TRANSVERSE OSCILLATIONS OF A RELATIVISTIC PARTICLE
BEAM IN A VACUUM CHAMBER WITH PERIODIC CROSS-SECTION
VARIATIONS (ELECTRODYNAMICS)

by

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ABSTRACT

A "transverse coupling impedance" can be defined which expresses the interaction of a transversely oscillating particle beam with its surrounding fields. The fields are found by solution of the wave equation with appropriate boundary conditions. Only the finite conductivity of the wall with enlarged radius is taken into account, all other surfaces are assumed to be perfectly conducting.

A pair of opposing cross-section variations forms a cavity which can lead to both TM ("hybrid") and TE resonances. Approximate expressions for the resonant frequencies and coupling impedances of these resonances are derived. These formulae are used to discuss the stability of the proton beam in the ISR which has a vacuum chamber that alternates in cross-section between circular and elliptical. It is found that the limitations are less severe than in the longitudinal case. For comparison, the transverse impedance of a RF particle separator structure is calculated by these approximate expressions, and found to agree reasonably well with earlier results obtained by computer and verified experimentally.
LIST OF SYMBOLS

a) Capital Letters

\( A_m, B_m \) coefficients in region I

\( \bar{A}_m = A_m I_1(\chi_m b), \bar{B}_m = B_m I_1(\chi_m b) \) column

\( \bar{A}_m = \bar{A}_m + \bar{V}_1, \quad \bar{B}_m = \bar{B}_m + \bar{V}_2 \) vectors

\( C_m, C_m' \) coefficients in region II

\( \bar{k}_m = \frac{\chi_m b}{\bar{\alpha}_m} \) diag. matrix \( C \)

\( D_{ss} = \frac{\bar{\alpha}_m}{\bar{g}_s b} \) diag. matrix \( D \)

\( \bar{E} \) electric field strength

\( E_s, F_s \) coefficients in region III

\( \bar{E}_s = E_s R_o (\bar{\Gamma}_s b) \) column

\( \bar{F}_s = -i \bar{F}_s (\bar{\Gamma}_s b) \) vectors

\( \bar{F} \) electromagnetic force

\( \bar{F}_m \) coefficients of force

\( F_i (\bar{\Gamma}' r) = Y_1 (\bar{\Gamma}' d) J_i (\bar{\Gamma}' r) - J_1 (\bar{\Gamma}' d) Y_i (\bar{\Gamma}' r) \)

\( G_i (\bar{\Gamma}' r) = Y_1 (\bar{\Gamma}' d) J_i (\bar{\Gamma}' r) - J_1 (\bar{\Gamma}' d) Y_i (\bar{\Gamma}' r) \)

\( F_1 (x) = Y_1 (\lambda x) J_1 (x) - J_1 (\lambda x) Y_1 (x) \)

\( G_1 (x) = Y_1 (\lambda x) J_1 (x) - J_1 (\lambda x) Y_1 (x) \)

\( \bar{G}(s) = R \bar{R}^{\dagger} (N^{\dagger} \bar{V}_1 - L^{\dagger} \bar{I}' \bar{V}_2) \) column

\( \bar{H}(s) = 2 M^{\dagger} \bar{V}_{(2)} \) vectors

\( \bar{H} \) magnetic field strength

\( I_{o,1}(x) \) modif. Besselfct. 1st kind

\( I_o \) DC beam current

\( I_{nm} = \frac{\bar{g}_m \bar{I}_{m'}}{\chi_m I_1(\chi_m b)} \) diag. matrix \( I \)

\( \bar{J} \) (volume) current density
Capitol Letters (cont'd)

\[ J_0, 1(x) \] regul. Besselct. 1st kind

\[ K \] surface current density

\[ K_0, 1(x) \] modif. Besselct. 2nd kind

\[ K_{ss} = \frac{2}{1 + \delta_{so}} \] diag. matrix K

\[ L_{(ps)} = MD - CN \] matrix L

\[ M_{ps} = \frac{2}{(\pi \alpha p)^2 - (\frac{\pi s}{2})^2} \begin{bmatrix} \sin \\ -\cos \end{bmatrix} \] matrix M

\[ M^+_{sp} = M^*_{ps} \] Hermitian conjugate \( M^+ \)

\[ N_{ps} = \frac{\pi \alpha p}{(\pi \alpha p)^2 - (\frac{\pi s}{2})^2} \begin{bmatrix} \sin \\ -\cos \end{bmatrix} \] matrix N

\[ N^+_{sp} = N^*_{ps} \] Hermitian conjugate \( N^+ \)

\[ P_{ss} = \frac{s}{2 \alpha \beta n} \] diag. matrix P

\[ Q \] (vertical) betatron number

\[ Q_{ss} = (1 + \frac{\alpha^2}{\Gamma^2 R_s}) \frac{\Gamma_s}{\beta k} \frac{R_1}{S_1} \Gamma_s \] diag. matrix Q

\[ R \] Machine radius

\[ R_i(xr) = K_1(\alpha R)(xR) + (-i) \Gamma_i K_1(\alpha R) \] \( K_i(xR) \)

\[ R_i(\Gamma R) = K_1(\Gamma R)(\alpha R) + (-i) \Gamma_i K_1(\Gamma R) \] \( K_i(\Gamma R) \)

\[ S_i(\Gamma R) = K_1(\Gamma R)(\alpha R) + (-i) \Gamma_i K_1(\Gamma R) \] \( K_i(\Gamma R) \)

\[ R_{mm} = \frac{\beta k}{\chi_m} \frac{R_1}{S_1} \chi_m \] diag. matrix \( R \)

\[ R_{ss} = \left( \frac{x R_1'}{R_1} \right) \Gamma_s b \] diag. matrix \( R \)

\[ S_{ss} = \left( \frac{x S_1'}{S_1} \right) \Gamma_s b \] diag. matrix \( S \)

\[ R'_{ss} = \frac{\beta k}{\Gamma_s} \frac{R_1'}{R_1} \Gamma_s b \] diag. matrix \( R' \)

\[ S'_{ss} = \frac{\beta k}{\Gamma_s} \frac{S_1'}{S_1} \Gamma_s b \] diag. matrix \( S' \)

\[ \mathcal{R} = \sqrt{\frac{\delta_{so}^2}{2 \sigma_1}} \] (normalized) surface resistivity
Capital Letters (cont'd)

\[ T_{ss} = \frac{\beta_k}{s} \begin{vmatrix} 1 & S \end{vmatrix} \quad \text{diag. matrix } T \]

\[ U_{ss} = \delta_{ss} \quad \text{diag. matrix } U \]

\[ U_{ss} = 1 \quad \text{unit matrix } U \]

\[ \vec{v}_{l,2m} = \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} (x b) \delta_{mn} \quad \text{column vectors} \]

\[ \vec{V}_{l,2m} = \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} \frac{1}{b} (x b) \delta_{mn} \]

\[ w_{ij}, w'_{ij} \quad \text{coefficients (Eq. 15.2/3)} \]

\[ Y_{1}(s) = KR^{-1} \begin{vmatrix} -1 & 0 \\ V_{1} & V_{2} \end{vmatrix} \quad \text{column vectors} \]

\[ Y_{2}(s) = -2M^{T} V_{2} \]

\[ Y_{0,1}(x) \quad \text{regul. Bessel fct. 2.kind} \]

\[ Z = (1+i) \mathcal{Z} \quad \text{(normal) surface impedance} \]

\[ Z_{o} = \sqrt{\mu_{o}/\varepsilon_{o}} \quad \text{free space impedance} \]

\[ Z_{c} \quad \text{(transv.) coupling impedance} \]

b) Small Letters:

\[ a \quad \text{beam radius} \]

\[ b \quad \text{tube radius} \]

\[ c \quad \text{light velocity} \]

\[ d \quad \text{cavity radius} \]

\[ \exp = e^{i k m (z-\sigma/2)} \]

\[ g \quad \text{cavity (gap) length} \]

\[ j_{q} \quad \text{q-th zero of } J_{1}(x) = 0 \]

\[ k = n/R \quad \text{(perturbation wave number)} \]

\[ k_{m} = m/R \quad \text{(space harm) wave number} \]

\[ m \quad \text{mode number (space harm)} \]

\[ n \quad \text{mode number (perturbation)} \]

\[ m, p \quad \text{indices from } -\infty \text{ to } +\infty \]

\[ r \quad \text{radial coordinate} \]
Small Letters (cont’d)

s
radial mode number

t
axial mode number

s,t
indices from 0(1) to ∞

t'
time coordinate

v_0 = β_0 c
beam velocity

x, y
transverse coord.

x_t
t-th zero of x_0 \to 2F_1

y_t
t-th zero of x_0 \to 2G_1

z
longitud. coord.

c) Greek Alphabet

α = \frac{g}{2πR}
geom. parameter

α_s = \frac{s}{2πg}
longit. propag. constant

β = \frac{ω}{kc}
(relat.) wave velocity

β_0 = \frac{v_0}{c}
(relat.) beam velocity

β_1 = 1 - \beta_0 \quad \{ abbreviations

β_2 = \beta_0 - \beta

γ = \frac{1}{\sqrt{1-β^2}}, \quad γ_0 = \frac{1}{\sqrt{1-β_0^2}}
energy factors

Γ_s = \sqrt{α^2 - \frac{ω^2}{c^2}}
radial propag. constant

δ_{ij}
Kronecker delta

ε = \frac{πb}{g}
geom. parameter

ε_0
free space permittivity

μ_0
free space permeability

λ = \frac{d}{b}
geom. parameter

λ = \frac{b}{d}

ρ_0
DC (volume) charge density

σ
surface charge density

σ'
conductivity

ε
oscillation amplitude

χ_m = \sqrt{\frac{k^2}{m} - \frac{ω^2}{c^2}}
radial propag. constant

χ = \frac{χ_m}{κ/γ}

Greek Alphabet (cont'd)

φ  azimuthal coord.
ω  (angular) frequency

MKS-units used throughout the report
Matrices are capital letters,
column vectors are capital with a bar.
INTRODUCTION

In a previous report\(^1\) we have treated the case of a longitudinally oscillating beam inside a corrugated wall. In addition to longitudinal oscillations, however, a particle beam can oscillate transversely. When the beam oscillates as a whole this is called the "dipole mode", which is most easily detectable because of the motion of the center of gravity of the beam. This mode is of primary interest since it is usually more dangerous to beam stability\(^2\) than the multipolar modes (monopole, quadrupole, etc.) where the beam particles move against each other without affecting the center of gravity.

We investigate a cylindrical particle beam oscillating vertically in the dipole mode in a surrounding vacuum chamber which is concentric to the beam, but of periodically varying diameter. Although the geometry is rotationally symmetric, the electromagnetic fields are not and we need in general all six field-components to fulfill the boundary conditions. Only at resonance one of the modes becomes dominant, and we can distinguish TM and TE resonances in the cavity, resp. hybrid modes in the axial region.

The transversely oscillating beam is replaced by a cylindrical beam with equivalent surface charges\(^5\). When the phase velocity of the perturbation differs from the beam velocity, we also find the existence of transverse (volume) currents, in addition to the longitudinal surface currents caused by the surface charges. The vacuum chamber walls are assumed to be perfectly conducting except for the rear wall of the cavity, which has a high but finite conductivity typical for actual metals. The finite conductivity of the other surfaces could be included in principle but would only cause a further increase of the already large amount of mathematics involved, without bringing any essentially new features into the theory.

The analysis is more complicated than that of the longitudinal case due to the presence of six non-vanishing field
components. We have to solve two wave-equations for both the electric and magnetic longitudinal field components, and we have twice as many undetermined coefficients in each region. The effect of finite conductivity of the outer cavity wall does not lead to a simple replacement of a real matrix by a complex one, but generates different complex matrices in each of the four final equations for the four coefficients in the beam and cavity regions.

Reduction of the system of four equations to two is possible without matrix inversion. If we calculate the desired coefficients in the beam region directly, the resulting matrix products contain slowly converging infinite sums. However, if we calculate the coefficients in the cavity region first, we find matrix products that are almost diagonal and rapidly converging. A simple approximation then brings about a tremendous simplification of the matrix equations, which then contain only diagonal matrices and separate in the case of resonance. This allows the derivation of explicit expressions for the resonant frequencies and for the resonant field-coefficients.

From the field coefficients in the beam region one can directly calculate the dispersion relation coefficients U and V defined previously\(^4\). However, in analogy to the longitudinal case it seems preferable to define a "coupling impedance" that is in principle proportional to the dispersion relation coefficients but allows a better visualization of the strength of interaction between beam and wall. We therefore utilize a recently proposed definition\(^5\) of a coupling impedance per unit height (ohms/meter) to express the stability criterion of a particle beam against dipole oscillations. This criterion is applied to the CERN Intersecting Storage Rings (ISR), for which the coupling impedance for the vacuum-chamber of alternating elliptic and circular cross-section is calculated. As a check on the theory, also the transverse coupling impedance of an RF-particle separator structure is calculated and compared with results obtained by different methods\(^6\)\(^7\).
PART I: Derivation of the Matrix Equations for the Field Coefficients

1. Beam Model
   - unperturbed beam: cylindrical, uniform charge density \( \rho_0 \), uniform velocity entirely longitudinal \( v_0 = \beta_0 c \)
   - vertical (dipole) perturbation of (angular) frequency \( \omega \)
     wave number \( k = n/R \) (=2\pi/wavelength), where \( n \) is the mode number (number of wavelengths in one period 2\pi R)
     the wave (phase) velocity is then \( \beta c = \frac{\omega}{k} \) \hfill (1.2)
   - (complex notation, the factor \( e^{-i\omega t} \) is suppressed).

2. Source Terms
   The vertical oscillation is replaced by an equivalent surface charge (see ref. 3, p. 2-5)
   \( \sigma = \rho_0 \xi \sin \varphi e^{ikz} \) \hfill (2.1)
   where \( \xi \) is the amplitude of oscillation. This leads to a surface current \( K_z = \sigma \beta_0 c \) or
   \( Z_0 K_z = \rho_0 \xi \frac{\beta_0}{\varepsilon_0} \sin \varphi e^{ikz} \) \hfill (2.2)
   and to a vertical volume current
   \( Z_0 J_y = i \frac{\rho_0 \xi}{\varepsilon_0} (\beta_0 - \beta) ke^{ikz} \) \hfill (2.3)
   (corresponding to cylindrical components \( J_r = J_y \sin \varphi, \quad J_\varphi = J_y \cos \varphi \))

   We normalize by putting \( \frac{\rho_0 \xi}{\varepsilon_0} = 1 \) \hfill (2.4)

   We want to check the continuity equation. Inside the beam, we have no perturbed charges and thus \( \text{div} \ J = 0 \), which is fulfilled as \( J_y \) does not depend on \( y \). On the beam surface, we have a radial current supplying extra charges or
   \( \text{Div} \ K + \frac{\partial \sigma}{\partial t} = J_r \) \hfill (2.5)
   which is easily verified.

   Since \( J_y \) is also independent of \( x \), curl \( J \) vanishes and we have no source terms in the wave eqs for \( E_z \) or \( H_z \); (ref. 3, p. 28).
3. **Space harmonics**

The solutions of the homogeneous wave equation for region I and II (see fig. 1) can be written (cf. ref. 1)

\[
E_Z^I = \sum_{m=-\infty}^{+\infty} A_m I_1(\chi_m r) \sin \phi e^{ik_m(z-g/2)} 
\]

\[
Z_0 H_Z^I = \sum_m B_m I_1(\chi_m r) \cos \phi e^{ik_m(z-g/2)} 
\]

and

\[
E_Z^{II} = \sum_m \left[ C_m I_1 + C_m' K_1 \right] \chi_m r \sin \phi e^{ik_m(z-g/2)} 
\]

\[
Z_0 H_Z^{II} = \sum_m \left[ D_m I_1 + D_m' K_1 \right] \chi_m r \cos \phi e^{ik_m(z-g/2)} 
\]

where

\[
k_m = \frac{m}{R} 
\]

\[
\chi_m = k \left( \frac{1}{\gamma} - \frac{\omega^2}{c^2} \right) 
\]

(note that \( k_n = \frac{n}{R} = k, \chi_n = \frac{k}{\gamma} = \chi \))

(summation over \( m \) is from \(-\infty\) to \(+\infty\) throughout the report).

We have shifted the origin in order to obtain equations with real coefficients for the field coefficients (cf. ref. 1).

We shall write "exp" for \( e^{ik_m(z-g/2)} \) in the following.

4. **Transverse Field Components**

- we require only the azimuthal components for matching.

- from Maxwell's equations we get for vacuum (\( \varepsilon = \varepsilon_0, \mu = \mu_0 \))

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) E_\phi = \frac{1}{r} \frac{\partial^2 E_\phi}{\partial \phi \partial z} - \frac{i\omega}{c} \frac{\partial (Z_0 H_z)}{\partial r} - \frac{i\omega}{c} (Z_0 J_r) \right) 
\]

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) Z_0 H_\phi = \frac{i\omega}{c} \frac{\partial E_\phi}{\partial r} + \frac{1}{r} \frac{\partial^2 (Z_0 H_z)}{\partial \phi \partial z} - \frac{\partial}{\partial z} (Z_0 J_r) 
\]
Here we have \( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} = -k_m^2 + \frac{\omega^2}{c^2} = -\chi_m^2 \) and \( \frac{\omega}{c} = \beta k \)

to get
\[
E^{I}_\varphi = -i \sum_m \left[ \frac{k}{\chi_m r} A_m I_{1} - \frac{\beta k}{\chi_m} B_m I_{1}' - i\beta \gamma^2 \delta_{mn} \right] \cos \varphi \exp \left[ \chi_m r \right] \\
Z_0 H^{I}_\varphi = -i \sum_m \left[ \frac{\beta k}{\chi_m} A_m I_{1}' - \frac{k}{\chi_m r} B_m I_{1} - i\beta \gamma^2 \delta_{mn} \right] \sin \varphi \exp \left[ \chi_m r \right] \\
E^{II}_\varphi = -i \sum_m \left[ \frac{m}{\chi_m^2 r} (C_m I_{1} + C_m 'K_{1}) - \frac{\beta k}{\chi_m} (D_m I_{1} + D_m 'K_{1}) \right] \cos \varphi \exp \left[ \chi_m r \right] \\
Z_0 H^{II}_\varphi = -i \sum_m \left[ \frac{\beta k}{\chi_m} (C_m I_{1} + C_m 'K_{1}) - \frac{k}{\chi_m^2 r} (D_m I_{1} + D_m 'K_{1}) \right] \sin \varphi \exp \left[ \chi_m r \right]
\]

where
\[
\beta_1 = 1 - \beta \beta_0 \\
\beta_2 = \beta_0 - \beta
\]

(prime on a function means derivative with respect to complete argument).
(exp stands for \( e^{ik_m(z-\varphi/2)} \))

5. **Matching at the Beam-Edge** \( r = a \):

\[
E^{II}_z = E^{I}_z \\
E^{II}_\varphi = E^{I}_\varphi \\
H^{II}_z = H^{I}_z - K_z \\
H^{II}_\varphi = H^{I}_\varphi + K_z
\]

(the conditions on radial components are redundant).

With \( K_\varphi = 0 \), the \( z \)-components yield (argument \( \chi_m a \))
\[
C_m I_{1} + C_m 'K_{1} = A_m I_{1} \\
D_m I_{1} + D_m 'K_{1} = B_m I_{1}
\]

while the \( \varphi \)-components give
\[
\frac{k}{\chi_m^2 r} (C_m I_{1} + C_m 'K_{1}) - \frac{\beta k}{\chi_m} (D_m I_{1} + D_m 'K_{1}) = \frac{k}{\chi_m^2 a} A_m I_{1} - \frac{\beta k}{\chi_m} B_m I_{1}' - i\beta \gamma^2 \delta_{mn}
\]
\[
\frac{\beta k}{\chi_m} (C_m I_{1} + C_m 'K_{1}) - \frac{k}{\chi_m^2 a} (D_m I_{1} + D_m 'K_{1}) = \frac{\beta k}{\chi_m} A_m I_{1}' - \frac{k}{\chi_m^2 a} B_m I_{1} - i(\beta \gamma^2 - \beta_0) \delta_{mn}
\]
A number of terms cancel with Eqs. (5.2). We find
\[
\beta_2 \gamma^2 - \beta_0 = \gamma^2 \left[ (\beta_0 - \beta) - \beta_0 (1 - \beta^2) \right] = - \beta \beta_1 \gamma^2
\]  
(5.4)
and thus
\[
C_m I_1' + C_m' I_1 = A_m I_1 + i \beta_1 \gamma_\delta mn
\]
(5.5)
\[
D_m I_1' + D_m' I_1 = B_m I_1 + i \beta_2 \gamma_\delta mn
\]
The Wronskian for modified Bessel function yields
\[
(K_1 I_1' - I_1 K_1') x = K_1 (I_0 - \frac{I_1}{x}) + I_1 (K_0 + \frac{K_1}{x}) = (K_1 I_0 + I_1 K_0) x = \frac{1}{x}
\]
(5.6)
and we get the solutions
\[
C_m = A_m + i \beta_1 k a K_1(\chi a)\delta mn
\]
(5.7)
\[
C_m' = - i \beta_1 k a I_1(\chi a)\delta mn
\]
\[
D_m = B_m + i \beta_2 k a K_1(\chi a)\delta mn
\]
(5.8)
\[
D_m' = - i \beta_2 k a I_1(\chi a)\delta mn
\]

6. Solutions at Slotmouth \( r = b \):

\[
B^\Pi_{Z} \big|_{r=b} = \sum_{m} \left[ A_m I_1(\chi_m r) + i \beta_1 k a R_1(\chi_m r) \delta mn \right] \sin \varphi \exp \chi_m b
\]
(6.1)
\[
Z_0 H^\Pi_{Z} \big|_{r=b} = \sum_{m} \left[ B_m I_1(\chi_m r) + i \beta_2 k a R_1(\chi_m r) \delta mn \right] \cos \varphi \exp \chi_m b
\]
and
\[
B^\Pi_{\varphi} \big|_{r=b} = - i \sum_{m} \left[ \frac{k_m}{\chi_m b} (A_m I_1 + i \beta_1 k a R_1 \delta mn) - \frac{\beta k}{\chi_m b} (B_m I_1 + i \beta_2 k a R_1 \delta mn) \right] \cos \varphi \exp \chi_m b
\]
\[
Z_0 H^\Pi_{\varphi} \big|_{r=b} = - i \sum_{m} \left[ \frac{\beta k}{\chi_m b} (A_m I_1 + i \beta_1 k a R_1 \delta mn) - \frac{k_m}{\chi_m b} (B_m I_1 + i \beta_2 k a R_1 \delta mn) \right] \sin \varphi \exp \chi_m b
\]
(6.2)
where \( R_1(\chi r) = K_1(\chi a) I_1(\chi r) + (-)^i I_1(\chi a) K_1(\chi r) \)  
(6.3)
Define: Column vectors
\[ \vec{\Lambda}_m = \Lambda_m \Lambda_1 \chi_m b \]
\[ \vec{\Lambda}_m = \Lambda_m \Lambda_1 \chi_m b \]
\[ \vec{V}_{1,2m} = i \beta_{1,2} \kappa_a \vec{R}_1(\chi b) \delta_{mn} \]  
(6.4)

Diagonal vectors
\[ C_{mm} = \frac{\beta_{km} \Lambda_1}{\chi_m b} \]
\[ I_{mm} = \frac{\beta_{km} \Lambda_1}{\chi_m b} \]
\[ B_{mn} = \frac{\beta_{km} \Lambda_1}{\chi_m b} \]  
(6.5)

to rewrite the fields at \( r = b \) (leaving off the indices)
\[ E\chi  \]
\[ Z_0 H\chi \]
(6.6)

\[ E\chi \]
\[ Z_0 H\chi \]
(6.7)

(matrices are given by capital letters)
(the bar above a capital stands for column vectors)
(the tilde below for symbols that are only used temporarily)

7. *Redefinition of Matrices*

We put
\[ \vec{\Lambda} = \vec{\Lambda} + \vec{V}_{1,2} \]
\[ \vec{B} = \vec{B} + \vec{V}_{1,2} \]  
(7.1)

and
\[ \vec{V}_{1,2} = (\vec{R}_1 - I) \vec{V}_{1,2} \]  
(7.2)

From the Wronskian we get
\[ (\vec{R}_1 \Lambda_1 - I_1 \Lambda_1) \chi b \frac{1}{\chi b} I_1(\chi a) \]  
(7.3)
and the components of $\vec{V}_{1,2}$ thus are

$$\vec{V}_{1,2} \propto i \beta_{1,2}^2 a \frac{1}{b} \frac{I_{1}(xb)}{I_{1}(xa)} \delta_{mn}$$  \hspace{1cm} (7.4)$$

(single non-vanishing element).

The field coefficients at the slot mouth now become

$$E_{Z}^{II} \Big|_{r=b} = \sum_{m} \tilde{A}_{m} \sin \varphi \ e^{ik_{m}(z-g/2)} \Big\} \hspace{1cm} (7.5)$$

$$Z_{0} H_{Z}^{II} \Big|_{r=b} = \sum_{m} \tilde{B}_{m} \cos \varphi \ e^{ik_{m}(z-g/2)}$$

$$E_{\varphi}^{II} \Big|_{r=b} = -i \sum_{m} \left[ C_{m} \tilde{A}_{m} - (i \tilde{B}_{m} + i \tilde{V}_{2m}) \right] \cos \varphi \ e^{ik_{m}(z-g/2)} \Big\} \hspace{1cm} (7.6)$$

$$Z_{0} H_{\varphi}^{II} \Big|_{r=b} = -i \sum_{m} \left[ (i \tilde{A}_{m} + \tilde{V}_{lm}) - C_{m} \tilde{B}_{m} \right] \sin \varphi \ e^{ik_{m}(z-g/2)}$$

(where we left off the double indices of diagonal matrices).

8. **Solution in Cavity Region**

Finite resistivity only at outer wall $r = d$

Perfect conducting sidewalls at $z = 0$ and $z = g$ ($E_{z} = E_{\varphi} = 0$)

$$E_{Z}^{III} = \sum_{s=0}^{\infty} \left( E_{s} R_{1} + E_{s}^{*} S_{1} \right) \Gamma_{s} \sin \varphi \ \cos \alpha_{s} z \Big\} \hspace{1cm} (8.1)$$

$$Z_{0} H_{Z}^{III} = \sum_{s} \left( P_{s} R_{1} + P_{s}^{*} S_{1} \right) \Gamma_{s} \cos \varphi \ \sin \alpha_{s} z$$

where

$$\alpha_{s} = \frac{\pi s}{g}$$

$$\Gamma_{s}^2 = \alpha_{s}^2 - \frac{\omega^2_{s}}{c^2}$$  \hspace{1cm} (8.2)$$
and

\[ \begin{align*}
R_i(\Gamma r) &= K_1(\Gamma r) I_1(\Gamma r) + (-)^i I_1(\Gamma r) K_1(\Gamma r) \\
S_i(\Gamma r) &= K'_1(\Gamma d) I_1(\Gamma r) + (-)^i I_1'(\Gamma r) K_1(\Gamma r)
\end{align*} \]  \hspace{1cm} (8.3)

with the following properties

\[ \begin{align*}
R_1(\Gamma d) &= S_1'(\Gamma d) = 0 \\
R_1'(\Gamma d) &= - S_1(\Gamma d) = \frac{1}{\Gamma d}
\end{align*} \]  \hspace{1cm} (8.4)

(the last line follows from the Wronskian).

(sum over \( s \) from 0 to \( +\infty \) throughout the report).

with

\[ \frac{\partial}{\partial z} \frac{2}{\epsilon} + \frac{\omega_0^2}{c^2} = - \alpha_s^2 + \beta_s^2 = - \Gamma_s \]

the azimuthal components become

\[ \begin{align*}
\frac{\partial^2}{\partial z^2} + \frac{\omega_0^2}{c^2} = 0
\end{align*} \]  \hspace{1cm} (8.5)

9. Boundary Conditions on Outer Cavity Wall \( r = d \):

\[ \begin{align*}
E_r &= iZ (Z_0 H_r) \\
E_\phi &= - iz (Z_0 H_\phi)
\end{align*} \]  \hspace{1cm} (9.1)

where

\[ Z = (1 + i) \mathcal{R} \]  \hspace{1cm} (9.2)

is a normalized surface impedance, derived from the normalized (dimensionless) surface resistance

\[ \mathcal{R} = \frac{\omega_s}{2\alpha} \]  \hspace{1cm} (9.3)

Eqs. (9.1) are only good to 1st order in \( \mathcal{R} \) (or \( Z \)).

Substitution of the field components yields with (8.4)
\[ E_s' = iZ \left( -\frac{i\beta k}{\Gamma_s} E_s - \frac{\alpha_s}{\Gamma_s \gamma d} F_s \right) \]

\[ -\frac{\alpha_s}{\Gamma_s \gamma d} E_s' + \frac{i\beta k}{\Gamma_s} F_s' = iZ F_s \]  

Solutions for \( E_s' \) and \( F_s' \) yields

\[ E_s' = Z \left[ -\frac{\beta k}{\Gamma_s} E_s + \frac{i\alpha_s}{\Gamma_s \gamma d} F_s \right] \]

\[ F_s' = Z \left[ \frac{i\alpha_s}{\Gamma_s \gamma d} E_s + \frac{\Gamma_s}{\beta k} \left( 1 + \frac{\alpha_s^2}{\Gamma_s \gamma d^2} \right) F_s \right] \]  

(9.4)

(9.5)

10. Fields at Slotmouth \( r = b \) (argument of functions \( R_1 \) and \( S_1' \) is \( \Gamma_s b \))

\[ E_{Z}^{III} | r = b = \sum_{s} \left\{ \left[ R_1 - \frac{\beta k Z}{\Gamma_s} S_1 \right] E_s + \frac{i\alpha_s^2}{\Gamma_s \gamma d} F_s \right\} \sin \phi \cos \alpha_s z \]  

(10.1)

\[ Z_0 H_{Z}^{III} | r = b = \sum_{s} \left\{ \frac{i\alpha_s^2}{\Gamma_s \gamma d} E_s + \left[ S_1 + \frac{\Gamma_s Z}{\beta k} \left( 1 + \frac{\alpha_s^2}{\Gamma_s \gamma d^2} \right) R_1 \right] F_s \right\} \cos \phi \sin \alpha_s z \]

\[ E_{\phi}^{III} | r = b = \sum_{s} \left\{ \frac{\alpha_s^2}{\Gamma_s \gamma d} \left[ R_1 - \frac{\beta k Z}{\Gamma_s} (S_1 + \frac{b \beta k}{\Gamma_s}) \right] E_s + \frac{i\beta k}{\Gamma_s \gamma d} \left( 1 + \frac{\alpha_s^2}{\Gamma_s \gamma d^2} \right) R_1' \right\} \sin \phi \cos \alpha_s z \]

(10.2)

\[ Z_0 \phi^{III} | r = b = \sum_{s} \left\{ -i \left[ \frac{\beta k}{\Gamma_s} R_1' - Z \left( \frac{\beta k^2}{\Gamma_s} \gamma d^2 \frac{\gamma d^2}{S_1} + \frac{\alpha_s^2}{\Gamma_s \gamma d^2} \right) R_1 \right] E_s + \frac{\alpha_s}{\Gamma_s \gamma d} x \right\} \sin \phi \cos \alpha_s z \]

Define:

Column vectors

\[ \bar{E}_s = E_s R_1 (\Gamma_s b) \]

\[ \bar{F}_s = iF_s S_1 (\Gamma_s b) \]  

(10.3)
Diagonal matrices

\[ D_{ss} = \frac{a_s}{r_s} R^2 \]

\[ \begin{align*}
R_{ss} &= \frac{\hat{b} k}{r_s} \frac{R_1}{R_2} \Gamma_s \Gamma_s \\
S_{ss} &= \frac{\hat{b} k}{r_s} \frac{S_1}{S_2} \Gamma_s \Gamma_s \\
T_{ss} &= \frac{\hat{b} k}{r_s} \frac{R_1}{S_1} \Gamma_s \Gamma_s \\
Q_{ss} &= (1 + \frac{\sigma_s}{\Gamma_s}) \frac{\Gamma_s}{\Gamma_1} \frac{R_1}{R_2} \frac{\Gamma_s}{\Gamma_s} \Gamma_s \Gamma_s \}
\end{align*} \] (10.4)

\[ \text{or } Q = (U + \lambda \frac{2d^2}{m^2})^{-1} \] (10.7)

with \( \lambda = \frac{b}{d} \) (10.8)

and the unit matrix \( U \)

the fields can be written

\[ E^\text{III}_{z} |_{r=b} = \sum_{s} \left[ (U-ZT) E_s - \lambda ZD E_s \right] \sin \phi \cos a_s z \] (10.9)

\[ Z^\text{III}_s \cos \phi \sin a_s z \]

\[ E^\text{III}_\phi |_{r=b} = \sum_{s} \left[ \left( [U-Z(T+\lambda R') ] E_s - [S' + Z(QR + \lambda D^2)] E_s \right) \right] \cos \phi \sin a_s z \] (10.10)

\[ Z^\text{III}_s \cos \phi \sin a_s z \]

11. Matching at Slotmouth \( r = b, 0 \leq z < g \)

and BC at perfectly conducting tube \( r = b, g < z < 2\pi R \)

\[ E^\text{II}_z = \left\{ \frac{E^\text{III}_z}{0} \right\}, \quad E^\text{II}_\phi = \left\{ \frac{E^\text{III}_\phi}{0} \right\} \text{ for } \{ 0 \leq z < g \} \quad \{ g < z < 2\pi R \} \] (11.1)

\[ H^\text{II}_z = H^\text{III}_z, \quad H^\text{II}_\phi = H^\text{III}_\phi \text{ for } 0 < z < g \] (11.2)

field components from 7) and 10) yield

\[ \sum_{m} A_m e^{ik_m(z-g/2)} = \sum_{S} \left[ (U-ZT) E_s - \lambda ZD E_s \right] \cos a_s z \] (11.3)
-12-

\[-i \sum_m [C_{m} - iB_{m} + V_{2}] e^{i k_m (z-g/2)} = \left\{ \sum_s \left[ U-Z(T+\Lambda R') \right] \bar{D}_{s} - [S'+Z(QR'+\Lambda D')] \bar{F}_{s} \right\} \sin \alpha_s z \]

(11.4)

\[\sum_m \bar{E}_m e^{i k_m (z-g/2)} = i \sum_s \left[ \Lambda Z \bar{E}_s + (U+ZQ) \bar{F}_s \right] \sin \alpha_s z \quad (11.5)\]

\[-i \sum_m \left[ I_{m} - C_{m} + V_{1} \right] e^{i k_m (z-g/2)} = -i \sum_s \left\{ R' - Z(S'T+\Lambda D') \right\} \bar{E}_s - [U+Z(Q+\Lambda S)] \bar{D}_s \right\} \cos \alpha_s z \quad (11.6)\]

12. Integrals and Orthogonality Properties

\[k_m = m/R, \quad \alpha_s = \pi s/g; \]

\[
\int_0^{2\pi R} e^{i (k_m - k_p) (z-g/2)} \, dz = 2\pi R \delta_{mp} \quad (12.1)
\]

\[
\int_0^{2\pi} \cos \alpha_s z \cos \alpha_t z \, dz = (1+\delta_{st}) \frac{g}{2} \delta_{s t} \quad \text{and} \quad (12.2)
\]

\[
\int_0^{2\pi} \sin \alpha_s z \sin \alpha_t z \, dz = (1-\delta_{st}) \frac{g}{2} \delta_{s t} \quad \text{and} \quad (12.3)
\]

\[
M_{ps} = \frac{i}{g} \int_0^{2\pi} e^{-i k_p (z-g/2)} \sin \alpha_s z \, dz = \frac{\pi s}{(\pi \alpha p)^2 - (\pi s)^2} \left\{ \begin{array}{ll}
\sin \alpha_s z & \text{s even} \\
-\cos \alpha_s z & \text{s odd}
\end{array} \right\} \quad (12.3)
\]

\[
N_{ps} = \frac{1}{g} \int_0^{2\pi} e^{-i k_p (z-g/2)} \cos \alpha_s z \, dz = \frac{\pi \alpha p}{(\pi \alpha p)^2 - (\pi s)^2} \left\{ \begin{array}{ll}
\sin \alpha_s z & \text{s even} \\
-\cos \alpha_s z & \text{s odd}
\end{array} \right\} \quad (12.4)
\]

Similarly

\[
\int_0^{2\pi} e^{i k_m (z-g/2)} \sin \alpha_s z \, dz = ig M^*_{mt} \quad (12.5)
\]

\[
\int_0^{2\pi} e^{i k_m (z-g/2)} \cos \alpha_s z \, dz = g N^*_{mt}
\]
13. Solution of Matching Equations

multiply (11.3) and (11.4) by $e^{-ik_p(z-g/2)}$, integrate from 0 to $2\pi R$;
multiply (11.5) by $\sin \alpha_t z$, (11.6) by $\cos \alpha_t z$, integrate from 0 to $g$

$$\bar{A}_p^{\pm}2\pi R = g \sum_s \left[ (U-ZT)\bar{E}_s^{\pm} - AZDE_s^{\pm} \right] N_{ps}$$

$$-i(CA_p - IB_p + \bar{V}_2)2\pi R = -ig \sum_s \left\{ \left[ U-Z(T+AR') \right] \bar{E}_s^{\pm} - \left[ S' + Z(QR' + \lambda D^2) \right] F_s^{\pm} \right\} \bar{M}_{ps}$$ (13.1)

$$ig \sum_m \bar{E}_{m^*}^{\pm} = i\bar{\xi}(1 - \delta_t) \left[ \lambda ZDE_t^{\pm} + (U + ZQ) F_t^{\pm} \right]$$

$$- ig \sum_m (I\bar{A}_m - C\bar{E}_m + \bar{V}_1) N_{mt} = -i\bar{\xi}(1 + \delta_t) \left\{ \left[ R' - Z(S'T + \lambda D^2) \right] \bar{E}_t^{\pm} - \left[ U + Z(Q + \lambda S') \right] \bar{D}_t^{\pm} \right\}$$ (13.2)

Define the diagonal matrix $K$

$$K_{tt} = \frac{2}{1 + \delta_t}$$ (13.3)

and the circumference factor $\alpha = \frac{\bar{\xi}}{2\pi R}$ (13.4)

Hermitian conjugate $N^+_tm = N^*_mt$, $M^+_tm = M^*_mt$

leave off summation (matrix notation)

$$\bar{A} = \alpha N \left[ (U-ZT)\bar{E} - AZDE \right]$$ (13.5)

$$CA - IB - \bar{V}_2 = \alpha N \left\{ \left[ U-Z(T+AR') \right] \bar{D}E - \left[ S' + Z(QR' + \lambda D^2) \right] F \right\}$$ (13.6)

$$2M^+D = AZDE + (U + ZQ) \bar{F}$$ (13.7)

$$K^{+}\left( IA - C\bar{E} + \bar{V}_1 \right) = \left[ R' - Z(S'T + \lambda D^2) \right] \bar{E} - \left[ U + Z(Q + \lambda S') \right] \bar{D}$$ (13.8)

(Since $M^+_{tm}$ vanishes for $t = 0$, the inverse of $\frac{1 - \delta_t}{2}$ is not required in 13.7, which is only valid for $t \neq 1$.)
15. Elimination of Coefficients

One can eliminate any two coefficients from Eqs. (13.5-8) without matrix inversion. Solving for \( \overline{A} \) and \( \overline{B} \) directly results in matrices of the type \( M R' M' \) etc., which have elements that decrease very slowly away from the maximum at location \( n, n \). As a result, very large matrices of the order of \( n \times n \) have to be inverted to solve for \( \overline{A} \) or \( \overline{B} \) (which are required for the coupling impedance).

Alternatively, one can solve for \( \overline{E} \) and \( \overline{F} \) first, and calculate \( \overline{A} \) and \( \overline{B} \) subsequently. This results in matrices of the type \( M'^+ I^{-1} M \) etc., which do not only decrease rapidly away from the maximum element at \( 0,0 \) but are further almost diagonal. The first property allows rapid numerical computation, the second the derivation of approximate formulae.

In order to simplify the task, we first add \( D \) times Eq. (13.7) to (13.8) to get

\[
K N^+(I \overline{A} + \overline{V}_1) + (2 D M^+ - K N^+ C) \overline{E} = (R' - Z S' T) \overline{E} - \lambda Z S' D \overline{F} \quad (14.1)
\]

Define

\[
L = M D - C N \quad (14.2)
\]

then

\[
2 K_{t t}^{-1} D M^+ - N^+ C = D M^+ - N^+ C = L^+ \quad (14.3)
\]
as \( 2 K_{t t}^{-1} \) = 1 except for \( t = 0 \), where \( M_{t m}^+ \) vanishes anyhow.

Thus we get

\[
K N^+ (I \overline{A} + \overline{V}_1) + K L^+ \overline{B} = (R' - Z S' T) \overline{E} - \lambda Z S D \overline{F} \quad (14.4)
\]

which takes the place of Eq. (13.8).

We further want to solve (13.6) for \( I \overline{B} \), which yields with (14.2)

\[
I \overline{B} = -\alpha \left[ L - Z (L T + \lambda M R' D) \right] \overline{E} + \alpha \left[ M S' + Z (\lambda L D + \lambda M Q R') \right] \overline{F} - \overline{V}_2 \quad (14.5)
\]
15. Matrix Equations for the Coefficients $\bar{E}$ and $\bar{F}$

Substitution of Eqs. (13.5) and (14.5) into (13.7) and (14.4) yields after collecting of terms

\[
\begin{align*}
(U - W_{11} - ZW_{11})\bar{E} & - (W_{12} - ZW_{12})\bar{F} = \bar{Y}_1 \\
(W_{21} - ZW_{21})\bar{E} & + (U - W_{22} - ZW_{22})\bar{F} = \bar{Y}_2
\end{align*}
\]

where

\[
\begin{align*}
W_{11} &= \alpha KR^{-1}(N^+IN - L^+L) \\
W_{12} &= \alpha KR^{-1}L^+LMS \\
W_{21} &= 2\alpha M^+L \\
W_{22} &= 2\alpha M^+LM^+S'
\end{align*}
\]

\[
\begin{align*}
W_{11}' &= \alpha KR^{-1}\left[(N^+IN - L^+L)T - AL^+LMR'D\right] - R^+S'T \\
W_{12}' &= \alpha KR^{-1}\left[(N^+IN - L^+L)A_D - L^+LMR'D\right] - R^+S'D \\
W_{21}' &= 2\alpha\left[M^+LT + AM^+LMR'D\right] - AD \\
W_{22}' &= 2\alpha\left[M^+LAD + M^+LMR'D\right] - Q
\end{align*}
\]

and

\[
\begin{align*}
\bar{Y}_1 &= KR^{-1}(N^+\bar{Y}_1 - L^+L\bar{Y}_2) \\
\bar{Y}_2 &= -2M^+L\bar{Y}_2
\end{align*}
\]

The two matrix Eqs. (15.1) with the definitions (15.2 - 4) constitute the system of equations which has to be solved for $\bar{E}$ and $\bar{F}$, from which we can find $\bar{A}$ and $\bar{E}$ afterwards with Eqs. (13.5) and (14.5).
PART II: Solutions for the Field Coefficients and the Coupling Impedance.


For a uniform vacuum chamber it has been customary \( \text{(3)} \) \( \text{(4)} \) to express the stability criterion in terms of the "dispersion relation coefficients" \( U \) and \( V \). For a vertically oscillating beam they are related to the average vertical force due to the electromagnetic field by \( \text{(4)} \)

\[
U + (1+i)V = \frac{Ncr_o}{\pi a^2 q_{0} y_{0}} \langle \hat{F}_y \rangle \tag{16.1}
\]

where

\[
\langle \hat{F}_y \rangle = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a rdr \left( \hat{E}_y + \hat{B}_0 \hat{H}_x \right) \tag{16.2}
\]

is the "amplitude" of the force, i.e. without the exponential factor \( e^{i(kz-\omega t)} \). The arbitrary normalization factor \( \frac{\rho_o}{\varepsilon_o} \) cancels and need not be considered in this definition.

For a vacuum chamber with variations in the longitudinal direction, the fields are given by infinite sums over all space-harmonics, and the concept of the "amplitude" has to be generalized by integrating along a particle trajectory. However, we prefer to introduce at this stage the "transverse coupling impedance", which can be defined \( \text{(5)} \) as the integral of the average force along a particle trajectory divided by the driving current \( I_o \)

\[
Z_L = \frac{1}{I_o} \int_0^T \frac{\rho_o}{\varepsilon_o} \langle \hat{F}_y \rangle e^{-i\omega t} \frac{cdt}{\beta_o c t} = \frac{z-\varepsilon/2}{\beta_o c t} \]

\[
= - \frac{iZ_o}{\eta \beta_o^2 a^2} \int_0^{2\pi R} \langle \hat{F}_y \rangle e^{-i\beta_o^2} \frac{dz}{\beta_o} \tag{16.3}
\]

which has actually the dimensions of "impedance per unit length", or \( \Omega/m \) in the mks system. An arbitrary phase factor has been chosen.
such that $Z_\perp$ is real for imaginary $<F>_y$.

17. Expressions for the Average Vertical Force

The vertical force on a unit charge is given by

$$F_y = E_y + \beta_o Z_\perp H_x$$

We can find the transverse field components from the longitudinal ones with the relations (cf. 4.1)

$$\left(\frac{\partial^2}{\partial z^2} + \frac{w^2}{c^2}\right) E_y = \frac{\partial^2 E_x}{\partial y \partial z} - \frac{i\omega}{c} \frac{\partial (Z_\perp H_z)}{\partial x} - \frac{i\omega}{c} (Z_\perp J_y)$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{w^2}{c^2}\right) Z_\perp H_x = -\frac{i\omega}{c} \frac{\partial E_z}{\partial y} + \frac{\partial^2 (Z_\perp H_z)}{\partial x \partial z} + \frac{\partial}{\partial z} (Z_\perp J_y)$$

With the expressions for the longitudinal field coefficients and the vertical current density we find after some arithmetic

$$E_y = -i \sum_m \left[ \frac{k m_{A_m}}{2 \chi_m} (I_0 - I_2 \cos 2\varphi) \chi_m r - \frac{\beta k m_{B_m}}{2 \chi_m} (I_0 + I_2 \cos 2\varphi) \chi_m r \right]$$

$$- i \beta_2 \gamma^2 \delta_{mn} \exp$$

$$Z_\perp H_x = i \sum_m \left[ \frac{\beta k m_{A_m}}{2 \chi_m} (I_0 - I_2 \cos 2\varphi) \chi_m r - \frac{k m_{B_m}}{2 \chi_m} (I_0 + I_2 \cos 2\varphi) \chi_m r \right]$$

$$- i \beta_2 \gamma^2 \delta_{mn} \exp$$

Leaving off the $\varphi$-dependent terms—which will cancel in the average anyhow—we find

$$F_y = -i \sum_m \left[ \left( k_m - \beta_0 \beta k \right) A_m + (\beta_0 k_m - \beta k) B_m \right] \frac{I_0 (x m r)}{2 \chi_m} + i \beta_2 \gamma^2 \delta_{mn} \exp$$

With Eq. 16.2 we thus get for the average vertical force

$$<F_y> = -i \sum_m \left[ F_m \frac{I_0 (x m a)}{\chi_m a} + i \beta_2 \gamma^2 \delta_{mn} \right] e^{ik_m (z-s/2)}$$
where
\[ F_m' = \frac{1}{\chi_m} \left[ (k_m - \beta_0 k) A_m + \beta_0 k_m B_m \right] \]  
(17.6)

The space harmonic with \( m = n \), the mode number of the perturbation, thus contributes proportional to
\[ F_n' = \gamma (\beta_1 A_n + \beta_2 B_n) \]  
(17.7)

18. The Coupling Impedance

From Eq. (16.3) we find
\[ Z_\perp = -\frac{Z_o}{\pi \beta_0 a^2} \sum_m \left[ \frac{F_m'}{\chi_m a} + \beta_2 \gamma \delta_{mn} \right] \int_0^2 e^{i(k_m - \beta_0 k)z} dz \]  
(18.1)

If we make the simplifying assumption that \( \beta = \beta_0 \), the integral vanishes for all terms with \( m \neq n \) and we get (as \( \beta_2 = 0 \))
\[ Z_\perp = \frac{2RZ_o}{\beta_0 a^2} F_n' \frac{I_1(\chi a)}{\chi a} \approx \frac{2RZ_o}{\beta_0 a^2} A_n \approx \frac{2RZ_o}{\beta_0 a^2} \tilde{A}_n \]  
(18.2)
when \( \chi a < 1 \).

A more reasonable assumption is
\[ \beta = \beta_0 \left( 1 - \frac{Q}{n} \right) \]  
(18.3)

which follows from \( \omega = (n-Q)\Omega \), the relation for the frequency of the slow wave on the beam. Then the integral yields
\[ Z_\perp = \frac{2RZ_o}{\pi \beta_0 a^2} \sin \Delta Q e^{i\pi \Delta Q} \sum_{m} \frac{m-n+Q}{m^2 - \beta_0 (n-Q)^2} \frac{I_1(\chi_m a)}{I_1(\chi_m b)} \]  
(18.4)

where \( \Delta Q \) is the non-integer part of \( Q \). The coupling impedance becomes, when we substitute \( \tilde{A}_n \) and \( \tilde{B}_n \) from Eqs. (6.4) and (7.1) and leave off all terms not containing \( \tilde{A}_n \) and \( \tilde{B}_n \) (which are the only ones to become large at resonances)
If we assume that the main contribution to the sum comes from the term \( m = n-Q \), we get for \( n \gg Q \)

\[
\begin{align*}
|z_l| &= \frac{2\beta^2 z_0}{\beta^2_0 n a^3} \frac{\sin\Delta Q}{\pi\Delta Q} \left( n + \beta_0^2 \gamma_0^2 \Delta Q \right) \tilde{A}_{n-Q} + \beta_0^2 \gamma_0^2 \Delta Q \tilde{B}_{n-Q} \\
&\times \frac{I_1(ax_{n-Q})}{I_1(bx_{n-Q})} \\
&\times \frac{I_1(ax_{n-Q})}{I_1(bx_{n-Q})}
\end{align*}
\]  

(18.6)

For \( Q \to 0 \), this expression tends correctly towards Eq. (18.2). In general, however, both \( \tilde{A}_m \) and \( \tilde{B}_m \) contribute to the coupling impedance.

19. Solution for a Smooth Vacuum Chamber

When \( \alpha \) tends towards zero, we expect to obtain the well-known formula for the "negative mass instability", i.e. the effect of a smooth, perfectly conducting vacuum chamber. From Eqs. (13.5) and (13.6) we find directly

\[
\begin{align*}
\tilde{A}_m &= 0 \\
\tilde{B}_m &= -I_{mm}^{-1} \tilde{v}_{2m}
\end{align*}
\]  

(19.1)

As \( \tilde{v}_{2m} \) has only a single non-vanishing element for \( m=n \), only \( \tilde{B}_n \) will be different from zero. We further need the unbarred coefficients

\[
\begin{align*}
A_m &= (\tilde{A}_m - \tilde{v}_{lm})/I_1(x_m b) \\
B_m &= (\tilde{B}_m - \tilde{v}_{2m})/I_1(x_m b)
\end{align*}
\]  

(19.2)

to derive

\[
\begin{align*}
A_m &= -i \beta_1 ka \frac{R_1(x_b)}{I_1(x_b)} \delta_{mn} \\
B_m &= -i \beta_2 ka \frac{R_1(x_b)}{I_1(x_b)} \delta_{mn}
\end{align*}
\]  

(19.3)
where we have made use of the relation

\[ \frac{B_{1}^{1}(xb)I_{1}(xb) - B_{1}(xb)I_{1}^{1}(xb)}{x^{b}} = \frac{I_{1}(xa)}{xa} \]  \hspace{1cm} (19.4)

which follows from the Wronskian for modified Bessel functions.

From Eq. (17.5) we now find the average force

\[ \langle F_{y} \rangle = -i\gamma(\beta_{1}A_{n} + \beta_{2}B_{n}) + \beta_{2}^{2}\gamma^{2} = \]

\[ = \gamma^{2} \left[ \beta_{2}^{2}(1 - \frac{I_{1}(xa)B_{1}(xb)}{I_{1}(xb)}) - \beta_{1} \frac{I_{1}(xa)B_{1}(xb)}{I_{1}(xb)} \right] \]  \hspace{1cm} (19.5)

With \( U = U_{0} \langle F_{y} \rangle \) (Eq. 16.1) and taking into account differences in notation \((a \rightarrow b, R_{b} \rightarrow -R_{b}, \beta_{b} \rightarrow s_{b})\) we find complete agreement with Eq. (8.3) of ref. (4) for \( R = 0 \). If we further use the small argument approximations of modified Bessel functions

\[ I_{1}(x) \approx \frac{x}{2}, \quad K_{1}(x) \approx \frac{1}{x}, \quad R_{1}(xb) = \frac{b}{2a}(1 - \frac{a^{2}}{b^{2}}), \]

\[ R_{1}'(xb) = \frac{1}{2xa}(1 + \frac{a^{2}}{b^{2}}) \]  \hspace{1cm} (19.6)

we get

\[ U = - \frac{Ncr_{0}}{2\pi a^{2}Q_{s}^{3}q_{0}^{3}} (1 - \frac{a^{2}}{b^{2}}) \]  \hspace{1cm} (19.7)

in agreement with the familiar "negative mass term" (3).

20. First Order Approximation

In order to find solutions for the field coefficients when \( \alpha \neq 0 \), we introduce a single approximation that reduces the formidable system of matrix equations given by Eqs. (15.1) - (15.4) quite drastically. Specifically, we assume that we may replace the Bessel functions in the expressions for the elements of the matrix \( I \) by their small argument approximations.
\[ I_{mn} = \beta k \frac{I'_1}{x_m} \frac{I'_1}{I'_1} \approx \frac{\beta k}{x_m^2 b} \] (20.1)

Naturally, this assumption is restrictive as \( |x_m b| \) will be large for large enough values of \( m \). On the other hand, the elements of the other matrices \( I \) is multiplied with fall off with large \( m \), and limit the error thereby committed. For the \( n \)-th element, the expression \( x_n^b = \frac{k b}{\gamma} \) will be small compared to unity for mode numbers

\[ n \leq \gamma \frac{R}{b} \] (20.2)

which is usually true up into the lower resonance region for high energy beams (cf. Eq. 21.7).

In the appendix we derive the expressions

\[
\begin{align*}
M^+N &= \frac{U-U^0}{2\alpha} \\
N^+N &= \frac{U+U^0}{2\alpha}
\end{align*}
\] (20.3)

from which we get

\[
\begin{align*}
M^+I^{-1}MD &= \frac{P}{2\alpha} (U-U^0) \\
N^+I'CM &= M^+I'C_N = \frac{P}{2\alpha} \\
M^+I'L &= L^+I'M = 0 \\
(N^+IN-L^+I'L)P &= \frac{D}{2\alpha} (U+U^0)
\end{align*}
\] (20.4)

with the diagonal matrices \( P, R, S \) defined by

\[
\begin{align*}
P_{ss} &= \frac{s}{2a\beta n} \\
R_{ss} &= \left( \frac{xR_1}{R_1} \right) \Gamma_s b \\
S_{ss} &= \left( \frac{xS_1}{S_1} \right) \Gamma_s b
\end{align*}
\] (20.5)
Eqs. (20.4) then yield

\[ W_{11} = R^{-1}, \quad W_{12} = W_{21} = 0, \quad W_{22} = (U-U^0)S \]

(20.6)

and

\[ \begin{align*}
W_{11}' &= R^{-1} (U-S)T \\
W_{12}' &= \lambda \overline{R}^{-1} (U-S)D \\
W_{21}' &= \lambda (R-U) D \\
W_{22}' &= R(U-U^0)Q-Q
\end{align*} \]  \tag{20.7}

If we further write for the RHS

\[ \bar{G} = RR^T(N^+\bar{V}_1 - L^+\bar{V}_2) \]

\[ \bar{H} = 2 M^+I^{-1}\bar{V}_2 \]

(20.8)

We find the approximate equations for the field coefficients \( \bar{E} \) and \( \bar{F} \) which we write as

\[ \begin{align*}
\begin{bmatrix}
(U-R) - Z(U-S)T \\
\lambda Z(U-S)D
\end{bmatrix} \bar{E} - \lambda Z(U-S)D\bar{F} &= -K\bar{G} \\
\lambda Z(U-R)D\bar{E} + \begin{bmatrix}
(U-S(U-U^0)) - Z(U-R(U-U^0)Q
\end{bmatrix} \bar{F} &= -\bar{H}
\end{align*} \]  \tag{20.9}

All matrices involved in these equations are diagonal, and their solution now becomes rather simple.

21. **TM-Resonances**

We can eliminate the coefficient \( \bar{F} \) from above equations. When we neglect terms of second order in \( Z \) (and of first order on the RHS), we find

\[ \bar{E}_S = \frac{-K_{ss} \bar{G}_S}{(1-R_{ss}) - Z(1-S_{ss})T_{ss}} \]

(21.1)

which can become very large for small \( Z \) if \( R_{ss} \) equals unity. The coefficients \( \bar{F}_S \) remain limited, and we thus may speak of TM resonances as far as the cavity region is concerned (Eqs. 13.5 and 14.5 show, on the other hand, that both \( \lambda \) and \( \bar{E} \)
will become large, and we may refer to these resonances also as "hybrid modes". It turns out that all solutions of \( R_{ss} = 1 \) are imaginary, thus it is preferable to rewrite the modified Bessel functions of imaginary argument as regular Bessel functions of real argument. We define

\[
\begin{align*}
F_i(\Gamma'r) &= Y_i(\Gamma'd)J_i(\Gamma'r) - J_i(\Gamma'd)Y_i(\Gamma'r) \\
G_i(\Gamma'r) &= Y'_i(\Gamma'd)J_i(\Gamma'r) - J'_i(\Gamma'd)Y_i(\Gamma'r)
\end{align*}
\]

and get for \( \Gamma' = i\Gamma \) the condition

\[
\left( x \frac{F'_i}{F_i} \right)_{\Gamma'_{sb}} = 1
\]

(21.3)

or, as \( F'_i = F_i - \frac{F_i}{x} \)

\[
\left( x \frac{F'_i}{F_i} \right)_{\Gamma'_{sb}} = 2
\]

(21.5)

This transcendental equation has an infinite number of solutions.
Calling \( x_t \) the \( t \)-th solution in ascending order, the resonance condition can be written

\[
\Gamma'_{sb} = x_t
\]

(21.6)

Substituting the definition \( \Gamma'^2 = \frac{\omega^2}{c^2} - \left( \frac{\pi_{ss}}{g} \right)^2 \) we thus find for the resonant frequencies, resp. mode numbers

\[
\begin{align*}
\omega_{s,t}^{(TM)} &= \frac{c}{b} \sqrt{x_t^2 + (\varepsilon s)^2} \\
\eta_{s,t}^{(TM)} &= \frac{R}{\beta b} \sqrt{x_t^2 + (\varepsilon s)^2}
\end{align*}
\]

(21.7)

where

\[
\varepsilon = \frac{\pi b}{g}
\]

(21.8)
is a fixed geometrical parameter. We can identify \( s \geq 0 \) as the axial, \( tl \) as the radial modenumber in the cavity (number of zero crossings). The quantity \( x_t \) is only a function of the ratio \( \lambda = b/d \), and is shown for \( t = 1 \) in Fig. 2. The condition for validity of the first approximation at the lowest TM resonance becomes with Eqs. (20.2) and (21.7)

\[
x_{1} \ll \beta y \quad (21.9)
\]

22. Resonant Field Coefficients

At resonance the quantity \( \bar{g}_s \) defined in Eq. (20.8) (see appendix B) becomes

\[
\bar{g}_s(t) = \frac{2\beta \alpha^2}{bg} \frac{\beta_0 x_t^2 + \beta_2 (es)^2}{x_t^2 + (es)^2} \begin{cases} \sin \pi \alpha n & \text{if } s \text{ even} \\ \cos \pi \alpha n & \text{if } s \text{ odd} \end{cases} \quad (22.1)
\]

To evaluate Eq. (21.1) we further need the expression for the resonant value of

\[
(1-S_{ss})T_{ss} = \frac{4 \lambda \beta k b}{\pi x_t^2 \bar{F}_1^2(x_t)} \quad (22.2)
\]

We finally express the normalized surface impedance \( \mathcal{R} \) by the skin-depth \( \delta \)

\[
\mathcal{R} = \frac{\beta k \delta}{2} \quad (22.3)
\]

and get

\[
\bar{F}_s(t) = -\frac{2\pi^2}{1+\delta_{so}} \frac{\beta \alpha^2 d}{bg\delta} \frac{[\beta_0 x_t^2 + \beta_2 (es)^2] x_t^2 \bar{F}_1^2(x_t)}{(x_t^2 + (es)^2)^2} \begin{cases} \sin \pi \alpha n & \text{if } s \text{ even} \\ \cos \pi \alpha n & \text{if } s \text{ odd} \end{cases} \quad (22.4)
\]

To lowest order in \( Z \), we get from Eq. (13.5) and (14.5) for negligible \( \bar{F} \)

\[
\bar{A}_n = \alpha N_{ns} \bar{E}_s \\
\bar{B}_n = -\alpha I_{nn}^{-1} I_{ns} \bar{E}_s \quad (22.5)
\]
At resonance, we find

\[ N_{ns} = \frac{2\beta^2}{x_t^2 + (\epsilon\delta)^2} \frac{b^2}{\gamma} \left\{ \sin \frac{\sin}{\cos} \right\} \pi \alpha_n \]

\[ L_{ns} = -\frac{2\beta^2}{x_t^2} \frac{\gamma^2}{g} \left\{ \sin \frac{\sin}{\cos} \right\} \pi \alpha_n \]

Neglecting non-resonant terms, we further have

\[ A_n = \frac{A_n}{L_1(x_b)} \approx \frac{2\gamma}{kb} A_n \]

\[ B_n = \frac{B_n}{L_1(x_b)} \approx \frac{2\gamma}{kb} B_n \]

and we thus get

\[ (A_n')_{s,t} = -\frac{4\pi\beta^3\gamma}{1 + \delta/\delta_0} \frac{a^2_d}{gR\delta} \frac{2\pi^2}{x_t^2 + (\epsilon\delta)^2} \left[ x_t^2 + (\epsilon\delta)^2 \right] \frac{\sin^2}{\cos^2} \pi \alpha_n \]

\[ (B_n')_{s,t} = -\frac{4\pi\beta^3\gamma}{1 + \delta/\delta_0} \frac{a^2_d}{gR\delta} \frac{\beta_0 x_t^2 + \beta_2 (\epsilon\delta)^2}{x_t^2 + (\epsilon\delta)^2} \left[ x_t^2 + (\epsilon\delta)^2 \right] \frac{\sin^2}{\cos^2} \pi \alpha_n \]

For \( s = 0 \) we see easily that \( (A_n')_{o,t} = \beta (B_n')_{o,t} \).

23. Coupling Impedance at TM-Resonances

To find the complete expression for the coupling impedance, we should actually calculate all coefficients \( \tilde{A}_m \) and \( \tilde{B}_m \) and substitute these values into Eq. (18.5). For a simpler expression we may also use Eq. (18.6) and assume that \( \tilde{A}_{n-Q} \approx \tilde{A}_n \), \( \tilde{B}_{n-Q} \approx \tilde{B}_n \).

However, here we will use the simplest form Eq. (18.2) which is actually only valid for \( \beta = \beta_0 \) or \( Q = 0 \). With \( I_1(x_a) \approx x_a^2 \) we find
\[
\left( Z_{TM} \right)_{s,t} = \frac{4\pi Z_0}{1+\delta_{s0}} \frac{\delta^2}{
abla_{s}^2} \frac{\gamma_{\perp}^2(x_t)}{x_t^2 + (e\delta)^2} \frac{x_t^4}{\left[ x_t^2 + (e\delta)^2 \right]^2} \frac{\sin^2}{\cos^2} \frac{2\pi n}{\gamma} \tag{23.1}
\]

For the lowest axial mode \((s = 0)\) this expression simplifies to

\[
\left( Z_{TM} \right)_{0,t} = 2\pi Z_0 \frac{\delta^2}{
abla_{s}^2} \frac{\gamma_{\perp}^2(x_t)}{x_t^2} \sin^2 \frac{gx_t}{2\delta b} = \frac{\pi Z_0}{2} \frac{\delta^2}{b^2} \frac{\gamma_{\perp}^2(x_t)}{x_t^2} \left( \frac{\sin \theta/2}{\theta/2} \right)^2 \frac{gx_t}{\beta b} \tag{23.2}
\]

where \(\sin \theta/2 \theta/2\) is the familiar transit time factor with \(\theta = \gamma g\), the transit angle across the cavity. We shall now investigate the limit of this expression for vanishing beam hole \(b \to 0\), or \(\lambda = \frac{d}{b} \to \infty\).

In appendix C we show that for large \(\lambda\)

\[
x_t \to \frac{j_t}{\lambda} \tag{23.3}
\]

\[
\gamma_{\perp}(x_t) \to \frac{1}{\pi \lambda J_0(j_t)}
\]

where \(j_t\) is the \(t\)-th zero of the Bessel function of first order.

We therefore get

\[
\left( Z_{TM} \right)_{0,t} \to \frac{Z_0}{2\pi J_0^2(j_t)} \frac{\delta^2}{
abla_{s}^2} \frac{\gamma_{\perp}^2(x_t)}{x_t^2 + (e\delta)^2} \left( \frac{\sin \theta/2}{\theta/2} \right)^2 \frac{gx_t}{\beta d} \tag{23.4}
\]

We compare this formula with the expression (5) for the lowest resonant mode in a closed cavity of quadratic cross-section \(2d \times 2d\) and length \(g\)

\[
Z_{TM} = \frac{4Z_0}{5} \frac{\delta^2}{
abla_{s}^2} \left( \frac{\sin \theta/2}{\theta/2} \right)^2 \tag{23.5}
\]

The only difference is the numerical factor \(4/5\) instead of

\[
\frac{1}{2\pi J_0^2(j_1)} = 0.98 \text{ which can be blamed on the different geometry.}
\]
The agreement is thus quite satisfactory, and we can answer two questions raised in connection with Eq. (23.5):

a) the effect of holes in the sidewalls is expressed by Eq. (23.2), and shall be discussed in more detail in the next section.

b) the effect of higher modes is expressed in Eq. (23.1). The factor $1 + \delta_{so}$ in the denominator increases $Z_1$ by a factor 2 for all modes with $s \neq 0$. On the other hand, the coupling impedance decreases with increasing $s$ due to the denominators $(x_t^2 + (es)^2)$ and $(x_t^2 + (\frac{gs}{\gamma})^2)$. But for long and shallow cavities, where $\varepsilon$ is small and $x_t$ large, this decrease is very slow initially, and we have to take twice the value given by Eq. (23.5) for the $s = 1$ mode (apart from the transit-time factors). For short and deep cavities the decrease with $s$ is much more rapid (and the transit time factor cooperates with the reduction) and then Eq. (23.5) can be used as an upper limit for all axial modes. For higher radial modes, the coupling impedance may initially increase as $F_{1}^{2}(x_t) x_t^4$ for large $\frac{gs}{\gamma}$, but will eventually decrease as $F_{1}^{2}(x_t)/x_t^2$.

24. The Lowest TM-Mode

Apart from the factor $\sin^2\theta$, Eq. (23.2) can be written

\[ (\hat{Z}_1)_{01} = 2\pi Z_0 \frac{\beta}{\varepsilon} \frac{F_{1}^{2}(x_1)}{x_1^2} \]  

(24.1)

where the skin depth is still a function of the resonant frequency $\omega_{01} = \frac{c x_1}{b}$

(24.2)

Substituting

\[ \delta = \sqrt{\frac{2}{\omega_{01} \sigma'}} = \sqrt{\frac{2b}{x_1 Z_0 \sigma'}} \]  

(23.4)

we get

\[ (\hat{Z}_1)_{01} = \pi Z_0 \sqrt{2 Z_0 \sigma'} \frac{\beta}{\varepsilon} \frac{F_{1}^{2}(x_1)}{x_1^{3/2}} \]  

(24.4)
Similarly Eq. (23.4) yields for the limit of large \( \lambda \)

\[
(\hat{Z}_1)_{01} = \frac{Z_0}{\pi} \sqrt{2\pi \sigma} \frac{\beta^2 / \alpha}{g} \frac{1}{j_1^{3/2} j_0^2(j_1)}
\]  

(24.5)

By dividing these two quantities, we get the function

\[
\bar{f}_1(\lambda) = \pi^2 \lambda^{1/2} j_1^{3/2} j_0^2(j_1) \frac{f_1^2(x_1)}{x_1^{3/2}}
\]  

(24.6)

which tends to unity for large \( \lambda \) and is shown in Fig. 3.

We further define the constant

\[
c_1 = \frac{Z_0 / 2\pi}{\pi j_1^{3/2} j_0^2(j_1)} = 2708 \Omega^{3/2}
\]  

(24.7)

to write the coupling impedance of the lowest mode as

\[
(\hat{Z}_1)_{01} = c_1 \bar{f}_1(\lambda) \frac{\beta^2 / \alpha}{g}
\]  

(24.8)

where we still have to attach the suppressed factor \( \sin^2 \).

We define

\[
Z_1(\lambda) = \frac{\lambda x_1}{j_1}
\]  

(24.9)

which is also shown in Fig. 3, and get then with

\[
D_1 = \frac{j_1 c}{2 \pi} = 1.8282 \times 10^8 \frac{m}{s}
\]  

(24.10)

the final equations

\[
\begin{align*}
\tau_{01}^\text{TM} &= D_1 \frac{Z_1(\lambda)}{d} \\
(\hat{Z}_1)_{01} &= c_1 \bar{f}_1(\lambda) \frac{\beta^2 / \alpha}{g} \sin^2 \left( \pi \frac{\tau_{01}}{\beta c} \right)
\end{align*}
\]  

(24.10)
In this form we see clearly that \( \Phi_1(\lambda) \) expresses the reduction of the coupling impedance by the hole in the sidewalls (resp. the end tubes), but its value is further influenced by the different transittance factor due to the change of the resonant frequency expressed by the factor \( Z_1(\lambda) \).

25. Coupling Impedance of the ISR Vacuum-Chamber

As a numerical example, we calculate the transverse coupling impedance of the ISR vacuum chamber consisting of sections of circular chambers enclosed by elliptical pipes (which we replace by equivalent circular ones). We assume the following parameters:

\[
b = .04, \; d = .08, \; g = 1.5 \; \text{m}, \; \sigma = 10^6 \text{Sm/m}, \; \beta \approx 1
\]  

(25.1)

to get for \( \lambda = 2 \)

\[
\Phi_1 = .1375, \; Z_1 = 1.1866
\]  

(25.2)

and hence

\[
\begin{align*}
\omega_{01}^{TM} &= 2.711 \; \text{GHz} \\
\left( Z_{01}^{TM} \right) &= 70.2 \; \text{k}\Omega/\text{m}
\end{align*}
\]  

(25.3)

when we ignore the factor \( \sin^2 \) which is approximately unity.

We further calculate for \( R = 150 \; \text{m} \)

\[
n = \frac{\omega R}{\beta c} = 8522
\]  

(25.4)

and

\[
\left( Z_{01}^{TM} / n \right) = 8.24 \; \Omega/\text{m}
\]

Because the cavity is long, the maximum value of \( (Z_{01}^{TM} / n) \) will be for the \( s = 1, \; t = 1 \) mode and be close to twice the above value:

\[
\left( \frac{Z_{01}^{TM}}{n} \right)_{\text{max}} = 16.5 \; \Omega/\text{m}
\]  

(25.5)
26. RF-Particle Separator

As a check on the validity of our approximations, we calculate the coupling impedance of an iris-loaded wave guide that has been measured and calculated exactly by matrix inversion methods previously (6) (7). We assume the following parameters:

\[ b = 0.0262, \, d = 0.055, \, g = 0.02, \, 2\pi R = 0.025; \, \sigma' = 58.10^6 \text{Sm/m} \] (6.1)

to get with \( \lambda = 0.476 \)

\[ f_0 = 3.86 \text{ GHz} \]

\[ (Z_{TM})_{01} = 20.3 \text{ \Omega/m} \] (6.3)

We can further improve on this calculation by including the neglected side and tubewall resistivity in the outer wall in an approximate manner. The area ratio of the various surfaces is

\[ \frac{A_c}{A_{tot}} = \frac{d \cdot g}{d \cdot g + (\frac{d}{2} - b^2) + (2\pi R - g)} = 0.308 \] (6.4)

If we assume that the complete losses are concentrated at the rear wall, the coupling impedance must be multiplied by the square root of this ratio. We get thus

\[ (Z_{TM})_{01} = 11.3 \text{ \Omega/m} \] (6.5)

This value can be compared with the shunt impedance (7)

\[ Z_S = \frac{E^2}{P_L} \]

From Fig. 5 of ref. (6) we find (for \( \alpha/\lambda = 0.262 \)) \( P_L = 7.1 \text{ MW/m} \). The authors had taken \( E = 10 \text{ MV/m} \), and we thus find

\[ Z_S = 14.1 \text{ \Omega/m} \]

for the assumed frequency \( f_0 = 3 \text{ GHz} \)
We see that the resonant frequency and the coupling impedance are within 30 o/o of the computer value. A better frequency criterion is derived in appendix D, which yields \( f_{01} = 3.11 \) GHz for the above parameters.

27. **TE-Resonances**

When we solve the set of Eqs. (20.9) for \( \tilde{F} \) and again neglect terms of second order in \( Z \)

\[
\tilde{F}_s = -\frac{\tilde{H}_s}{(1-S_{ss}) + Z(1-R_{ss})Q_s} (sz1) \tag{27.1}
\]

The coefficient \( \tilde{F}_s \) thus becomes large when \( S_{ss} = 1 \), or (cf. section 21)

\[
\left( x \frac{G_2}{G_1} \right) \gamma_s \beta_b = 2 \tag{27.2}
\]

We call \( y_t \) the \( t \)-th solution of this transcendental equation, and thus find for the resonant frequency

\[
\omega_{st} = \frac{c}{b} \sqrt{y_t^2 + (\epsilon s)^2} \tag{27.3}
\]

Again we can calculate the value of \( \tilde{F}_s \) at resonance (cf. section 22) to obtain

\[
\tilde{F}_s(t) = -2\pi^2 \beta^2 \beta_2 \frac{\alpha^2 bd}{R_b G_3} \frac{ns}{y_t^2 + (\epsilon s)^2} \frac{y_t \gamma^2 (y_t)^2 (\sin \gamma t)}{y_t^4 (\lambda s)^2} \{ \cos \} \tag{27.4}
\]

The lowest order in \( Z \), we find from Eqs. (13.5) and (14.5)

\[
\bar{A}_n = 0 \]

\[
\bar{B}_n = -\alpha I_{mn}^{-1} M_{ns} \tilde{F}_s(t) \tag{27.5}
\]

and hence
\[ (B_n)_{s,t} = -4\pi \beta^2 \beta_2 \frac{a}{s_g R_0} (\eps s)^2 \left[ \frac{y_t^2 + (\eps s)^2}{y_t^2 (\lambda s)^2} \right] \left[ \frac{1}{y_t^2 + (\eps s)^2} \right] \left( y_t^4 \gamma \frac{\gamma_t}{\gamma} \right) \sin^2 \frac{\sin^2 \theta}{\cos^2 \theta} \]  

(27.6)

In the simple approximation \( \beta = \beta_0 \) we have \( \beta_2 = 0 \) and thus \( A_n = B_n = 0 \), and the coupling impedance given by Eq. (18.2) vanishes. Including the effect of a finite \( Q \) value would yield a finite contribution proportional to \( \beta_2 \frac{Q^2}{n^2} \) which will remain small when \( n \geq Q \).

28. Stability Criteria

In analogy to the form of the longitudinal stability criterion given in ref. (8) we rewrite the formulae derived in ref. (5) as

\[ \left| \frac{z_n}{n} \right| \leq \pi \left( \frac{E_0}{m_0 c^2} \right) \frac{Q n \lambda}{R_0} \left( \frac{\Delta p}{m_0 c} \right) \]  

(28.1)

where \( E_0 \) is the rest energy \( E_0 = 9.382 \times 10^8 \text{V for protons} \),

\( Q \) is the betatron number

\( \eta = \frac{p}{\Omega} \frac{d\Omega}{dp} = \frac{1}{\gamma^2 - \gamma^{-2}} \)

\( R \) is the machine radius

\( I_0 \) is the beam current

and \( \left( \frac{\Delta p}{m_0 c} \right) \) is the full normalized momentum spread at half height,

which will be assumed to be equal to half the spread at the base.

For the ISR, we have \( Q = 8.7, R = 150, I_0 = 20A, \left( \frac{\Delta p}{p} \right) = 2 \text{ o/o} \)

For \( \gamma = 30 \) with \( \gamma_{tr} = 8.96 \) we get \( \eta = 1.132 \times 10^{-2} \) and

\[ \left( \frac{\Delta p}{m_0 c} \right) = \beta \frac{\gamma}{2} \left( \frac{\Delta p}{p} \right) = 0.3 \]  

(28.2)

We find then

\[ \left| \frac{z_n}{n} \right| \leq 29 \text{ kΩ/m} \]  

(28.3)

for the full stack. For the first injected pulse we get similar numbers, as beam current and momentum spread are approximately
proportional (and we neglect the form factor taking the stack-
shape into account). However, for smaller values of $\gamma$ the stability
limit will decrease because of the decrease of $\eta$ (assuming that
$\frac{A_0}{m_0 c}$ remains unchanged). We get for $\gamma = 10$

$$|\frac{Z}{n}| \leq 6.23 \text{ k}\Omega/m \quad (28.4)$$

If we take the values of the coupling impedance of the
circular vacuum chambers between elliptical pipes from Eq. (25.5)
and multiply by 150 for the approximate number of these elements we find

$$\frac{Z}{n} = 2.48 \text{ k}\Omega/m \quad (28.5)$$

which is well below the stability limit even for $\gamma = 10$.
However, this is entirely due to the inclusion of the effect of the holes in the sidewalls of the cavities, which reduced the closed-
cavity value by the factor $\tilde{J}_1 = -1.375 = 1/7.3$. Without this
effect the impedances would be $\frac{Z}{n} \approx 18 \text{ k}\Omega/m$, and could cause
problems at energies close to transition ($\gamma < 12$).
29. **Conclusions**

The coupling impedance of a transversely oscillating particle beam in a vacuum chamber with periodic enlargements is found to increase significantly over its low frequency value at TM-resonances of the cavity (corresponding to hybrid modes in the axial region), while TE-resonances appear to be of minor importance.

Approximate expressions for the resonant frequencies and coupling impedances have been derived, and are given in Eq. (24.10) for the lowest TM mode. Although this "first approximation" is of limited accuracy, it yields satisfactory values for the coupling impedance of a RF-particle separator structure that has been computed before. It is therefore used with some confidence to estimate the coupling impedance of the circular-to-elliptic cross-section variations of the ISR vacuum chamber. It is found that the impedance lies well below the stability limit, with a safety factor increasing from 3 to 12 for $\gamma$ between 10 and 30. The simple closed cavity model, on the other hand, would yield instability for $\gamma$ below 12.

In general, we conclude that transverse resonances are less dangerous than longitudinal ones, and that the introduction of damping resistors - which was prompted by the latter - will eliminate all danger of transverse resonances.

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(4) B. Zotter: High Frequency Effects on the Transverse Resistive Wall Instability, CERN 69-16 (June 1969)

(5) H. Hereward, unpublished notes; see also
   W. Schnell: A Relation Between Longitudinal and Transverse Instability Thresholds, CERN-ISR-RP-70-7


(9) H. Hahn: Deflecting Modes in Circular Iris-Loaded Waveguides, Rev. Sci. Instr. 34, 1094-1100 (October 1963)
APPENDIX A: Evaluation of Matrix Products

1) \( (M^tM)_st = \sum_{m=-\infty}^{+\infty} \frac{\left( \frac{\pi s}{2} \right) \left( \frac{\pi t}{2} \right)}{\left( \pi \alpha m \right)^2 \left( \frac{\pi^2}{2} \right)^2 \left( \pi \alpha m \right)^2 - \left( \frac{\pi t}{2} \right)^2} \frac{\sin}{\icos} \frac{\sin}{-\icos} \frac{\pi \alpha m}{\pi \alpha m} \) \quad (A1)

- upper (resp. lower) sign of first bracket for even (resp. odd) values of \( s \), of the second bracket for even (resp. odd) values of \( t \).

- When \( s \) or \( t \) (or both) are zero, the sum vanishes because of the product \( st \) in the numerator.

- When the parity of \( s \) unequals the parity of \( t \), the terms are odd in \( m \). Since also the \( m=0 \) term vanishes, the sum is zero.

a) \( s \neq t \): \( (M^tM)_st = \frac{st}{\pi^2 \alpha^2 (s^2 - t^2)} \sum_{m=-\infty}^{+\infty} \left[ \frac{1}{m^2 + \left( \frac{s}{2\alpha} \right)^2} - \frac{1}{m^2 - \left( \frac{t}{2\alpha} \right)^2} \right] \frac{\sin^2}{\cos^2} \frac{\pi \alpha m}{\pi \alpha m} \) \quad (A2)

From Jolley we derive the summation formulae

\[
\begin{align*}
\sum_{m=-\infty}^{+\infty} \frac{\sin^2 m\theta}{m^2 - a^2} &= \frac{\pi}{a} \frac{\sin a \theta}{\sin a \pi} \sin a (\pi - \theta) \\
\sum_{m=-\infty}^{+\infty} \frac{\cos^2 m\theta}{m^2 - a^2} &= \frac{\pi}{a} \frac{\cos a \theta}{\sin a \pi} \cos a (\pi - \theta)
\end{align*}
\] \quad (A3)

Since \( a\theta = \frac{\pi s}{2} \) or \( \frac{\pi t}{2} \), \( \sin a \theta \) (resp. \( \cos a \theta \)) vanishes for even (resp. odd) \( s \) and \( t \), and the sums are all zero.

b) \( s = t \): \( (M^tM)_{ss} = \frac{s^2}{4 \pi^2 \alpha^2} \sum_{m=-\infty}^{+\infty} \frac{1}{m^2 - \left( \frac{s}{2\alpha} \right)^2} \frac{\sin^2}{\cos^2} \frac{\pi \alpha m}{\pi \alpha m} \) \quad (A4)

By a limiting process we get from (A3)

\[
\begin{align*}
\sum_{m=-\infty}^{+\infty} \frac{\sin^2 m\theta}{m^2 - a^2} &= \frac{\pi}{2a \sin a \pi} \left[ \sin a (\pi \theta) + a \theta \sin a \pi - \sin a (\pi - \theta) \right] \\
\sum_{m=-\infty}^{+\infty} \frac{\cos^2 m\theta}{m^2 - a^2} &= \frac{\pi}{2a \sin a \pi} \left[ \cos a (\pi \theta) + \cos a (\pi - \theta) \right]
\end{align*}
\] \quad (A5)
The terms multiplied by \( \sin \alpha \theta \) (resp. \( \cos \alpha \theta \)) vanish when 
\( \alpha \theta = \frac{\pi s}{2} \) for even (resp. odd) \( s \). We get \( \pm \frac{\pi \theta}{2a^2} \cos \alpha \theta \) for the 
two sums in (A5), and (A4) yields \( \frac{1}{2a} \) in either case.

c) We can combine these results to yield
\[
M^+M = \frac{1}{2a} (U - U^0)
\]  
(A6)

where \( U \) is the unit matrix, and \( U^0_{st} = \delta_{st} \delta_{s0} \delta_{t0} \) has a single 
non-vanishing element of value one at \( s = t = 0 \).

2) \[
(N^+N)_{st} = \sum_{m=-\infty}^{+\infty} \frac{(\pi a m)^2}{(\pi s)^2 \left(\frac{\pi t}{2}\right)^2} \left\{ \sin \frac{\pi \alpha m}{\pi s} \right\} \left\{ \sin \frac{\pi \alpha m}{\pi t} \right\} \left\{ \cos \frac{\pi \alpha m}{\pi s} \right\} \left\{ -\cos \frac{\pi \alpha m}{\pi t} \right\} 
\]  
(A7)

has been calculated in Ref. (1). The result is
\[
N^+N = \frac{1}{2a} (U + U^0)
\]  
(A8)

or, as
\[
N^+N = \frac{1}{2} N^+N = \frac{1}{a} \left( N^+N \right)^{-1} = K^{-1}
\]  
(A9)

We further draw attention to the fact that neither \( MM^+ \) nor \( NN^+ \) 
(inverted order) are diagonal matrices. Defining
\[
V_{pq} = \frac{\sin \pi \alpha (p-q)}{\pi \alpha (p-q)}
\]  
(A10)

we get
\[
MM^+ = \frac{V}{2}
\]
\[
NN^+ = \frac{1}{2} (V + NU^0N^+ + N^+U^0V)
\]  
(A11)

One can further conclude that neither \( M \) nor \( N \) have proper inverses, 
as a left inverse exists without being a right one. Multiplication 
of a matrix equation by a left inverse of \( M \) or \( N \) thus leads to a 
loss of information (corresponding to multiplication with a matrix 
with vanishing determinant).
3) Approximations

The assumption \( I_{pp} = \frac{k}{\chi_p} \) yields the following expressions:

\[
(M^+I^M)_{st} = \frac{\pi^2 b s t}{4\beta n R} \sum_{m=-\infty}^{+\infty} \frac{m^2 - \beta^2 n^2}{(\pi \alpha m)^2 \left(\frac{n s}{2}\right)^2} \left(\frac{2}{\pi \alpha m}\right) \left(\frac{2}{\pi \alpha m}\right) \left\{ \sin \left(\frac{\pi \alpha m}{2}\right) \sin \left(\frac{\pi \alpha m}{2}\right) \right\} \tag{A12}
\]

This can be evaluated with the expressions for \( M^+M \) and \( N^+N \) to give

\[
(M^+I^M)_{st} = \frac{\Gamma^2 b}{2\alpha^2} \delta_{st} \tag{A13}
\]

or with the diagonal matrices

\[
D_{ss} = \frac{\alpha_s}{r^2_b}, \ P_{ss} = \frac{s}{2\alpha^2 n} \tag{A14}
\]

we get

\[
M^+I^M = \frac{PD^1}{2\alpha} \tag{A15}
\]

b) With

\[
C_{pp} = \frac{k}{\chi_p} \frac{P}{b^2} \tag{A16}
\]

we get

\[
(I^1C)_{pq} = \frac{P}{\beta n} \delta_{pq}
\]

and

\[
(N^+I^1CM)_{st} = \frac{t}{2\alpha^2 n} \sum_{m=-\infty}^{+\infty} \frac{(\pi \alpha m)^2}{(\pi \alpha m)^2 \left(\frac{\pi s}{2}\right)^2} \left(\frac{2}{\pi \alpha m}\right) \left(\frac{2}{\pi \alpha m}\right) \left\{ \sin \left(\frac{\pi \alpha m}{2}\right) \sin \left(\frac{\pi \alpha m}{2}\right) \right\} \tag{A17}
\]

From the expression for \( N^+N \) we find (leaving off \( U^0 \) as the expression vanishes for \( s = 0 \) anyhow)

\[
N^+I^1CM = M^+I^1CN = \frac{P}{2\alpha} \tag{A18}
\]

(the Hermitian conjugate of a real, diagonal matrix is the matrix itself).
c) With (A16) we further get \( \mathbf{I}^\dagger (c^2 - I^2) \mathbf{p}_q = \frac{\mathbf{p}_q}{\mathbf{b}_k} \) (A19)

and thus

\[
\mathbf{N}^+ \mathbf{I}^\dagger (c^2 - I^2) \mathbf{N} = \frac{\mathbf{I}^\dagger}{\mathbf{a}_k \mathbf{b}_k} 
\] (A20)

d) We now find with \( \mathbf{L} = \mathbf{M} - \mathbf{C} \mathbf{N} \)

\[
\mathbf{M}^+ \mathbf{I}^\dagger \mathbf{L} = \mathbf{M}^+ \mathbf{I}^\dagger (\mathbf{M} - \mathbf{C} \mathbf{N}) = \frac{\mathbf{P}}{2\alpha} - \frac{\mathbf{P}}{2\alpha} 
\] (A21)

or

\[
\mathbf{M}^+ \mathbf{I}^\dagger \mathbf{L} = \mathbf{L}^+ \mathbf{I}^\dagger \mathbf{M} = 0 
\] (A22)

e) Finally we evaluate

\[
\mathbf{N}^+ \mathbf{I}^\dagger - \mathbf{L}^+ \mathbf{I}^\dagger \mathbf{L} = \mathbf{D} \mathbf{M}^+ \mathbf{I}^\dagger \mathbf{C} \mathbf{N} + \mathbf{N}^+ \mathbf{I}^\dagger \mathbf{C} \mathbf{M} \mathbf{D} - \mathbf{D} \mathbf{M}^+ \mathbf{I}^\dagger \mathbf{M} - \mathbf{N}^+ \mathbf{I}^\dagger (c^2 - I^2) \mathbf{N} 
\]

\[
= \frac{\mathbf{P}}{2\alpha} + \frac{\mathbf{P}}{2\alpha} - \frac{\mathbf{P}}{2\alpha} - \frac{\mathbf{I}^\dagger}{\mathbf{a}_k \mathbf{b}_k} 
\] (A23)

The elements of the diagonal matrix become

\[
(\mathbf{N}^+ \mathbf{I}^\dagger - \mathbf{L}^+ \mathbf{I}^\dagger \mathbf{L})_{ss} = \frac{\mathbf{I}^\dagger}{2\mathbf{a}_s \mathbf{b}_s} \left[ \frac{s^2}{s^2 - (2\mathbf{a}_s \mathbf{n})^2} - 1 \right] = \frac{\mathbf{I}^\dagger}{\mathbf{a}_s} \frac{\mathbf{a}_s}{\mathbf{b}_s} 2\mathbf{a}_s \mathbf{n} \frac{2\mathbf{b}_s}{s} 
\] (A24)

or

\[
\mathbf{N}^+ \mathbf{I}^\dagger - \mathbf{L}^+ \mathbf{I}^\dagger \mathbf{L} = \frac{\mathbf{I}^\dagger}{\mathbf{a}_s} \mathbf{D} \mathbf{P}^{-1} 
\] (A25)

(note that the two constituents of the sum cannot be evaluated by themselves).
APPENDIX B: The Quantity \( L = MD - CN \)

from definitions

\[
L_{ps} = \frac{\pi b}{(\pi \alpha \rho)^2 - \left(\frac{\pi s}{2}\right)^2} \left\{ \begin{array}{c} \sin \rho \alpha \rho \left(\frac{s}{g}\right)^2 \sin \rho \alpha \rho \left(\frac{p}{R}\right)^2 \cos \rho \alpha \rho \left(\frac{s}{g}\right)^2 \cos \rho \alpha \rho \left(\frac{p}{R}\right)^2 \end{array} \right\} \\
= \frac{g}{2b} \left(\frac{\pi s}{g \Gamma_s}\right)^2 - \left(\frac{p}{R \Gamma s}\right)^2
\]

The bracket becomes

\[
\left(\frac{1}{\Gamma_s^2 \chi_p^2} \left[ \frac{\pi^2 s^2}{g^2} \left(\frac{p^2}{R^2} - \beta^2 \kappa^2 \right) - \frac{p^2}{R^2} \left(\frac{\pi^2 s^2}{g^2} - \beta^2 \kappa^2 \right) \right] = \frac{4 \beta^2 k^2}{g \chi_p^2 \Gamma_s^2} \left(\frac{\pi s}{2}\right)^2
\]

and thus

\[
L_{ps} = \frac{2}{bg} \frac{\beta^2 \kappa^2}{\chi_p \Gamma_s^2} \left\{ \begin{array}{c} \sin \rho \alpha \rho \left(\frac{s}{g}\right)^2 \sin \rho \alpha \rho \left(\frac{p}{R}\right)^2 \cos \rho \alpha \rho \left(\frac{s}{g}\right)^2 \cos \rho \alpha \rho \left(\frac{p}{R}\right)^2 \end{array} \right\} = \frac{2 \beta^2 g}{(\pi^2 b)} \frac{1}{(p^2 - \beta^2 n^2) (s^2 - (2 \alpha \beta n)^2)} \left\{ \begin{array}{c} \sin \rho \alpha \rho \left(\frac{s}{g}\right)^2 \sin \rho \alpha \rho \left(\frac{p}{R}\right)^2 \cos \rho \alpha \rho \left(\frac{s}{g}\right)^2 \cos \rho \alpha \rho \left(\frac{p}{R}\right)^2 \end{array} \right\}
\]

At resonance for \( p = n \) given by (20.10)

\[
L_{np} = \frac{2 \beta^2 y^2}{b g \left\{ \begin{array}{c} \sin \rho \alpha \rho \left(\frac{s}{g}\right)^2 \sin \rho \alpha \rho \left(\frac{p}{R}\right)^2 \cos \rho \alpha \rho \left(\frac{s}{g}\right)^2 \cos \rho \alpha \rho \left(\frac{p}{R}\right)^2 \end{array} \right\}
\]

\[
\text{(B4)}
\]
APPENDIX C: Asymptotic Expressions for $x_q$ and $F_1(x_q)$

The eq. $\left(\frac{xF_0}{F_1}\right) = 2$ can be written

$$x\left[Y_1(\lambda x) J_0(x) - J_1(\lambda x) Y_0(x)\right] = 2\left[Y_1(\lambda x) J_1(x) - J_1(\lambda x) Y_1(x)\right] \quad (01)$$

We know from numerical solutions that $x$ becomes small for large $\lambda$. We thus may use the small argument approximations for $x$ (not for $\lambda x$)

$$x\left[Y_1(\lambda x) + J_1(\lambda x) \frac{2}{\pi x} \right] = 2\left[Y_1(\lambda x) \frac{x}{2} - J_1(\lambda x) \frac{2}{\pi x}\right] \quad (02)$$

This equation is fulfilled for $J_1(\lambda x) = 0$ or

$$x = \frac{j_{1q}}{\lambda} \quad (03)$$

where $j_{1q}$ is the $q$-th zero of $J_1$.

We then have approximately

$$F_1(x_q) = Y_1(j_{1q}) \frac{j_{1q}}{2\lambda} \quad (05)$$

The Wronskian for Bessel functions

$$Y_0(x) J_1(x) - Y_1(x) J_0(x) = \frac{2}{\pi x} \quad (06)$$

yields for $x = j_{1q}$

$$Y_1(j_{1q}) J_0(j_{1q}) = -\frac{2}{\pi j_{1q}} \quad (07)$$

and hence

$$F_1(x_q) = -\frac{1}{\pi \lambda j_{1q}} \quad \frac{1}{j_{1q}(j_{1q})} \quad (08)$$
APPENDIX D: Improved Criterion for the Resonant Frequency

If we limit ourselves to a single harmonic \((s=0)\) in the cavity, the matrix element \(W_{11}^{(0)}\) becomes

\[
W_{11}^{(0)} = \alpha \frac{F}{P_{11}} \sum_{p} \left[ \frac{1}{\text{op}} \frac{1}{\text{pp}} \frac{1}{\text{po}} \right] \left( \frac{\text{op}}{\text{pp}} \frac{1}{\text{po}} \right)
\]

Substitution of the expressions for the matrices yields

\[
W_{11}^{(0)} = \alpha \left( \frac{F}{P_{11}} \right) \sum_{p} \frac{x^2 H_p - (2\alpha \epsilon_p)^2 / H_p \left( \frac{\sin \beta \epsilon_p}{\beta \epsilon_p} \right)^2}{x^2 - (2\alpha \epsilon_p)^2}
\]

where \(H_p = \left( \frac{x_{11}}{j_{11}} \right) \chi_p \cdot b\)

The resonance condition is then \(W_{11}^{(0)} = 1\) or

\[
\left( \frac{F}{P_{11}} \right) = 1 + \alpha \sum_{p} \frac{x^2 H_p - (2\alpha \epsilon_p)^2 / H_p \left( \frac{\sin \beta \epsilon_p}{\beta \epsilon_p} \right)^2}{x^2 - (2\alpha \epsilon_p)^2}
\]

We can evaluate the sum on the RHS numerically and thus get an improved criterion (2nd approximation). If we assume that \(|\chi_p \cdot b| \ll 1\), we find \(H_p \approx 1\) and

\[
\left( \frac{F}{P_{11}} \right) = 1 + \alpha \sum_{p} \left( \frac{\sin \beta \epsilon_p}{\beta \epsilon_p} \right)^2 = 2
\]

is the first approximation result. If we limit the sum to the single term \(p = n\), but evaluate \(H\) to one more power: \(H_n = 1 - \frac{(xb)^2}{4}\), we get

\[
\left( \frac{F}{P_{11}} \right) = 1 + \alpha \left( 1 - \frac{x^2}{2\beta} \left( \frac{\sin \frac{\pi x}{2\beta \epsilon}}{\frac{\pi x}{2\beta \epsilon}} \right)^2 \right)
\]

in agreement for \(\beta = 1\) with the frequency criterion for \(v_p = c\) given in ref. 9. This formula has been used to obtain the improved value of \(f_{11}\) in section 26.
Figure Captions:

Fig. 1: Geometry of vacuum chamber with transversely oscillating particle beam.

Fig. 2A: Lowest solutions of the transcendental equations
\[ \frac{F(x)}{F_1(x)} = 2 \quad \text{and} \quad \frac{G(y)}{G_1(y)} = 2 \]
as function of the ratio of chamber to cavity radius \( \lambda = b/d \).

Fig. 2B: The values of \( F_1(x) \) and \( G_1(y) \) as function of \( \lambda \).

Fig. 3: Correction functions for the lowest TM-mode:
\( \Phi_1(\lambda) \) versus \( \lambda \) for the coupling impedance, and
\( 1/Z_1(\lambda) \) versus \( \lambda \) for the resonant frequency.
Fig. 2B