$, defined by the capability to distinguish two spectral peaks spaced by $\Delta f_{\text{res}}$, is proportional to the FWHM of the $\text{sinc}(x)$ distribution ($\approx 0.9 N$) as shown for a simple and zero-padded DFT in Fig. 11 for a signal consisting of two harmonic oscillations spaced by $\Delta f_{\text{res}}$ and sampled with $N = 1024$ samples. The spectra were computed with no additional windowing. It is visible that the zero-padding efficiently removes the spectral representation error, thus revealing the two spectral peaks otherwise compromised by the binning of the standard DFT.

Provided that there are no other peaks in the immediate vicinity of the tune ($Q_0 \pm \frac{2}{N}$), an intermediate solution is to interpolate the true peak centre using the central binned frequency $\frac{k}{N}$, its estimated magnitude $M_k$, and estimated amplitudes of its neighbouring bins $M_{k\pm1}$. Some of the proposed estimates are

- no interpolation:
  $$q_{\text{frac}} \approx \frac{k}{N};$$  \hfill (54)

- barycentre ($l \geq 1$) fit:
  $$q_{\text{frac}} \approx \frac{1}{N} \frac{M_{k-1}^l(k - 1) + M_k^l(k) + M_{k+1}^l(k + 1)}{(M_{k-1}^l + M_k^l + M_{k+1}^l)};$$  \hfill (55)

- parabolic fit [16]:
  $$q_{\text{frac}} \approx \frac{1}{N} \left( k + 0.5 \frac{M_{k+1} - M_{k-1}}{2M_k - M_{k-1} - M_{k+1}} \right);$$  \hfill (56)
Equations (55)–(57) are qualitative type fits and implicitly assume an additional windowing prior to computing the DFT, while Eq. (58) assumes a perfect sinusoidal input signal and thus interpolates the expected $\text{sinc}(x)$ distributed tune peak. The latter assumption is often inaccurate with respect to real-world beam signals, as the observed tune signal is usually damped and often affected by frequency shifts during the acquisition.

Another approach is to use the functional dependence of the beam response and to fit the spectral tune peak to the Cauchy–Lorentz distribution in the frequency domain given in Eq. (39), or to fit the tune response directly to either Eq. (31) or (32) in the time domain. Figure 12 shows an exemplary tune spectrum fit of a damped kick-type response using a parabolic [Eq. (56)], Gaussian [Eq. (57)] and Cauchy–Lorentz type fit [Eq. (39)]. All three interpolations perfectly fit the highest and its neighbouring bins. However, it is visible that the Cauchy–Lorentz distribution provides a better fit and is thus a better description of the resonance as seen at the additional control sample points (orange points). This is not too surprising, as a Gaussian function always provides better fits of a Gaussian distribution than a parabolic function. It is important to point out that the DFT-based peak interpolated tune fits are merely a trade-off of reduced numerical complexity for a given precision and, due to the given assumption, always perform less precisely than a $\chi^2$ fit using the physical dependence of the beam oscillation signal. Nevertheless, the interpolation-based methods often give sufficiently precise resolutions for most operational requirements.

In order to study the interpolation methods more systematically, the following frequency error is defined:

$$|\Delta f_{\text{err}}| = \frac{Q_0 - Q_{\text{est}}}{\text{bin width}},$$  \hspace{1cm} (59)

describing the difference between the true $Q_0$ and estimated tune peak $Q_{\text{est}}$ normalized to the first-order DFT bin width $1/N$. According to Eq. (59), the error of the simple DFT-based tune estimate given by Eq. (54) is symmetrical with respect to the bin centre and less than 0.5. Quantitative estimates of $|\Delta f_{\text{err}}|$ for given type tune signals and interpolation methods are shown in Fig. 13. As also seen in Fig. 10(a), doubling and zero-padding the samples of the input signal is beneficial with respect to the sampled DFT spectra representation and was thus in combination with a Hann window used for the barycentre, parabolic, and Gaussian interpolation type fits. The interpolation error $|\Delta f_{\text{err}}|$ depends on
the true frequency location in the bin. However, since the true frequency is a priori unknown, the true figure of merit is the worst-case error of $|\Delta f_{\text{err}}|$ shown in Fig. 13. Figure 13(a) shows the residual tune errors for a perfect, noise-free, non-damped sinusoidal tune oscillation signal. As expected, all interpolation-based methods have errors of less than 0.5 times the bin width. In particular, the ‘NAFF’-type interpolation shows good results as the spectra correspond to the sinc $(x)$ distribution upon which this estimate is based. The $\chi^2$ fit using Eq. (32) nevertheless provides a better estimate, essentially limited by the numerical precision of the fitting algorithm, since it does not rely on the DFT frequency binning and associated error shown in Fig. 10. Figure 13(b) shows the same noise-free sinusoidal-type signal, damped with a time constant of 100 turns and starting only after 100 turns, as is typical for kick-type beam responses. While the $\chi^2$ fit — including the turn number the decay starts as a free parameter — provides the best results, the ‘NAFF’-type error significantly deteriorates up to the level where the interpolation is worse than applying no interpolation. This is not surprising since the damped, finite-length input signal does not match its implicit assumption of a perfect harmonic oscillation.

Figures 13(c) and 13(d) show the same signal including noise. The noise levels $\sigma_t$ are 0.1% and 1% with respect to the maximum turn-by-turn amplitude. It is visible that the residual maximum tune reconstruction error $\Delta f_{\text{res}}$ and noise propagation $\Delta f_{\text{noise}}$ as seen in Figs. 13(c) and 13(d) are additive. The absolute error $\Delta f$ of the frequency reconstruction can thus in units of $f_{\text{rev}}$ be written as

\[ \Delta f = \Delta f_{\text{res}} + \Delta f_{\text{noise}} , \]

\[ \Delta f = \frac{1}{N} \cdot c_1 + \sqrt{\frac{2}{N} \frac{\sigma_t}{A}} \cdot c_2 , \]

with $A$ being the maximum amplitude of the signal. Owing to the different dependences on $N$, for small signal-to-noise ratios $\sigma_t/A$, the tune frequency resolution is often limited by noise. The numeric constants $c_1$ and $c_2$, dependent on the given interpolation type, can be derived, for example, from the worst-case interpolation errors shown in Fig. 13. They are summarized for finite-length, exponentially damped harmonic oscillations in Table 1. Besides the given dependence on $1/N$ and $1/\sqrt{N}$ in Eq. (61), these coefficients are fairly independent of the given assumed tune input signal — provided the signal

---

**Fig. 12:** FFT interpolation principle, showing a Gaussian, parabolic, and Cauchy–Lorentz type fit
performs exponentially or in a Gaussian damped harmonic oscillation of the form given in Eqs. (32) and (31), or has similar spectrum characteristics. If the discrete Fourier spectra representation is not limited by the finite binning, the interpolation shows only little dependence on the additionally applied windowing functions. Please note that the tune resolution is given by the finite width of the sinc($x$) distribution which itself depends on the finite length of the input signal only, as shown in Fig. 11, and may only become worse by additional window functions.

### 3.1.2 Lomb periodogram

Another class of the non-Fourier-based frequency estimation approach is given by least-squares spectral analysis methods, as proposed, for example by Vanícek, later referred to as Lomb periodograms [18–20]. As a particularity, the Lomb periodogram does not require that the data samples $(t_j, X_j)$ be equally...
spaced, but allows gaps. The corresponding transform can be written as

\[ P_x(\omega) = \frac{1}{2} \left( \frac{\sum_j X_j \cos(\omega(t_j - \tau))}{\sum_j \cos^2(\omega(t_j - \tau))} + \frac{\sum_j X_j \sin(\omega(t_j - \tau))}{\sum_j \sin^2(\omega(t_j - \tau))} \right) \]  

(62)

with the constant phase component \( \tau \) being matched so that it fulfils

\[ \tan 2\pi \tau = \frac{\sum_j \sin 2\omega t_j}{\sum_j \cos 2\omega t_j}. \]  

(63)

Similar to the DTFT, the Lomb periodogram is a priori defined for arbitrary continuous frequencies \( \omega \). However, in real life, discrete test frequencies are chosen. There are many other similar least-squares fitting methods. Nearly all have in common that the analysed signal is usually a perfect harmonic oscillation but with some measurement constraints due to unequal sampling and large noise floors. Their spectrum and frequency resolutions are comparable with respect to each other and similar to the one obtained using the DTFT.

### 3.2 Special tune diagnostic instrumentation

The biggest challenge in measuring tunes of a circular accelerator is the dynamic range of the processed signals, as the small signal related to transverse beam oscillations is carried by large and short pulses. Many accelerators exploit standard BPMs with turn-by-turn acquisition capabilities to record residual or excited tune oscillations. An introduction and overview of beam instrumentation, in particular BPMs, can be found in Refs. [21–24]. While BPMs are proven solutions for most beam diagnostics, there exist more efficient, optimized techniques for tune specific measurements such as Schottky monitors (Ref. [25]) or the BBQ principle, further discussed below.

In order to elaborate on the deficiencies of using BPMs for tune measurements it is useful to discuss their basic principles. As expressed by Eq. (6), BPMs usually measure the combined trajectory offset of closed-orbit perturbation, momentum specific offset, and the turn-by-turn varying betatron-oscillation, while only the latter carries the turn-by-turn tune dependent signal. In this context, the tune independent closed-orbit and momentum dependences are commonly re-grouped as ‘systematic offsets’, in the technical jargon often called common mode. With respect to tune measurement, because of the finite range and binning of the analog-to-digital converter (ADC), these systematic offsets reduce the available range and thus the resolution otherwise available for resolving the tune oscillations.

In addition to the beam intrinsic offsets, the pick-up and analog signal-processing of the beam signal itself usually introduces additional systematic offsets that prevail even for perfectly centred beams. For relativistic beams, the longitudinal beam image charge density travelling along the vacuum chamber is nearly identical to the longitudinal bunch profile, since the opening angle of the spherical equipotential surface of the electric field of a single charge is proportional to \( 1/\gamma_{rel} \). The distribution of the image charge on the vacuum chamber boundaries usually depends on the beam position and its transverse and longitudinal distribution. For an off-centre beam, the image charge density on a wall segment at an angle \( \theta \) and vacuum chamber radius \( R \) can be written as

\[ J_{\text{image}}(r, \phi, R, \theta) = \frac{J_{\text{bunch}}(r, \phi)}{2\pi R} \frac{1 - \left( \frac{r}{R} \right)^2}{1 + \left( \frac{r}{R} \right)^2 - 2\frac{r}{R} \cos(\theta - \phi)}, \]  

(64)

with \((r, \phi)\) the beam displacement with respect to the centre as illustrated in Fig. 14(b) and \(J_{\text{bunch}}(r, \phi)\) the charge distribution. The current travelling through a given wall segment, as indicated in Fig. 14(b), is given by the integral

\[ I_{\text{segment}} = \int_{-\psi}^{+\psi} J_{\text{image}}(r, \phi, R, \theta) d\theta, \]  

(65)
with $\psi$ the opening angle of the selected wall segment, $(r, \phi)$ the coordinate of the charge, and $(R, \theta)$ the point of observation located on the vacuum chamber wall with radius $R$. As shown in Ref. [24], Eq. (64) can be expanded into

$$J_{\text{image}}(r, \phi, R, \theta) = \frac{J_{\text{bunch}}(r, \phi)}{2\pi R} \left[ 1 + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^2 \cos n(\theta - \phi) \right].$$

Inserting this expansion into Eq. (65) and integrating over the ‘left’ and ‘right’ electrode areas indicated in Fig. 14(b) leads to the following currents flowing through the ‘left’ ($I_L$), ‘right’ ($I_R$) segments, respectively:

$$I_L = \frac{I_w}{2\pi} \left[ 2\psi + 2\frac{x_0}{R} \sin \psi + \frac{x_0^2 - y_0^2}{R^2} \sin(2\psi) + \text{h.o.} \right],$$

$$I_R = \frac{I_w}{2\pi} \left[ 2\psi - 2\frac{x_0}{R} \sin \psi + \frac{x_0^2 - y_0^2}{R^2} \sin(2\psi) + \text{h.o.} \right],$$

with $I_w = I_w(t)$ the longitudinal bunch current density, $x_0$ the horizontal, and $y_0$ the vertical bunch position in Cartesian coordinates with respect to the BPM centre. The corresponding terms for the ‘top’ ($I_T$) and ‘bottom’ ($I_B$) segments are similar, with the exception of the swapped $x$ and $y$ coordinates.

For BPMs, one typically defines a sum ($\sum$) and difference signal ($\Delta$) given as

$$\Delta := (I_L - I_R) \cdot Z_0 \quad \text{and} \quad \sum := (I_L + I_R) \cdot Z_0$$

with $I_L$ and $I_R$ the current at the left and right, respectively, and $Z_0$ the transfer impedance of the pick-up, defined by Eq. (69). Comparing with Eqs. (67) and (68), it is visible that the difference signal is proportional to the position $z$ and the longitudinal bunch current density $I_w$, while the sum signal essentially depends on the bunch current density. The sum signal is typically used in BPMs to normalize the difference signal in order to make the orbit measurement

$$z_{\text{norm}} = \frac{\Delta}{\sum} = \frac{I_L - I_R}{I_L + I_R}.$$
insensitive to the actual bunch current. This requirement is less important for tune measurements, as the analysis is dependent on the oscillation frequency rather than the absolute scale of the oscillations. As illustrated in Fig. 15, the button or strip-line based difference signal is typically generated using a $\Sigma-\Delta$ hybrid (also known as 180° hybrid), implementing Eq. (69) in the analog domain, in order to minimize the range of the signal that needs to be processed by an ADC. Sometimes the analog signal is amplified to adapt the signal to the maximum ADC range, band-pass and low-pass filtered in order to select given bunch harmonics and to minimize potential aliasing prior to the sampling. In the absence of closed orbit or momentum related offsets, the pre-amplifier could be adjusted to such a theoretic amplification level that even microscopic nanometre-scale oscillations would correspond to a sufficient number of ADC bits, limited only by the electronic noise of the components prior to the ADC. For example, the nominal 1 ns long LHC bunches induce on the electrodes of the $Q$ measurement stripline pick-ups some 50 V. For the 80 mm pick-up diameter and 1 $\mu$m beam oscillation amplitudes the pick-up output pulses are modulated by a few mV.

While most $\Sigma-\Delta$ hybrids usually provide good sum and difference estimates for a wide range of signal frequencies, even the best hybrids always have a small intrinsic bleed-through of the intensity signal into the difference signal, of a few per cent in the optimal hybrid frequency range. In addition, while Eq. (69) assumes a constant impedance $Z_0$ of the pick-up pairs, due to variation in the production process, electrodes typically differ on the per cent level causing a systematic offset of the delta signal for otherwise perfectly centred beams. These bleed-throughs and the above discussed beam offsets are pulsed signals and are thus difficult to remove using analog circuits and ultimately limit the maximum allowed pre-amplification without saturating the ADC. The delta signal is dependent on the bunch intensities, and for accelerators that have to accommodate large ranges of beam intensities, amplification is typically implemented using several gain stages for an optimal use of the ADC range.

As visible in Eqs. (67) and (68), the transverse signal is intrinsically a modulation of the longitudinal carrier signal $I_w$. Thus, in addition to the intrinsic intensity signal bleed-through, longitudinal oscillations and instabilities are also visible in the transverse plane. The most common signals that have their origin in the longitudinal plane are bunch length oscillations and bunch time-of-arrival (synchrotron) oscillations, typically found in the lower part of the beam response spectra, as seen in Figs. 9(a) and 9(b).

An efficient way to filter the betatron modulation signal $x_\beta$ from its inconvenient carrier is the diode-based BBQ technique described in Ref. [26]. Its principle is shown in Fig. 16, with the simplified signal waveforms sketched above the corresponding circuit paths. The BBQ can be considered as a fast ‘sample-and-hold’ circuit, self-triggered at the bunch maxima and ‘held’ by the parallel capacitor that acts as an analog ‘memory’ of the bunch peak amplitude seen by the pick-up. The purpose of the parallel resistors is to discharge the capacitors so that the next bunch with potentially smaller amplitude also contributes to the detector output signal. The self-triggered sampling by the peak detectors at the bunch
repetition rate intrinsically down-converts the beam energy from the GHz to the baseband, which for large accelerators or small energies is in the kHz range. In this frequency regime, the signals can be efficiently and cost-effectively processed by audio-frequency components. Systematic closed-orbit and intensity bleed-through signals, which are otherwise pulsed signals, are converted into a static voltage signal at 0 Hz also referred to as ‘DC’. Being at 0 Hz, these systematic common-mode signals can be efficiently rejected by a capacitor put in series to the signal path. The frequency of this AC coupling is typically chosen to sufficiently reject the low frequency components close to DC while minimizing the attenuation at tune modulation frequency.

For practical purposes, this scheme makes saturation of the detector impossible and facilitates a very high dynamic ADC range and related sensitivity, essentially limited by the analog noise of the peak detector and down-stream analog processing chain. As shown in the CERN SPS, this scheme can achieve resolutions of the order of a few nanometres for peak bunch signals of the order of 50 V and a half-aperture of about 80 mm. For comparison, since the SPS transverse bunch sizes are of the order of a few millimetres and provided that there are no other significant noise sources, the required excitation levels can be kept to a minimum — if required at all — and can thus be considered insignificant with respect to emittance blow-up and machine performance. In most cases the BBQ sensitivity is sufficient to observe the residual, often nm-sized, tune oscillations anyway present in the beam spectrum. An exemplary beam spectrum of the LHC is shown for a pilot-bunch (one bunch, \( \approx \cdot 10^9 \) protons/bunch, \( \sigma_t \approx 0.4 \text{ ns r.m.s.} \)) in Fig. 17(a) and zoomed to the tune resonance in Fig. 17(b). The tune signal-to-noise ratio of about 10–20 dB is visible, which is typically adequate for tune peak detection and interpolation techniques, and consequently tune correction. The rejection of signals for frequencies below \( \approx 0.1 f_{\text{reg}} \) due to the series capacitor and following filtering is visible. In addition, the vertical tune eigenmode \( Q_1 \) can be seen in the horizontal spectrum due to the presence of non-vanishing betatron-coupling. The synchrotron side-band peaks next to the tune peak seen in Fig. 17(b) are caused by beam momentum modulations driven by the RF cavities. While the particular LHC BBQ setup is based on a 30 cm long strip-line having a half-aperture of 40 mm, the BBQ detection scheme is a priori not limited to a given transverse pick-up type.

The 3D technique as described above yields an ‘averaged’ tune for all bunches, with those of dominating amplitudes contributing more than smaller ones. This can be improved with a preceding fast, large signal gate, selecting only bunches of interest, however, at the cost of potentially reduced sensitivity and increased system complexity. Such a solution is being studied.

3.3 Phase-locked-loop method

As a working principle, PLL control systems continuously adjust the phase \( \varphi \) and frequency \( f_e \) of their reference exciter to match and track changes of the betatron tunes. A classic application of PLLs is to measure the chromaticity by modulating the beam momentum using the RF frequency while tracking...
Fig. 17: Residual non-excited horizontal LHC Beam 2 tune spectra. The spectra are normalized to the maximum detected tune amplitude, indicated by diamonds.

Fig. 18: LHC tune phase-locked-loop schematics

the tune. The modulation amplitude is then proportional to the linear machine chromaticity while the barycentre is a measure of the unperturbed tunes.

In good approximation, the tune resonance can be described by a second-order harmonic oscillator. Thus the tune resonance is found once the phase between excited and measured oscillation equals $\pi/2$. The corresponding PLL block diagram is shown in Fig. 18.

The system mixes the digitized beam signal with the sine and cosine components of the excitation signal and uses low-pass filters in order to remove the $2\pi f_e$ frequency component that is created in the mixing process. The resulting signals are passed through a rectangular-to-polar converter that separates signal phase and amplitude, which can then be treated further by two independent controllers. The actual control input $\Delta \varphi$ is shifted by $\pi/2$ to zero in order to obtain a bipolar signal around the tune resonance. Negative phases thus indicate that the excitation frequency is below the tune resonance, and positive phases indicate the opposite. The control input is further compensated for other non-beam-related contributions to the measured phase shift, such as constant lag due to data processing, cable transmission delays, analog pre-filters suppressing dominant harmonics (e.g., revolution frequency), and the response of the beam exciter itself. The individual figures in Fig. 19 illustrate the commissioning steps, usually involving a BTF measurement each, prior to closing the loop and locking on to the tune.
resonance. An uncompensated BTF is shown in Fig. 19(a). While the magnitude response shows the expected shape, it can be seen that the phase is affected by several phase wrap-arounds due to the intrinsic sampling and processing phase lag. Counting the number of wrap-arounds, one can roughly estimate the number of samples \( n_s \) the signal is delayed and subtract the corresponding phase delay \( e^{jn_s \Omega T} \) from the phase response, leading to the BTF shown in Fig. 19(b). The next step is to compensate for the remaining phase slope by subtracting \( d\varphi/df \). Then \( d\varphi/df \) can be estimated by measuring the slope beyond the BTF core (e.g., \( f \gg |q - \Gamma| \)). Figure 19(c) shows the corresponding BTF including the sampling delay and phase-slope compensation. It is visible that the phase zero-crossing is not aligned with the tune peak seen in the amplitude response. The BTF including the correction for this systematic phase mismatch is shown in Fig. 19(d). The speed of the BTF is usually adjusted in order to improve the resolution of the scan. For comparison, Fig. 20 shows a typical compensated BTF taken with the LHC-type PLL system at the CERN SPS. In the case of the LHC tune PLL, the system is implemented on the same FPGA-based data acquisition board as the continuous FFT processing chain, providing operational flexibility and redundancy in case of hardware failure [27].

Providing operation at or close to the phase-lock condition, the phase control loop dynamics can be linearized and reduced to a first-order system with the open loop gain \( K_0 \) depending on the slope of the phase response at the location of the tune, and the inverse time constant \( \tau \) on the bandwidth of the low-pass filter. The LHC PLL controller design is based on Youla’s affine parametrization, an optimal control approach, and yields a simple proportional–integral controller [28]:

\[
D_\varphi(s) = K_p + K_i \frac{1}{s} \quad \text{with} \quad K_p = K_0 \frac{\tau}{\alpha}, \quad K_i = K_0 \frac{1}{\alpha}.
\]  

The proportional \( K_p \) and integral gain \( K_i \) are coupled. The inverse of the free parameter \( \alpha \) defines the effective closed-loop bandwidth. Depending on the operational scenario (‘gain-scheduling’), the single parameter \( \alpha \) thus facilitates the trade-off between a fast tracking PLL (\( f_{bw} \approx 6 \text{ Hz} \)) but with larger measurement resolution (\( \Delta Q_{\text{res}} \approx 10^{-3} \)) and a PLL having smaller measurement resolution (\( \Delta Q_{\text{res}} \approx 10^{-4} - 10^{-6} \)) but also smaller tracking bandwidth.

Figure 21 shows a typical PLL-based tune trace measured in the SPS using a LHC-type beam that is accelerated from 26 GeV/c (time: 0 s) to a final momentum of 450 GeV/c (6.5 s) and resonantly extracted using the lattice sextupoles. The tune trace (blue), residual phase error (red), and amplitude signal (green) are shown. The initial fractional tune \( q_h \approx 0.76 \) is quickly changed below (6.5–7.5 s) and slowly brought towards the third-order resonance (\( q_h = 0.66 \)) before the beam is eventually fully extracted and the beam signal lost (\( \approx 8.3 \) s). The less than \( \pi/2 \) phase error and the non-vanishing amplitude response indicate that the PLL was capable of tracking tune changes of up to \( \Delta Q \approx 0.1 \) within about 200 to 300 milliseconds which is orders of magnitude faster than the expected tune changes shown in Fig. 4(a). Figure 21(b) illustrates the other mode of operation of a slow but precise tune tracking. Here, the impedance induced tune-shifts that were caused by closing and re-opening of the jaws of the LHC-prototype collimator installed in the CERN SPS are shown. The tune resolution is of the order of at least \( 10^{-6} \).

### 3.4 Betatron coupling

As described in Ref. [29], a robust and reliable tune PLL and in particular any control of orbit, tune, and chromaticity also requires the measurement and control of global betatron coupling. The classic method of measuring the absolute of the betatron coupling parameter \(|C^-|\) uses eigenmode definition, given in Eq. (16). It is visible that the residual distance of the eigenmodes \(|Q_1 - Q_2|\) equals the coupling parameter \(|C^-|\), if the unperturbed tunes are set to the same working point \( q_x = q_y \). Usually, since the true working point is unknown, the unperturbed tunes are moved and crossed with respect to each other. An illustration of the method is shown in Fig. 22. While this closed-tune approach provides a robust estimate of the absolute coupling, it cannot be used to reconstruct the coupling phase \( \xi \), required for the compensation using the skew-quadrupoles.
Another approach, reconstructing both the real and imaginary part of the betatron coupling, tracks the regular and cross-term amplitudes $A_{1/2,x/y}$, indicated in Fig. 23, of the perturbed eigenmode ‘1’ and ‘2’ and computes the ratios

$$r_1 = \frac{A_{1,y}}{A_{1,x}}, \quad r_2 = \frac{A_{2,x}}{A_{2,y}},$$

(72)

using either an DFT or PLL-based tune measurement. As shown in Ref. [29], using the ratio definitions given in Eqs. (72), the coupling parameter is given as

$$|C^-| = |Q_1 - Q_2| \cdot \frac{2\sqrt{r_1 r_2}}{(1 + r_1 r_2)} ,$$

(73)

$$\Delta = |Q_1 - Q_2| \cdot \frac{1 - r_1 r_2}{(1 + r_1 r_2)} ,$$

(74)

$$\xi = \frac{1}{2} \left( \text{atan2}(A_{1,y}, A_{1,x}) + \text{atan2}(A_{2,x}, A_{2,y}) \right) ,$$

(75)

**Fig. 19:** Individual BTF scans and corresponding phase compensation prior to and in view of a tune PLL operation. The horizontal frequency axes in Figs. 19(c) and 19(d) are zoomed around the resonance for better visibility.
TUNE AND CHROMATICITY DIAGNOSTICS

Fig. 20: Typical compensated beam-transfer-function taken with the LHC-type PLL system at the CERN SPS

and using Eq. (16)

\[ q_x = Q_1 + \frac{1}{2} \Delta - \frac{1}{2} \sqrt{\Delta^2 + |C^-|^2}, \]

\[ q_y = Q_2 - \frac{1}{2} \Delta + \frac{1}{2} \sqrt{\Delta^2 + |C^-|^2}. \]

An example betatron-coupling measurement taken in the SPS using a chirp-type excitation is shown in Fig. 24. The horizontal and vertical tunes were crossed by ‘accident’, providing an opportunity to verify and compare the closed-tune and eigenmode amplitude ratio-based approach. The perturbed and crossing unperturbed tunes are visible in the lower left part of the plot.

4 Chromaticity diagnostics

4.1 Classic slow \( \Delta p/p \) modulation based methods

The most fundamental chromaticity diagnostic method derives directly from the chromaticity definition provided in Eq. (17), describing the chromaticity as the linear (quadratic, cubic, etc.) dependence of the momentum-induced tune shift. For the sextupole-compensated chromaticity and bunched beams, the required momentum shifts are typically induced by shifting the accelerator’s RF frequencies \( \Delta f/f \) according to Eq. (78)

\[ \frac{\Delta p}{p} = \left( \frac{1}{\gamma^2} - \alpha_c \right) \frac{\Delta f}{f} := \eta \cdot \frac{\Delta f}{f}, \]

with \( \gamma \) being the relativistic boost factor, \( \alpha_c \) the momentum compaction factor, and \( \eta \) the so-called slip-factor defined by the same equation. The point where \( \eta = 0 \) marks the so-called transition (energy) and thus the momentum compaction factor is also often written with a similar gamma notation: \( \alpha_c = 1/\gamma_0^2 \). It is visible that for low-energy accelerators typically operating below transition (\( \eta < 0 \)), the RF frequency needs to be increased for an increasing momentum shift, and for high-energy (and practically all lepton accelerators) operating above transition (\( \eta > 0 \)), the RF frequency needs to be decreased.
(a) CERN SPS: PLL fast tracking example

(b) CERN SPS: PLL precision tracking example

Fig. 21: LHC PLL tune tracking examples in the SPS. The additional measurements (non-blue) in Fig. 21(b) are based using the same BBQ pick-up but using FFT spectra.
Fig. 22: Closest-tune approach schematic. The unperturbed horizontal (red, dashed) and vertical (blue, dashed) tunes, eigenmodes ‘1’ (red) and ‘2’ (blue), and closest distance equal to $|C|$ are indicated.

Fig. 23: Coupling schematic. The rotation angle $\xi$ of the tune eigenmodes $Q_1$ and $Q_2$ and their projections with respect to the unperturbed horizontal and vertical reference system are indicated.

Figures 25(a) and 25(b) show an exemplary dependence of the tune on beam momentum for the CERN PS and LEP accelerators. The tunes were measured using kick-type beam excitations and the momentum varied from one tune measurement to the next. The circumference of the captured beam is defined by the accelerator’s RF frequency.

Another possible way to measure chromaticity at injection is to use a known injection momentum mismatch, given by the preceding accelerator, and the equilibrium momentum of the circulating beam while measuring the injection and circulating beam tunes. The calculation of chromaticity is done as in the other cases. The injection momentum mismatch $\Delta p/p$ can be estimated using, for example, the $N$
Fig. 24: CERN SPS coupling reconstruction example using the scheme described in Ref. [29]. The crossing of the tunes in the frequency range $[0.35, 0.48]$ is visible. The increased low-frequency spectra and $Q_s$ sidebands of the revolution signal are visible due to longitudinal instabilities.

Fig. 25: Linear and non-linear tune dependence on momentum. The linear (straight-line fits) and non-linear chromaticity contributions are visible. Courtesy H. Burkhardt and R. Steerenberg.
BPMs distributed over the ring and computing the by the dispersion function $D_i$ weighted estimate

$$\frac{\Delta p}{p} \bigg|_{\text{est}} = \frac{\sum_{i=0}^{N} D_i \Delta z_i}{\sum_{i=0}^{N} D_i^2},$$

with $\Delta z_i$ being the difference between the first-turn and circulating beam position and $i$ denoting the index of a given BPM. Figure 26 shows an example taken in the LHC showing the injection and circulating tune spectrum. The tune shift of $\Delta Q \approx 0.007$ due to the injection momentum mismatch is visible. Using a momentum shift of $\Delta p_p \approx 2 \cdot 10^{-4}$, this related to a chromaticity of about 30. The synchrotron sidebands due to the large chromaticity are visible and can be used as another possibility for estimating chromaticity as discussed in Section 4.4.

Applying a momentum shift using Eq. (78) effectively shifts the radial orbit, thus forcing the beam to oscillate and sample the off-centre fields of, for example, the sextupoles leading to the total chromaticity expression given in Eq. (25). A method to measure the natural chromaticity of the machine is to change the momentum by adjusting the main dipole field

$$\frac{\Delta p}{p} \approx \frac{\Delta B}{B}.$$  

Provided the RF frequency is kept constant on the so-called central frequency, a change of the main dipole field will change only the beam momentum while the beam continues traversing through the centre of all magnets and thus not sample higher-order fields. Having by-passed the effect of sextupoles, for example, the tune changes are given by the natural chromaticity created by the quadrupoles only.

### 4.2 Continuous $\Delta p/p$ modulation based methods

Compared to other past and present machines, the LHC has the tightest requirements on chromaticity diagnostics and control. Owing to persistent currents, the related decay and snap-back phenomenon (inherent to superconducting magnets) as well as other perturbation sources, values of orbit, energy, $Q$, and $Q'$ may exceed LHC beam stability requirements by orders of magnitude, with significant drift-rates as shown in Fig. 4(b). Further discussion thus uses the LHC as archetypical example for fast chromaticity trackers.
Fig. 27: $\Delta p/p$ as a function of RF modulation frequency $f_{\text{mod}}$. Upper limits imposed by the RF and the LHC cleaning system are indicated.

4.2.1 Slow vs. fast RF modulation

The LHC RF cavities can provide a maximum RF voltage of 16 MV per beam. However, most of this voltage is required to provide a sufficiently large bucket area for nominal LHC bunch intensities, and thus only a small fraction, some 0.25–0.5 MV, is actually available for $Q'$ related modulation. In addition to RF system driven constraints, the non-zero dispersion at the collimator location relates the transverse amplitude constraints to an effective limit on $\Delta p/p$ of the order of $10^{-5}$. The corresponding $Q'$ tracking parameter range is shown in Fig. 27.

Modulation frequencies equal to or close to the synchrotron tune $Q_s$ are excluded by longitudinal emittance preservation, which essentially leaves two choices for RF modulation-based $Q'$ tracking methods:

- modulation frequencies well above $Q_s$, as proposed in Refs. [30, 31]. As visible in Fig. 27, their use in the LHC is impractical due to limited available RF power.
- modulation frequencies well below $Q_s$, commonly referred to as the classical method, which tracks the $Q'$ dependent tune changes $\Delta Q$ as a function of momentum modulation $\Delta p/p$. The underlying relation, also defining the unit of $Q'$, is given by

$$\Delta Q := Q' \cdot \frac{\Delta p}{p}. \quad (81)$$

In most accelerators, the classic $Q'$ measurement relies on $\Delta p/p$ modulations typically of the order of $10^{-4}$–$10^{-3}$. As visible in Eq. (81), a $Q'$ resolution of 1 would require a $Q$ measurement resolution of the same order of magnitude ($10^{-4}$).

4.2.2 LHC $Q'$ tracker prototype

Owing to the fast drift rates and long periods over which these drifts need to be monitored and controlled, the LHC requires a continuous measurement of $Q'$ and, consequently, the momentum-driven tune changes. For short time-scales, these tune changes can, for non-zero modulation amplitudes, be approximated by

$$Q(t) = +\beta_0 + \beta_1 \cdot \sin(\omega_m t) + \beta_2 \cdot \cos(\omega_m t) + \beta_3 \cdot t + \beta_4 \cdot t^2 + \beta_5 \cdot t^3 \quad (82)$$
The modulation amplitude, and $\beta_3, \ldots, \beta_5$ higher order correction terms that compensate for linear, quadratic and cubic tune drifts unrelated to the RF modulation, but which are fast with respect to the modulation frequency. There are several suitable techniques for reconstructing the average tune and its modulation. They can be grouped into

- Classic amplitude demodulation: The average tune is first removed using a high-pass filter and then — similar to a PLL — multiplied with a sinusoidal reference signal. The higher-order mixing products are rejected through a low-pass filter. The modulation amplitude can be reconstructed according to Eq. (83) using the in- and out-of-phase amplitudes. The high- and low-pass cut-off frequencies need to be well below $f_{mod}$, which ultimately limits this scheme’s performance in the presence of fast tune drifts owing to the phase lag introduced by the filter.

- Linear regression: The modulation frequency $\omega_m$, the time between tune measurements and consequently the harmonic and polynomial terms in Eq. (82) are constant. Thus this problem can be transformed to a system of linear equations for each tune measurement $Q_i$ with $\beta_0, \ldots, \beta_5$ being free parameters. The collection of $N$ measurements can be rewritten in matrix form as

$$
(Q_1, \ldots, Q_N)^T = \mathbf{R} \cdot (\beta_0, \ldots, \beta_5)^T. \tag{84}
$$

Here $N$ should cover at least half or multiple periods. A universal solution to Eq. (84) can be found by computation of the pseudo-inverse matrix $\mathbf{R}^{-1}$. Using a Singular-Eigenvalue-Decomposition (SVD), potential singularities can be identified as very small or vanishing eigenvalues and eliminated in the inversion process by setting their inverse to zero. While SVD has a cubic complexity, the final computation to reconstruct the average tune and modulation amplitude consists of a simple matrix–vector multiplication.

- Chi-square fitting: This method also allows the reconstruction of non-linear parameters, such as the frequency. For the above case, the solutions using linear regression and chi-square fitting are identical. This method was mainly used as a cross-check in offline analysis.

The feasibility of the $Q'$ measurement with the unprecedented small momentum modulation, compatible with nominal LHC operation, has been demonstrated at the CERN SPS [32]. Two typical measurements are shown in Figs. 28(a) and 28(b). The reconstruction was based on a sliding window covering two oscillation periods of the tune modulation ($f_{mod} = 0.5$ Hz) and shifted for each measurement by the sampling interval. The achieved $Q$ resolution was of the order of $10^{-6}$, resulting in a chromaticity resolution of better than 1 unit. As illustrated in Fig. 28(b), the $Q'$ tracking loop was able to cope with chromaticity values up to 36 units, which provides some margin for operation in a regime where a classic $Q$ kicker or chirp-based measurement using BPMs would fail because of the very fast decoherence times. The momentum shift has been controlled by changes to the SPS-RF frequency reference. While the achieved tune resolution would have supported smaller momentum modulation amplitudes, the ultimate lower limit of $\Delta p/p = 1.8 \cdot 10^{-5}$ was given by the minimum quantization of the RF frequency changes. The tune steps are caused by feed-down effects due to the off-centre orbit in the lattice sextupoles, also causing transients in the chromaticity reconstructions because of the trimming-induced tune drifts being faster than those due to RF modulation. For the LHC system, the time-scales of $Q$ and $Q'$ drifts are expected to be small compared to the targeted modulation frequencies of between 1 and 5 Hz. In case of problems and assuming relaxed RF and cleaning system tolerances, the measurement can be improved by increasing the amplitude or frequency of the modulation.
4.3 Head–Tail phase shift

From the tune width point of view, it would be a priori sufficient to keep chromaticity small. However, as first observed in the ACO and Adone accelerators and described in Ref. [13], negative chromaticity in accelerators operating above transition may contribute to the so-called head–tail instability. This instability is driven by the inherent non-zero machine impedance and amplified by longitudinal oscillations of the individual particles inside the bunch. These oscillations can be described by

$$\frac{\Delta p}{p}(t) = \frac{\hat{\Delta p}}{p} \cdot \sin (\omega_s \cdot n + \varphi_i)$$ \hspace{1cm} (85)

with $\frac{\Delta p}{p} / p$ the maximum deviation of a given off-momentum particle, $n$ the turn count, and $\omega_s$ the synchrotron oscillation frequency. The growth time $\tau_{HT}$ of this head–tail instability for a bunch slice located at $\hat{\tau}$ has been estimated to

$$\frac{1}{\tau_{HT}} \propto \frac{N_b}{E} \cdot \frac{\hat{\tau} Q'}{Q^2 \cdot (\alpha_c - 1/\gamma_{rel}^2)}$$ \hspace{1cm} (86)

with $N_b$ the number of particles in the bunch, $E$ the beam energy, and $\alpha_c$ the momentum compaction factor. The momentum slip factor $\alpha_c - 1/\gamma_{rel}^2$ is positive below transition and negative above transition. Therefore machines operating above transition, practically all lepton accelerators and most high-energy hadron accelerators, favour a slightly positive working point for the chromaticity $Q'$.

The mechanism can be transferred to head–tail damping and therefore be used to estimate $Q'$ as first proposed in Ref. [33]. In this case, a given bunch slice $\hat{\tau}$ performs transverse oscillations after a transverse kick excitation that can be written as

$$y(n) = A \cos \left[ \omega_{\delta}\cdot n + \frac{Q' \omega_0 \hat{\tau}}{\eta} \cdot \left\{ \cos(2\pi Q_s n) - 1 \right\} \right].$$ \hspace{1cm} (87)

Figure 29 shows a simulated colour-coded transverse bunch oscillation as a function of $n$ and position in the bunch for $Q' = 0$, $Q' = 1$, and $Q' = 2$. The Landau detuning due to the head–tail frequency shift has been omitted for better visibility. The quickening of the head’s (e.g., $\tau = +0.5$ ns) motion in the first half and, consequently, the slowing in the second half synchrotron period and the vice versa motion of the tail (e.g., $\tau = -0.5$ ns) is visible for non-zero chromaticities. Please note that the chromaticities and...
The chromaticity dependent synchrotron phase modulation $\Delta \psi$ can be recovered via demodulation of the signal at the tune frequency, using the Fourier–Hilbert transform or via the quadrature of the signal $y(n)$ and successive low-pass filtering to remove the mixing product. In terms of practical application, it proves useful to compute the difference between two or more slices in the head and tail of the bunch as this also minimizes possible remaining tune frequency components and other higher-order dependencies.

In the case of the Hilbert transform, the phase argument of Eq. (87) can be recovered via

$$\psi(n) = \text{atan2} \left( \mathcal{H}(y(n)), y(n) \right) = \omega \beta_0 n + \Delta \psi(n)$$

(88)

with $\mathcal{H}(y(n))$ being the discrete Hilbert transform of the signal $y(n)$, and

$$\text{atan2}(a,b) = \begin{cases} 
\varphi \cdot \text{sgn}(a) & b > 0 \\
\frac{\pi}{2} \cdot \text{sgn}(a) & b = 0 \\
(\pi - \varphi) \cdot \text{sgn}(a) & b < 0 
\end{cases}$$

(89)

with $\text{sgn}(x)$ being the signum function. Taking the phase difference from the transverse head and tail motion, removing the tune frequency dependence, and expressing Eq. (88) in terms of $Q'$ yields

$$Q' = \frac{-\eta \Delta \psi_{HT}(n)}{\omega_0 \Delta \tau (\cos(2\pi Qs) - 1)} = \frac{-\eta \Delta \psi_{HT}|_{\text{max}}}{\omega_0 \Delta \tau},$$

(90)

with $\Delta \tau$ being the distance of the selected head- and tail-slice, $\Delta \psi_{HT}(n)$ being the phase-difference between head and tail bunch slice as a function of turn number $n$, and $\psi_{HT}|_{\text{max}}$ its maximum.

The principle has been tested and implemented at the CERN SPS, Tevatron, and LHC using long strip-line couplers to separate direct and reflected signals, followed by a $\Sigma$–$\Delta$ hybrid to minimize the common mode (intensity) signal. Since the head–tail shift method relies on the ability to resolve the bunch and distinguish between its head- and tail-specific oscillations, usually a much higher bandwidth is required than for orbit-type measurements. In the case of the SPS and LHC accelerators, the typical r.m.s. bunch length is of the order of 0.5 ns which makes it necessary to use fast real-time sampling scopes with analog bandwidths of a few GHz. At these frequencies, it is important to compensate for non-beam related effects that may contribute to the effective measured phase $\Delta \psi(n)$, such as intrinsic errors of the pick-up, hybrid, cable, and scope transfer functions. Figure 30 shows an experimental example of a compensated head–tail oscillation. In the shown example the head–tail oscillations were measured using the instability driven by negative chromaticity (the CERN SPS operates typically above transition) instead of a kick. In any case, because of the symmetries of the head–tail mechanism, the oscillation behaves similarly for self-driven instability growth or damping after a kick-type beam excitation.

While this method is ideal for obtaining fast chromaticity estimates without further need of momentum modulation, it typically requires large transverse kicks that may cause significant emittance blow-up, and is also intrinsically limited by some higher-order effects that may dominate or deteriorate the observed chromaticity dependent phase shift:

- non-chromaticity induced damping: synchrotron radiation, machine impedance, amplitude detuning, and other higher order effects driving head–tail instabilities;
- low-synchrotron tune, as the decoherence is more dominant and the signal may have vanished before one complete synchrotron oscillation period;
- RF bucket non-linearities, and, consequently, dependence of the synchrotron tune on the individual particle’s amplitude.
Fig. 29: Head–Tail simulation: the evolution of the bunch-slice oscillation amplitude (colour-coded) is shown as the function of position in the bunch (fast ns scale) and number of turns (horizontal scale). For better visibility, the bunch was sampled several times per turn. The Landau detuning has been omitted for better visibility of the head–tail effect.
– Similar to other BPM-based tune systems, the head–tail phase shift method with the present exploitation of using fast sampling scopes requires large kick amplitudes of the order of one to two r.m.s. beam widths that may contribute to emittance blow-up.

Current experiments are trying to tackle the latter limitation by deploying a BBQ-like detection scheme with the aim of removing the inconvenient common mode signal and boosting the system’s resolution to operate within negligible excitation amplitudes.

4.4 Synchrotron sidebands

Another method to estimate the chromaticity without modulating the momentum is based on the spectral amplitude of the tune and its synchrotron sidebands for bunched beams. These arise because of the intrinsic longitudinal and corresponding momentum modulation driven by the accelerator’s RF system. As shown in Section 2.3, the momentum driven tune shifts depend on the chromaticity and the single-turn tune \( \frac{dQ}{dn} \), also known as \( \text{instantaneous frequency} \ 2\pi \cdot \frac{df}{dn} \), is given thus by

\[
\frac{dQ}{dn} = Q_0 + Q' \frac{\Delta p}{p} (n) = Q_0 + Q' \frac{\Delta p}{p} \cos(\omega_s n) \tag{91}
\]

with \( n \) the turn count, and \( \omega_s = 2\pi Q_s \) the synchrotron tune of the longitudinal momentum oscillation. Integrating Eq. (91), the momentum modulated betatron oscillation can be written as

\[
\Delta z(n) = z_0 \cdot \cos \left( 2\pi \left[ Q_0 \cdot n + \frac{Q'}{\omega_s} \frac{\Delta p}{p} \cdot \sin(\omega_s n) \right] + \varphi_\beta \right) \tag{92}
\]
with \( z_0 \) being the normalized oscillation amplitude and \( \varphi_\beta \) being the initial oscillation phase. The momentum modulation can be rewritten as

\[
\cos (\omega_c t + B \sin(\omega_m t)) = \sum_{n=-\infty}^{+\infty} J_n(B) \cdot \cos((\omega_c + \omega_m \cdot n)t)
\]

with \( J_n(x) \) being the Bessel function of the first kind, describing the spectral tune \( (n = 0) \) and synchrotron sideband \( (n \geq \pm 1) \) amplitudes

\[
S_n(Q') = J_n \left( \frac{Q' \Delta p}{\omega_s \cdot p} \right)
\]

as shown for a synchrotron tune of \( Q_s = 70 \) Hz and momentum spread \( \Delta p/p = 10^{-3} \) in Fig. 31. It is visible, that — to first order — the tune amplitude is constant and the first \( (n = 1) \) depending linearly on the chromaticity. The small non-linearity and amplitude calibration dependence can be mitigated using the ratio between tune and first \( (n = \pm 1) \) synchrotron sidebands for the chromaticity estimate

\[
Q' \propto \frac{S_1}{S_0}
\]

with the calibration factor depending only on the synchrotron tune and momentum spread. As can be seen in Fig. 31, this ratio is linear for a wide range of chromaticities. As visible in Eq. (94), the linear regime becomes larger for larger synchrotron tunes and smaller momentum spreads. The synchrotron tune \( \omega_s \) can easily be derived from the same spectra by computing the average distances between the tune and individual synchrotron sidebands. The momentum spread is usually obtained using the longitudinal bunch profile for a given (usually known) accelerating voltage. Figure 32 shows exemplary LHC circulating beam spectra with particularly strong synchrotron sidebands. Using the measured \( Q_s \approx 70 \pm 2 \) Hz and \( \Delta p/p \approx 10^{-3} \), according to the dependence shown in Fig. 31, the chromaticity is estimated to be \( |Q'| = 30 \). Please note that in contrast to previously described methods, this method cannot differentiate between positive or negative chromaticities.

### 4.5 Tune width dependence on chromaticity

Classic tune PLL designs (Refs. [34, 35]) often model the PLL as a first-order process defined by the phase detector’s filter time constant and open loop gain \( K_0 \) that depends on the angle of the phase slope at
Fig. 32: Exemplary LHC circulating beam spectra showing synchrotron sidebands, Chauchy–Lorentz fit (red) and poly-line interpolation (green) indicating the width of the tune resonance

Fig. 33: SPS driven beam response example

the location of the tune resonance. In the presence of varying chromaticity, the open loop gains $K_0$ and thus the optimal controller parameter are functions of chromaticity itself. Using linear control design only, this cross-dependence implies either a controller design that is optimal for large chromaticities, which becomes sensitive to noise and unstable for low values of chromaticity, or a controller design that is optimal for small chromaticities but lags behind the real tune for large values of chromaticity. This dependence can itself be used to estimate chromaticity changes, as first shown at the CERN SPS [36, 37].

In order to exploit this dependence, the beam response is excited at two additional frequencies in the vicinity of the primary exciter used for tracking the tune. While the phase detection is the same as for the central exciter, their frequencies are kept constant with respect to the central exciter. For better phase measurement resolution and minimization of spectral leakage between side and central exciters, the side-exciters’ phase signals are subjected to a tighter low-pass filtering. Figure 33 shows a typical SPS beam response function measurement, indicating the lower (red vertical line) and upper (blue vertical line) side-exciters around the tune (dashed black line). In this example, the side-exciters frequencies were set to be $\Delta Q_{sex} \approx 5 \cdot 10^{-4}$ apart from the tune. The BTF was scanned with a slow scanning speed of about $\Delta Q/\Delta t \approx 3 \cdot 10^{-4} \text{s}^{-1}$.
In the given tune resonance approximation, the measured unnormalized side-exciters phase and tune width $\Delta Q$ are related through

$$\tan(\varphi) = \frac{\Delta Q \cdot Q_0 Q_{\text{sex}}}{Q_{\text{sex}}^2 - Q_0^2}$$

with $Q_0$ being the tune and $Q_{\text{sex}}$ the side-exciters frequency normalized by the revolution frequency. In order to obtain similar chromaticity values, the tune width has been normalized by the average r.m.s. bunch momentum spread similar to the unbunched beam case. Using the same formalism, it can be shown that the inverse amplitude response at the resonance frequency is also proportional to the tune width. Figure 34 shows the combined results of the tune-width-based chromaticity measurement obtained using the phase of the lower (red) and upper (blue) side-exciters. In addition, the inverse amplitude response of the central exciter (black) has been superimposed. Since the amplitude response depends directly on the excitation strength, tune-width-based chromaticity estimated using the inverse amplitude response was scaled by a constant factor and shifted to match the phase-based one. The visible steps correspond to relative increases of $\Delta Q' = 2.6$ units of the machine chromaticity. The orbit was not perfectly centred inside all lattice sextupoles, and thus each chromaticity change also induced a small tune change that caused the visible transients due to the lag of the tune PLL. While the measured property changes linearly with chromaticity, the absolute measured chromaticity may be biased due to non-chromaticity-related, higher-order effects widening the tune peak via the Landau detuning mechanism. This bias can be removed using a cross-calibration with respect to a classical momentum-modulation-based chromaticity measurement.

### 4.6 Decoherence-based methods

Another possibility to exploit the head–tail instability’s dependence on chromaticity is a direct measure of its growth or damping time $1/\gamma_{HT}$ after a transverse kick excitation. While the time constant depends linearly on the chromaticity, as visible in Eq. (86), it is also affected by the intensity and energy of the given bunch that have to be measured by other means. Figure 35 gives an example for such a measurement, taken at LEP, showing the head–tail decoherence time-constant as a function of beam current and chromaticity [38]. It can be seen that the decoherence, as measured at LEP, is primarily dependent on the chromaticity and bunch current. While the decoherence-based method provides a simple, direct means to measure chromaticity, similar to the previously discussed PLL side-exciters method, the decoherence may be affected by other non-chromaticity-related effects such as synchrotron radiation and Landau detuning effects due to octupole magnets.
Fig. 35: Chromaticity-dependent decoherence at LEP [38]

Acknowledgements
The author expresses his thanks and gratitude to the following people for their valuable contributions to the lecture materials and to this document: A. Boccardi, H. Burkhardt, P. Cameron, M. Gasior, R. O. Jones, A. S. Müller, H. Schmickler, R. Steerenberg, J. Wenninger, and F. Zimmermann.

References


[34] O. Berrig et al., The Q-loop: a function driven feedback system for the betatron tunes during the LEP energy ramp, CERN SL-98-03, 1998.