Appendix A
List of Symbols, Notation, and Useful Expressions

In this appendix the reader will find a more detailed description of the conventions and notation used throughout this book, together with a brief description of what spinors are about, followed by a presentation of expressions that can be used to recover some of the formulas in specified chapters.

A.1 List of Symbols

\[ \kappa \equiv \kappa_{ijk} \quad \text{Contorsion} \]
\[ \xi \equiv \xi_{ijk} \quad \text{Torsion} \]
\[ K \quad \text{Kähler function} \]
\[ K^I_j \equiv g^I_j \equiv G^I_j \quad \text{Kähler metric} \]
\[ W(\Phi) \quad \text{Superpotential} \]
\[ V \quad \text{Vector supermultiplet} \]
\[ \Phi, \phi \quad \text{(Chiral) Scalar supermultiplet} \]
\[ \phi, \varphi \quad \text{Scalar field} \]
\[ V(\phi) \quad \text{Scalar potential} \]
\[ \chi \equiv \gamma^0 \chi^\dagger \quad \text{For Dirac 4-spinor representation} \]
\[ \psi^\dagger \quad \text{Hermitian conjugate (complex conjugate and transposition)} \]
\[ \phi^* \quad \text{Complex conjugate} \]
\[ [M]^T \quad \text{Transpose} \]
\[ \{ , \} \quad \text{Anticommutator} \]
\[ [ , ] \quad \text{Commutator} \]
\[ \theta \quad \text{Grassmannian variable (spinor)} \]
\[ j_{ab}, j_{AB} \quad \text{Lorentz generator (constraint)} \]
\[ q_X, X = 1, 2, \ldots \quad \text{Minisuperspace coordinatization} \]
\[ \pi^{\mu\alpha\beta} \quad \text{Spin energy–momentum} \]
<table>
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<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>( S^{\mu\alpha\beta} )</td>
<td>Spin angular momentum</td>
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<td>( D )</td>
<td>Measure in Feynman path integral</td>
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<tr>
<td>([ , ]_P \equiv [ , ])</td>
<td>Poisson bracket</td>
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<td>([ , ]_D )</td>
<td>Dirac bracket</td>
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<td>( F )</td>
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<td>( V (\beta^+, \beta^-) )</td>
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<td>( Z^{IJ} )</td>
<td>Central charges</td>
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<td>( ds )</td>
<td>Spacetime line element</td>
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<tr>
<td>( ds )</td>
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<td>( e_\mu )</td>
<td>Coordinate basis</td>
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<td>( e_a )</td>
<td>Orthonormal basis</td>
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<td>(^{(3)} V )</td>
<td>Volume of 3-space</td>
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<td>( \psi^{(a)}_\mu )</td>
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<td>( \pi^{ij} )</td>
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<td>( J^j )</td>
<td>Lorentz rotation generator</td>
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<td>( K_i )</td>
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<td>( P^\mu )</td>
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<td>( \varepsilon_{\mu\nu\lambda\sigma} )</td>
<td>Four-dimensional permutation</td>
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<tr>
<td>( \varepsilon_{ijk} )</td>
<td>Three-dimensional permutation</td>
</tr>
<tr>
<td>( S_A )</td>
<td>SUSY generator (constraint)</td>
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<td>( \mathcal{H}_\perp )</td>
<td>Hamiltonian constraint</td>
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<td>( \mathcal{H}_i )</td>
<td>Momentum constraints</td>
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<td>( G )</td>
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<td>( \Psi )</td>
<td>Wave function (state) of the universe</td>
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<tr>
<td>( \Omega )</td>
<td>ADM time</td>
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<tr>
<td>( R )</td>
<td>Spinor-valued curvature</td>
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</table>
A.1 List of Symbols

$\beta_{ij}(\beta_+, \beta_-)$ Anisotropy matrices (Misner–Ryan representation) in Bianchi cosmologies

$p_{ij}$ Canonical momentum to $\beta^{ij}$

$H_{\text{ADM}}$ ADM Hamiltonian

$K_{ij}$ Extrinsic curvature

$(^4)R$ Four-dimensional spacetime curvature

$(^3)R$ Three-dimensional spatial curvature

$\Lambda$ Cosmological constant

$\Gamma^{\mu}_{\nu\lambda}$ Christoffel connection coefficients

| and $(^3)\overline{\nabla}_i$ Four-dimensional spacetime covariant derivative with respect to the 3-metric $h_{ij}$, no spin connection and no torsion, with Christoffel term (vectors, tensors)

; and $(^4)\overline{\nabla}_\mu$ Four-dimensional spacetime covariant derivative with respect to the 4-metric $g_{\mu\nu}$, no spin connection and no torsion, with Christoffel term (vectors, tensors)

$\parallel$ Covariant derivative with respect to the 3-metric, including spin connection (vectors, tensors)

$\mathcal{T}$ Covariant derivative with respect to the 4-metric $g_{\mu\nu}$, including spin connection (vectors, tensors)

Dot over symbol Proper time derivative

$d\Omega^2_3$ Line element of spatial sections

$k$ Spatial curvature index

$a(t) \equiv e^{\alpha(t)}$ FRW scale factor

$N$ Lapse function

$N^i$ Shift vector

$\mu, \nu, \ldots$ World spacetime indices

$a, b, \ldots$ Condensed superindices (either bosonic or fermionic)

$a, b, \ldots$ Local (Lorentz) indices

$\hat{a}, \hat{b}, \ldots = 1, 2, 3, \ldots$ Local (Lorentz) spatial indices

$i, j, k$ Spatial indices

$h_{ij}$ Spatial 3-metric

$g_{\mu\nu}$ Spacetime metric

$\eta^{ab}$ Lorentz metric

$\omega^\mu \equiv \{\omega^0, \omega^i\}$ One-form basis

$A, A'$ Two-spinor component indices (Weyl representation)

$[a]$ Four-spinor component indices, e.g., Dirac representation
l, J, ...  Kähler indices
J, j  Representation label of SU(2) (spin state)
M, N  Elements of SL(2, C)
$\mathcal{L}^a_b$  Lorentz (transformation) matrix representation
$p_i^{AA'}$  Momentum conjugate to $e_i^{AA'}$
$\mathcal{H}_{AA'}$  Hamiltonian and momentum constraints condensed (two-spinor components)
S  Lorentzian action
L  Lagrangian
I  Euclidean action
S  Superspace
M  Four-dimensional spacetime manifold
$\Sigma_i$  Three-dimensional spatial hypersurface
t  Coordinate time
$\tau$  Proper time
$n \equiv \{n^\mu\}$  Normal to spatial hypersurface
${(4)D}_v \equiv D_v$  Four-dimensional spacetime covariant derivative with respect to the 4-metric $g_{\mu\nu}$, with spin connection and torsion, no Christoffel term (spinors [vectors, tensors])
$i$ and $D_v$  Four-dimensional spacetime covariant derivative with respect to the 3-metric $h_{ij}$, with spin connection and torsion, with Christoffel term (spinors [vectors, tensors])
$\parallel$  Three-dimensional analogue of $D_v$ (spinors [vectors, tensors])
${(3)D}_j \equiv \nabla_j \equiv \nabla_j$  Three-dimensional analogue of $D_v$ (spinors [vectors, tensors])
${(3s)D}_j \equiv \nabla_j \equiv \tilde{\nabla}_j$  Three-dimensional analogue of $D_v$, no torsion (spinors [vectors, tensors])
$\mathcal{M}_{ab}, \mathcal{M}_{AB}$  Lagrange multipliers for the Lorentz constraints (Lorentz and 2-spinor indices)
$\omega^A_{B\mu}, \omega^a_{b\mu}, \omega^{AA'BB'}$  Spin connections
$\psi^A_{\mu}$  Gravitino field
$n^\mu, n^{AA'}$  Normal to the hypersurfaces
$\varepsilon^{AB}$  Metric for spinor indices (Weyl representation)
$\sigma^{AA'}$  Pauli matrices with two-component spinor indices (Weyl representation)
$\sigma$  Infeld–van der Waerden symbols
A.2 Conventions and Notation

Throughout this book we employ $c = 1 = \hbar$ and $G = 1 = M_P^{-2}$, with $k \equiv 8\pi G$ unless otherwise indicated. In addition, we take:

- $\mu, \nu, \ldots$ as world spacetime indices with values 0, 1, 2, 3,
- $a, b, \ldots$ as local (Lorentz) indices with values 0, 1, 2, 3,
• \(i, j, k\) as spatial indices with values 1, 2, 3,
• \(A, A'\) as 2-spinor notation indices with values 0, 1 or 0', 1', respectively,
• \([a]\) as 4-spinor component indices, e.g., the Dirac representation, with values 1, 2, 3, 4.

In this book we have also chosen the signature of the 4-metric \(g_{\mu\nu}\) (or \(\eta_{ab}\)) to be \((- , + , + , +)\). Therefore, the 3-metric \(h_{ij}\) on the spacelike hypersurfaces has the signature \((+ , + , +)\), which gives a positive determinant. Thus the 3D totally anti-symmetric tensor density can be defined by \(\varepsilon^{0123} \equiv \varepsilon^{123} = 1\).

### A.3 About Spinors

As the fellow explorer may already have noticed (or will notice, if he or she is venturing into this appendix prior to probing more deeply into some of the chapters) there are some idiosyncrasies in the mathematical structures of SUSY and SUGRA which are essentially related to the presence of fermions (and hence of the spinors which represent them). But why is this, and what are the main features the spinors determine? In fact, why are spinors needed to describe fermions? What are spinors and how do they come into the theory? This section is devoted (in part) to introducing this issue.

Before proceeding, for completeness let us just indicate that a tetrad formalism is indeed mandatory for introducing spinors.\(^1\) The tetrad corresponds to a massless spin-2 particle, the graviton [1], i.e., corresponding to two degrees of freedom.\(^2\)

#### Note A.1
Extending towards a simple SUSY framework, with only one generator (labeled \(N = 1\) SUSY), the corresponding (super)multiplet will also contain a fermion with spin 3/2, the gravitino, as described in Chap. 3 of Vol. I, where the reader will find elements of SUSY (not in this appendix).

#### A.3.1 Spinor Representations of the Lorentz Group

Local Poincaré invariance is the symmetry that gives rise to general relativity [2]. It contains Lorentz transformations plus translations and is in fact a semi-direct product of the Lorentz group and the group of translations in spacetime. The Lorentz

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\(^1\) Tensor representations of the general linear group \(4 \times 4\) matrices \(GL(4)\) behave as tensors under the Lorentz subgroup of transformations, but there are no such representations of \(GL(4)\) which behave as spinors under the Lorentz (sub)group (see the next section).

\(^2\) In simple terms, the tetrad has 16 components, but with four equations of motion, plus four degrees of freedom to be removed due to general coordinate invariance and six due to local Lorentz invariance, this leaves \(16 - (4 + 4 + 6) = 2\).
group has six generators: three rotations $J^a$ and three boosts $K^a$, $a, b = 1, 2, 3$ with commutation relations [3–5]:

$$[J^a, J^b] = i\epsilon_{abc} J^c, \quad [K^a, K^b] = -i\epsilon_{abc} K^c, \quad [J^a, K^b] = i\epsilon_{abc} K^c.$$  \hspace{1cm} (A.1)

The generators of the translations are usually denoted $P_\mu$, with\(^3\)

$$[P_\mu, P_\nu] = 0, \quad [J_i, P_j] = i\epsilon_{ijk} P_k.$$  \hspace{1cm} (A.2)

$$[J_i, P_0] = 0, \quad [K_i, P_j] = -iP_0\delta_{ij}, \quad [K_i, P_0] = -iP_i.$$  \hspace{1cm} (A.3)

Or alternatively, defining the Lorentz generators $L_{\mu\nu} \equiv -L_{\nu\mu}$ as $L_{0i} \equiv K_i$ and $L_{ij} \equiv \epsilon_{ijk} J_k$, the full Poincaré algebra reads

$$[P_\mu, P_\nu] = 0,$$  \hspace{1cm} (A.4)

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i\eta_{\nu\rho} L_{\mu\sigma} + i\eta_{\mu\rho} L_{\nu\sigma} + i\eta_{\nu\sigma} L_{\mu\rho} - i\eta_{\mu\sigma} L_{\nu\rho},$$  \hspace{1cm} (A.5)

$$[L_{\mu\nu}, P_\rho] = -i\eta_{\rho\mu} P_\nu + i\eta_{\rho\nu} P_\mu.$$  \hspace{1cm} (A.6)

Let us focus on the mathematical structure of (A.1). The usual tensor (vector) formalism and corresponding representation is quite adequate to deal with most situations of relativistic classical physics, but there are significant advantages in considering a more general exploration, namely from the perspective of the theory of representations of the Lorentz group. The spinor representation is very relevant indeed. For this purpose, it is usual to discuss how, under a general (infinitesimal) Lorentz transformation, a general object transforms linearly, decomposing it into irreducible pieces. To do this, we take the linear combinations

$$J^\pm_j \equiv \frac{1}{2} (J_j \pm iK_j),$$  \hspace{1cm} (A.7)

and the Lorentz algebra separates into two commuting SU(2) algebras:

$$[J^\pm_i, J^\pm_j] = i\epsilon_{ijk} J^\pm_k, \quad [J^\pm_i, J^\mp_j] = 0,$$  \hspace{1cm} (A.8)

i.e., each corresponding to a full angular momentum algebra. A few comments are then in order:

\(^3\) Note that we will be using either world (spacetime) indices $\mu$, or local Lorentz indices $a, b, \ldots$, and also world (space) indices $i$, or local spatial Lorentz indices $\hat{a}, \hat{b}, \ldots$, which can be related through the tetrad $e^\mu_i$ (see Sect. A.2).
The representations of SU(2) are known, each labelled by an index \( j \), with \( j = 0, 1/2, 1, 3/2, 2, \ldots \), giving the spin of the state (in units of \( \hbar \)). The spin-\( j \) representation has dimension \((2j + 1)\).

The representations of the Lorentz algebra (A.8) can therefore be labelled by\(^4\) \((j_-, j_+)\), the dimension being \((2j_+ + 1)(2j_- + 1)\), where we find states with \( j \) taking integer steps between \([j_+ - j_-] \) and \( j_+ + j_- \).

- In particular, \((0, 0)\) is the scalar representation, while \((0, 1/2)\) and \((1/2, 0)\) both have dimension two for spin \(1/2\) and constitute spinorial representations.\(^5\)

- In the \((1/2, 0)\) representation, \( J^- \equiv \{\mathcal{J}^- \}_{i=1,...} \) can be represented by \( 2 \times 2 \) matrices and \( J^+ = 0 \), viz., \( J^- = \bar{\sigma} \setminus 2 \), which implies\(^6\) \( J = \bar{\sigma} \setminus 2 \), but \( K = i\bar{\sigma} \setminus 2 \), where we henceforth use\(^7\) the four \( 2 \times 2 \) matrices \( \sigma_{\mu} \equiv (1, \sigma^i) \), with \( \sigma_0 \) usually being the identity matrix and \( \bar{\sigma} \equiv \sigma_i, i = 1, 2, 3 \), the three Pauli matrices:

\[
\sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(A.9)

satisfying \([\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k\). Note the relation between Pauli matrices with upper and lower indices, namely \( \sigma^0 = -\sigma_0 \) and \( \sigma^i = \sigma_i \), using the metric \( \eta_{\mu\nu} \).

- In the \((0, 1/2)\) representation we have \( J = \sigma \setminus 2 \) but \( K = -i\sigma \setminus 2 \).

This means that we have two types of spinors, associated respectively with \((0, 1/2)\) and \((1/2, 0)\), which are inequivalent representations. These will correspond in the following to the primed spinor \( \bar{\psi}_{A'} \) and the unprimed spinor \( \psi_A \), where \( A = 0, 1 \) and \( A' = 0, 1' \) [6–9].

The ingredients above conspire to make the group\(^8\) SL(2,\( \mathbb{C} \)) the universal cover\(^9\) of the Lorentz group. If \( M \) is an element of SL(2,\( \mathbb{C} \)), so is \( -M \), and both produce the same Lorentz transformation.

\(^4\) The pairs \((j_-, j_+)\) of the finite dimensional irreducible representations can also be extracted from the eigenvalues \( j_\pm (j_\pm + 1) \) of the two Casimir operators \( J_\pm^2 \) (operators proportional to the identity, with the proportionality constant labelling the representation).

\(^5\) The representation \((1/2, 1/2)\) has components with \( j = 1 \) and \( j = 0 \), i.e., the spatial part and time component of a 4-vector. Moreover, taking the tensor product \((1/2, 0) \otimes (0, 1/2)\), we can get a 4-vector representation.

\(^6\) It should be noted from \( J = \sigma \setminus 2 \) that a spinor effectively rotates through half the angle that a vector rotates through (spinors are periodic only for \( 4\pi \)).

\(^7\) The description that follows is somewhat closer to [10, 9, 11], but different authors use other choices (see, e.g., [3, 5, 12, 13]), in particular concerning the metric signature, \( \sigma_0 \), and some spinorial elements and expressions. See also Sect. A.4.

\(^8\) The letter S stands for ‘special’, indicating unit determinant, and the L for ‘linear’, while \( \mathbb{C} \) denotes the complex number field, whence SL(2,\( \mathbb{C} \)) is the group of \( 2 \times 2 \) complex-valued matrices.

\(^9\) So the Lorentz group is SL(2,\( \mathbb{C} \))/\( \mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) consists of the elements 1 and \( -1 \). Note also that SU(2) is the universal covering for the spatial rotation group SO(3). There is a two-to-one mapping of SU(2) onto SO(3).
A spinor is therefore the object carrying the basic representation of SL(2, C), fundamentally constituting a complex 2-component object (e.g., 2-component spinors, or Weyl spinors)

\[ \psi_A \equiv \begin{bmatrix} \psi_{A=0} \\ \psi_{A=1} \end{bmatrix} \]

transforming under an element \( M \) according to

\[ \psi_A \longrightarrow \psi'_A = M^B_A \psi_B, \quad M \equiv \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \in \text{SL}(2, \mathbb{C}), \quad (A.10) \]

with \( A, B = 0, 1 \), labelling the components. The peculiar feature is that now the other 2-component object \( \overline{\psi}_A' \) transforms as

\[ \overline{\psi}_A' \longrightarrow \overline{\psi}'_A' = M^{* B'}_{A'}, \quad (A.11) \]

which is the primed spinor introduced above, while the above \( \psi_A \) is the unprimed spinor.

**Note A.2** The representation carried by the \( \psi_A \) is \((1/2, 0)\) (matrices \( M \)). Its complex conjugate \( \overline{\psi}_A' \) in \((0, 1/2)\) is *not* equivalent: \( M \) and \( M^* \) constitute inequivalent representations, with the complex conjugate of (A.10) associated with (A.11), and \( \overline{\psi}_A' \) identified with \( (\psi_A)^* \).

**Note A.3** There is no unitary matrix \( U \) such that \( N = U M U^{-1} \) for matrices \( N \equiv M^* \). We have instead \( N = \xi M^* \xi^{-1} \) with

\[ \xi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \equiv -i \sigma_2. \]

This follows from the (formal) relation \( \sigma_2 \sigma^* \sigma_2 = -\sigma \), which suggests writing \( \xi \) in terms of a known matrix (appropriate for 2-component spinors), viz., \( \sigma_2 \).

It is from this feature applied to a Dirac 4-spinor

\[ \psi \equiv \begin{pmatrix} \chi \\ \overline{\eta} \end{pmatrix}, \]

that Lorentz invariants are obtained as \((i \sigma^2 \chi)^T \chi\). (This is the Weyl representation, where \( \chi, \overline{\eta} \) are 2-component spinors transforming with \( M \) and \( N \), respectively.) To be more concrete, with
\( \chi \equiv \chi_A = \begin{pmatrix} \chi_{A=0} \\ \chi_{A=1} \end{pmatrix} \),

we put

\[
\chi^A = \begin{pmatrix} \chi^0 \\ \chi^1 \end{pmatrix} = i\sigma^2 \chi = \begin{pmatrix} \chi_1 \\ -\chi_0 \end{pmatrix},
\]

whence the invariant is \( \chi^A \chi_A \). (Note that \( \chi^0 \chi_0 + \chi^1 \chi_1 = \chi_1 \chi_0 - \chi_0 \chi_1 = -\chi_1 \chi_1 - \chi_0 \chi^0 = -\chi_A \chi_A \), which is quite different from the situation for spacetime vectors and tensors \( A_\mu A^\mu = A^\mu A_\mu \)! This can be simplified by introducing new matrices with \( \chi^A = \epsilon^{AB} \chi_B \), where (formally!) \( \epsilon^{AB} \equiv i\sigma^2 \).

Likewise for

\[
\overline{\eta} \equiv \overline{\eta}^{A'} = \begin{pmatrix} \overline{\eta}^{0'} \\ \overline{\eta}^{1'} \end{pmatrix},
\]

\[\overline{\eta}^{A'} \overline{\eta}_{A'}, \overline{\eta}^{A'} = \epsilon^{A'B'} \overline{\eta}_{B'}, \epsilon^{A'B'} \equiv i\sigma^2, \text{ etc.}\] Of course, \( \epsilon^{AB} \equiv i\sigma^2, \epsilon^{A'B'} \equiv i\sigma^2 \), are different objects acting on different spinors and spaces. The presence of the matrix representation is formal for computations.

The primed and unprimed index structure\(^{10}\) carries to the generators, e.g., \( J = \sigma \backslash 2 \) and \( K = i\sigma \backslash 2 \), with the four \( \sigma_\mu \) matrices

\[
(\sigma^\mu)_{AA'} = \{-1, \sigma^i\}_{AA'}, \quad (A.12)
\]

and

\[
(\overline{\sigma}^\mu)^{A'A} \equiv \epsilon^{A'B'} \epsilon^{AB} (\sigma^\mu)_{BB'} \equiv (1, -\sigma^i)^{A'A}, \quad (A.13)
\]

where indices are raised by the antisymmetric 2-index tensors \( \epsilon^{AB} \) and \( \epsilon_{AB} \) given by\(^{11}\)

\[
\epsilon^{AB} = \epsilon^{A'B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \epsilon_{AB} = \epsilon_{A'B'} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (A.14)
\]

\(^{10}\) Note that (A.12) and (A.13) can used to convert an O(1,3) vector into an SL(2, \( \mathbb{C} \)) mixed bispinor \( b_{AB'} \).

\(^{11}\) Note that, in a formal matrix form, \( \epsilon^{AB} = \epsilon^{A'B'} = i\sigma^2, \epsilon_{AB} = \epsilon_{A'B'} = -i\sigma^2 \).
with which:

\[ \psi^A = \varepsilon^{AB} \psi_B, \quad \psi_A = \varepsilon_{AB} \psi^B, \quad \overline{\psi}^{A'} = \varepsilon^{A'B'} \overline{\psi}_{B'}, \quad \overline{\psi}_A = \varepsilon_{A'B'} \overline{\psi}^{B'} . \]  

(A.15)

Note A.4  
Note the difference between \((\sigma^\mu)_{AA'}\) and \((\overline{\sigma}^\mu)_{A'A}\), i.e., the lower, upper and respective order of indices, arising from \(\varepsilon^{AB} = \varepsilon^{A'B'} = i\sigma_2\) and \(\varepsilon_{AB} = \varepsilon_{A'B'} = -i\sigma^2\). To be more precise, the (‘natural’) lower (upper) \(A'A\) indices on \((\sigma^\mu)_{AA'}, (\overline{\sigma}^\mu)_{A'A}\) come from covariance, e.g., analysing how the matrices

\[ \gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix} \]

transform under a Lorentz transformation (the Weyl representation here). 
Further relations [9], in particular between \(\sigma^\mu\) and \(\overline{\sigma}^\mu\), can be written

\[ \sigma_{\mu CD} \overline{\sigma}^{\mu A'B'} = -2 \delta_{C}^{B} \delta_{D}^{A'} , \quad \text{(A.16)} \]
\[ \text{Tr} (\sigma^\mu \overline{\sigma}^\nu) \equiv \sigma^\mu_{AB} \overline{\sigma}^{\nu B'A} = -2 g^{\mu \nu} , \quad \text{(A.17)} \]

constituting two completeness relations.

In essence, it all involves the statement that \(\sigma^\mu_{AB}\) is Hermitian, and that, for a real vector, its corresponding spinor is also Hermitian.

Note A.5  
In addition, the following points may be of interest:

- Concerning (A.14) and (A.15), it must be said that this is a matter of convention.
- Unprimed indices are always contracted from upper left to lower right (‘ten to four’), while primed indices are always contracted from lower left to upper right (‘eight to two’). This rule does not apply when raising or lowering spinor indices with the \(\varepsilon\) tensor.
- However, other choices can be made (see Sect. A.4) [6, 14, 7–9], and this is what we actually follow from Chap. 4 of Vol. I onwards. In particular,

\[ \varepsilon^{AB} = \varepsilon^{A'B'} = \varepsilon_{AB} = \varepsilon_{A'B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad \text{(A.18)} \]

with which [note the position of terms and order of indices, in contrast with (A.14)]:
\[ \psi^A = \epsilon^{AB} \psi_B \, , \quad \psi_A = \psi^B \epsilon_{BA} \, , \quad \overline{\psi}^{A'} = \epsilon^{A'B'} \overline{\psi}_{B'} \, , \quad \overline{\psi}_{A'} = \overline{\psi}^{B'} \epsilon_{B'A'} \, . \]  

(A.19)

With (A.15) [or (A.19)], scalar products such as \( \psi \chi \) or \( \overline{\psi} \overline{\chi} \) can be employed. Of course, when more than one spinor is present, we have to remember that spinors anticommute.\(^{12} \) Therefore, for 2-component spinors, \( \psi_0 \chi_1 = -\chi_1 \psi_0 \), and also, e.g., \( \psi_0 \overline{\chi}_1 = -\overline{\chi}_1 \psi_0 \). In more detail,

\[ \psi \chi = \psi^A \chi_A = \epsilon^{AB} \psi_B \chi_A = -\epsilon^{AB} \psi_A \chi_B \]

\[ = -\psi_A \chi^A = \chi^A \psi_A = \chi \psi \, , \]  

(A.20)

\[ \overline{\psi} \overline{\chi} = \overline{\psi}^{A'} \overline{\chi}^{A'} = \ldots = \overline{\chi}^{A'} \overline{\psi}^{A'} = \overline{\chi} \overline{\psi} \, , \]

(A.21)

\[ (\psi \chi)^\dagger = (\psi^A \chi_A)^\dagger = \overline{\chi}^{A'} \overline{\psi}^{A'} = \overline{\chi} \overline{\psi} = \overline{\overline{\psi} \overline{\chi}} \, , \]  

(A.22)

as well as

\[ \psi \sigma^{\mu} \overline{\chi} = \psi^A \sigma^{\mu}_{AB'} \overline{\chi}^{B'} \, , \quad \overline{\psi} \overline{\sigma}^{\mu} \chi = \overline{\psi}^{A'} \sigma^{\mu A'B} \chi_B \, , \]

(A.23)

from which, e.g.,

\[ \chi \sigma^{\mu} \overline{\psi} = -\overline{\overline{\psi}} \overline{\sigma}^{\mu} \chi \, , \]

(A.24)

\[ \chi \sigma^{\mu} \overline{\sigma}^{\nu} \psi = \psi \sigma^{\nu} \sigma^{\mu} \chi \, , \]

(A.25)

\[ (\chi \sigma^{\mu} \overline{\psi})^\dagger = \psi \sigma^{\mu} \chi \, , \]

(A.26)

\[ (\chi \sigma^{\mu} \overline{\sigma}^{\nu} \psi)^\dagger = \overline{\psi} \sigma^{\nu} \sigma^{\mu} \chi \, , \]

(A.27)

### A.3.2 Dirac and Majorana Spinors

The above (Weyl) framework will often be used for most of this book [in particular, (A.18) and (A.19)], but if the reader goes on to study the fundamental literature on SUSY and SUGRA, other representations will be met, some of which are also adopted by a few authors in SQC. Let us therefore comment on that (see Chap. 7).

In particular, we present the Dirac and Majorana spinors (associated with another representation).

---

\(^{12} \) The components of two-spinors are Grassmann numbers, anticommuting among themselves. Therefore, complex conjugation includes the reversal of the order of the spinors, e.g., \( (\zeta \psi)^* = (\zeta^A \psi_A)^* = (\psi_A)^* (\zeta^A) = \overline{\psi} \overline{\zeta} = \overline{\overline{\psi} \overline{\zeta}} \).
A.3 About Spinors 219

A 4-component Dirac spinor is made from a 2-component unprimed spinor and a 2-component primed spinor via

\[
\psi_{[a]} \equiv \begin{pmatrix} \psi_A \\ \overline{\chi}^{A'} \end{pmatrix},
\]

transforming as the reducible \((1/2, 0) \oplus (0, 1/2)\) representation of the Lorentz group, hence with

\[
\begin{pmatrix} \psi_A \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \overline{\chi}^{A'} \end{pmatrix}
\]

as Weyl spinors. \(^{13}\)

But is there any need for a Dirac spinor (apart from in the Dirac equation)? The point is that, under a parity transformation \([3, 5]\), the boost generators \(K\) change sign, whereas the generators \(J\) do not. It follows that the \((j, 0)\) and \((0, j)\) representations are interchanged under parity, whence 2-component spinors are not sufficient to provide a full description. The Dirac 4-component spinor carries an irreducible representation of the Lorentz group extended by parity.

Note A.6 In addition, from

\[
\psi_{[a]} \equiv \begin{pmatrix} \chi_A \\ \overline{\eta}^{A'} \end{pmatrix} \implies \psi_{[a]}^{\dagger} = \begin{pmatrix} \overline{\chi}^{A'}, \eta^A \end{pmatrix},
\]

(A.28)

the adjoint Dirac spinor is written as (Weyl representation)

\[
\overline{\psi}_{[a]} \equiv -\psi^{\dagger} \gamma^0 \begin{pmatrix} \eta^A, \overline{\chi}^{A'} \end{pmatrix} \implies \overline{\psi}_{[a]}^{T} = \begin{pmatrix} \eta^A \\ \overline{\chi}^{A'} \end{pmatrix}, \quad \gamma^0 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(A.29)

Note that \(\overline{\psi}_1 \psi_2 = \eta_1 \chi_2 + \overline{\chi}_1 \overline{\eta}_2\) where the right-hand side is a Weyl 2-spinor representation, noticing that in the left-hand side we have Dirac 4-spinors, with the bar above either \(\psi_1\) or \(\chi_1\) concerning different configurations and operations, and subscripts 1 and 2 merely labeling different Dirac spinors.

Note A.7 The charge conjugate spinor \(\psi^c\) is defined by

\[\text{13 There are also chiral Dirac spinors. These constitute eigenstates of } \gamma^5 \text{ and behave differently under Lorentz transformations [see (A.33) and (A.34)].}\]
\[ \psi^c \equiv C \psi^T = \begin{pmatrix} -i \sigma^2 \eta^A \\ i \sigma^2 \overline{\chi}^A' \end{pmatrix} = \begin{pmatrix} \eta_A \\ \overline{\chi}^A' \end{pmatrix}, \]  

(A.30)

where

\[ C \equiv \begin{bmatrix} \varepsilon_{AB} = -i \sigma^2 & 0 \\ 0 & e^{A'B'} = i \sigma^2 \end{bmatrix}. \]  

(A.31)

**Note A.8** The Majorana condition \( \psi = \psi^c \) gives \( \chi = \eta \):

\[ \psi = \psi^c = \begin{pmatrix} \chi_A \\ \overline{\chi}^A' \end{pmatrix}. \]  

(A.32)

The importance of the Majorana condition is that it defines the *Majorana spinor*. This is invariant under charge conjugation, therefore constituting a relation (a sort of ‘reality’ condition) between \( \psi \) and the (complex) conjugate spinor \( \psi^\dagger \). A Majorana spinor is a particular Dirac spinor, namely

\[ \begin{pmatrix} \psi_A \\ \overline{\psi}^{A'} \end{pmatrix}, \]

with only half as many independent components. A Majorana spinor is therefore a Dirac spinor for which the Majorana condition (A.32) is implemented: \( \psi = \psi^c \), i.e., \( \overline{\chi}^{A'} = \psi^{\dagger}_A \).

Of course, whenever we have (Dirac or Majorana) 4-spinors, a related matrix framework must be used, in the form of the Dirac matrices (here in the chiral or Weyl representation):

\[ \gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix}, \quad \gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \left( \gamma^5 \right)^2 = 1, \]  

(A.33)

with \( \gamma_5 \) inducing the projection operators \( (1 \pm \gamma_5)/2 \) for the chiral components \( \psi_A \), \( \overline{\chi}^{A'} \) of \( \psi \), determining their helicity as right (positive) or left (negative). But there are many other options [9]:

- **Standard**\(^\text{14} \)** canonical basis:

\[ \gamma_0 \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma_i \equiv \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}. \]  

(A.34)

\(^{14}\) The *standard* (Dirac) representation has \( \gamma_0 \) appropriate to describe particles (e.g., plane waves) in the rest frame.
A.3 About Spinors

- A Majorana basis, where $\gamma_i^* = -\gamma_i$:
  \[
  \gamma_0 \equiv \begin{bmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, \quad \gamma_1 \equiv \begin{bmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{bmatrix}, \\
  \gamma_2 \equiv \begin{bmatrix} i1 & 0 \\ 0 & -i1 \end{bmatrix}, \quad \gamma_3 \equiv \begin{bmatrix} 0 & -i\sigma^1 \\ -i\sigma^1 & 0 \end{bmatrix}.
  \] (A.35)

- A basis in which the $\gamma$ matrices are all real and where $N = 1$ SUGRA allows a real Rarita–Schwinger (gravitino) field (see Sect. 4.1 of Vol. I) [15]:
  \[
  \gamma_0 \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_2 \equiv \begin{bmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{bmatrix}, \\
  \gamma_3 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \begin{bmatrix} 0 & i\sigma^3 \\ -i\sigma^3 & 0 \end{bmatrix}.
  \] (A.36)

Furthermore,
\[
\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}.
\] (A.37)

Through (A.33), the Lorentz generators then become
\[
\mathcal{L}^{\mu\nu} \longrightarrow \frac{i}{2} \gamma^{\mu\nu},
\]
where
\[
\gamma^{\mu\nu} \equiv \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \frac{1}{2} \begin{bmatrix} \sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu & 0 \\ 0 & \sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu \end{bmatrix},
\] (A.38)

and $\{,\}$ and $[,]$ denote the anticommutator and commutator, respectively, As expected from the representation $(1/2, 0) \oplus (0, 1/2)$, this determines that the $\psi_A$ and $\overline{\chi}^{A'}$ spinors transform separately. The specific generators are also rewritten $i\sigma^{\mu\nu}$ for $\psi_A$ and $i\overline{\sigma}^{\mu\nu}$ for $\overline{\chi}^{A'}$, with
\[
(\sigma^{\mu\nu})_A^B \equiv \frac{1}{4} \big[ \sigma^{\mu}_{AC} \overline{\sigma}^{\nu C'B} - (\mu \leftrightarrow \nu) \big],
\] (A.39)
\[
(\overline{\sigma}^{\mu\nu})_B^{A'} \equiv \frac{1}{4} \big[ \overline{\sigma}^{\mu A'C} \sigma^{\nu CB'} - (\mu \leftrightarrow \nu) \big].
\] (A.40)
A.4 Useful Expressions

In this section we present a set of formulas which will help to clarify some properties of SUSY and SUGRA, and help also to simplify some of the expressions in the book. The initial emphasis is on the 2-component spinor, but we will also indicate, whenever relevant, some specific expressions involving 4-component spinors.

In a 2-component spinor notation, we can use a spinorial representation for the tetrad. This then allows us to deal with the indices of bosonic and fermionic variables in a suitable equivalent manner. To be more precise, in flat space we therefore associate a spinor with any vector by the Infeld–van der Waerden symbols $\sigma_a^{AA'}$, which are given by \[\sigma_0^{AA'} = -\frac{1}{\sqrt{2}} \mathbf{1}, \quad \sigma_i^{AA'} = \frac{1}{\sqrt{2}} \sigma_i.\] (A.41)

Here, $\mathbf{1}$ denotes the unit matrix and $\sigma_i$ are the three Pauli matrices (A.9). To raise and lower the spinor indices, the different representations of the antisymmetric spinorial metric $\varepsilon_{AB}$, $\varepsilon_{AB}$, $\varepsilon_{A'B'}$, and $\varepsilon_{A'B'}$ can be used [see Note A.5 and (A.18)]. Each of them can be written as the same matrix, given by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\] (A.42)

Hence, the spinorial version of the tetrad reads

\[e_\mu^{AA'} = e_\mu^a \sigma_a^{AA'}, \quad e^a_\mu = -\sigma^a_{AA'} e^{AA'}_\mu.\] (A.43)

Generally, for any tensor quantity $T$ defined in a (curved) spacetime, a corresponding spinorial quantity is associated through $T \rightarrow T^{AA'} = e^{AA'}_\mu T^\mu$, with the inverse relation given by $T^\mu = -e^{\mu}_{AA'} T^{AA'}$.

Equipped with this feature, the foliation of spacetime into spatial hypersurfaces $\Sigma_t$ from a tetrad viewpoint (spinorial version) is straightforward:

- The future-pointing unit normal vector $n \rightarrow n^\mu$ has a spinorial form given by

\[n^{AA'} = e^{AA'}_\mu n^\mu.\] (A.44)

- The tetrad (A.43) is thereby decomposed into timelike and spatial components $e_0^{AA'}$ and $e_i^{AA'}$.

\[15\] We henceforth condense and follow the structure in Note A.5 (see Chap. 4 of Vol. I).
Moreover, the 3-metric is written as

\[ h_{ij} = -e_{A\alpha}^i e_{A\alpha}^j. \]  \hspace{1cm} (A.45)

This metric and its inverse are used to lower and raise the spatial indices \( i, j, k, \ldots \).

- From the definition of \( n^{A\alpha} \) as a future-pointing unit normal to the spatial hypersurfaces \( \Sigma_t \), we can further retrieve

\[ n^{A\alpha}_A e_{A\alpha}^i = 0 \quad \text{and} \quad n^{A\alpha}_A n^{A\alpha} = 1, \]  \hspace{1cm} (A.46)

which allow us to express \( n^{A\alpha} \) in terms of \( e_{i}^{A\alpha} \) (see Exercise 4.3 of Vol. I).

- Using the lapse function \( \mathcal{N} \) and the shift vector \( \mathcal{N}^i \), the timelike component of the tetrad can be decomposed according to

\[ e_{0}^{A\alpha} = \mathcal{N} n^{A\alpha} + \mathcal{N}^i e_{i}^{A\alpha}. \]  \hspace{1cm} (A.47)

- Other useful formulas\(^\text{16}\) involving the spinorial tetrad \( e_{i}^{A\alpha} \) with the unit vector in spinorial form \( n^{A\alpha} \) can be found in Sect. A.4.1.

The reader and fellow explorer of SQC should note that these expressions do indeed allow one to considerably simplify, e.g., the differential equations for the bosonic functionals of the wave function of the universe (see, e.g., Chap. 5 of Vol. I).

### A.4.1 Metric and Tetrad

Using the above definitions [6], we have the following relations for the timelike normal vector \( n^{A\alpha}_A \) and the tetrad \( e_{i}^{A\alpha} \):

\[ n^{A\alpha}_A n^{B\beta}_B = \frac{1}{2} \varepsilon_{A\alpha}^{B\beta}, \]  \hspace{1cm} (A.48)

\[ n^{A\alpha}_A n^{B\alpha}_B = \frac{1}{2} \varepsilon_{A\alpha}^{B\alpha}, \]  \hspace{1cm} (A.49)

\[ e_{AA'}^i e_{j}^{AB'} = -\frac{1}{2} h_{ij} \varepsilon_{A\alpha}^{B\beta} - i \sqrt{h} \varepsilon_{ijk} n^{A\alpha} e_{AB'}^{i} , \]  \hspace{1cm} (A.50)

\[ e_{AA'}^i e_{j}^{BA'} = -\frac{1}{2} h_{ij} \varepsilon_{A\alpha}^{B\alpha} - i \frac{1}{\sqrt{h}} \varepsilon_{ijk} n^{A\alpha} e_{BA'}^{i} , \]  \hspace{1cm} (A.51)

\[ e_{AA'}^i e_{B}^{BB'} = n^{A\alpha}_A n^{B\beta}_B - \varepsilon_{AB\alpha} e_{A\alpha}^{B'} . \]  \hspace{1cm} (A.52)

\(^{16}\) Equations (A.44), (A.45), (A.46), and (A.47) can be contrasted with (A.60), (A.61), (A.62), (A.63), (A.64), and (A.65).
From (A.50) and (A.51), by contracting with $\epsilon^{ijl}$, we obtain
\begin{align}
 n_{AA'} e^{AB'l} &= - n^{AB'} e^{k}_{AA'} = \frac{i}{2\sqrt{h}} \epsilon^{ijl} e_{AA'}^{i} e^{AB'}_{j}, \quad (A.53) \\
n_{AA'} e^{BA'l} &= - n^{BA'} e^{k}_{AA'} = - \frac{i}{2\sqrt{h}} \epsilon^{ijl} e_{AA'}^{i} e^{BA'}_{j}. \quad (A.54)
\end{align}

In addition, we have
\begin{align}
g^{\mu\nu} e^{AA'}_{\mu} e^{BB'}_{\nu} &= - \epsilon^{AB} \epsilon^{A'B'}, \quad (A.55) \\
g_{\mu\nu} &= - e^{AA'}_{\mu} e^{BB'}_{\nu} \epsilon^{AB} \epsilon^{A'B'}, \quad (A.56)
\end{align}

\begin{align}
2e_{AA'}(\mu e_{BA'}^{\nu}) &= g^{\mu\nu} \epsilon^{AB} \epsilon^{A'B'}, \quad (A.57) \\
2e_{AA'}(\mu e_{AB'}^{\nu}) &= g_{\mu\nu} \epsilon^{A'B'}, \quad (A.58)
\end{align}

The normal projection of the expression $\epsilon^{ilm} D_{B}^{B'} m_{j} D_{A'}^{A' kl}$, used throughout Chap. 4, is given by [17]
\begin{align}
E_{Bjk}^{CAB'i} &\equiv n^{AA'} e^{ilm} D_{B}^{B'} m_{j} D_{A'}^{A' kl} = \frac{-2i}{\sqrt{h}} \epsilon^{ilm} D_{B}^{B'} m_{j} e_{l}^{CD'} e_{DD'}^{k} n_{A}^{A'} \\
&= \frac{2i}{\sqrt{h}} \epsilon^{ilm} D_{B}^{B'} m_{j} e_{l}^{CD'} e_{D}^{A} e_{D'}^{k}. \quad (A.59)
\end{align}

Now, regarding the 4-spinor framework in Sects. 4.1.1 and 4.1.2 of Vol. I,
\begin{align}
N_{i} &= e_{0a} e_{a}^{i}, \quad (A.60) \\
N &= N_{j} N^{j} - e_{0a} e_{a}^{0}, \quad (A.61) \\
e_{0}^{a} &= - \frac{n^{a}}{N}, \quad (A.62) \\
e_{0a} &= \gamma^{m} n_{a} + \gamma^{m} e_{ma}, \quad (A.63) \\
n^{s} e_{m} &= 0, \quad (A.64) \\
\epsilon^{ijk} \gamma_{s} \gamma_{i} &= 2i \gamma^{\perp} \sigma^{ik}, \quad (A.65)
\end{align}

and we have the notation
\begin{align}
-A_{\perp} &= A^{\perp} = -n^{\mu} A_{\mu} = N A^{0} = \frac{1}{N} \left( A_{0} - N^{i} A_{i} \right), \quad (A.66) \\
A^{i} &= A^{i} - \frac{N^{i}}{N} A^{\perp}. \quad (A.67)
\end{align}
A.4 Useful Expressions

A.4.2 Connections and Torsion

In the second order formalism applied to $N = 1$ SUGRA [13], the tetrad $e_\mu^{CD'}$ and gravitinos $\psi_\mu^C$, $\psi^D_\nu$ determine the explicit form of the connections.

Four-Dimensional Spacetime

We will use

$$\omega^{ab}_\mu = \omega^{[ab]}_\mu \rightarrow \omega^{AA'BB'}_\mu = \omega^{AB}_\mu \varepsilon^{A'B'} + \overline{\omega^{AB}_\mu} \varepsilon^{AB}, \quad (A.68)$$

where

$$\omega^{AB}_\mu = \omega^{(AB)}_\mu = \tilde{\omega}^{AB}_\mu + \kappa^{AB}_\mu, \quad (A.69)$$

while $\tilde{\omega}^{AB}_\mu$ is the spinorial version of the torsion-free connection form

$$\tilde{\omega}^{ab}_\mu = e^a_\nu \partial_{[\mu} e^b_{\nu]} - e^b_\nu \partial_{[\mu} e^a_{\nu]} - e^{a\nu} e^b_{\mu} e_{c\nu} \partial_{[\nu} e^c_{\rho]} , \quad (A.70)$$

and $\kappa^{AB}_\mu = \kappa^{(AB)}_\mu$ is the spinorial version of the contorsion tensor $\kappa^{ab}_\mu = \kappa^{[ab]}_\mu$, with

$$\kappa^{AA'BB'}_\mu = e^{AA'}_\mu e^{BB'}_\nu \kappa^{\nu\rho}_\mu = \kappa^{AB}_\mu \varepsilon^{A'B'} + \overline{\kappa^{AB}_\mu} \varepsilon^{AB}. \quad (A.71)$$

Furthermore, explicitly, it will in this case relate to the (4D spacetime) torsion through

$$\xi^{AA'}_{\mu \nu} = -4\pi i G \overline{\psi^A_{[\mu} \psi^A_{\nu]}}, \quad (A.72)$$

whose tensorial version is

$$\xi^{\rho}_{\mu \nu} = -e^{\rho}_{AA'} \kappa^{AA'}_{\mu \nu}. \quad (A.73)$$

The contorsion tensor $\kappa$ is defined by

$$\kappa_{\mu \nu \rho} = \xi_{\nu \mu \rho} + \xi_{\rho \mu \nu} + \xi_{\mu \nu \rho}. \quad (A.74)$$

Spatial Representation

Within a 3D (spatial surface) representation, the novel ingredient is the expansion into $n_{AA'}$ and $e^{AA'}_i$. In more detail, the spin connection is written in the form

$$(3) \omega^{AA'BB'}_i = (3) \omega^{AA'BB'}_i + (3) \kappa^{AA'BB'}_i, \quad (A.75)$$
and then decomposed into primed and unprimed parts:

\[
(3)\omega_i^{A'A'B'B'} = (3)\omega_i^{AB} e^{A'B'} + (3)\omega_i^{A'B'} e^{AB} .
\] (A.76)

Using the antisymmetry \((3)\omega_i^{A'A'B'B'} = -(3)\omega_i^{B'B'AA'}\), we then obtain the symmetries \((3)\omega_i^{AB} = (3)\omega_i^{BA}\) and \((3)\omega_i^{A'B'} = (3)\omega_i^{B'A'}\) and the explicit representations

\[
(3)\omega_i^{AB} = \frac{1}{2} (3)\omega_i^{A'B'} , \quad (3)\omega_i^{A'B'} = \frac{1}{2} (3)\omega_i^{B'A'} .
\] (A.77)

Analogous relations hold for \((3)s\omega_i^{A'A'B'B'}\) and \((3)\kappa_i^{A'A'B'B'}\). Furthermore, we now have

\[
(3)\omega_i^{AB} = (3)s\omega_i^{AB} + (3)\kappa_i^{AB} ,
\] (A.78)

where \((3)s\omega_i^{AB}\) is the spinorial version of the spatial torsion-free connection form

\[
(3)s\omega_i^{ab} = \left( e^{b j} \partial [ j e^a_i] - \frac{1}{2} e^{a j} e^{b k} e^c_i \partial_j e_{ck} - \frac{1}{2} e^{a j} n^b n^c \partial_j e_{ci} - \frac{1}{2} n^a \partial_i n^b \right) -(a \leftrightarrow b) ,
\] (A.79)

and \(\kappa_i^{AB}\) is the spinorial version of the spatial contorsion tensor \(17\), with

\[
3\kappa_{AA'B'B'} = e^j_{AA'} e^{BB'} \kappa_{jki} = 3\kappa_{AB'i} e^{A'B'} + 3\kappa_{A'B'i} e^{AB} .
\] (A.81)

The three-dimensional torsion-free spin connection \((3)s\omega_i^{A'A'B'B'}\) can therefore be expressed in terms of \(n^{AA'}\) and \(e_i^{AA'}\) as

\[
(3)s\omega_i^{AA'B'B'} = e^{BB'} j \partial [ j e_i^{AA'}] - \frac{1}{2} \left( e^{AA'} j e^{BB'} e_i^{CC'} j \partial_j e_{CC'i} + e^{AA'} j n^{BB'} n^{CC'} j \partial_j e_{CC'i} + e^{AA'} j n^{BB'} \right) - e^{AA'} j \partial [ j e_i^{BB'}] + \frac{1}{2} \left( e^{BB'} j e^{BB'} e_i^{CC'} j \partial_j e_{CC'i} + e^{BB'} j n^{AA'} n^{CC'} j \partial_j e_{CC'i} + e^{BB'} j \partial [ j n^{AA'}] \right).
\] (A.82)

\[\text{17 The 3D contorsion is simply obtained by restricting the 4D quantity:}\]

\[
(3)\kappa_{ijk} = \kappa_{ijk} ,
\] (A.80)

with the spinorial contorsion \((3)\kappa^{AA'B'B'} = e^{AA'} j e_{k}^{BB'} (3)\kappa_{jki} = -(3)\kappa^{BB'AA'i}.\)
A.4.3 Covariant Derivatives

Still referring to a second order formalism, the tetrad $e^{CD'}_\mu$ and the gravitinos $\psi^C_\mu$, $\psi^D'_\nu$ constitute the action variables, with which a specific covariant derivative $D_\mu$ is associated, acting only on spinor indices (not spacetime, with which $\Gamma$ would be associated).

**Four-Dimensional Spacetime**

To be more precise, we use

$$D_\mu e^{AA'}_v = \partial_\mu e^{AA'}_v + \omega^A_B e^{BA'}_v + \omega^{A'}_{B'} e^{AB'}_v,$$

(A.83)

$$D_\mu \psi^A_v = \partial_\mu \psi^A_v + \omega^A_B \psi^B_v,$$

(A.84)

with $\omega^A_B$ the connection form (spinorial representation) as described in Sect. A.4.2 above.

**Spatial Representation**

Subsequently, we have the spatial covariant derivative $(3) D_j$ acting on the spinor indices, where, for a generic tensor in spinorial form,$^{19}$

$$(3) D_j T^{AA'} = \partial_j T^{AA'} + (3) \omega^A_B T^{BA'} + (3) \omega^{A'}_{B'} T^{AB'},$$

(A.85)

with $(3) \omega^A_B$ and $(3) \omega^{A'}_{B'}$ the two parts of the spin connection [see (A.77)], using the decomposition

$$(3) \omega^A_B BB' = (3) s_i \omega^A_i BB' + (3) c_i \omega^A_i BB'$$.  

(A.86)

A.4.4 (Gravitational) Canonical Momenta

Related to the presence of (covariant) derivatives in the action of $N = 1$ SUGRA, we retrieve the momenta conjugate to the tetrad $e^{CD'}_\mu$ and gravitinos $\psi^C_\mu$, $\psi^D'_\nu$. The latter require the use of Dirac brackets, which are discussed elsewhere (see Appendix B and Chap. 4 of Vol. I), so we present here some elements regarding the former.

---

18 Notice that $D_\mu e^{AA'}_v = \xi^{AA'}_\mu$, where $\xi^{AA'}_\mu$ is the spinor version of the torsion.

19 $T^{AA'...ZZ'} = e^{AA'}_\mu ... e^{ZZ'}_v T^{...\mu v}$. 

For the gravitational canonical momentum, we can write
\[ p_{AA'}^i = \frac{\delta S_{N=1}^{N}}{\delta e_i^{AA'}} \rightarrow p^{ji} = -e^{AA'}_j p_{AA'}^i, \quad p^{\perp i} = n^{AA'} p_{AA'}^i. \]  
(A.87)
where we can also use (as in the pure gravitational sector, see Sect. 4.2 of Vol. I)
\[ p^{(ij)} = -2\pi^{ij}. \]  
(A.88)
This then leads us to the issue of curvature.

### A.4.5 Curvature
In fact, we can express the curvature (e.g., the intrinsic curvature or second fundamental form \( K_{ij} \)) in spinorial terms and through the (gravitational) canonical momentum \( p_{AA'}^i \), with the assistance of (A.87) and (A.88). Let us recall that
\[ \pi^{ij} \sim -\frac{\hbar^{1/2}}{2k^2} \left\{ K^{(0)}(ij) - \left[ \text{tr} K^{(0)} \right] h^{ij} \right\} \]
\[ \sim -\frac{\hbar^{1/2}}{2k^2} \left\{ \left[ K^{(ij)} - \tau^{(ij)} \right] - h^{ij}(K - \tau) \right\}, \]  
(A.89)
\[ K_{ij} = -e^a_i \partial_j n_a + n_a \omega^a_j e_{bi}, \]  
(A.90)
but now emphasizing the influence of the torsion:
\[ K_{(ij)} = \frac{1}{2N} \left( \mathcal{N}_{ij} + \mathcal{N}_{ji} - h_{ij,0} \right) - 2\xi_{(ij)}^{\perp}, \]  
(A.91)
\[ K_{[ij]} = \xi_{\perp ij} \equiv n^\mu \xi_{\mu ij}. \]  
(A.92)

### Four-Dimensional Spacetime
The spinor-valued curvature is given by
\[ \mathcal{R}_{AB}^{\mu\nu} = \mathcal{R}_{AB}^{(AB)} = 2 \left( \partial_{[\mu} \omega_{\nu]}^{AB} + \omega_{C[\mu}^{A} \omega_{\nu]}^{B} \right), \]  
(A.93)
\[ \mathcal{R} = e_{AA'}^{\mu} e_{B}^{A'} \mathcal{R}_{AB}^{\mu\nu} + e_{A}^{A'} \mu e_{AB'}^{\nu} \mathcal{R}_{AB'}^{\mu\nu}. \]  
(A.94)
\[ \pi^{ij} \] and additional terms in \( p^{\perp i} \) will appear in the Lorentz constraint \( J_{ab} \leftrightarrow J_{AB} e_{A'B'} + J_{A'B'} e_{AB}. \)
Spatial Representation

The components of the 3D curvature in terms of the spin connection read

\[
(3)^R_{ij}^{AB} = 2 \left[ \partial_i (3)^{\omega^A}_{j}^{\phantom{A}B} + (3)^{\omega^A}_{C|i} \left(3^{\omega^B}_{j}^{\phantom{B}C} \right) \right], \\
(3)^R_{ij}^{A'B'} = 2 \left[ \partial_i (3)^{-\omega^A'}_{j}^{\phantom{A'}B'} + (3)^{-\omega^A'}_{C|i} \left(3^{-\omega^B'}_{j}^{\phantom{B'}C'} \right) \right].
\]

Because of the symmetry of \( (3)^{\omega^A}_{i}^{\phantom{A}B} = 0 \) and \( (3)^{-\omega^A'}_{i}^{\phantom{A'}B'} = 0 \), the chosen notation \( (3)^{\omega^A}_{i}^{\phantom{A}B} \) and \( (3)^{-\omega^A'}_{i}^{\phantom{A'}B'} \) is unambiguous. The horizontal position of the indices does not need to be fixed. The scalar curvature is given by

\[
(3)^R = e^i_{AA'}e^j_{BB'} \left[ (3)^R_{ij}^{AB} \varepsilon^{A'B'} + (3)^R_{ij}^{A'B'} \varepsilon^{AB} \right].
\]

The same procedure performed on \( (3s)^{\omega^A}_{i}^{AA'B'B'} \) leads to the torsion-free scalar curvature:

\[
(3s)^R_{ij}^{AB} = 2 \left[ \partial_i (3s)^{\omega^A}_{j}^{\phantom{A}B} + (3s)^{\omega^A}_{C|i} \left(3s^{\omega^B}_{j}^{\phantom{B}C} \right) \right], \\
(3s)^R_{ij}^{A'B'} = 2 \left[ \partial_i (3s)^{-\omega^A'}_{j}^{\phantom{A'}B'} + (3s)^{-\omega^A'}_{C|i} \left(3s^{-\omega^B'}_{j}^{\phantom{B'}C'} \right) \right],
\]

and

\[
(3s)^R = e^i_{AA'}e^j_{BB'} \left[ (3s)^R_{ij}^{AB} \varepsilon^{A'B'} + (3s)^R_{ij}^{A'B'} \varepsilon^{AB} \right].
\]

A.4.6 Decomposition with Four-Component Spinors

We now present, in a somewhat summarized manner, a few helpful formulas for the \( 3 + 1 \) decomposition of a Cartan–Sciama–Kibble (CSK) theory, and in particular, Einstein gravity with torsion (see Chap. 2 of Vol. I), using 4-component spinors [15]:

- The torsion is given by

\[
\xi^{\mu \nu \lambda} = \frac{1}{2} \left( \Gamma^{\mu \nu \lambda} - \Gamma^{\nu \mu \lambda} \right),
\]

where \( e_\lambda \) are basis vectors, which we will take as a coordinate basis with \( e_i \) tangent to a spacelike hypersurface and the normal \( n \) with components \( n_\mu = (N^0, 0, 0) \), and where the components of the metric are as in (2.6), (2.7), and (2.8) of Vol. I. Then we can write
\[ \mathbf{e}_{\mu;\nu} = e^\lambda \Gamma_{\mu \nu}^\lambda, \]  
(A.100)\\
\[ \Gamma_{\mu \nu}^\lambda = \Gamma_{\mu \nu}^{(0)\lambda} - \kappa_{\mu \nu}^\lambda, \]  
(A.101)\\
\[ \kappa_{\mu \nu}^\lambda = \xi_{\mu \nu}^\lambda - \xi_{\mu}^\lambda \nu + \xi_{\nu}^\lambda \mu, \]  
(A.102)

with \( \Gamma_{\mu \nu}^{(0)\lambda} \) the Christoffel symbols and \( \kappa_{\mu \nu}^\lambda \) the contorsion tensor.

- The extrinsic curvature \( K_{ij} \) can then be computed and retrieved from \( n_{;j} \) by [see (2.9) of Vol. I]

\[ K_{ji} = - N (4) \Gamma_{ji}^0 = \frac{1}{2N} (-h_{ji,0} + \mathcal{N}_{ji} + \mathcal{N}_{ij}) - \kappa_{ji\perp}, \]  
(A.103)\\
\[ K_{(ji)} = \frac{1}{2N} (-h_{ji,0} + \mathcal{N}_{ji} + \mathcal{N}_{ij}) + \tau_{(ji)}, \]  
(A.104)\\
\[ K_{[ji]} = \xi_{ij\perp}, \quad \tau_{ji} \equiv 2 \xi_{j\perp i}, \]  
(A.105)

where | denotes the derivative involving the Christoffel symbols for \( h_{ij} \).

- The curvature is then obtained from

\[ (4) R = (3) R - K_{ij} K^{ij} + K^2 - 2 (4) R_{\perp}^\alpha \perp \alpha, \]  
(A.106)\\
\[ (4) R_{\perp}^\alpha \perp \alpha = (n^\alpha n^\beta_{\perp\alpha} - n^\beta n^\gamma_{\perp\gamma})_{\beta} - K^{ij} K_{ij} + K^2 - 2 \xi_{\alpha\beta} \perp n_{\alpha\beta}. \]  
(A.107)

- Finally, or almost, the Lagrangian density is

\[ \sqrt{h} L = N h^{1/2} \left[ (3) R - K_{ij} K^{ij} + K^2 - 2 \left( n^\gamma n^\alpha_{\perp\alpha} - n^\beta n^\gamma_{\perp\gamma} \right)_{\beta} + 4 \xi_{\alpha\beta} \perp n_{\alpha\beta} \right], \]  
(A.108)

from which the first two lines of (4.17) of Vol. I are obtained.

From the above, the Hamiltonian and constraints are\(^{21}\)

\[ H_m = -2 h_{mi} \pi_{ik}^i, \]  
(A.109)\\
\[ J^a b \equiv p^k e^b_k - p^k e^a_k, \]  
(A.110)\\
\[ H_{\perp} \sim h^{-1/2} \left( \pi_{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - h^{1/2} (3) R + h^{1/2} \left[ \tau_{(ij)} \tau_{(ij)} - \tau^2 \right] \]  
(A.111)\\
\[ + \sqrt{h} \xi_{ij\perp} \xi_{ij\perp} - 2 h^{1/2} \tau_{[ij]} \xi_{[ij]} + \sqrt{h} q^k \rho_k - 2 h^{1/2} \rho_{[i}^i + 2 h^{1/2} \rho_{j}^j \rho_i, \]

\(^{21}\) The reader should notice that, with the above alone, i.e., no gravitino (matter) action, we have \( P_{\mu\nu}^{\lambda} = 0 \) for the conjugate to torsion \( \xi_{\mu\nu\lambda} \), whose conservation leads to \( \xi_{\mu\nu\lambda} = 0 \).

But if the Rarita–Schwinger field is present in an extended action, with Lagrangian terms, e.g.,

\[ \epsilon^{\lambda\mu\nu\rho} \bar{\psi}_{\lambda} \gamma_{\gamma} \gamma_{\mu} D_{\nu} \psi_{\rho}, \]

then

\[ \xi_{\mu\nu\lambda} = - \frac{1}{4} \bar{\psi}_{\mu} \gamma_{\lambda} \psi_{\nu}, \]

and torsion cannot then be ignored.
A.4 Useful Expressions

\[ H = \mathcal{N} \mathcal{H}_\perp + \mathcal{N}^i \mathcal{H}_i + \mathcal{M}_{ab} \mathcal{J}^{ab}. \]  

(A.112)

A.4.7 Equations Used in Chap. 4

We employ several derivatives of functionals with respect to the tetrad \( e^{AB'} \) [17]. Two of them are:

\[ \varepsilon^{ilmn} a^{ij \, ij} (D_B \frac{B'}{mj} D^{C}_{A'kl}) \]  

(A.113)

and

\[ n^{AA'} \frac{\delta}{\delta e^{AB'}_j} D^{B'B'}_{ij}. \]  

(A.114)

First we need an explicit form for \( \frac{\delta n^{AA'}}{\delta e^{BB'}_j} \). With (4.109) of Vol. I, which expresses \( n^{AA'} \) in terms of the tetrad, the relation \( n^{AA'} e_{AA'} = 0 \) implies

\[ 0 = e^{CC'} i \frac{\delta n^{AA'}}{\delta e^{BB'}_j} e_{AA'} = n^{CC'} n^{AA'} \frac{\delta n^{AA'}}{\delta e^{BB'}_j} - \varepsilon_{A'C} \varepsilon_{A'C'} \frac{\delta n^{AA'}}{\delta e^{BB'}_j} + e^{CC'} e_{BB'} n^{BB'}. \]  

(A.115)

In addition, we have

\[ \frac{\delta n^{AA'}}{\delta e^{BB'}_j} = \frac{\delta n^{CC'}}{\delta e^{BB'}_j} n^{CC'} n^{AA'} e^{BB'} \]  

(A.116)

whence

\[ \frac{\delta n^{AA'}}{\delta e^{BB'}_j} = e^{AA'} e_{BB'}. \]  

(A.117)

We then use the derivative of the determinant \( h \) of the three-metric:

\[ \frac{\partial h}{\partial h_{ij}} = h^{ij} h. \]  

(A.118)

Hence,

\[ \frac{\delta h}{\delta e^{AA'}_i} = -2 h e^{AA'}. \]  

(A.119)

Using this and (A.48), (A.49), (A.50), (A.51), and (A.52), we can calculate the expressions
\[ n^{AA'} e^{ilm} \frac{\delta}{\delta e^{AB'}_j} (D_B^{B'} m_j D^{C}_{A'kl}) \]

\[ = -4n^{AA'} e^{ilm} \frac{\delta}{\delta e^{AB'}_j} \left( \frac{1}{h} \epsilon_{Bj}^{D'} \epsilon_{DD'm}^{E'} m^{DB'} e^{C'E'}_l \epsilon_{EE'k} \right) \]

\[ = \epsilon_{B}^C \delta_{ik}^j \left( 1 - 1 + \frac{1}{2} - \frac{1}{2} \right) + \frac{2i}{\sqrt{h}} \left( 2e^{C'B'}_i \epsilon_{BB'k} + e^{i}_{BB'} e^{CB'}_k \right) \]

\[ = -\frac{3i}{\sqrt{h}} \delta_{ik}^j \epsilon_{B}^C - 2h_{ij} \epsilon_{jkl} n^{CB'} \epsilon_{BB'}^j , \] (A.120)

and

\[ n^{AA'} \frac{\delta}{\delta e^{AB'}_j} D^{B'}_{ij} = -2in^{AA'} \frac{\delta}{\delta e^{AB'}_j} \left( \frac{1}{\sqrt{h}} \epsilon_{ij}^{B'C'} \epsilon_{CC'k} n^{CB'} \right) \]

\[ = -\frac{2i}{\sqrt{h}} n^{AA'} n^{BC'} \epsilon_{AC'I} . \] (A.121)

Finally, we can write a transformation rule to express derivatives in terms of the tetrad \( e_i^{AA'} \) as derivatives in terms of the three-metric \( h_{ij} \). Let \( \mathcal{F}[e] \) be a functional depending on the tetrad, and note that \( h_{ij} \) can be expressed in terms of the tetrad, since we have the relation \( h_{ij} = -e_i^{AA'} e_{AA'}^j \). Moreover, there is of course no inverse relation. We thus restrict the functional \( \mathcal{F} \) by demanding that it can be written in the form \( \mathcal{F}[h_{ij}] \). Consequently, using the chain rule, we find for the transformation of the functional derivatives:

\[ \frac{\delta \mathcal{F}}{\delta e_i^{AA'}} = \frac{\delta \mathcal{F}}{\delta h_{jk}} \frac{\delta h_{jk}}{\delta e_i^{AA'}} = -\frac{\delta \mathcal{F}}{\delta h_{jk}} \epsilon_{BC}^B \epsilon_{B'C'}^C \frac{\delta e^{BB'}_j e^{CC'}_k}{\delta e_i^{AA'}} \]

\[ = -\frac{\delta \mathcal{F}}{\delta h_{jk}} \epsilon_{AC}^B \epsilon_{A'C'}^C \frac{\delta e^{BB'}_j e^{CC'}_k}{\delta e_i^{AA'}} - \frac{\delta \mathcal{F}}{\delta h_{ji}} \epsilon_{BA}^B \epsilon_{B'A'}^A \frac{\delta e^{BB'}_j}{\delta e_i^{AA'}} \]

\[ = -2 \frac{\delta \mathcal{F}}{\delta h_{ij}} e_{AA'}^j . \] (A.122)

Using \( e_i^{AA'} e_{AA'}^j = -\delta_j^i \), the inverse relation for an arbitrary functional \( \mathcal{G}[h_{ij}] \) is simply

\[ \frac{\delta \mathcal{G}}{\delta h_{ij}} = \frac{1}{2} e_{AA'}^j \frac{\delta \mathcal{G}}{\delta e_i^{AA'}} . \] (A.123)

Note that it is always possible to rewrite \( \mathcal{G}[h_{ij}] \) in the form \( \mathcal{G}[e] \).
References

Appendix B
Solutions

Problems of Chap. 2

2.1 The Born–Oppenheimer Approximation and Gravitation

The Born–Oppenheimer approximation [1–4] was first developed for molecular physics. It was then extended to quantum gravity. In brief, in the form of a recipe, we have the following:

- We use the simple Hamiltonian
  \[ H = \frac{p^2}{2M} + \frac{p^2}{2m} + V(R, r) \equiv \frac{p^2}{2M} + h, \]
  with \( M \gg m \).
- \( R, r \) denote variables for heavy \((M)\) and light \((m)\) particles, respectively.
- The purpose is to solve (approximately) the stationary Schrödinger equation \( H\Psi = E\Psi \). Then:
  - Assume (and this is one of the main ingredients!) that the spectrum (in terms of eigenvalues and eigenfunctions) of the light particle is known for each configuration \( R \) of the heavy particle, i.e., \( h|n; R \rangle = \varepsilon(R)|n; R \rangle \).
  - Carry out the expansion \( \Psi = \sum_k \Psi_k(R)|k; R \rangle \).
- Returning to the Schrödinger equation, after multiplying by \( \langle n; R \mid \), obtain an equation for the wave functions \( \Psi_n \):
  \[
  - \frac{\hbar^2}{2M} \nabla_R^2 + \varepsilon_n(R) - E \right] \Psi_n(R) = \frac{\hbar^2}{M} \sum_k \langle n; R \mid \nabla_R k; R \rangle \nabla_R \Psi_k(R) + \frac{\hbar^2}{2M} \sum_k \langle n; R \mid \nabla_R^2 k; R \rangle \Psi_k(R). \]
Now neglect the off-diagonal terms\(^1\) in (B.1):

\[
\left\{ \frac{1}{2M} \left[ -i\hbar \nabla - hA(R) \right]^2 - \frac{\hbar^2 A^2}{2M} - \frac{\hbar^2}{2M} \left( n; R|\nabla_R^2 n; R \right) + \varepsilon_n(R) \right\} \Psi_n(R) = E \Psi_n(R).
\]

(B.2)

The reader should note that the momentum of the slow particle has been shifted: \(P \rightarrow P - \hbar A\), where \(A(R) \equiv -i \langle n; R|\nabla_R n; R \rangle\). This constitutes a Berry connection.\(^2\)

Concerning the application of the Born–Oppenheimer expansion to gravitation, let us add the following:

1. An expansion with respect to the large parameter \(M\) leads to satisfactory results if the relevant mass scales of non-gravitational fields are much smaller than the Planck mass.

2. Regarding the non-gravitational fields:
   a. Without them, an \(M\) expansion is fully equivalent to an \(h\) expansion, i.e., to the usual WKB expansion for the gravitational field.
   b. If non-gravitational fields are present, the \(M\) expansion is analogous to a Born–Oppenheimer expansion: large (nuclear) mass \(\rightarrow M\), small (electron) mass \(\rightarrow \) mass scale of the non-gravitational field.

### 2.2 Hamilton–Jacobi Equation and Emergence of ‘Classical’ Spacetime

If a solution \(S_0\) to the Hamilton–Jacobi equation is known, we can extract

\[
\pi^{ij} = M \frac{\delta S_0}{\delta h_{ij}},
\]

(B.3)

from which \(\dot{h}_{ij}\) is recovered in the form

\[
\dot{h}_{ij} = -2\mathcal{N} K_{ij} + \mathcal{N}_{ij} + \mathcal{N}_{ji},
\]

(B.4)

with \(K_{ij}\) the components of the extrinsic curvature of 3-space, viz.,

\[
K_{ij} = -16\pi G G_{ijkl} \pi^{kl}.
\]

(B.5)

Note that a choice must be made for the lapse function and the shift vector in order for \(\dot{h}_{ab}\) to be specified. Once this and an ‘initial’ 3-geometry have been chosen, (B.4)

---

\(^1\) Neglecting the off-diagonal terms corresponds to a situation in which an interaction with environmental degrees of freedom allows the \(\Psi_k(R) |k; R\) to decohere from each other (see Sects. 2.2.2 and 2.2.3).

\(^2\) If the fast particle eigenfunctions are complex, the connection \(A\) is nonzero.
can then be integrated (for a thorough explanation see [5]). A ‘complete’ spacetime will follow, associated with a chosen foliation and a choice of coordinates on each member of this foliation [1]:

- Combine all the trajectories in superspace which describe the same spacetime into a sheaf [6].
- The lapse and shift can be chosen in such a way that these curves comprise a sheaf of geodesics.
- Superposing expressions in the form of WKB solutions (bearing the gravitational field), one specific trajectory in configuration space, i.e., one specific spacetime, will be described by means of a wave packet.

However, there is another point to note here. One must also discuss an expansion of the momentum constraints in powers of $M$. In fact, $S_0$ satisfies the three equations

$$\left(\frac{\delta S_0}{\delta h_{ab}}\right)_{|b} = 0 . \quad \text{(B.6)}$$

The Hamilton–Jacobi equation and (B.6) are equivalent to the $G_{00}$ and $G_{0i}$ parts of the Einstein equations, while the remaining six field equations can be found by differentiating (B.4) with respect to $t$ and eliminating $S_0$ by making use of (B.4) and the Hamilton–Jacobi equation.

### 2.3 Factor Ordering and Semi-Classical Gravity

The terms in (2.19) are independent of the factor ordering chosen for the gravitational kinetic term. Any ambiguity in the factor ordering would have to come from the gravity–matter coupling. But let us elaborate on this [1–4].

Let us introduce a linear derivative term (parametrized by $\xi_{ij}$) that represents an ambiguity in the factor ordering of the kinetic term. This will then appear as we ‘enter the range’ of the Schrödinger equation. In the Wheeler–DeWitt equation, we now have:

$$\left( -\frac{\hbar^2}{2M} G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{jk}} + \xi_{ij} \frac{\delta}{\delta h_{ij}} + MU^g + H^m_\perp \right) \Psi = 0 . \quad \text{(B.7)}$$

Note that (2.11) can also be written as

$$G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta \mathcal{K}}{\delta h_{kl}} - \frac{1}{2} G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} \mathcal{K} = 0 , \quad \text{(B.8)}$$

but with the linear term we have instead

---

3 Equations (B.6) will follow from the Hamilton–Jacobi equation and the Poisson bracket relations between the constraints $\mathcal{H}$ and $\mathcal{H}_i$ [7].
\[ G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta \mathcal{K}}{\delta h_{kl}} - \frac{1}{2} \left( G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta S_0}{\delta h_{ij}} \right) \mathcal{K} = 0 \]  \quad \text{(B.9)}

At order \( M^0 \) an equation also involving \( S_1 \) is

\[ G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta S_1}{\delta h_{kl}} - \frac{i \hbar}{2} \left( G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta S_0}{\delta h_{ij}} \right) + \mathcal{H}_m = 0 \],  \quad \text{(B.10)}

but with the linear term we can write

\[ G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta S_1}{\delta h_{kl}} - \frac{i \hbar}{2} \left( G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta S_0}{\delta h_{ij}} \right) + \mathcal{H}_m = 0 \]  \quad \text{(B.11)}

With regard to (2.15), we now get

\[ 0 = G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta S_2}{\delta h_{kl}} - \frac{1}{2} \left( G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} + \frac{i \hbar}{2} \left( G_{ijkl} \frac{\delta^2 S_0}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta S_0}{\delta h_{ij}} \right) \right) + \mathcal{H}_m = 0 \],  \quad \text{(B.12)}

If we now further require \( \tilde{\sigma}_2 \) to satisfy

\[ G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta \tilde{\sigma}_2}{\delta h_{kl}} - \frac{\hbar^2}{K^2} G_{ijkl} \frac{\delta \mathcal{K}}{\delta h_{ij}} \frac{\delta D}{\delta h_{kl}} + \frac{\hbar^2}{2K} \left( G_{ijkl} \frac{\delta^2 \mathcal{K}}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta \mathcal{K}}{\delta h_{ij}} \right) = 0 \],

we obtain

\[ G_{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta \tilde{\eta}}{\delta h_{kl}} = \frac{\hbar^2}{2 \mathcal{F}} \left( - \frac{\hbar^2}{K} G_{ijkl} \frac{\delta \mathcal{F}}{\delta h_{ij}} \frac{\delta \mathcal{K}}{\delta h_{kl}} + G_{ijkl} \frac{\delta^2 \mathcal{F}}{\delta h_{ij} \delta h_{kl}} + \zeta_{ij} \frac{\delta \mathcal{F}}{\delta h_{ij}} \right) + \frac{i \hbar}{\sqrt{\hbar}} \frac{\delta \tilde{\eta}}{\delta \phi} \frac{\delta \mathcal{F}}{\delta \phi} + \frac{i \hbar}{2 \sqrt{\hbar}} \frac{\delta^2 \tilde{\eta}}{\delta \phi^2} \].  \quad \text{(B.13)}

Hence, the Schrödinger equation with quantum gravitational corrections, including the linear term, is now

\[ i \hbar \frac{\delta \Theta}{\delta \tau} = \mathcal{H}_m^\perp \Theta + \frac{\hbar^2}{M \mathcal{F}} G_{ijkl} \left( \frac{1}{K} \frac{\delta \mathcal{K}}{\delta h_{ij}} \frac{\delta \mathcal{F}}{\delta h_{kl}} - \frac{1}{2} \frac{\delta^2 \mathcal{F}}{\delta h_{ij} \delta h_{kl}} - \frac{1}{2} \zeta_{ij} \frac{\delta \mathcal{F}}{\delta h_{ij}} \right) \Theta \].  \quad \text{(B.14)}

We decompose derivatives of \( \mathcal{F} \) into normal and tangential components to the constant \( S_0 \) hypersurfaces. In more detail (see also Sect. 4.2.6),

\[ G_{ijkl} \frac{\delta \mathcal{F}}{\delta h_{ij}} = - \frac{i}{\hbar} G_{ijkl} \mathcal{D}^j \mathcal{H}_m^\perp \mathcal{F} \bigoplus G_{mnop} \frac{\delta \mathcal{F}}{\delta h_{mn}} \ell^m_{op} \ell^l_{kl} \].  \quad \text{(B.15)}
The first term on the right-hand side is the normal component, with

\[ U^{ij} \equiv \left( G^{ijkl} \frac{\delta S_0}{\delta h_{ij}} \frac{\delta S_0}{\delta h_{kl}} \right)^{-1} = -\frac{1}{2U} \frac{\delta S_0}{\delta h_{ij}}. \]  

(B.16)

The second term is the tangential component and \( \ell_{ij} \) is a unit vector tangent to the \( S_0 \) constant hypersurface, satisfying \( U^{ij} \ell_{ij} = 0 \). Here we use

\[ G^{mnop} \frac{\delta F}{\delta h_{mn}} \ell_{op} \ell_{kl} \equiv a_{mn}. \]

We then write (see Sect. 4.2.6), using (B.14),

\[ \frac{h^2}{2MF} \left( -\dot{G}_{ijkl} \frac{\delta^2 F}{\delta h_{ij} \delta h_{kl}} + \frac{2}{K} \dot{G}_{ijkl} \frac{\delta K}{\delta h_{ij}} \frac{\delta F}{\delta h_{kl}} - \xi_{ij} \frac{\delta F}{\delta h_{ij}} \right) \Theta \equiv C_n \Theta + C_t. \]  

(B.17)

where we have

\[ C_n \equiv -\frac{1}{4MUF} \left[ (\mathcal{H}_m^\perp)^2 \mathcal{F} - ih \dot{G}_{ijkl} \frac{\delta S_0}{\delta h_{kl}} \left( -\frac{\delta \mathcal{H}_m^\perp}{\delta h_{ij}} + \frac{\delta U}{\delta h_{ij}} \mathcal{H}_m^\perp \mathcal{F} \right) \right] \Theta \]

\[ = -\frac{1}{4MUF} \left[ (\mathcal{H}_m^\perp)^2 \mathcal{F} - ih \left( -\frac{\delta \mathcal{H}_m^\perp}{\delta \tau} + \frac{\delta U}{U \delta \tau} \mathcal{H}_m^\perp \mathcal{F} \right) \right] \Theta. \]  

(B.18)

The reader should note that it is due to the conservation law that there are no ambiguities here, as all terms cancel. Moreover,

\[ C_t \equiv -\frac{h^2}{4MUF} \left[ \frac{2}{K} \frac{\delta K}{\delta h_{kl}} \left( G^{mnop} \frac{\delta F}{\delta h_{mn}} \ell_{op} \right) \ell_{kl} + \frac{\delta G_{ijkl}}{h_{kl}} \left( G^{mnop} \frac{\delta F}{\delta h_{mn}} \ell_{op} \right) \ell_{ij} \right. \]

\[ - \left. \frac{\delta}{\delta h_{kl}} \left( G^{mnop} \frac{\delta F}{\delta h_{mn}} \ell_{op} \right) \ell_{kl} - \xi_{ij} G^{mnop} \frac{\delta F}{\delta h_{mn}} \ell_{op} \ell_{ij} \right] \Theta. \]  

(B.19)

For the normal components [1],

\[ ih \frac{\delta \Theta}{\delta \tau} = \mathcal{H}_m^\perp \Theta + \frac{4\pi G}{\sqrt{h}} \left( \mathcal{H}_m^\perp \right)^2 \Theta + ih 4\pi G \frac{\delta}{\delta \tau} \left( \mathcal{H}_m^\perp \right) \Theta. \]  

(B.20)
2.4 Quantum Gravity Corrections and Unitarity vs. Non-Unitarity

The second correction term in (2.19) is pure imaginary. This may lead to a ‘complex’ situation. From the Wheeler–DeWitt equation, we get [1, 3]

$$\frac{1}{M} \frac{\delta}{\delta h_{ij}} \left( \Psi^* \frac{\delta}{\delta \tilde{h}_{kl}} \Psi \right) + \frac{1}{\sqrt{\hbar}} \frac{\delta}{\delta \phi} \left( \Psi^* \frac{\delta}{\delta \phi} \Psi \right) = 0.$$  \hspace{1cm} (B.21)

Applying the \(M\) expansion, at the level of the corrected Schrödinger equation we obtain

$$0 = \frac{\delta}{\delta \tau} (\Theta^* \Theta) + \frac{\hbar}{2i \sqrt{\hbar}} \frac{\delta}{\delta \phi} \left( \Theta^* \frac{\delta}{\delta \phi} \Theta \right) - \frac{i \hbar}{2M} \frac{\delta}{\delta h_{ij}} \left( \Theta^* \frac{\delta}{\delta h_{kl}} \psi \right) - \frac{1}{K \delta h_{ij}} \psi \frac{\delta}{\delta h_{kl}} \Theta \right]. \hspace{1cm} (B.22)$$

It is the term between square brackets that is of interest, being proportional to \(M^{-1}\). Functionally integrating this equation over the field \(\psi\) and assuming that \(\Theta \to 0\) for large field configurations:

$$\frac{d}{dt} \int D\phi \Theta^* \Theta = 8\pi G \int D\phi \int d^3 x \psi^* \frac{\delta}{\delta \tau} \left\{ \frac{\mathcal{H}_m}{\sqrt{\hbar}} \left[ (^{(3)}R - 2\Lambda) \right] \right\} \psi. \hspace{1cm} (B.23)$$

A violation of the Schrödinger conservation law thus comes from the above-mentioned term in (2.19).

2.5 De Sitter Space and Quantum Gravity Corrections

The following is a summary of a rather more detailed explanation presented in [3]:

- Apply a conformal transformation and write the 3-metric as \(h_{ij} = h^{1/3} \tilde{h}_{ij}\).
- The Hamilton–Jacobi equation becomes

$$- \frac{3 \sqrt{\hbar}}{16} \left( \frac{\delta S_0}{\delta \sqrt{\hbar}} \right)^2 + \tilde{h}_{ik} \tilde{h}_{jl} \frac{\delta S_0}{\delta \tilde{h}_{kl}} = 2\sqrt{\hbar} \left[ (^{(3)}R - 2\Lambda) \right] = 0. \hspace{1cm} (B.24)$$

- Set \(^{(3)}R = 0\) and look for a solution of the form \(S_0 = S_0(\sqrt{\hbar})\), viz.,

$$S_0 = \pm 8 \sqrt{\frac{\Lambda}{3}} \int \sqrt{\hbar} d^3 x \equiv \pm 8 H_0 \int \sqrt{\hbar} d^3 x. \hspace{1cm} (B.25)$$
- The local time parameter in configuration space is

\[
\frac{\delta}{\delta \tau} = -\frac{3}{8} \frac{\sqrt{h} \delta S_0 \delta}{\sqrt{h} \delta \sqrt{h}} = \sqrt{3h} \frac{A}{\sqrt{h}}. \tag{B.26}
\]

- From the conservation law ‘chosen’ for \( K \), it follows that \( \delta K / \delta \tau = 0 \), i.e., it is constant. Then the functional Schrödinger equation is (setting \( \bar{\hbar} = 1 \))

\[
i \dot{\Theta} = \int d^3x \left[ -\frac{1}{2a^3} \frac{\delta^2}{\delta \phi^2} + \frac{a}{2} (\nabla \phi)^2 + \frac{a^3}{2m^2} \phi^2 \right] \Theta. \tag{B.27}
\]

- Use the Gaussian ansatz

\[
\Theta \equiv \tilde{N}(t) \exp \left[ -\frac{1}{2} \int \tilde{k} \tilde{\Omega}(\tilde{k}, t) \tilde{\chi}(\tilde{k}) \tilde{\chi}(\tilde{k}) \right].
\]

Two equations follow, but the relevant one is

\[
y'' + \frac{2a'}{a} y' + (m^2 a^2 + k^2) y = 0, \tag{B.28}
\]

where \( \tilde{\Omega} = -\frac{1}{a^3} \frac{\dot{\phi}}{\phi} \), and primes denote differentiation with respect to \( \tilde{\eta} \), a conformal time coordinate.

- To the next order, there are corrections to the Schrödinger equation:

\[
i \hbar \frac{\delta \Theta}{\delta \tau} = \mathcal{H}_\perp \Theta - \frac{2\pi G}{\sqrt{h} A} (\mathcal{H}_\perp)^2 \Theta - i \hbar \frac{2\pi G}{A} \frac{\delta}{\delta \tau} \left( \mathcal{H}_\perp \sqrt{h} \right) \Theta. \tag{B.29}
\]

The second correction term is a source of non-unitarity. It produces an imaginary contribution to the energy density of the order of magnitude \( 29G \hbar^2 H_0^3 / 480\pi V c^5 \) (reinstating \( c \)), indicating a possible instability of the system whose associated time scale is \( (H_0^{-1} / t_{pl})^3 \). The first correction term in (B.29) causes a shift

---

4 This allows one to recover \( \frac{\partial a^3}{\partial t} = \int d^3y \frac{\delta \sqrt{h}(x)}{\delta \phi(y)} = 3H_0 a^3 \rightarrow a(t) = e^{H_0 t} \).

5 The momentum representation \( \phi(x) = \int \frac{d^3k}{(2\pi)^3} \psi(k)e^{ikx} \) is employed. Gaussian states are used to describe generalised vacuum states. The special form here is due to the Hamiltonian being quadratic, and the Gaussian form is preserved in time.

6 Proceeding from \( i \dot{\Omega} = \frac{\Omega^2}{a^3} - a^3 \left( m^2 + k^2 \right) \).

7 We have \( dt = ad\tilde{\eta} \rightarrow a(\tilde{\eta}) = -\frac{1}{H_0 \tilde{\eta}}, \quad \tilde{\eta} \in (0, -\infty) \).

8 The prediction can be made more concrete through the following elements:
\[ \langle \mathcal{H}_m^\perp \rangle \rightarrow \langle \mathcal{H}_m^\perp \rangle - \frac{2\pi G}{3a^3 H_0^2} \langle (\mathcal{H}_m^\perp)^2 \rangle \rightarrow \langle \mathcal{H}_m^\perp \rangle - \frac{841}{1382400\pi^3} G h H_0^6 a^3 . \]  
\hspace{2cm} \text{(B.30)}

### 2.6 The Bunch–Davies Vacuum and Quantum Cosmology

Take the simplified Gaussian ansatz (see above)

\[ \tilde{\Psi} = \tilde{N}(t)e^{-\tilde{\Omega}(t)f_p^2/2} , \]  
\hspace{2cm} \text{(B.31)}

and insert it in (2.33) to get equations for \( \tilde{N} \) and \( \tilde{\Omega} \):

\[ i\dot{\tilde{N}} = \frac{\tilde{\Omega}^2}{2a^3} , \]  
\hspace{2cm} \text{(B.32)}

\[ i\dot{\tilde{\Omega}} = \frac{\tilde{\Omega}^2}{a^3} - a^3 \left( m^2 + \frac{n^2}{a^2} \right) . \]  
\hspace{2cm} \text{(B.33)}

Using

\[ \tilde{\Omega} \equiv -ia^3 \frac{\dot{y}}{y} \equiv -ia^2 \frac{y'}{y} , \]  
\hspace{2cm} \text{(B.34)}

and the conformal time \( \eta \), with \( dt = a d\eta \), we get

\[ y'' - \frac{2}{\eta} y' + \left( \frac{m^2}{H^2 \eta^2} + n^2 \right) y = 0 , \]  
\hspace{2cm} \text{(B.35)}

where we take \( a(\eta) = -1/H \eta \). We can obtain [3]

\begin{itemize}
  \item Note that \( \langle (\mathcal{H}_m^\perp)^2 \rangle \) is divergent, rather like \( \langle T_{\mu\nu}(x) \rangle \), i.e., a four-point correlation function.
  \item But use the Bunch–Davies vacuum state (an adiabatic vacuum state) to compute the expectation value [8]. The Bunch–Davies vacuum arises as a particular solution of (B.28). It is de Sitter SO(3,1) invariant, reducing to the Minkowski vacuum at early times, i.e., \( \tilde{\Omega} \rightarrow \sqrt{k^2 + m^2} \) as \( t \rightarrow -\infty \), where the metric is essentially static and one can put \( \dot{\tilde{\Omega}} = 0 \) and \( a = 1 \) in \( i\tilde{\Omega} \). Then

\[ \langle \mathcal{H}_m^\perp \rangle \approx \frac{29h H_0^4 a^3}{960\pi^2} . \]

The fluctuation of the Bunch–Davies state will be (quantum mechanically) zero, and

\[ \langle (\mathcal{H}_m^\perp)^2 \rangle \approx \langle \mathcal{H}_m^\perp \rangle^2 . \]
\end{itemize}
\[\tilde{\Omega} \simeq -\frac{i}{\eta^2 H^2 \left(\frac{3-2v}{2\eta} + \frac{n Z_{v-1}}{Z_v}\right)} \longrightarrow -\frac{i}{\eta^2 H^2 \left(\frac{m^2}{3\eta H^2} + \frac{n Z_{v-1}}{Z_v}\right)} \longrightarrow \frac{n^2 a^2}{n^2 + a^2 H^2} (n + iaH) + i m^2 a^3, \quad (B.36)\]

where we have used the usual inflationary limit \(m^2/H^2 \ll 9/4, v \simeq 3/2 - m^2/3H^2\), not choosing a real Bessel function for \(Z\) so that the Gaussian can be normalized. We take complex Hankel functions. If we choose a de Sitter invariant, i.e., invariant under SO(3,1), this reduces to the Minkowski vacuum at early times, i.e., \(\tilde{\Omega}\) tends to \(\sqrt{n^2 + m^2}\) as \(t \rightarrow -\infty\), where the metric is essentially static and one can put \(\dot{\tilde{\Omega}} = 0\) and \(a = 1\). This is the Bunch–Davies vacuum.

**Problems of Chap. 3**

### 3.1 SUSY Breaking Conditions

Take a state \(|f\rangle\) for which (see Chap. 3 of Vol. I)

\[\langle f | P_0 | f \rangle \sim \frac{1}{4} \|S_A | f \rangle\|^2 + \frac{1}{4} \|S_A^\dagger | f \rangle\|^2 \geq 0. \quad (B.37)\]

If \(|f\rangle\) is invariant under all SUSY generators, i.e., \(S_A | f \rangle = 0\), then necessarily \(\langle f | P_0 | f \rangle = 0\). Conversely, if \(\langle f | P_0 | f \rangle > 0\), not all \(S_A\) and \(S_A^\dagger\) can annihilate the state \(|f\rangle\). This non-invariance of the vacuum state implies that a spontaneous breaking of the (super)symmetry has occurred.

### 3.2 No-Scale SUGRA

An interesting Kähler class of potentials are those for which we obtain *no-scale* SUGRA (extracted from [9–12] within the SUGRA limits of string theory), by which we mean that the effective potential for the hidden sector is flat (i.e., constant), wherein the Planck mass is the only input mass, to fix features such as the hierarchy problem, with subsequent non-gravitational radiative corrections to fix the degeneracy:

- A trivial situation is described by

\[G = -3 \ln (\phi + \phi^*) \implies V = 0, \quad \forall \phi,\]

where \(\phi\) is the hidden sector scalar. This simple solution has a simple flat potential (no scale) model, where the potential \(V\) vanishes identically, making the vacuum expectation value \(\langle \phi \rangle\) arbitrary, with an arbitrary value for the mass.
of the gravitino (and the cosmological constant). These models arise naturally in (super)string compactifications, where the size of the compact dimensions is not fixed \[13\]. Changing them thus costs no energy and a moduli space can be defined. They correspond to Re$\phi$. In a (SUGRA) extension of the standard model, the visible sector (of the chiral multiplets) is connected to a SUSY breaking (hidden) sector. This is all very model dependent.

- An extension is

$$G = -3 \ln \left( \phi + \phi^* - \varphi_r^* \varphi_r \right) + \ln |W|^2, \quad W \sim d_{pqr} \varphi_p \varphi_q \varphi_r, \quad \varphi_r \sim \mathcal{T}_{ars} \varphi_s,$$

where $\phi$ is now in the visible sector and $\varphi$ is in the hidden sector. With $d_a \equiv G^r \mathcal{T}_{ars} \varphi_s$, this leads to

$$V \sim \left( \phi + \phi^* - \varphi_r^* \varphi_r \right)^{-2} \left| \frac{\partial W}{\partial \varphi_r} \right|^2 + \frac{1}{2} \text{Re}(d_a d_b),$$

whose minimum requires

$$\frac{\partial W}{\partial \varphi_r} = d_a = 0, \quad \forall r, a \quad \Rightarrow \quad V_{\text{min}} = 0.$$

### 3.3 Finite Energy State Degeneracy

On the one hand, from Exercise 3.1, the energy vanishes only if $S|f\rangle = S^\dagger|f\rangle = 0$. On the other hand, let us take a state $f'$, associated with a given energy. Since $S$ commutes with the Hamiltonian and cannot change the energy of such a state, with $S|f'\rangle = \ell|f'\rangle$, we have $0 = S^2|f'\rangle = \ell S|f'\rangle = \ell^2|f'\rangle$ leading to $S|f'\rangle$. Hence, if a state $|f\rangle$ is unique, it must satisfy $S|f\rangle = S^\dagger|f\rangle = 0$, have zero energy, and necessarily be the ground state (see Note 3.6 and \[14–19\]).

### 3.4 Pair States

Notice that the energy eigenvalues of both $\mathcal{H}_1$ and $\mathcal{H}_2$ are positive semi-definite, i.e., $E_n^{(1,2)} \geq 0$ [see (3.40) and \[20\]):

$$E = \langle \psi | \mathcal{H} | \psi \rangle \sim \langle S \psi | S \psi \rangle + \langle S^\dagger \psi | S^\dagger \psi \rangle \geq 0.$$

The Schrödinger equation for $\mathcal{H}_1$ is

$$\mathcal{H}_1 \psi_n^{(1)} = \mathcal{A}^\dagger \mathcal{A} \psi_n^{(1)} = E_n^{(1)} \psi_n^{(1)}.$$  \hspace{1cm} \text{(B.38)}

Multiplying both sides from the left by $\mathcal{A}$, we get

$$\mathcal{H}_2 \left( \mathcal{A} \psi_n^{(1)} \right) = \mathcal{A} \mathcal{A}^\dagger \mathcal{A} \psi_n^{(1)} = E_n^{(1)} \left( \mathcal{A} \psi_n^{(1)} \right).$$  \hspace{1cm} \text{(B.39)}
Similarly, the Schrödinger equation for $A_2$, viz.,

$$A_2 \psi_2^{(2)} = A A^\dagger \psi_2^{(2)} = E_2^{(2)} \psi_2^{(2)},$$

implies

$$H_1 \left( A^\dagger \psi_2^{(2)} \right) = A^\dagger A A^\dagger \psi_2^{(2)} = E_2^{(2)} \left( A^\dagger \psi_2^{(2)} \right).$$

We may argue as follows:

- If $A \psi_0^{(1)} \neq 0$, the argument goes through for all the states including the ground state, and hence all the eigenstates of the two Hamiltonians are paired, i.e., they are related by

$$E_n^{(2)} = E_n^{(1)} > 0,$$

$$\psi_n^{(2)} = \left[ E_n^{(1)} \right]^{-1/2} A \psi_n^{(1)},$$

where $n = 0, 1, 2, \ldots$.

- If $A \psi_0^{(1)} = 0$, then $E_0^{(1)} = 0$ and this state is unpaired, while all other states of the two Hamiltonians are paired. It is then clear that the eigenvalues and eigenfunctions of the two Hamiltonians $H_1$ and $H_2$ are related by

$$E_n^{(2)} = E_{n+1}^{(1)}, \quad E_0^{(1)} = 0,$$

$$\psi_n^{(2)} = \left[ E_{n+1}^{(1)} \right]^{-1/2} A \psi_{n+1}^{(1)},$$

$$\psi_{n+1}^{(1)} = \left[ E_n^{(2)} \right]^{-1/2} A^\dagger \psi_n^{(2)},$$

where $n = 0, 1, 2, \ldots$.

Note also the following points:

- Adequate state wave functions must vanish at $x = \pm \infty$. From the above discussion and (3.42), one cannot have both $A \psi_0^{(1)} = 0$ and $A^\dagger \psi_0^{(2)} = 0$. Only one of the two ground state energies can be zero.

9 Note that one would have

$$\psi_2^{(2)} \simeq \exp \left[ \frac{\sqrt{2m}}{h} \int^x W(y)dy \right].$$
For $\mathcal{A}\psi_0^{(1)} = 0$, since the ground state wave function of $\mathcal{H}_1$ is annihilated by the operator $\mathcal{A}$, this state has no SUSY partner. Knowing all the eigenfunctions of $\mathcal{H}_1$, we can determine the eigenfunctions of $\mathcal{H}_2$ using the operator $\mathcal{A}$. And conversely, using $\mathcal{A}^\dagger$, we can reconstruct all the eigenfunctions of $\mathcal{H}_1$ from those of $\mathcal{H}_2$ except for the ground state.

For (3.42), the normalizable zero energy ground state is associated with $\mathcal{H}_1$, and $W$ is positive (negative) for large positive (negative) $x$, implying a zero fermion number.

If we cannot find normalizable functions like (3.42) and (B.48), then $\mathcal{H}_1$ does not have a zero eigenvalue, and SUSY is broken. For $W$ of the form $wx^n$, if $w$ is positive and $n$ odd, we get a normalizable state, but not the reverse, i.e., for $w$ negative and $n$ even. With $W_\pm \equiv \lim_{x \to \pm\infty} W$, if $\text{sign}(W_+) = \text{sign}(W_-)$, SUSY is broken, whereas if $\text{sign}(W_+) = -\text{sign}(W_-)$, SUSY remains intact, i.e., we can also write

$$\Delta = \frac{1}{2} \left[ \text{sign}(W_+) - \text{sign}(W_-) \right].$$

The Witten index does not depend on the details of the superpotential being invariant under deformations that maintain their asymptotic behavior, something that is further discussed in the context of topological invariance.

If SUSY remains intact, and if the potential $V(x)$ provides an exactly solvable situation with $n$ bound states and ground state energy $E_0$, one can extend to a $V_1(x) = V(x) - E_0$ setting, whose ground state energy is zero. One can obtain all the $n - 1$ eigenstates of $\mathcal{H}_2$, then obtain all the $n - 2$ eigenstates of $\mathcal{H}_3$. In this way, new exactly solvable settings are possible.

Problems of Chap. 4

4.1 $N=1$ SUGRA Action, SUSY Transformations, and Time Gauge

Let us for this discussion rewrite the $N = 1$ SUGRA as (see [21–23] for more details)

$$S_2 = \frac{1}{2k^2} \int_M d^4 x \, e R,$$

$$S_{3/2} = S_{3/2}' + \tilde{S}_{3/2}'$$

(B.49)
\[
\int_{\mathcal{M}} d^4x \, \varepsilon_{\mu \nu \rho \sigma} \left( \nabla^\mu e_{A A'}^\nu D^\rho \psi^A_\sigma - \psi^A_\mu e_{A A'}^\nu D^\rho \nabla^\sigma e_{A A'}^\nu \right), \tag{B.50}
\]

\[
S^B_2 = \frac{1}{k^2} \int_{\partial \mathcal{M}} d^3x \, \hat{e} K , \tag{B.51}
\]

\[
S^B_{3/2} = \frac{1}{2} \int_{\partial \mathcal{M}} d^3x \, \varepsilon^{ijk} \psi_{A_i} e_{A A'} e_{A A'}^j \nabla^A_\sigma \nabla^A_\tau , \tag{B.52}
\]

where \( e = \text{det} \left[ e^a_\mu \right] \), \( \hat{e} = \text{det} \left[ \hat{e}^a_\mu \right] \), \( \hat{a} = 1, 2, 3 \), and the integration is over a space-time manifold \( \mathcal{M} \) with boundary \( \partial \mathcal{M} \). The boundary terms, labelled by the subscript \( B \), are necessary for dealing with quantum amplitudes of transitions and also to obtain the correct amplitudes, with prescribed data, leading to classical solutions, i.e., with \( \delta S = 0 \) providing the classical field equations \([24, 25]\).

Recall that the curvature scalar \( R \) is given in terms of the tetrad field \( e \) and the spin connection \( \omega_{ab}^\mu \) by

\[
R(e, \omega) = e_a^\mu e_b^\nu R_{\mu \nu}^{ab}(\omega) , \tag{B.53}
\]

that is,

\[
R_{\mu \nu}^{ab}(\omega) = \partial_\mu \omega_{ab}^\nu - \partial_\nu \omega_{ab}^\mu + \omega_{ac}^\mu \omega_c^b \nu - \omega_{ac}^\nu \omega_c^b \mu . \tag{B.54}
\]

The variation is then

\[
\delta S(e, \psi, \overline{\psi} \omega(e, \psi, \overline{\psi})) = \delta e \frac{\delta S}{\delta e} \left( e_\psi, \overline{\psi} \omega \right) + \delta \psi \frac{\delta S}{\delta \psi} \left( e_\psi, \overline{\psi} \omega \right) + \delta \overline{\psi} \frac{\delta S}{\delta \overline{\psi}} \left( e_\psi, \overline{\psi} \omega \right) + \frac{\delta \omega(e, \psi, \overline{\psi})}{\delta e} \delta e , \tag{B.55}
\]

with the help of the chain rule. It is useful to recall that \( \omega_{ab}^\mu \) satisfies its own field equations, so one can drop the last term in (B.55) after inserting \( \omega_{ab}^\mu(e, \psi, \overline{\psi}) \) into the action. Let us then vary the SUGRA action with respect to the tetrad and the gravitinos. In list form, we have the following points:

- Take

\[
\delta e = e \delta (\ln \text{det} \left[ e^a_\mu \right]) = e \delta (\text{Tr} \ln \left[ e^a_\mu \right]) = e e_a^\mu \delta e^a_\mu , \tag{B.56}
\]

where

\[
\delta e_a^\nu = - (\delta e^b_\mu) e_b^\nu e_a^\mu . \tag{B.57}
\]
It follows that
\[
\delta(eR) = (\delta e) R + e(\delta R) = e \left[ e_a^\mu (\delta e^\nu_\mu) R + \delta(e_a^\mu e_b^\nu) R^{ab}_{\mu\nu} \right] = e \left[ (\delta e^a_\mu) e_a^\mu R + 2 e_a^\mu (\delta e_b^\nu) R^{ab}_{\mu\nu} \right] = e \delta e_a^\mu (e_a^\mu R - 2 e_a^\nu e_b^\mu e_c^\rho R^{cb}_{\rho\nu}) .
\] (B.58)

In addition,
\[
\delta(\hat{e}K) = \hat{e} \left[ \delta K + \hat{e}^{-1} (\delta \hat{e}) K \right] = \hat{e} \left[ \omega^{0\hat{a}} i \delta e_{\hat{a}}^i + e_{\hat{b}}^i (\delta e_{\hat{b}}^j) \omega^{0\hat{a}} i e_{\hat{a}}^j \right] = \hat{e} \omega^{0\hat{a}} i (e_{\hat{a}}^j e_{\hat{b}}^i - e_{\hat{a}}^i e_{\hat{b}}^j) \delta e_{\hat{b}}^j .
\] (B.59)

Note that we have used the time gauge here, and we are assuming it to be preserved under the SUSY transformations\(^{10}\) (discussed at length in Exercises 5.1–5.3).\(^{11}\)

Moreover\(^{12}\)
\[
\delta S'_{3/2} = \frac{1}{2} \int_M d^4 x \varepsilon^{\mu\nu\rho\sigma} \left[ (\delta \bar{e}A')_\mu e_{AA'}_\nu D_\rho \psi^A_\sigma + \bar{e}A'_{\mu} (\delta e_{AA'}_\nu) D_\rho \psi^A_\sigma \right] = \frac{1}{2} \int_M d^4 x \varepsilon^{\mu\nu\rho\sigma} \left( 2 k^{-1} D_\mu \bar{e}A' + \bar{A}A'_{B'} B' \psi^B_\mu e_{AA'}_\nu D_\rho \psi^A_\sigma \right) + \text{a volume integral}
\]
\[
= \frac{1}{2} \int_M d^4 x \varepsilon^{\mu\nu\rho\sigma} \left[ 2 k^{-1} \partial_\mu (\bar{e}A' e_{AA'}_\nu D_\rho \psi^A_\sigma) - 2 k^{-1} \bar{e}A' D_\mu (e_{AA'}_\nu D_\rho \psi^A_\sigma) \right] + \text{a volume integral ,}
\]
\[
= \frac{1}{k} \int_{\partial M} d^3 x \varepsilon^{ijk} \bar{e}A'_i e_{AA'} e_{A'A} D_j \psi^A_k + \text{a volume integral} .
\] (B.60)

\(^{10}\) \(\varepsilon^{\mu\nu\rho\sigma} D_\rho D_\sigma \bar{e}A' = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} [D_\rho, D_\sigma] \bar{e}A' = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{R}A'_{B'} B'_{\rho\sigma} \bar{e}B'.\)

\(^{11}\) In brief, the usual SUGRA action is extended by the boundary terms, namely (B.52). The new feature are also the \(A^A_B\) and \(\bar{A}^A'_{B'}\) terms in (B.60) and (B.61), which correspond to the Lorentz transformation terms.

\(^{12}\) \(D_\mu\) agrees with \(\partial_\mu\) when it acts on objects with no free spinor indices.
\[
\delta S_{3/2} = -\frac{1}{2} \int_M d^4x \epsilon^{\mu \nu \rho \sigma} \left[ \psi^A_{\mu} (\delta e_{AA'})_{\mu} D_\rho \bar{\psi}^{A'}_{\sigma} + \psi^A_{\mu} e_{AA'} \mu D_\rho (\delta \bar{\psi}^{A'}_{\sigma}) \right] \\
= -\frac{1}{2} \int_M d^4x \epsilon^{\mu \nu \rho \sigma} \psi^A_{\mu} e_{AA'} \mu D_\rho (2k^{-1} D_\sigma \bar{e}^{AA'} + \bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_\sigma) \\
+ \text{a volume integral} \\
= -\frac{1}{2} \int_M d^4x \epsilon^{\mu \nu \rho \sigma} \psi^A_{\mu} e_{AA'} \mu D_\rho (\bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_\sigma) + \text{a volume integral} \\
= -\frac{1}{2} \int_M d^4x \epsilon^{\mu \nu \rho \sigma} \left[ \partial_\rho (\psi^A_{\mu} e_{AA'} \mu \bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_\sigma) - D_\rho (\psi^A_{\mu} e_{AA'} \mu \bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_\sigma) \right] \\
+ \text{a volume integral} \\
= -\frac{1}{2} \int_{\partial M} d^3x \epsilon^{ijk} \psi^A_{i} e_{AA'} \mu \bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_k + \text{a volume integral}. \quad (B.61)
\]

- Let us address two questions the reader may be wondering about:
  
  - Note that we are taking left-handed SUSY transformations [25], i.e., \( \epsilon = 0 \), which could be the setup for the amplitude to go from prescribed data \((e^A_{iAA'}, \bar{\psi}_A^j)\) on an initial surface to data \((e^A_{iAA'}, \bar{\psi}_A^j)\) on a final surface. The gravitinos filling in between \((e^A_{iAA'}, \bar{\psi}_A^j)\) on an initial surface and data \((e^A_{iAA'}, \bar{\psi}_A^j)\) on a final surface would now be independent of \((\psi^A_{i}, \psi^A_{j})\).
  
  - With \( n^\mu \) the future-pointing unit vector normal to the boundary \( \partial M \), if we assume\(^\text{13} \) that the three spacelike axes of the tetrad lie in \( \partial M \), so that

\[
n_a = e_{a\mu} n^\mu = (1, 0, 0, 0), \quad (B.62)
\]

the extrinsic curvature has the simple form

\[
K = h^{ij} n_a \omega^{ab} j e_{bi} = \omega^{a\hat{i}} i e_{\hat{i}}. \quad (B.63)
\]

The corresponding action of \( N = 1 \) SUGRA is subsequently invariant, up to a boundary term, if local supersymmetry transformations become

\[
\delta e_{AA'}^\mu \equiv -i k (\epsilon^A \bar{\psi}_{A'}^\mu + \bar{\psi}_{A'}^\mu \psi^A_{\mu}) + \Lambda^A B e_{BA'}^\mu + \bar{\Lambda}^{AA'}_B e_{AB'}^\mu \quad (B.64)
\]

\[
\delta \psi^A_{\mu} \equiv 2k^{-1} D_\mu e^A + \Lambda^A B \psi^B_{\mu} \quad , \quad (B.65)
\]

\[
\delta \bar{\psi}^{A'}_{\mu} \equiv 2k^{-1} D_\mu \bar{e}^{A'} + \bar{\Lambda}^{AA'}_B \bar{\psi}^{B'}_{\mu} \quad , \quad (B.66)
\]

\(^\text{13} \) This gauge choice is known as the time gauge.
where

\[ \Lambda^A_B \equiv -2i n^{AA'} n_{CC'} \bar{e} C^C i e_{BA'} i , \]  

(B.67)

and its Hermitian conjugate

\[ \bar{\Lambda}^{A'}_{B'} \equiv -2i n^{AA'} n_{CC'} \bar{e} C^C i e_{AB} i , \]  

(B.68)

and where \( \varepsilon^A \) and \( \bar{\varepsilon}^{A'} \) are spacetime-dependent anticommuting fields. The inclusion of the Lorentz terms in (B.64) ensures the preservation of the gauge defined in (B.62) (see Exercises 5.1–5.3).

The reader should note that the variation of the tetrad does not depend on \( D_\mu \varepsilon \) or \( D_\mu \bar{\varepsilon} \) [see (B.64)]. Thus the volume Einstein–Hilbert sector does not involve any derivatives of \( \varepsilon \) or \( \bar{\varepsilon} \), and so cannot be integrated by parts to give a surface term. Hence, since the theory is supersymmetric, its variation must cancel with the volume terms arising from the variation of the gravitino action. From the remarks following (B.64), we are only interested in surface terms, that is, we will integrate by parts to extract all surface terms in \( \delta S_{3/2} \) which involve \( D_\mu \varepsilon \) or \( D_\mu \bar{\varepsilon} \).

Collecting equations (B.59), (B.60), (B.61), with the volume integral terms vanishing in this SUSY context, the total variation under these local supersymmetry transformations is

\[ \delta S = \frac{1}{k} \int_{\partial \mathcal{M}} d^3 x \varepsilon^{ijk} \varepsilon^{A} e_{AA'} i (D_j \psi^A k) - \frac{1}{2} \int_{\partial \mathcal{M}} d^3 x \varepsilon^{ijk} \bar{\Lambda}^{A'} B' i e_{AA'} j \psi^A k + \frac{1}{k^2} \int_{\partial \mathcal{M}} d^3 x \bar{\omega}_0 (e_{\bar{a}}^i e_{\bar{b}}^i - e_{\bar{a}}^i e_{\bar{b}}^j) \delta e^\bar{b} j , \]  

(B.69)

provided that the gauge condition (B.62) is preserved.

Finally, note the generic expression [25] retrieved under left-handed SUSY transformations:

\[ \delta S = \frac{2}{k} \int_{\partial \mathcal{M}} d^3 x \varepsilon^{ijk} \varepsilon^{A'} e_{AA'} i (3s) D_j \psi^A k , \]  

(B.70)

where \((3s) D_j \) is the torsion-free spatial covariant derivative.

4.2 \( S_0 \) Must Depend on the Gravitino \( \psi^A i \)

With the SUSY constraints and the ansatz

\[ \Psi = \exp \left[ i (S_0 G^{-1} + S_1 + S_2 G + \cdots) / \hbar \right] , \]

obtain to lowest order \( G^0 \),
This must hold for arbitrary fields $\psi^A_i$ and $e_{iA}'$. Otherwise we would get the condition $\varepsilon^{ijk} e_{AA'i} (3s) D_j \psi^A_k + 4\pi i \psi^A_i \frac{\delta S_0}{\delta e_{iA}'} = 0$, which cannot hold for all fields.

Assume then that $S_0$ does not depend on the gravitino field $\psi^A_i$. Integrating (B.71) over space with an arbitrary continuous spinorial test function $\varepsilon^A(x)$ leads to

$$I_0 = \int d^3x [\varepsilon^{ijk} e_{AA'i} (3s) D_j \psi^A_k + 4\pi i \psi^A_i \frac{\delta S_0}{\delta e_{iA}'}] = 0.$$  

(B.72)

With the replacement $\psi^A_i \mapsto \psi^A_i \exp \{\phi(x)\}$ and $\bar{e}^A(x) \mapsto \bar{e}^A(x) \exp \{-\phi(x)\}$, this becomes [26]

$$I'_0 = \int d^3x [\varepsilon^{ijk} e_{AA'i} (3s) D_j (\psi^A_k \exp \phi) + 4\pi i \psi^A_i \frac{\delta S_0}{\delta e_{iA}'} \exp \phi] = 0.$$  

(B.73)

Now for $\Delta I_0 = I_0 - I'_0 = 0$.

$$\Delta I_0 = \int d^3x [\varepsilon^{ijk} e_{AA'i} (x) \bar{e}^A(x) \psi^A_k (x) \partial_j \phi(x)] = 0,$$  

(B.74)

which cannot hold for all fields.

To higher orders, the calculation for $n \geq 1$ leads to

$$[\Psi]^{-1} S_{AA'} \Psi \overset{O(G^n)}{=} -4\pi i G^n \psi^A_i \frac{\delta S_n}{\delta e_{iA}'} = 0.$$  

(B.75)

So can we choose $S_n$ not to depend on the bosonic field $e_{iA}'$? This would be very restrictive: no proper bosonic limit would exist. We therefore dismiss this hypothesis. Hence, to satisfy (B.75), it is necessary to introduce a dependence on the gravitino field at each order. This means that we must have $S^n \equiv S^n[e, \psi]$ for all $n$ [27].

### 4.3 Obtaining Equation (4.13)

Write the fermionic part of the last line as
\[-\frac{1}{2} \hbar (3s) D_l \left( \epsilon^{ijk} \psi_{A_j} D^B A'^k \frac{\delta}{\delta \psi^B_l} \right) = \frac{1}{2} \epsilon^{ijk} (3s) D_l (\psi_{A_j} \overline{\psi}^{A'k}) \]
\[= \frac{1}{2} \epsilon^{ijk} \left[ (3s) D_l \psi_{A_j} \right] \overline{\psi}^{A'k} + \psi_{A_j} \left[ (3s) D_l \overline{\psi}^{A'k} \right] \]
\[= \frac{1}{2} \epsilon^{ijk} \left[ (3s) D_l \psi_{A_k} \right] \overline{\psi}^{A'i} - \psi_{A_l} \left[ (3s) D_l \overline{\psi}^{A'k} \right] \, . \]

(B.76)

Comparing the above with the third line written in the form
\[\frac{1}{2} \epsilon^{ijk} \left( (3s) D_j \psi_{A_k} \right) \overline{\psi}^{A'i} + \psi_{A_l} \left( (3s) D_j \overline{\psi}^{A'k} \right) \, , \]

(B.77)

the terms containing \((3s) D_j (\overline{\psi}^{A'k})\) will cancel out. Finally, the normal projection of the remaining term containing \((3s) D_j \psi^A_k\) is given by
\[n^{AA'} i \hbar \epsilon^{ijk} \left[ (3s) D_j \psi_{Ak} \right] D^B A'^i \frac{\delta}{\delta \psi^B_l} \frac{h}{\sqrt{\hbar}} \epsilon^{ijk} e^A C^l \left[ (3s) D_j \psi_{Ak} \right] \frac{\delta}{\delta \psi^B_l} \, . \]

(B.78)

4.4 Decomposition of the Hamilton–Jacobi Equation into Bosonic and Fermionic Sectors

For the WKB wave functional, write
\[\Psi[e, \psi] \equiv \exp \left( \frac{i}{\hbar} B_0 G^{-1} \right) \exp \left[ \frac{i}{\hbar} (F_0 G^{-1} + S_1 + \cdots) \right] \, . \]

(B.79)

Then select the pure bosonic part \(B_0\) satisfying
\[4\pi i n^{AA'} D_{ij}^{BB'} \frac{\delta B_0}{\delta e_{ij}^{AB'}} \frac{\delta B_0}{\delta e_{ij}^{BA'}} + U = 0 \, . \]

(B.80)

i.e., corresponding to the Hamilton–Jacobi equation (4.22). This solution \(B_0\) can then be used to determine the condition for the part \(F_0\) involving fermions:

\[0 = 4\pi i \left( \psi^B \frac{\delta F_0}{\delta e_{ij}^{AB'}} \frac{\delta F_0}{\delta e_{ij}^{BA'}} + \psi^B \frac{\delta B_0}{\delta e_{ij}^{AB'}} \frac{\delta B_0}{\delta e_{ij}^{BA'}} \right) + \frac{i}{\sqrt{\hbar}} \epsilon^{ijk} e^A C^l \left[ (3s) D_j \psi_{Ak} \right] \frac{\delta F_0}{\delta \psi^B_l} \, . \]

(B.81)
If a solution $S_0$ of (4.20) is to be found [omitting the term (4.26)], together with $F_0 = S_0 - B_0$, then the above equation is satisfied.

### 4.5 The DeWitt Supermetric $G^{(s)}_{ab}$ Should Contain the DeWitt Metric $G_{ijkl}$

Take arbitrary ‘vectors’

$$v^a = \left[ \begin{array}{c} B^l_{AB} \\ F_D^l \end{array} \right], \quad \tilde{v}^b = \left[ \begin{array}{c} \tilde{B}^i_{BA'} \\ \tilde{F}_C^k \end{array} \right],$$

and obtain (see Chap. 4)

$$G^{(s)}_{ab} v^a \tilde{v}^b = -4\pi i \left( n_{AB} D_{ij}^{BB'} + n_{BB'} D^{AA'}_{ij} \right) B_{AB'}^i \tilde{B}_{BA'}^i + 4\pi i n_{BB'} \psi_j^{C} e^{ijk} D_C^{A'}_D F_D^l \tilde{B}_{BA'}^i + 4\pi i n_{AA'}^{A'} \psi_i^{B} e^{ilm} D_B^{B'} D_{m}^{C} A'_{kl} B_{AB'}^i \tilde{F}_C^k .$$

(B.83)

Let

$$B_{AB'}^i = \frac{\delta a}{\delta e^{AB'}}, \quad \tilde{B}_{BA'}^i = \frac{\delta b}{\delta e^{BA'}} ,$$

where $a[e]$ and $b[e]$ are two arbitrary functionals (that can also be written as $a[h_{ij}]$ and $b[h_{ij}]$). Then for the first line on the right-hand side of (B.83) (see Appendix A),

$$4\pi i \left( n_{AB} D_{ij}^{BB'} \frac{\delta a}{\delta e^{AB'}_j} \frac{\delta b}{\delta e^{BA'}_i} + n_{BB'} D^{AA'}_{ij} \frac{\delta a}{\delta e^{AA'}_j} \frac{\delta b}{\delta e^{BA'}_i} \right) = -32\pi G_{iljk} \frac{\delta a}{\delta h_{ij}} \frac{\delta b}{\delta h_{jk}} .$$

(B.84)

Therefore, for quantities that can be written in terms of the three-metric $h_{ij}$ and the gravitino, the block $B$ contains the usual DeWitt metric. To be more precise, it is not exactly the usual DeWitt metric due to the change of the fundamental bosonic field from $h_{ij}$ to $e_{ij}^{AA'}$. Rather it is a tetrad version of it.

### 4.6 Towards a Conservation Law?

Condition (4.34) can be rewritten as [27]
\[ n^{AA'} \frac{\delta}{\delta e_{AB}^j} \left( D_{ij}^{BB'} \frac{\delta S_0}{\delta e_{BA'}^i} K \right) = n^{AA'} \frac{\delta S_0}{\delta e_{AB}^j} D_{ij}^{BB'} \frac{\delta K}{\delta e_{BA'}^i} \]

\[ -\psi_i^B \frac{\delta S_0}{\delta e_{AB}^j} \epsilon^{CAB'i} \frac{\delta K}{\delta \psi_j^C} - \psi_i^B \epsilon^{ilm} D_B B_j^{mi} D^{C}_{Akl} \frac{\delta S_0}{\delta \psi_k^C} K \]

\[ - \left( \frac{3i}{\sqrt{\hbar}} \psi_k^C + \psi_j^B \epsilon_{jkl} n^{CB'} \epsilon^{BB'} \right) \frac{\delta S_0}{\delta \psi_k^C}. \quad (B.85) \]

If the right-hand side of (B.85) is equal to zero, then it can be interpreted as a conservation law (see [1] and Sect. 2.1, and also [28–31, 2, 32, 3, 4, 34]). But here, the physical context is SuperRiem(\(\Sigma\)). A conservation law can only be established in the highly unrealistic and special case of (i) a vanishing dependence of \(S_0\) and \(K\) on the gravitino and (ii) the assumption that \(S_0[e]\) and \(K[e]\) can be expressed in the form \(S_0[h_{ij}]\) and \(K[h_{ij}]\). In fact, (B.85) then implies the conservation equation

\[ n^{AA'} \frac{\delta}{\delta e_{AB}^j} \left( D_{ij}^{BB'} \frac{\delta S_0}{\delta e_{BA'}^i} K^{-2} \right) = 0. \quad (B.86) \]

However, as explained in Sect. 4.2.1 (see also Sect. 4.1), \(S_0\) ought to depend on the gravitino [27].

**Problems of Chap. 5**

**5.1 Time Gauge and SUSY Transformations**

The variation of the supergravity action can induce boundary terms. Focusing on extracting a finite quantum amplitude, we aim at exact invariance by adding boundary correction terms. This has so far been impossible for full supergravity, but for homogeneous Bianchi A models, the precise invariance of the action under the subalgebra can be restored by adding an appropriate boundary correction to the action; more precisely, under the subalgebra of left-handed SUSY generators (see [35, 21, 36, 22, 23]). Let us take a non-diagonal Bianchi class A ansatz in the time gauge. In order to preserve this gauge, the transformation laws must be modified by adding a Lorentz generator term to the supersymmetry generators. In fact, the time gauge represents a boundary fixing property (see Exercise 4.1). To preserve it, all bosonic generators must be symmetries of the boundaries, i.e., not induce translations normal to it. Any anticommutator of any fermionic generators should induce bosonic generators tangential to the boundary. Consequently, the tetrad axes must remain within a 3D boundary, whence the gauge condition \(e^0_i = 0\) will be preserved under a supersymmetry transformation.

The general SUSY transformations will not preserve this gauge. One way forward is through a combination of SUSY transformations added to appropriate Lorentz transformations:
\[ \delta e^{AA'}_\mu = -i k \left( \varepsilon^A \bar{\psi}^{A'}_\mu + \bar{\psi}^A \psi^{A'}_\mu \right) + \Lambda^A_B e^{BA'}_\mu + \bar{\Lambda}^{A'}_{B'} e^{A'B'}_\mu , \]  

(B.87)

for some parameters \( \Lambda \) which are linear in \( \varepsilon^A \) and \( \varepsilon^A' \), such that \( \delta e^0 = 0 \) or \( n_{AA'} \delta e^{AA'} = 0 \).

A solution is

\[ \Lambda^A_B = -2i k n_{CC'} \varepsilon^{C'} \bar{e}^{BA'}_i , \]  

(B.88)

where we have \( \Lambda^{A}_A = 0 \) and \( \Lambda^{AB} = \Lambda^{BA} \). Thus the extended supersymmetry transformations are

\[ \delta e^{AA'}_\mu = -i k (\varepsilon^A \bar{\psi}^{A'}_\mu + \bar{\psi}^A \psi^{A'}_\mu) + \Lambda^A_B e^{BA'}_\mu + \bar{\Lambda}^{A'}_{B'} e^{A'B'}_\mu , \]  

(B.89)

\[ \delta \psi^A_\mu = 2k^{-1} D_\mu \varepsilon^A + \Lambda^A_B \psi^B_\mu , \]  

(B.90)

\[ \delta \bar{\psi}^{A'}_\mu = 2k^{-1} D_\mu \bar{\varepsilon}^{A'} + \bar{\Lambda}^{A'}_{B'} \bar{\psi}^{B'}_\mu . \]  

(B.91)

The addition of the Lorentz term to the SUSY transformations subsequently induces new SUSY generators in terms of the Lorentz rotation generators \( J_{BC}, J_{B'C'} \), given by

\[ S_A(\text{new}) \equiv S_A + \phi^{AB} J_{BC} , \]  

(B.92)

\[ S_A'(\text{new}) \equiv S_A' + \bar{\phi}^{A'B'C'} J_{B'C'} , \]  

(B.93)

where specifically here

\[ \phi^{AB} \equiv \phi^{(AB)} = i k^2 n_{CC'} \varepsilon^{C'} i e^A_{B'} , \]  

(B.94)

and \( \bar{\phi}^{A'B'} \) is the Hermitian conjugate of \( \phi^{AB} \).

5.2 Variation of N=1 SUGRA Action, Bianchi Models, and Boundary Terms

Let us now see what the action of SUGRA (or rather, its variation for left-handed SUSY transformations) becomes for Bianchi A models. The restriction to left-handed SUSY transformations means also that \( \varepsilon \) and \( \bar{\varepsilon} \) are no longer Hermitian conjugates of each other; \( e^{AA'}_\mu \) is no longer required to be Hermitian and \( \psi^A_\mu \) and \( \bar{\psi}^{A'}_\mu \) become independent quantities [35, 21, 36, 22, 23].

For spatially homogeneous cosmologies, (B.69) takes the form

\[ \delta S = \frac{1}{k} \int_{\partial M} d^3 x \varepsilon^{pq} \bar{e}^{A'}_{AA'} e_{AA'} (D_q \psi^{A'}_r) - \frac{1}{2} \int_{\partial M} d^3 x \varepsilon^{pq} \bar{A}^{A'}_{B'} \bar{\psi}^{B'}_p e_{AA'} \psi^A_r 
+ \frac{1}{k^2} \int_{\partial M} d^3 x h^{1/2} \omega \bar{\delta}_{pq} (e^a_{\tilde{a}} e^b_{\tilde{b}} - e^a_{\tilde{a}} e^b_{\tilde{b}}) \delta e^b_{\tilde{b}} , \]  

(B.95)
where $e^{\hat{a}}_p$, $e^{AA'}_p$, $\psi^A_p$, and $\overline{\psi}^{A'}_p$ are defined by

\begin{align}
  e^{\hat{a}}_i &\equiv e^{\hat{a}}_p(t)\omega^p_i, & e^{AA'}_i &\equiv e^{AA'}_p(t)\omega^p_i, \\
  \psi^A_i &\equiv \psi^A_p(t)\omega^p_i, & \overline{\psi}^{A'}_i &\equiv \overline{\psi}^{A'}_p(t)\omega^p_i,
\end{align}

(B.96)

with $\omega^p = \omega^p_i dx^i$. Now, after a rather lengthy calculation using

\begin{align}
  e_{a\mu} &\equiv \left( \begin{array}{cc} N & 0 \\ N' e_{\hat{a}i} & e_{\hat{a}i} \end{array} \right),
\end{align}

we obtain the following:

- For the torsion-free spin connection\(^{14}\)

\begin{align}
  (s)\omega^{\hat{a}\hat{b}}_p &\equiv 2 e^{[\hat{b}\hat{a}]}_c \hat{e}^\epsilon_d e^d_p - Q^\epsilon_d e^{\hat{a}\hat{b}}_d e^\epsilon_p, \\
  (s)\omega^{\hat{a}0}_p &\equiv \frac{1}{2N} (e^{\hat{a}}_p + e^{\hat{a}q} e^{\hat{b}}_p e^{\hat{b}q}) + \frac{Nq}{N} \left( Q^{\hat{a}\hat{b}}_c e_{\hat{b}c\hat{d}} e^\epsilon_p e^d_q - Q_c^{\hat{b}\hat{a}\hat{c}\hat{d}} e_b e^d_q \right).
\end{align}

(B.98)

(B.99)

- For the contorsion,

\begin{align}
  \kappa^{\hat{a}\hat{b}}_p &\equiv i k^2 \left( e^{[\hat{b}\hat{a}]\sigma_A A'}_{A'p} \overline{\psi}^{A'}_p [\psi^A_q] + \frac{1}{2} e^{\hat{a}q} e^{\hat{b}r} e_{AA'} e_{AA'} [\psi^A_q] \right), \\
  \kappa^{\hat{a}0}_p &\equiv i k^2 \left[ \frac{1}{N} \sigma^{\hat{a}}_{AA'} \overline{\psi}^{A'}_p [\psi^A_0] - \frac{Nq}{N} \sigma^{\hat{a}}_{AA'} + e^{\hat{a}q} e_{AA'} \right] \overline{\psi}^{A'}_p [\psi^A_q] \\
  &+ \frac{1}{N} e^{\hat{a}q} e_{AA'} e_{AA'} [\psi^A_q] - \frac{N'}{N} e^{\hat{a}q} e_{AA'} e_{AA'} [\psi^A_q],
\end{align}

(B.100) \hspace{1cm} (B.101)

- Subsequently,

\begin{align}
  (s)\omega^{AB}_p = \frac{1}{N} n^A_{A'} \sigma^{AB}_{A'p} \left[ i N \left( e^{\hat{a}p} Q^{\hat{b}}_b - 2 e_{bp} Q^{\hat{b}}_p \right) + \frac{1}{2} (e^{\hat{a}p} + e^{\hat{a}q} e^{\hat{b}q}) \\
  &+ Nq \left( Q^{\hat{a}\hat{b}}_c e_{\hat{b}c\hat{d}} e^\epsilon_p e^d_q - Q_c^{\hat{b}\hat{a}\hat{c}\hat{d}} e_b e^d_q \right) \right].
\end{align}

(B.102)

\(^{14}\) $Q^{\hat{a}\hat{b}} = Q^{(\hat{a}\hat{b})} \equiv e^{\hat{a}p} m^{pq} e^{\hat{b}q} (\det [e^{\hat{a}p}])^{-1}$.\]
\[ \kappa^{AB}_{\mu} = \frac{i k^2}{2N} \left[ N e_{CA'}^q e^{(AC)A'}_{[p]q} + \frac{N}{2} e^A_{\;\;\;A'} e^{BA' r} e_{DD'}^r \bar{\psi}_{D'}^{[q] q} \right] 
\]

\[ -n^{(A} e_{CA'}^{A')}_{[p]q} n_{A'}^{(B} e^{BA'}_{q]q} + n^{(A} e_{CA'}^{A')}_{[p]q} n_{A'}^{(B} e^{BA'}_{q]q} \right] . \tag{B.103} \]

- And last but not least,

\[ \delta e^A_{\hat{\mu}} = -\sigma^A_{\hat{\mu}} \delta e^{AA'}_{\mu} \]

\[ = i k \sigma^A_{\hat{\mu}} e^{AA'}_{\mu} + k e^{\hat{a} bc} e_b^q e_{\hat{c}}^r n^{AA'} e^{A'} \psi^A_{\mu} . \tag{B.104} \]

- Substituting (B), (B.103), and (B.104) into (B.95), we obtain

\[ \delta S = \varsigma \det [e^A_{\mu}] e^{A'} \psi^A_s \left\{ \frac{1}{k} \sigma^{AA'} e^s_{\hat{a}} Q^{\hat{a} \hat{b}} + \frac{i}{4kN} \left( e^{AA'} q e^r_{\hat{a}} e_{\hat{a}}^s + \delta^{AA'} r h^{ts} - 2 e^{AA'} s q e^q_{\hat{a}} q \right) \right. \]

\[ + \frac{i}{2kN} \left( N^p \sigma^{\hat{a} bc} e^s_{\hat{a}} e^d_{\hat{a}} q Q^{\hat{a} \hat{b} \hat{c}} e_{\hat{a} \hat{c} \hat{d}} \right) + N^p \sigma^{\hat{a} bc} e^d_{\hat{a}} q Q^{\hat{a} \hat{b} \hat{c}} e_{\hat{a} \hat{c} \hat{d}} \right) \]

\[ + \frac{k}{2} \left( n^{AA'} \chi_{pq0} h_{pq} h^{ts} - \frac{1}{2} e^{AA'} q \chi_{pq0} h^{ps} \right) \]

\[ + \frac{k}{2N} \left[ e^{AA'} q \chi_{p0q} h^{pq} - e^{AA'} q \chi_{q0r} h^{ts(r h^q p)q} \right] \]

\[ + \frac{kN^p}{2N} \left[ e^{AA'} q \chi_{p0q} h^{qr} + e^{AA'} q \chi_{pq0} h^{s(r h^q)q} \right] \right\} . \tag{B.105} \]

where

\[ \varsigma \equiv \int d^3x \det [\omega^i_{\mu}] \]

and

\[ \chi_{pq0} \equiv \bar{\psi}_{A'}^{[p]q} e_{AA'} \]

\[ \chi_{p0q} \equiv \bar{\psi}_{A'}^{[p]0} e_{AA'}q , \]

\[ \chi_{pq0} \equiv \bar{\psi}_{A'}^{[p]q} n_{AA'} \]. \tag{B.106}
5.3 Invariance of \( N=1 \) SUGRA Action, Bianchi Models, and Boundary Terms

It is possible to restore the invariance of the action of \( N=1 \) SUGRA restricted to the Bianchi class A setting, under a subalgebra of (left!) SUSY generators [35, 21, 36, 22, 23]. If we add

\[
S^B' \equiv \frac{1}{\kappa^2} \sum_{pq} m^{pq} h_{pq} , \quad W_\psi \equiv i \varepsilon^{pqr} \bar{\psi}^A' \psi^A q e_{AA'} r ,
\]

(B.108)

then invariance of the \( N=1 \) SUGRA action is restored under left-handed SUSY transformations.

To check this, besides the transformation for \( e_{AA'} p \) and \( \hat{e}_{A p} \), we also need [23]

\[
\delta \psi^A' p = -\frac{2i}{\kappa} \bar{\psi}^A' n^{AA'} \sigma_{\hat{a}AB'} \left( e_{\hat{a} b} Q_{\hat{b}}^b - 2 e_{\hat{b} b} Q_{\hat{a}}^a \right)
\]

\[
- \frac{1}{\kappa N} \bar{\psi}^A' n^{AA'} \sigma_{\hat{a}AB'} \left( e_{\hat{a} p} + e_{\hat{a} q} e_{\hat{b} p} \hat{e}_{\hat{b} q} \right)
\]

\[
- \frac{2}{\kappa N} \bar{\psi}^A' n^{AA'} \sigma_{\hat{a}AB'} \left( e_{\hat{b} p} e_{\hat{c} q} Q_{\hat{a}}^d \hat{e}_{\hat{d} b c} + \varepsilon_{\hat{p} \hat{q}} e_{\hat{b} p} e_{\hat{q} \hat{b} q} \hat{e}_{\hat{d} c} \hat{e}_{\hat{a} d} \right)
\]

\[-i k \bar{e}^B' e_{AC'} r \varepsilon_{(A'C')} \bar{\psi}^{B'} \left( p \psi^A r \right) - \frac{1}{2} ik \bar{e}^B' e_{AA'} r e_{DD'} p \bar{\psi}^{D'} \left[ r \psi^D \right]
\]

\[+ \frac{i k}{N} \bar{e}^B' n A \left( A' \psi^B \right) \left( p \psi^A r \right) - \frac{i k}{N} \bar{e}^B' n A \left( A' \psi^B \right) \left( p \psi^A q \right)
\]

\[-i k \bar{e}^B' n^{AA'} e_{AB'} q e_{CC'} p \bar{\psi}^{C'} \left[ q \psi^C r \right] + \frac{i k}{N} \bar{e}^B' n^{AA'} e_{AB'} r e_{CC'} p \bar{\psi}^{C'} \left[ r \psi^C q \right]
\]

\[-2i k \bar{e}^{C'} n^{AA'} n_{CC'} \psi^C q e_{AB'} q \bar{\psi}^{B'} \left( p \right).
\]

(B.109)

The significance of the above is that the existence of a non-trivial, boundary-preserving (sub)algebra allows the theory to be SUSY invariant and admits a Nicolai map.15

But the addition of the boundary term also presents a significant bonus which underlies the (perhaps crucial) developments in Sect. 5.2.3 of Vol. I and Sect. 4.1.1

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15 Each bosonic component of the wave function will evolve under supersymmetry according to a type of Fokker–Planck equation. In Bianchi class A cosmoiologies, for the quantum states that permit Nicolai maps, just two states appear, corresponding to the empty and filled fermion states (see Sects. 3.3 and 3.4).
of this volume: it simplifies the canonical formulation of the theory compared with
the formulation presented in Chap. 4 of Vol. I. In particular, it leads to simplifications
in the new Dirac brackets, whereupon the quantization provides a wider range of
potential tools [37–40].

5.4 Quantization of Bianchi Models and Boundary Terms

The exact invariance of the $N = 1$ SUGRA Bianchi A theory under the action of the
left-handed generators is due to (B.107). For right-handed SUSY transformations
($\bar{\epsilon} = 0$), the correction term would have been the Hermitian conjugate. Note that
$W_\psi$ is just the fermion number $F$, and recall that only the Euclidean version of a
supersymmetric theory is known to admit a Nicolai map. Hence we take

$$S_E^{B'} \equiv \frac{d}{d\tau} \left( W_g + \frac{1}{2} W_\psi \right) ,$$

(B.110)

$$\bar{S}_E^{B'} \equiv - \frac{d}{d\tau} \left( W_g + \frac{1}{2} W_\psi \right) .$$

(B.111)

With $S_E^{B'} = - \bar{S}_E^{B'}$, this can be written [23]

$$L \rightarrow L_\pm \equiv L \pm \frac{d}{d\tau} \left( \frac{1}{k^2} \xi m^{pq} h_{pq} + \frac{1}{2} i \xi \varepsilon^{pqr} \bar{\psi}^{A'} p \psi^A e_{AA'} \right) .$$

(B.112)

Now with $I$ is the Euclidean action introduced in Sect. 2.7 of Vol. I, define

$$I_\pm \equiv I \pm \left( W_g + \frac{1}{2} W_\psi \right) ,$$

(B.113)

from which there follows an expression for the new momentum $p^{\pm AA'}_p$:

$$p^{\pm AA'}_p \equiv \frac{\partial I_\pm}{\partial e_{AA'}^p}$$

$$= -i p_{AA'}^p \pm \frac{1}{2} i \xi \varepsilon^{pqr} \bar{\psi}^{A'} q \psi_{Ar} \mp \frac{2}{k^2} \xi m^{pq} e_{AA'}^q .$$

(B.114)

Then the following Dirac brackets become possible among the variables $e_{AA'}^p$, $p^{\pm AA'}_p$, $\psi^A_p$, and $\bar{\psi}^{A'}_p$:
\[
\left\{ e^{AA'}_p, e^{BB'}_q \right\}_D = 0 , \quad (B.115)
\]
\[
\left\{ e^{AA'}_p, p^{\pm}_{BB'} q \right\}_D = e^A_B e^{A'}_{B'} \delta^q_p , \quad (B.116)
\]
\[
\left\{ p^{+}_{AA'} p, p^{+}_{BB'} q \right\}_D = 0 , \quad (B.117)
\]
\[
\left\{ p^{-}_{AA'} p, p^{-}_{BB'} q \right\}_D = 0 , \quad (B.118)
\]
\[
\left\{ \psi^A_p, \psi^B_q \right\}_D = 0 , \quad (B.119)
\]
\[
\left\{ \overline{\psi}^A_{p}, \overline{\psi}^{B'}_q \right\}_D = 0 , \quad (B.120)
\]
\[
\left\{ \psi^A_p, \overline{\psi}^{A'}_q \right\}_D = \frac{1}{\sigma} D^{AA'}_{pq} , \quad (B.121)
\]
\[
\left\{ e^{AA'}_p, \psi^B_q \right\}_D = 0 , \quad (B.122)
\]
\[
\left\{ e^{AA'}_p, \overline{\psi}^{B'}_q \right\}_D = 0 , \quad (B.123)
\]
\[
\left\{ p^{+}_{AA'} p, \psi^B_q \right\}_D = 0 , \quad (B.124)
\]
\[
\left\{ p^{-}_{AA'} p, \overline{\psi}^{B'}_q \right\}_D = 0 , \quad (B.125)
\]
\[
\left\{ p^{+}_{AA'} p, \overline{\psi}^{B'}_q \right\}_D = -i \varepsilon_{prs} \overline{\psi}^{A'}_{s} D_{A} B'_{r} q , \quad (B.126)
\]
\[
\left\{ p^{-}_{AA'} p, \psi^B_q \right\}_D = i \varepsilon_{prs} \psi^A_{s} D^{B'}_{A'} q r , \quad (B.127)
\]

where (B.117), (B.118), (B.124), and (B.125) are indeed much simpler then the ones without (B.113).

The Hamiltonian becomes (see Exercise 5.1)

\[
H \equiv -e^{AA'}_0 \mathcal{H}_{AA'} + \psi^A_0 S_{A(new)} + \overline{S}_{A'(new)} \overline{\psi}^{A'}_0
- \left( \omega_{AB0} + \psi^C_0 \phi_{CAB} \right) \mathcal{J}^{AB} - \left( \overline{\omega}_{A'B'0} + \overline{\psi}^{C'}_0 \phi_{C'A'B'} \right) \overline{\mathcal{J}}^{A'B'} , \quad (B.128)
\]
where we now have
\begin{align}
\mathcal{J}_{AB} &= i e(A^Ap) p^- B)A'p , \\
\bar{\mathcal{J}}_{A'B'} &= i e(A^p p^+ AB')p . 
\end{align}
(B.129)
(B.130)
together with
\begin{align}
\mathcal{S}'_{A(new)} &= -\frac{1}{2} k^2 \psi^A p^+ AA'p + \bar{\phi}_A^{B'C'} \bar{\mathcal{J}}_{B'C'} , \\
\mathcal{S}_{A(new)} &= \frac{1}{2} k^2 \psi A' p^- AA'p + \phi_A^{BC} \bar{\mathcal{J}}_{BC} . 
\end{align}
(B.131)
(B.132)
The reader can indeed confirm that the resulting SUSY constraints become simpler.
Note also that, in $\mathcal{H}_{AA'} \equiv -n_{AA'} \mathcal{H}_{\perp} + e_{AA'} p \mathcal{H}_p$, the form of $\mathcal{H}_{AA'}$ is complicated, but we can write
\begin{align}
\mathcal{H}_{AA'} &\simeq \frac{2}{k^2} \left\{ \mathcal{S}_{A(new)}', \mathcal{S}_{A'}(new) \right\}_D + \text{terms proportional to } \mathcal{J} \text{ and } \bar{\mathcal{J}} \\
&= \frac{k^4}{4} \left( \frac{1}{\varsigma} p^- AB'p p^+ BA'q D^{BB'}_{qp} + i e^{pqr} p^+ BA's D^B_{B'sq} \psi_A \bar{\psi}^B_p + i e^{pqr} p^- AB's D^B_{B'qs} \bar{\psi}_A^r \psi^B_p + \frac{1}{2} \varsigma h^{1/2} n_{BB'} \bar{\psi}_A^r \psi^B_p \bar{\psi}^B_{s} \psi_A[p \bar{\psi}^B_{s}] \right) . 
\end{align}
(B.133)
Quantum mechanically [23], we can obtain an explicit representation of the differential operator $\hat{\mathcal{S}}_{A(new)}'$ as
\begin{align}
\hat{\mathcal{S}}'_{A(new)} &= \frac{1}{2} \hbar k^2 \psi^A p \frac{\partial}{\partial e^{AA'}} - \frac{1}{2} \hbar k^2 h^{1/2} n_{CA'} \psi^C q D^{AC'}_{qp} \frac{\partial}{\partial e^{AC'}} , 
\end{align}
(B.134)
while the differential operator $\hat{\mathcal{S}}_{A(new)}$ is represented by
\begin{align}
\hat{\mathcal{S}}_{A(new)} &= \frac{1}{2} \hbar k^2 \psi A' p \frac{\partial}{\partial e^{AA'}} + \frac{1}{2} \hbar k^2 h^{1/2} n_{AC'} \bar{\psi}^C q D^{CA'}_{q} \frac{\partial}{\partial e^{CA'}} . 
\end{align}
(B.135)

corresponding to the $\pm$ in the action in (B.113). So $\hat{\mathcal{S}}_{A(new)} \pm$ and $\hat{\mathcal{S}}'_{A(new)} \pm$ are related to the original operators $\hat{\mathcal{S}}_{A(new)}$ and $\hat{\mathcal{S}}'_{A(new)}$ by the following transformations (see Sect. 3.3):

\[16\text{ In the following, } \tau \text{ should be considered as a mere parameter, although in a path integral presentation, it would be the Euclidean time.}\]
\[ \hat{S}_{A(new)}^+ \equiv e^{-W_b/h} \hat{S}_{A(new)} e^{W_b/h}, \]  
\[ \hat{S}_{A'(new)}^+ \equiv e^{-W_b/h} \hat{S}_{A'(new)} e^{W_b/h}, \]  
\[ \hat{S}_{A(new)}^- \equiv e^{W_b/h} \hat{S}_{A(new)} e^{-W_b/h}, \]  
\[ \hat{S}_{A'(new)}^- \equiv e^{W_b/h} \hat{S}_{A'(new)} e^{-W_b/h}. \]  

The transformed wave functions \( \Psi^+, \Psi^- \) are therefore associated with wave functions \( \Psi(e, \psi, \tau), \Psi(e, \overline{\psi}, \tau) \) by

\[ \Psi^+(e, \psi, \tau) = e^{-W_b/h} \Psi(e, \psi, \tau), \]  
\[ \Psi^-(e, \overline{\psi}, \tau) = e^{W_b/h} \Psi(e, \overline{\psi}, \tau). \]  

In summary, we may take the (Euclidean) representations of the operators \( p_{AA'}^+, \overline{\psi}_{A'}^p \) acting on a wave function \( \Psi(e^{AA'} p, \psi^A p) \) by means of

\[ \hat{p}_{AA'}^p = -\hbar \frac{\partial}{\partial e^{AA'} p}, \]  
\[ \overline{\psi}_{A'}^p = \frac{\hbar}{\xi} D^{AA'} q_p \frac{\partial}{\partial \psi^A q}, \]

where

\[ \hat{S}_{A'(new)}^+ = \hat{S}_A^+ + \overline{\phi}_{A'} B'C' \hat{J}_{B'C'}, \]  
\[ \hat{S}_{A(new)}^- = \hat{S}_A^- + \phi_{A B C} \hat{J}_{BC}, \]

and

\[ \hat{S}_A^+ = \frac{1}{2} \hbar k^2 \psi_A^p p \frac{\partial}{\partial e^{AA'} p}, \]  
\[ \hat{S}_A^- = -\frac{1}{2} \hbar k^2 \overline{\psi}_{A'}^p p \frac{\partial}{\partial e^{AA'} p}. \]

**Problems of Chap. 6**

6.1 Spinor Symmetry

Since \( e_{AA'}^i n^{AA'} = 0 \), it follows that \((2\hbar)^{1/2} i e_{A'A}^i n^{BA'}\) is symmetric.
6.2 FRW with Cosmological Constant

The (dimensionally) reduced Lagrangian and Hamiltonian can be expressed, after integrating over the spatial coordinates, as

\[ L = \dot{\omega} \xi + \dot{\theta}_A \eta^A - \mathcal{H} \left( \omega, \xi ; \theta_A, \eta^A \right), \quad (B.148) \]

\[ \mathcal{H} = 4\sqrt{12V} \xi^{1/2} \left( i\omega - \omega^2 + \frac{\gamma'^2}{12} \xi - \frac{1}{4} \ell' \gamma' \theta_A \theta^A \right) + 4\sqrt{12V} \xi^{-1/2} \eta_A \left[ 2(i - \omega) \theta^A + \frac{\gamma'}{3\ell'} \eta^A \right], \quad (B.149) \]

where the gauge \( A_{AB0} = \psi_0 = N^i = 0, N = 1 \) was used. The classical evolution is given by [41, 24, 42, 43]

\[ \frac{d\omega}{dt} = [\omega, H] = \frac{\gamma'^2}{3} \sqrt{12V} \xi^{1/2}, \quad (B.150) \]

\[ \frac{d\theta_A}{dt} = -8\sqrt{12V} \xi^{-1/2} (i - \omega) \theta^A, \quad (B.151) \]

\[ \frac{d\sigma}{dt} = 4\sqrt{12V} \left[ -\sigma^{1/2} (i - 2\omega) \theta^A + 2\xi^{-1/2} \eta_A \theta^A \right], \quad (B.152) \]

\[ \frac{d\eta}{dt} = 4\sqrt{12V} \xi^{-1/2} \left[ \frac{1}{2} \ell' \gamma' \theta^A - 2(i - \omega) \eta^A \right], \quad (B.153) \]

with the relations (from the Hamilton–Jacobi equation)

\[ \xi = \frac{\partial Y}{\partial \omega} = \frac{12}{\gamma'^2} \left( -i\omega + \omega^2 + \frac{1}{4} \ell' \gamma' \theta_A \theta^A \right), \quad (B.154) \]

\[ \eta^A = \frac{\partial Y}{\partial \theta_A} = -\frac{6\ell'}{\gamma'} (i - \omega) \theta^A, \quad (B.155) \]

\[ Y = -\frac{12}{\gamma'^2} \left[ \frac{i\omega^2}{2} - \frac{\omega^3}{3} + \frac{1}{4} \ell' \gamma' (i - \omega) \theta_A \theta^A \right], \quad (B.156) \]

where \( \ell', \gamma' \) are rescalings of \( \Lambda = -6\gamma^2 \) and \( \ell = \sqrt{2/3}. \)

A comment is now in order. In (B.150) and (B.152), the presence of the Grassmann variables suggests that the solution process begins with the bosonic sector, i.e., obtaining an anti-de Sitter background, namely, an \( H^4 \) hyperboloid, for a negative cosmological constant (see Sect. 5.1.3 of Vol. I). The following step (to the
next order) is to solve (B.151), (B.153) which are linear in fermions, providing the evolution of the gravitino in the background (recall the analysis and discussion in Sect. 4.2). As expected, we consistently insert these results into (B.150), (B.152), thereby obtaining fermionic corrections to $\omega$ and $\zeta$. For more details, the reader should consult [24, 44, 42, 43].

6.3 $N=2$ Hidden SUSY and Ashtekar’s Variables

Another element that emerges is the (hidden) $N=2$ supersymmetrization of Bianchi cosmologies straight from bosonic general relativity, using Ashtekar’s new variables (see Chap. 6). In the following, we summarize the main features of [45].

Restricting to a minisuperspace at a pure gravity level, the constraint equations (not with a 2-spinor form) are:

$$\mathcal{J}^i = \mathcal{D}_a E^{ai}, \quad (B.157)$$
$$C_b = E^{ai} F_{ib}, \quad (B.158)$$
$$\mathcal{H} = \lder_{ijk} E^{ai} E^{bj} \tilde{R}_{ab}, \quad (B.159)$$

where $R_{ab}^i$ is the curvature of $A^i_a$, and $\mathcal{D}_a$ the covariant derivative formed with $A^i_a$. Restricting to Bianchi cosmologies, we introduce a basis of vectors $X^a_i$ [see (6.54)–(6.57)], and then expand into new variables:

$$\tilde{E}^a_j \equiv \tilde{E}^i_j X^a_i, \quad (B.160)$$
$$\tilde{A}^i_a = \tilde{A}^i_j X^a_j. \quad (B.161)$$

Dropping the diacritic to simplify the notation, the quantities $E^i_j$ and $A^i_j$ only depend on time. Inserting these substitutions into the constraint equations [41], we obtain

$$\mathcal{J}^i \equiv C^k_{jk} E^{ij} + \lder_{ijk} A^m_{mj} E^m_k, \quad (B.162)$$
$$\mathcal{H}_k = -E^i_j A^i_m C^m_{jk} + \lder_{imn} E^i_j A^j_{jm} A^k_{kn}, \quad (B.163)$$
$$\mathcal{H} = \lder_{ijk} C^p_{mn} E^m_i E^n_j A^k_p + E^n_i E^n_j (A^i_m A^j_n - A^j_n A^i_m). \quad (B.164)$$

Moreover, the constraint equations can be written in a more compact and rather interesting fashion. Introducing the variables

$$Q^i_j \equiv E^i_k A^k_j, \quad (B.165)$$

the Hamiltonian constraint for all Bianchi class A models can be written as

\[17\] Further restricting to diagonal models, $E^i_j = \text{diag}(E^1, E^2, E^3)$ and $A^i_j = \text{diag}(A_1, A_2, A_3)$. 


\[ H = Q^* i Q^k i - Q^* i Q^j j \]  \hspace{1cm} (B.166)

All the dynamics of all class A \textit{diagonal} models can be further summarized by

\[ H = \overline{Q} Q_2 + \overline{Q} Q_3 + \overline{Q} Q_1 + \overline{Q} Q_3 + \overline{Q} Q_2 \]
\[ = G^{ij} \overline{Q} Q_j \]  \hspace{1cm} (B.167)

where \( i, j = 1, \ldots, 3 \), and

\[ G^{ij} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]  \hspace{1cm} (B.168)

Using the Misner–Ryan parametrization with \( \beta_{ij} \) given by

\[ \beta_{ij} = \text{diag}(\beta_+, \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \]

it is possible to find solutions of the form [46]

\[ \Psi[\alpha, \beta_\pm] = C[\alpha, \beta_\pm] \exp[-I(\alpha, \beta_\pm)] \]

This involves requiring \( I \) to be the solution of the Euclidean Hamilton–Jacobi equation for this particular model. The relation with the Ashtekar variables is that, if we have a wave function \( \Psi_A \) given in terms of the Ashtekar variables [47], we can reconstruct the wave function in terms of the Misner–Ryan variables by choosing \( \Psi_{MR} = \exp(\pm iI_A)\Psi_A \), where \( I_A \) is given by \( I_A = 2i \int E^a_i \Gamma^i a d^3 x \), with \( I_A \) equal to \( \mp iI \) for the Bianchi IX case. \( \Gamma^i a \) is the corresponding (compatible) spin connection.

But where is the (‘hidden’ \( N = 2 \)) SUSY? The above Hamiltonian suggests (see Chap. 8 of Vol. I) introducing the expressions

\[ S \equiv \psi^i Q_i \]  \hspace{1cm} (B.169)
\[ \overline{S} \equiv \overline{\psi}^i Q^*_i \]  \hspace{1cm} (B.170)

for the SUSY constraints, with

\[ \overline{\psi}^i \psi^j + \psi^i \overline{\psi}^j = G^{ij} \]  \hspace{1cm} (B.171)

and \( G \) as given above. The Hamiltonian constraint becomes

\[ H = \frac{1}{2} (S \overline{S} + \overline{S} S) \]  \hspace{1cm} (B.172)

In quantum mechanical terms, one realization is \( \Psi[A, \eta] \), with
\[ E_i \psi [A] = \frac{\partial}{\partial A_i} \psi [A] , \quad \text{(B.173)} \]

\[ A_i \psi [A] = A_i \psi [A] , \quad \text{(B.174)} \]

together with \( \psi^i = \eta^i, \overline{\psi}^i = G^{ij} \partial / \partial \eta^j \). The equations become

\[ \mathcal{S} \Psi [A, \eta] = Q_i \eta^i \psi [A, \eta] , \quad \text{(B.175)} \]

\[ \overline{\mathcal{S}} \Psi [A, \eta] = \hat{Q}_i^* \frac{\partial}{\partial \eta^i} \psi [A, \eta] , \quad \text{(B.176)} \]

and using the reality conditions, we get

\[ \eta^i A_i \frac{\partial}{\partial A_i} \psi [A, \eta] = 0 , \quad \text{(B.177)} \]

\[ (A_k - 2 \Gamma_k) \frac{\partial}{\partial A_k} G_{ki} \frac{\partial}{\partial \eta_i} \psi [A, \eta] = 0 . \quad \text{(B.178)} \]

The only solution admitted by this system of equations is \( \Psi [A] = \text{constant} \), but it has a nontrivial form in terms of the Misner–Ryan variables.

### Problems of Chap. 7

#### 7.1 An Avant Garde (Non-SUSY) Square Root Formulation

Take the corresponding Hamiltonian constraint to be

\[ - p_\Omega^2 + p_+^2 + p_-^2 = 0 , \quad \text{(B.179)} \]

with the convention in [48] (see Sect. 7.2 for more details on the expressions used here). With the quantum correspondence

\[ p_\Omega \longrightarrow -\imath \hbar \frac{\partial}{\partial \Omega} , \quad p_+ \longrightarrow -\imath \hbar \frac{\partial}{\partial \beta_+} , \quad p_- \longrightarrow -\imath \hbar \frac{\partial}{\partial \beta_-} , \quad \text{(B.180)} \]

we may write

\[ \frac{\partial^2 \psi}{\partial \Omega^2} = \frac{\partial^2 \psi}{\partial \beta_+^2} + \frac{\partial^2 \psi}{\partial \beta_-^2} . \quad \text{(B.181)} \]
Now ‘linearize’ the square operator
\[
\frac{\partial^2 \Psi}{\partial \beta_+^2} + \frac{\partial^2 \Psi}{\partial \beta_-^2},
\]
to obtain
\[
\frac{i}{\partial \Omega} \Psi = \pm \left( \frac{\partial^2 \Psi}{\partial \beta_+^2} + \frac{\partial^2 \Psi}{\partial \beta_-^2} \right)^{1/2} \rightarrow \frac{i}{\partial \Omega} \Psi = i \alpha_+ \frac{\partial \Psi}{\partial \beta_+} - i \alpha_- \frac{\partial \Psi}{\partial \beta_-}, \tag{B.182}
\]
where \( \alpha_\pm \) are matrices satisfying \( \alpha_\pm^2 = 1 \) and which anticommute! The minimal rank for such matrices is two. Then taking \( \alpha_\pm \) to be the Pauli matrices, namely \( \alpha_+ \equiv \sigma_1, \alpha_- \equiv \sigma_2 \), \( \Psi \) must be a two-component ‘vector’ (spinor). Writing
\[
\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \tag{B.183}
\]
we obtain plane wave solutions
\[
\Psi_\pm \sim \exp \left[ i \left( p_+ \beta_+ + p_- \beta_- - E \Omega \right) \right], \tag{B.184}
\]
where \( \pm (p_+ \beta_+ + p_- \beta_-)^{1/2} = E \) and \( p_+, p_-, \Omega \) are constants. The two spinor degrees of freedom were described in [48] as ‘disturbing’. In fact, the two solutions correspond to expanding (\( E > 0 \)) and contracting (\( E < 0 \)) universes, but how should one then interpret the ‘spin’ states \( \Psi_\pm \) in terms of physical attributes of the universe? The reader should remember that this was in 1972! Researchers had to wait for SUGRA in order to obtain a clearer view (see Sect. 7.2). The above is a possible representation within SQC, i.e., SUGRA as applied to cosmology, using the matrix representation for fermionic momenta! It is altogether astonishing how this emerged back in 1972, before the full development of SUSY and SUGRA.

7.2 From the Lorentz Constraints to the Lorentz Conditions

We find that \( \Psi_{III} = -(\mathcal{J}_{12})^{-1} \mathcal{J}_{13} \Psi_{IV} \). Then show that \( \mathcal{J}_{12} \Psi_{II} = \mathcal{J}_{23} \Psi_{IV} \), whence
\[
\mathcal{J}_{23} = \mathcal{J}_{12} (\mathcal{J}_{13})^{-1} \mathcal{J}_{23} (\mathcal{J}_{12})^{-1} \mathcal{J}_{13},
\]
\[
\mathcal{J}_{12} = \mathcal{J}_{23} (\mathcal{J}_{13})^{-1} \mathcal{J}_{12} (\mathcal{J}_{23})^{-1} \mathcal{J}_{13},
\]
\[
\mathcal{J}_{13} = \mathcal{J}_{23} (\mathcal{J}_{12})^{-1} \mathcal{J}_{13} (\mathcal{J}_{23})^{-1} \mathcal{J}_{12}. \tag{B.185}
\]
Using \( \mathcal{J}_A = \varepsilon_{0ABC} \mathcal{J}^{BC} / 2 \), the result follows.
7.3 From the Lorentz Constraints to the Non-diagonal ‘Missing’ Equations

First obtain the equations of motion and Lorentz constraints for the diagonal Bianchi type IX cosmological model [49]. Then, e.g., substitute the $J^{01}$ and the $J^{23}$ component expressions in the $(0, 1)$ component of the Einstein–Rarita–Schwinger equation. This reduces to

$$
(-3\dot{\beta}_+ + \sqrt{3}\dot{\beta}_- ) \bar{\psi}_2 \gamma^2 \bar{\psi}_1 = 0 .
$$  \hspace{1cm} (B.186)

At the quantum level (in this matrix representation approach!), (B.186) yields the same result as the Lorentz constraints. Now with this result, we can analyze the $(2, 3)$ component of the Einstein–Rarita–Schwinger equation. The last term is once more $\bar{\psi}_2 \gamma^2 \bar{\psi}_1$ and the first and second terms cancel out. The first of the three remaining terms is quadratic in the gravitino field components. The other two terms can also be reduced to quadratic terms in $\bar{\psi}_i$, multiplied by linear combinations of $\dot{\Omega}, \dot{\beta}_+$, and $\dot{\beta}_-$, with the help of the equations for $\bar{\psi}_i$. The same procedure is applied to, e.g., the $(0, 2)$ and $(0, 3)$ components of the Einstein–Rarita–Schwinger equation. Substituting into all the $i \neq j$ equations, and the $\bar{\psi}_i$ equations are used once more. The quadratic expressions resulting from this procedure for all the equations $i \neq j$ are:

$$
\bar{\psi}_1 \gamma^0 \gamma^1 \gamma^2 \phi_1, \hspace{1cm} \bar{\psi}_2 \gamma^0 \gamma^2 \gamma^3 \bar{\psi}_1,
$$

$$
\bar{\phi}_1 \gamma^0 \gamma^1 \gamma^3 \phi_1, \hspace{1cm} \bar{\phi}_1 \gamma^0 \gamma^1 \gamma^3 \bar{\psi}_2,
$$

$$
\bar{\psi}_1 \gamma^0 \gamma^2 \gamma^3 \phi_1, \hspace{1cm} \bar{\phi}_3 \gamma^0 \gamma^2 \gamma^3 \bar{\psi}_1,
$$

$$
\bar{\psi}_2 \gamma^0 \gamma^1 \gamma^2 \phi_2, \hspace{1cm} \bar{\phi}_3 \gamma^1 \gamma^2 \gamma^3 \bar{\psi}_1,
$$

$$
\bar{\psi}_2 \gamma^0 \gamma^1 \gamma^3 \phi_2, \hspace{1cm} \bar{\phi}_1 \gamma^0 \gamma^1 \gamma^2 \bar{\psi}_3,
$$

$$
\bar{\psi}_2 \gamma^0 \gamma^2 \gamma^3 \phi_2, \hspace{1cm} \bar{\phi}_3 \gamma^0 \gamma^1 \gamma^2 \bar{\psi}_2,
$$

$$
\bar{\psi}_3 \gamma^0 \gamma^1 \gamma^2 \phi_3, \hspace{1cm} \bar{\phi}_2 \gamma^0 \gamma^1 \gamma^3 \bar{\psi}_3,
$$

$$
\bar{\psi}_3 \gamma^0 \gamma^1 \gamma^3 \phi_3, \hspace{1cm} \bar{\phi}_3 \gamma^1 \gamma^2 \gamma^3 \bar{\psi}_2,
$$

$$
\bar{\psi}_3 \gamma^0 \gamma^2 \gamma^3 \phi_3, \hspace{1cm} \bar{\phi}_1 \gamma^1 \gamma^2 \gamma^3 \bar{\psi}_3 .
$$

(B.187)

These expressions at the quantum level (i.e., proceeding from the variables $\dot{\psi}_i$ to $\chi_i$) lead to the same wave function that has been obtained by applying the $J^{ab}$ constraints. Therefore the equations $i \neq j$ yield the same result as the Lorentz constraints $J^{ab}$. 


Problems of Chap. 8

8.1 Invariance of the Action Under $N=2$ Conformal SUSY

For the the action (8.39), we get (see also [50])

$$\delta S = \frac{1}{2} \int \left[ \frac{f}{N} G_{XY}(q^Z) \dot{q}^X \dot{q}^Y + fN U(q^Z) \right] dt , \quad (B.188)$$

under the reparametrization $t' \rightarrow t + f(t)$ with

$$\delta q^X(t) = f(t) \dot{q}^X , \quad \delta N(t) = (fN)' . \quad (B.189)$$

Hence, we get a total derivative.

We are interested in the action (8.41), whose variation is

$$\delta S_{\text{grav}} = \frac{i}{2} \int \left\{ D_\eta \left[ L D_\eta \left( \frac{G_{XY}}{2N} D_\eta Q^X D_\eta Q^Y + W(Q^Z) \right) \right] \right\} d\eta d\bar{\eta} dt , \quad (B.190)$$

under (8.7), (8.8), (8.12), and

$$\delta Q = L \dot{Q} + \frac{i}{2} D_\eta L D_\eta Q + \frac{i}{2} D_\eta L D_\eta Q . \quad (B.191)$$

This is a total derivative, so the action is indeed invariant, as claimed.

8.2 Dirac Brackets in $N=2$ Conformal SUSY

From $\psi(t), \chi(t)$, and $\lambda(t)$, we have the following second class constraints:

$$p_\psi \equiv \pi_\psi + \frac{i}{3} \bar{\psi} = 0 , \quad (B.192)$$

$$p_{\bar{\psi}} \equiv \pi_{\bar{\psi}} + \frac{i}{3} \lambda = 0 , \quad (B.193)$$

$$p_I(\chi) \equiv \pi_I(\chi) - \frac{i}{2 \kappa^2} g_{IJ} \dot{\chi}^J = 0 , \quad (B.194)$$

$$p_T(\chi) \equiv \pi_T(\chi) - \frac{i}{2 \kappa^2} g_{IJ} \dot{\chi}^J = 0 , \quad (B.195)$$
\[ \Pi_I(\lambda) \equiv \pi_I(\lambda) - \frac{i}{2\kappa^2} G_{IJ} \dot{\chi}^J = 0 , \quad (B.196) \]

\[ \mathbf{p}_T(\lambda) \equiv \pi_T(\lambda) - \frac{i}{2\kappa^2} G_{IJ} \dot{\chi}^J = 0 , \quad (B.197) \]

where \( \pi_\psi = dL/d\dot{\psi} \), \( \pi_I(\chi) = dL/d\dot{\chi}^I \), and \( \pi_I(\lambda) = dL/d\dot{\lambda}^I \) are the momenta conjugate to the anticommuting variables \( \psi(t) \), \( \chi(t) \), and \( \lambda(t) \), respectively.

Then write the matrices

\[ C_{\psi\psi} = \frac{2}{3} i , \quad C_{TJ}(\chi) = -\frac{i}{\kappa^2} G_{TJ} , \quad C_{TJ}(\lambda) = -\frac{i}{\kappa^2} G_{TJ} , \quad (B.198) \]

and their inverses

\[ C_{\psi\psi}^{-1} = -\frac{3}{2} i , \quad C_{TJ}^\dagger(\chi) = i\kappa^2 G_{TJ} , \quad C_{TJ}^\dagger(\lambda) = i\kappa^2 G_{TJ} . \quad (B.199) \]

The Dirac brackets \([ \cdot , \cdot ]_D\) are (recall Chap. 4 of Vol. I)

\[ [f, g]_D = [f, g] - [f, \pi_i] (C^{-1})^i_k [\mathbf{p}_k, g] , \quad (B.200) \]

and allow elimination of the momenta conjugate to the fermionic variables, leaving us with the following non-zero Dirac bracket relations:

\[ [a, \pi_a]_D = [a, \pi_a]_D = -1 , \quad (B.201) \]

\[ [\phi_I, \pi_{\phi}^J]_D = [\phi_I, \pi_{\phi}^J] = -\delta_{IJ} , \quad (B.202) \]

\[ [\bar{\phi}_T, \pi_{\bar{\phi}}^J]_D = [\bar{\phi}_T, \pi_{\bar{\phi}}^J] = -\delta_{TJ} , \quad (B.203) \]

\[ [\chi^I, \chi^\dagger]_D = -i\kappa^2 G^{I\dagger} , \quad (B.204) \]

\[ [\lambda^I, \lambda^\dagger]_D = -i\kappa^2 G^{I\dagger} , \quad (B.205) \]

\[ [\psi, \bar{\psi}]_D = \frac{3}{2} \mathbf{1} . \quad (B.206) \]

### 8.3 FRW Wave Function in N=2 Conformal SUSY

With \( k = 0, 1 \), take the action to be (see [51, 52])
\[ S = S_{\text{FRW}} + S_{\text{mat}} , \]

\[ S_{\text{FRW}} = \int 6 \left( -\frac{1}{2k^2} \frac{A}{N} D_\eta A D_\eta A + \frac{\sqrt{k}}{2k^2} A^2 \right) d\eta d\bar{\eta} dt , \]

\[ S_{\text{mat}} = \int \left[ \frac{1}{2} \frac{A^3}{N} D_\eta \Phi D_\eta \Phi - 2A^3 \bar{W}(\Phi) \right] d\eta d\bar{\eta} dt , \]

(B.207)

where we use

\[ \Phi = \phi(t) + i\eta \bar{\psi}'(t) + i\bar{\eta} \chi'(t) + F'(t) \eta \bar{\eta} \]

(B.208)

for the component of the (scalar) matter superfields \( \Phi(t, \eta, \bar{\eta}) \), with \( \Phi^+ = \Phi \).

As often indicated, after the integration over the Grassmann complex coordinates \( \eta \) and \( \bar{\eta} \), and making the following redefinition of the 'fermion' fields (Grassmann variables)

\[ \psi(t) \rightarrow \frac{1}{3} a^{-1/2}(t) \psi(t) , \quad \chi(t) \rightarrow a^{-3/2}(t) \chi(t) , \]

we find the Lagrangian in which the field \( F(t) \) is auxiliary, and they can be eliminated with the help of their equations of motion. In addition, we use the Taylor expansion for the superpotential:

\[ \bar{W}(\Phi) = W(\phi) + \frac{\partial \bar{W}}{\partial \phi}(\Phi - \phi) + \frac{1}{2} \frac{\partial^2 \bar{W}}{\partial \phi^2}(\Phi - \phi)^2 + \cdots , \]

with \( \Phi \) as given above.\(^{18}\) In terms of components of the superfields \( A, N, \) and \( \Phi \), the Lagrangian then reads (somewhat more simply than in Chap. 8)

\[ L = -\frac{3}{k^2} \frac{a(Da)^2}{N} + \frac{1}{3} \frac{1}{\sqrt{k}} \psi_D \psi + \frac{\sqrt{k}}{k^2} a^{-1/2}(\psi_0 \psi - \psi_0 \bar{\psi}) \\
+ \frac{1}{3} Na^{-1} \sqrt{k} \bar{\psi} \psi + \frac{3k}{k^2} Na + \frac{a^3}{2} \frac{(D\phi)^2}{N} - i \bar{\chi} D\chi \\
- \frac{3}{2} \sqrt{k} Na^{-1} \bar{\chi} \chi - k^2 N W(\phi) \bar{\psi} \psi - 6\sqrt{k} N W(\phi) a^2 \\
- Na^3 V(\phi) + \frac{3}{2} k^2 N W(\phi) \bar{\chi} \chi + \frac{i k}{2} D\phi(\bar{\psi} \chi + \psi \bar{\chi}) \\
- 2N \frac{\partial^2 W(\phi)}{\partial \phi^2} \bar{\chi} \chi - kN \frac{\partial \bar{W}(\phi)}{\partial \phi}(\psi \bar{\chi} - \psi \bar{\chi}) + \frac{k^2}{4} a^{-3/2}(\psi_0 \psi - \psi_0 \bar{\psi}) \bar{\chi} \\
- ka^{3/2}(\psi_0 \psi - \psi_0 \bar{\psi}) W(\phi) + a^{3/2} \frac{\partial \bar{W}(\phi)}{\partial \phi}(\psi_0 \chi - \psi_0 \bar{\chi}) , \]

(B.209)

\(^{18}\) This series expansion ends at this point, in the second term, since \( (\Phi - \phi) \) is nilpotent.
where
\[ D a = \dot{a} - \frac{i k}{6} a^{-1/2}(\psi_0 \bar{\psi} + \bar{\psi}_0 \psi), \quad D \phi = \dot{\phi} - \frac{i}{2} a^{-3/2}(\bar{\psi}_0 \chi + \psi_0 \bar{\chi}), \]
are the supercovariant derivatives, and
\[ D \psi = \dot{\psi} - \frac{i}{2} V \psi, \quad D \chi = \dot{\chi} - \frac{i}{2} V \chi, \]
are the U(1) covariant derivatives. The potential for the homogeneous scalar fields is
\[ V(\phi) = 2 \left[ \frac{\partial W(\phi)}{\partial \phi} \right]^2 - 3 k^2 W^2(\phi), \quad (B.210) \]
and consists of two terms. One is the potential for the scalar field in the case of global supersymmetry. The potential (B.210) is not positive semi-definite, in contrast with what happens in standard supersymmetric quantum mechanics. Indeed, the present model describing the minisuperspace approach to supergravity coupled to matter, allows supersymmetry breaking when the vacuum energy is equal to zero \( V(\phi) = 0 \).

Quantization requires for the Dirac brackets
\[ [\psi, \bar{\psi}]_D = -\frac{3}{2} i, \quad [\chi, \bar{\chi}]_D = -i, \quad (B.211) \]
whence
\[ \{\psi, \bar{\psi}\} = -\frac{3}{2}, \quad \{\chi, \bar{\chi}\} = 1, \quad (B.212) \]
\[ [a, \pi_a] = -i, \quad [\phi, \pi_\phi] = -i. \quad (B.213) \]
Antisymmetrizing, i.e., writing a bilinear combination in the form of the commutators, e.g.,
\[ \bar{\chi} \chi \rightarrow \frac{1}{2} [\bar{\chi}, \chi], \]
this leads eventually to eigenstates of the Hamiltonian with four components:
\[ \Psi(a, \phi) = \begin{bmatrix} \Psi_1(a, \phi) \\ \Psi_2(a, \phi) \\ \Psi_3(a, \phi) \\ \Psi_4(a, \phi) \end{bmatrix}, \quad (B.214) \]
using instead \((\psi S - \overline{\psi} S) \Psi\) and \((\chi S - \chi \overline{S}) \Psi\), with a matrix representation for \(\psi, \overline{\psi}, \chi,\) and \(\overline{\chi}\). Then only \(\Psi_1\) or \(\Psi_4\) have the right behaviour when \(a \to \infty\), because the other components are infinite as \(a \to \infty\). We thus get the partial differential equations

\[
-a^{-1/2} \frac{\partial}{\partial a} - 6W(\phi)a^{3/2} + 6\sqrt{k} M_p^2 a^{1/2} + \frac{3}{4} a^{-3/2} \]
\[
\Psi_4 = 0 , \quad (B.215)
\]
\[
\left[ \frac{\partial}{\partial \phi} + 2a^3 \frac{\partial W(\phi)}{\partial \phi} \right] \Psi_4 = 0 , \quad (B.216)
\]

\[
-a^{-1/2} \frac{\partial}{\partial a} + 6W(\phi)a^{3/2} - 6\sqrt{k} M_p^2 a^{1/2} + \frac{3}{4} a^{-3/2} \]
\[
\Psi_1 = 0 , \quad (B.217)
\]
\[
\left[ \frac{\partial}{\partial \phi} - 2a^3 \frac{\partial W(\phi)}{\partial \phi} \right] \Psi_1 = 0 , \quad (B.218)
\]

with solutions

\[
\Psi \longrightarrow \Psi_4(a, \phi) \simeq a^{3/4} \exp \left[ -2w(\phi) a^3 + 3\sqrt{k} M_p^2 a^2 \right] , \quad (B.219)
\]

\[
\Psi \longrightarrow \Psi_1(a, \phi) \simeq a^{3/4} \exp \left[ 2w(\phi) a^3 - 3\sqrt{k} M_p^2 a^2 \right] . \quad (B.220)
\]

These are eigenstates of the Hamiltonian with zero energy and also with zero fermionic number.

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