THEORY OF THE DIPOLE-OCTUPOLE WIGGLER

PART II: COUPLING OF PHASE AND BETATRON OSCILLATIONS

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ABSTRACT

Dipole-octupole wigglers create a non-linear coupling through the damping terms of the Langevin equations describing the effect of quantum excitation on synchrotron and horizontal betatron oscillations in electron storage rings. By extension of the methods of Part I of this report, we obtain the appropriate Fokker-Planck equation and demonstrate that, when certain synchro-betatron resonances are avoided, adequate stability can be achieved simultaneously in both planes. The bunch lengthening described in Part I is accompanied by a narrowing of the transverse distribution.

The example of LEP Version II is considered in numerical detail and it is shown how a suitable wiggler could reduce the peak current at injection by a factor of 4 without obvious adverse effects. Recommendations are given as a prelude to detailed magnet design.

Finally we review the effects discussed in the language of bifurcation theory.
1. INTRODUCTION

In Part I \(^1\), we calculated the effect of a dipole-octupole wiggler on the energy distribution of particles in an electron storage ring. The synchrotron integral \(I_2\) was made to depend on \(\varepsilon\) through the transverse displacement \(D_w \varepsilon\) in the non-linear magnetic field of the wiggler. This resulted in an effective damping partition with a dependence on \(\varepsilon\). The horizontal betatron oscillations cause an additional transverse displacement, which was neglected in Part I and Ref. 2, but couples the two modes through the non-linear damping.

Our purpose here is to take this into account by generalising the methods of Part I. In Section 2 we find the dependence of \(J_c\) and \(J_x\) on the energy deviation \(\varepsilon\) and betatron displacement \(x_B\) by an extension of the calculation of Ref. 2. These are used to write a pair of coupled non-linear stochastic differential equations describing the time evolution of the system. Again, the Krylov-Bogoliubov method simplifies the equations and leads to a non-oscillatory Fokker-Planck equation for the distribution in synchro-betatron phase space. Although our pedestrian treatment might be given more dash by the use of the Lie transform formalism, it would probably be less transparent.

When a class of synchro-betatron resonances is avoided, the disparity of the two fundamental frequencies allows us to factorise the (quasi-) equilibrium distribution into one of the form found in Part I for the energy deviation and a slightly modified Gaussian for the betatron displacement. A further constraint on the choice of wiggler strengths is imposed by the need to maintain adequate transverse quantum lifetime. Nevertheless we see that it is possible to reduce the peak current in LEP (at least for some range of energies above injection) by a factor of 4. The bunch is lengthened by a factor of 3 but the lifetime may be kept above 10 h.

A practical prescription for the choice of wiggler strengths is presented, and finally we give a brief qualitative survey of the stability properties over the whole "control space" of wiggler parameters.

2. DAMPING PARTITION FUNCTIONS

To calculate the dependence of damping partitions on both \(\varepsilon\) and \(x_B\) we have to expand the energy loss per turn by radiation to third order in the total orbit deviation:

\[
x = x_B + D_w \frac{AE}{E_0} \equiv D_w (\delta + \varepsilon)
\]

(assuming all particles are highly relativistic) in the contribution from the wiggler. This yields
\[ U = \frac{2}{3} \frac{\tau E_0^3}{(m_0 c^2)^3} (1 + \varepsilon)^2 \left[ \frac{2\pi}{\rho_0} + \frac{L_{\mathcal{W}}}{\rho_0^2} + \frac{K''D^3 (\varepsilon + \delta)^3}{6} \right], \tag{2} \]

where we use notations defined in Refs. 1 and 2. Neglecting higher-order terms, we retain

\[ \frac{1}{E_0} \frac{dU}{\varepsilon} = U_0 \frac{2(1 + \varepsilon) + \left(1 + \frac{L_{\mathcal{W}} D_0}{2\pi\rho_0^2}\right)^{-1} \frac{L_{\mathcal{W}}}{2\pi\rho_0^2} \frac{K''D^3 (\varepsilon + \delta)^2}{2\pi\rho_0^2}}{E_0}, \tag{3} \]

with

\[ U_0 = \frac{2}{3} \frac{\tau E_0^3}{(m_0 c^2)^3} \frac{2\pi}{\rho_0} \left(1 + \frac{L_{\mathcal{W}} D_0}{2\pi\rho_0^2}\right). \tag{4} \]

The term linear in \( \varepsilon \) can be neglected because it averages to zero over a phase oscillation. (It can be checked that this leads to no inconsistency later.) The higher-order terms in \( \varepsilon \) and \( \delta \) are justifiably neglected when the sum of the magnitudes of the angles of deflection in the wiggler is significantly smaller than unity. We have now

\[ J_\varepsilon = -a + b(\varepsilon + \delta)^2 \tag{5} \]

where

\[ b = \frac{L_{\mathcal{W}} K''D_0}{2\pi\rho_0} \frac{U_L}{U_0}, \tag{6} \]

where \( U_0 \) is the total energy loss per turn for particles with \( \varepsilon = 0 \) and \( U_L \) is the energy loss in the lattice dipoles; \( a \) is determined by the rest of the machine, including dipole-quadrupole wigglers. The sacro Sanctum sum rule\(^3\) for \( J_\varepsilon \) and \( J_\delta \) still holds and yields

\[ J_\delta = 3 + a - b(\varepsilon + \delta)^2. \tag{7} \]

We do not consider coupling of the vertical betatron oscillations.

3. **STOCHASTIC DIFFERENTIAL EQUATIONS FOR COUPLED OSCILLATIONS**

The equations governing the evolution of the processes \( \varepsilon \) and \( \delta \) are

\[ \dot{\varepsilon} - \lambda \dot{\varepsilon} \left[1 - \mu(\varepsilon + \delta)^2\right] \dot{\varepsilon} + \Omega^2 \varepsilon = \lambda \Omega^2 \varepsilon(t) \tag{8} \]

\[ \ddot{\delta} + \lambda \ddot{\delta} \left[1 + 3/a - \mu(\varepsilon + \delta)^2\right] \ddot{\delta} + \omega^2 \delta = -\lambda \omega^2 \delta(t), \tag{9} \]

where

\[ \mu = \frac{b}{a}, \quad \lambda = \frac{P_0 a}{4\varepsilon_0}, \tag{10} \]
\( \omega \) is the betatron frequency, and \( \psi \) is a dimensionless constant which characterises the relative strength of the effect of quantum fluctuations on the betatron oscillations. Its value will be calculated later. The apparently eccentric disposition of the parameter \( \lambda \) in Eqs. (8) and (9) is for convenience in carrying out our perturbation expansion.

Processes \( A, \Delta, \phi, \psi \) are defined by

\[
A = (\epsilon^2 + \Omega^{-2}\xi^2)^{\frac{\gamma}{2}} \\
\Delta = (\delta^2 + \omega^{-2}\delta^2)^{\frac{\gamma}{2}} \\
\phi = -\tan^{-1}\left(\frac{\delta}{\Omega}\right) - \Omega t \\
\psi = -\tan^{-1}\left(\frac{\delta}{\Omega\omega}\right) - \omega t
\]  

and their evolutions in time are given by

\[
\dot{A} = \lambda F(A, \Delta, \phi, \psi, t) = \frac{\lambda \Omega}{2} A \left\{ \left(1 - \frac{\lambda A^2}{4} - \frac{\mu A^2}{2}\right) - \left(1 - \frac{\mu A^2}{2}\right) \cos 2\phi \\
- \frac{\mu A^2}{2} \cos 2\psi + \frac{\mu A^2}{4} \cos 4\phi \\
- \frac{\mu A}{2} \left[ \cos (\phi + \psi) + \cos (\phi - \psi) - \cos (3\phi + \psi) - \cos (3\phi - \psi) \right] \\
+ \frac{\mu A^2}{4} \left[ \cos 2(\phi + \psi) + \cos 2(\phi - \psi) \right] \right\} - \lambda \Omega \xi(t) \sin \phi
\]  

\[
\dot{\Delta} = \lambda G(A, \Delta, \phi, \psi, t) = -\frac{\lambda \Omega}{2} \Delta \left\{ \left(1 + \frac{3}{a} - \frac{\lambda A^2}{4} - \frac{\mu A^2}{2}\right) - \frac{\lambda A^2}{2} \cos 2\phi \\
- \left(1 + \frac{3}{a} - \frac{\lambda A^2}{4}\right) \cos 2\psi + \frac{\mu A^2}{4} \cos 4\psi \\
+ \frac{\mu A}{2} \left[ \cos (\phi + \psi) - \cos (\phi - \psi) + \cos (\phi + 3\psi) + \cos (\phi - 3\psi) \right] \\
+ \frac{\mu A^2}{4} \left[ \cos 2(\phi + \psi) + \cos 2(\phi - \psi) \right] \right\} - \frac{\lambda \Omega^2}{\omega} \xi(t) \sin \psi
\]
\[ \dot{\phi} = \lambda \kappa(A, \Delta, \psi, \psi, t) \]
\[ = \frac{\lambda \Omega}{2} \left\{ \left[ 1 - \frac{\mu A^2}{2} \right] \sin 2\phi - \frac{\mu A^2}{4} \sin 4\phi \right. \]
\[ - \frac{\mu A}{2} \left[ \sin (\phi + \psi) + \sin (\phi - \psi) + \sin (3\phi + \psi) + \sin (3\phi - \psi) \right] \]
\[ - \frac{\mu A^2}{4} \left[ \sin 2(\phi + \psi) + \sin 2(\phi - \psi) \right] \right\} - \frac{\lambda \Omega}{A} \xi(t) \cos \phi \quad (14) \]

\[ \dot{\psi} = \lambda L(A, \Delta, \phi, \psi, t) \]
\[ = \frac{\lambda \Omega}{2} \left\{ \left[ 1 + \frac{3}{a} - \frac{\mu A^2}{2} \right] \sin 2\psi - \frac{\mu A^2}{4} \sin 4\psi \right. \]
\[ - \frac{\mu A}{2} \left[ \sin (\phi + \psi) - \sin (\phi - \psi) - \sin (\phi - 3\psi) + \sin (\phi + 3\psi) \right] \]
\[ - \frac{\mu A^2}{4} \left[ \sin 2(\phi + \psi) - \sin 2(\phi - \psi) \right] \right\} - \frac{\lambda \Omega^2}{\omega \Omega} \xi(t) \cos \psi , \quad (15) \]

where

\[ \phi(t) = \Omega t + \phi(t) , \quad \psi(t) = \omega t + \psi(t) . \quad (16) \]

We refrain from writing out the typographically similar averaged equations for functions \( A(t), \Delta(t), \phi(t), \psi(t) \) in full detail:

\[ \dot{A}(t) = \lambda \langle F \rangle = \lambda \bar{F}(A(t), \Delta(t), \phi(t), \psi(t)) , \quad \text{etc.} \quad (17) \]

As in Part I, we transform to starred variables in order to eliminate oscillatory terms from the equations of motion\(^4\)

\[ A = A^* + \lambda u(A^*, \Delta^*, \phi^*, \psi^* ; \lambda) \]
\[ \Delta = \Delta^* + \lambda v(A^*, \Delta^*, \phi^*, \psi^* ; \lambda) \]
\[ \phi = \phi^* + \lambda p(A^*, \Delta^*, \phi^*, \psi^* ; \lambda) \]
\[ \psi = \psi^* + \lambda q(A^*, \Delta^*, \phi^*, \psi^* ; \lambda) \],

where the generating functions are expanded as power series in \( \lambda \):

\[ u = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n u_n(A^*, \Delta^*, \phi^*, \psi^* ) , \quad \text{etc.} \quad (19) \]

These variables have time evolutions

\[ A^* = \lambda \bar{F}^*(A^*, \Delta^*, \phi^*, \psi^* ; \lambda) \]
\[ = \sum_{n=1}^{\infty} \lambda^n u_n(A^*, \Delta^*, \phi^*, \psi^* ) , \quad \text{etc.} \quad (20) \]
It must be that
\[ \lambda^* + \lambda \dot{\lambda} = \lambda \bar{F}(\lambda^* + \lambda \mu^*, \lambda^* + \lambda \nu^*, \lambda^* + \lambda \rho^*, \lambda^* + \lambda q^*) , \quad \text{etc.} \quad (21) \]

If we now substitute for the derivatives of the generating functions
\[ \dot{\lambda} = \frac{2u}{\partial A} \lambda^* + \frac{2u}{\partial \Delta^*} \mu^* + \frac{2u}{\partial \nu^*} (\nu^* + \Omega) + \frac{2u}{\partial \rho^*} (\rho^* + \psi^*) , \quad \text{etc.} , \quad (22) \]
we find
\[ \dot{\bar{F}}^* + \Omega \dot{\lambda} = \frac{2u}{\partial \phi^*} \varphi^* = \bar{F}(\lambda^* + \lambda \mu^* , \ldots) - \lambda \frac{2u}{\partial A} \bar{F}^* \]
\[ - \lambda \frac{\partial \bar{F}}{\partial \varphi^*} - \lambda \frac{\partial \bar{G}}{\partial \Delta^*} - \lambda \frac{\partial \bar{L}}{\partial \psi^*} \quad (23) \]
and similar equations for \( \bar{G}^*, \bar{R}^*, \bar{L}^* \).

Substituting the series expansions (19) and (20), and equating coefficients of powers of \( \lambda \), we find
\[ \lambda^* : F_1^* + \Omega \frac{2u_1}{\partial \phi^*} + \omega \frac{2u_1}{\partial \psi^*} = \bar{F}(\lambda^*, \Delta^*, \phi^*, \psi^*) \quad (24) \]
\[ \lambda^1 : F_2^* + \Omega \frac{2u_2}{\partial \phi^*} + \omega \frac{2u_2}{\partial \psi^*} = \bar{F} \frac{\partial u_1}{\partial A} + \frac{\partial \bar{F}}{\partial \varphi^*} u_1 + \frac{\partial \bar{F}}{\partial \Delta^*} \nu_1 + \frac{\partial \bar{F}}{\partial \psi^*} p_1 + \frac{\partial \bar{F}}{\partial \psi^*} q_1 \]
\[ - \frac{\partial u_1}{\partial A} \bar{F}^* - \frac{\partial u_1}{\partial \phi^*} \bar{F}^* \cdot \bar{F}^* - \frac{\partial u_1}{\partial \Delta^*} \bar{G}^* - \frac{\partial u_1}{\partial \psi^*} \bar{L}^* , \quad \text{etc.} \quad (25) \]
The residual ambiguity in the transformations is removed by requiring
\[ F_1^* = \varphi \bar{F} = \frac{\Omega A}{2} \left( 1 - \frac{1}{4} \frac{\mu A^2}{\Delta^2} \right) \equiv F_1^*(A, \Delta) , \quad (26) \]
\[ G_1^* = \eta \bar{G} = - \frac{\Omega A}{2} \left( 1 + \frac{3}{4} \frac{\mu A^2}{\Delta^2} \right) \equiv G_1^*(A, \Delta) , \]
\[ K_1^* = \eta \bar{K} = 0 , \quad (27) \]
\[ L_1^* = \eta \bar{L} = 0 . \]

The complementary part of Eq. (24) gives a first-order partial differential equation for \( u_1 \). Integration of this condition gives
\[ u_1 = \frac{\Omega \Delta}{2} \left\{ -\frac{1}{2\Omega} \left( 1 - \frac{\mu \Delta^2}{2} \right) \sin 2\phi - \frac{\mu \Delta^2}{4\omega} \sin 2\psi \\
+ \frac{\mu \Delta^2}{16\omega} \sin 4\phi - \frac{\mu \Delta^2}{2} \left[ \frac{\sin (\phi + \psi)}{\Omega + \omega} + \frac{\sin (\phi - \psi)}{\Omega - \omega} \right] \\
- \sin (3\phi + \psi) - \sin (3\phi - \psi) \right\} \\
+ \frac{\mu \Delta^2}{4} \left[ \frac{\sin 2(\phi + \psi)}{2(\Omega + \omega)} + \frac{\sin 2(\phi - \psi)}{2(\Omega - \omega)} \right] \}
+ \bar{u}_1(A, \Delta, \omega, \phi - \omega \psi), \] (28)

where \( \bar{u}_1 \) is an arbitrary differentiable function of three variables. There are similar results for \( v_1, p_1, q_1 \), which we do not write out.

In the next order of approximation, Eq. (25) gives

\[ F_2^* = \mathcal{P} \left\{ \frac{\partial F}{\partial A} u_1 + \frac{\partial F}{\partial A} v_1 + \frac{\partial F}{\partial A} p_1 + \frac{\partial F}{\partial A} q_1 - \frac{3u_1}{\partial A} F_1 + \frac{3u_1}{\partial A} G_1 \right\} \]

= 0

since every product in the first four terms is of a cosine times a sine and has no non-oscillatory part. Neither do the last two terms contribute, because

\[ F_1^* = \mathcal{P} F_1^*, \quad G_1^* = \mathcal{P} G_1^* \]

and

\[ \frac{3u_1}{\partial A} = Q \frac{3u_1}{\partial A}, \quad \frac{3u_1}{\partial A} = Q \frac{3u_1}{\partial A}. \] (30)

Similarly,

\[ G_2^* = 0 \] (31)

while some calculation shows that

\[ K_2^*(A, \Delta) = -\frac{5}{8} \left\{ -1 + \frac{3}{2} \mu \Delta^2 - \mu^2 A^2 \Delta^2 - \frac{1}{4} \mu^4 A^4 - \frac{11}{32} \mu^2 A^4 \right\} \\
+ \frac{\Omega^2}{64(\Omega^2 - \omega^2)} \left\{ (3\Omega + 5\omega) \mu^2 A^2 \Delta^2 + \Omega \mu^2 \Delta^2 \right\} \\
+ \frac{3\Omega^2}{4(9\Omega^2 - \omega^2)} \mu^2 A^2 \Delta^2, \] (32)
\[ L_2^*(A, \Delta) = -\frac{\Omega^2}{8\omega} \left[ \frac{3}{2} \left( 1 + \frac{3}{\delta} \right) \mu \Delta^2 + \left( 1 + \frac{3}{\delta} \right) \mu A^2 - \mu^2 A^2 \Delta^2 \right. \]
\[ \left. - \frac{11}{32} \mu^2 \Delta^4 - \frac{1}{4} \mu^2 A^2 - \frac{1}{8} \left( 1 + \frac{3}{\delta} \right)^2 \right\} \]
\[ + \frac{\Omega^2}{16(\Omega^2 - \omega^2)} \left\{ (\Omega + 3\omega) \mu^2 A^2 \Delta^2 + \frac{\Omega}{4} \mu^2 A^2 \right\} \]
\[ + \frac{\Omega^2 (\Omega - 3\omega)}{16(\Omega^2 - 9\omega^2)} \mu^2 A^2 \Delta^2 \] (33)

We have used the fact that there is no way of constructing non-zero \( u_1, \tilde{u}_1, \tilde{p}_1, \tilde{q}_1 \) with any of them having the same \( \Phi \) and \( \Psi \) dependence as any term in \( \partial \Phi/\partial A, \ldots, \partial \Psi/\partial \Phi \).

We can now write down the non-oscillatory equations of motion for the corresponding deterministic system:
\[ \dot{A}^* = \frac{\lambda \Omega A^*}{2} \left( 1 - \frac{\mu A^*}{4} - \frac{\mu A^*}{2} \right) + \mathcal{O}(\lambda^3), \] (34)
\[ \dot{\Delta}^* = -\frac{\lambda \Omega \Delta^*}{2} \left( 1 + \frac{3}{\delta} - \frac{\mu A^*}{4} - \frac{\mu A^*}{2} \right) + \mathcal{O}(\lambda^3), \] (35)
\[ \dot{\psi}_1^* = \lambda^2 K_2(A^*, \Delta^*) + \mathcal{O}(\lambda^3) \] (36)
\[ = \ldots, \]
\[ \dot{\psi}_2^* = \lambda^2 L_2^*(A^*, \Delta^*) + \mathcal{O}(\lambda^3) \] (37)
\[ = \ldots, \]

and reconstruct the original quantities
\[ e(t) = A^* \cos \phi^* + \lambda u_1(A^*, \Delta^*, \phi^*, \psi^*) \cos \phi^* \]
\[ - \lambda A^* \tilde{p}_1(A^*, \Delta^*, \phi^*, \psi^*) \sin \phi^* + \mathcal{O}(\lambda^3) \] (38)
\[ \delta(t) = \Delta^* \cos \psi^* + \lambda v_1(A^*, \Delta^*, \phi^*, \psi^*) \cos \psi^* \]
\[ - \lambda \Delta^* \tilde{q}_1(A^*, \Delta^*, \phi^*, \psi^*) \sin \psi^* + \mathcal{O}(\lambda^3). \] (39)

Explicit expressions have been obtained for these but they are too long to be given here.

The fluctuating equations simplify to
\[ \hat{A}^* = \frac{\lambda \Omega^*}{2} \left( 1 - \frac{\mu A^*}{4} - \frac{\mu A^*}{2} \right) - \lambda \Omega \xi(t) \sin \phi^* \]

\[ \equiv \lambda \hat{F}(A^*, \Delta^*, \phi^*, \psi^*, \xi) , \]

\[ \hat{\Delta}^* = \frac{\lambda n A^*}{2} \left( 1 + \frac{\mu A^*}{4} - \frac{\mu A^*}{2} \right) - \lambda \omega^2 \xi(t) \sin \psi^* \]

\[ \equiv \lambda \hat{G}(A^*, \Delta^*, \phi^*, \psi^*, \xi) , \]

\[ \hat{\phi}^* = \lambda^2 k_2^* (A^*, \phi^*) - \frac{\lambda \Omega}{A^*} \xi(t) \cos \phi^* \]

\[ \equiv \lambda \hat{K}(A^*, \phi^*, \psi^*, \xi) , \]

\[ \hat{\psi}^* = \lambda^2 L_2^* (A^*, \psi^*) - \frac{\lambda \omega^2}{\omega^2} \xi(t) \cos \psi^* \]

\[ \equiv \lambda \hat{L}(A^*, \phi^*, \psi^*, \xi) . \]

We have relied on the same argument as in Part I to drop terms of order \( \lambda^2 \) containing \( \xi \). As we did there, we now drop asterisks and tildes.

We note that the shifts of the synchrotron and betatron tunes given by Eqs. (36) and (37) are small but now have a fairly complicated dependence on both amplitudes.

3. THE FOKKER-PLANCK EQUATION

If a set of random variables \( x_i \) satisfy evolution equations

\[ \dot{x}_i = \lambda f_i(x, t) , \quad i = 1, \ldots \]

then Ref. 5, § 4.9, gives the form of the corresponding Fokker-Planck equation

\[ \dot{W} = -\lambda \sum_i \left[ \frac{\partial}{\partial x_i} \left( \left< f_i \right> + \lambda \sum_j \int_{-\infty}^{0} \left< \frac{\partial f_i}{\partial x_j} f_j \right> \, dt \right) \right] W \]

\[ + \lambda^2 \sum_i \sum_j \frac{\partial^2}{\partial x_i \partial x_j} \left( \int_{-\infty}^{0} \left< f_i f_j \right> \, dt \right) W \]

where \( \left<,\right> \) is as defined in Part I. Applying this, with \( x = (A, \Delta, \phi, \psi) \), gives in our case the evolution of the distribution function in synchro-betatron phase space. As in Part I, we first eliminate oscillatory terms from the transport coefficients. Several of these may be taken over from the calculation in Part I and the remaining non-vanishing ones are
\[ \int_0^\infty \{ \xi, \xi_0^* \} \, \mathrm{d} \tau = \frac{v^2 \Omega^*}{\omega^2} \int_0^\infty \{ \xi \xi_0^* \} \sin \omega \sin (\omega \tau + \psi) \, \mathrm{d} \tau \]

\[ \frac{\rho}{\omega^2} \frac{v^2 \Omega^*}{8 \omega^2} S(\omega), \]  

\[ \int_0^\infty \{ L, L_0 \} \, \mathrm{d} \tau \frac{v^2 \Omega^*}{8 \omega^2 \Delta} S(\omega), \]  

\[ \int_0^\infty \left\{ \frac{3G}{\partial \psi^*} L_0 \right\} \, \mathrm{d} \tau \frac{v^2 \Omega^*}{8 \omega^2 \Delta} S(\omega), \]  

where

\[ S(\omega) = 2 \int_0^\infty \{ \xi \xi_0^* \} \cos \omega \tau \, \mathrm{d} \tau \]  

is the spectral density of the quantum noise at the betatron frequency.

Our Fokker-Planck equation is then

\[ \frac{2}{\lambda \Omega} \dot{W} = -\frac{\partial}{\partial \Delta} \left[ \frac{1}{4} \left( 1 - \frac{u \Delta^2}{4} - \frac{u \Delta^2}{4} \right) \Delta W + \frac{\lambda G \Omega(\Omega)}{4 \Delta} W \right] \]

\[ + \frac{\Delta W}{\Delta} \left[ \frac{\lambda G \Omega(\Omega)}{4 \Delta} W \right] \]

\[ + \frac{\Omega^2}{64(\Omega^2 - \omega^2)} \left\{ (3\Omega + 5\omega) \mu^2 A^2 \Delta^2 + \Omega \mu^2 \Delta^2 \right\} W \]

\[ + \frac{3 \Omega^3}{4(9\Omega^2 - \omega^2)} \mu^2 A^2 \Delta^2 W \]

\[ + \frac{\lambda G \Omega}{4 \Delta} \left[ \frac{3}{2} \left( 1 + \frac{3}{4} \Delta^2 \right) \Delta W + \left( 1 + \frac{3}{4} \Delta^2 \right) \mu A^2 - \mu^2 A^2 \Delta^2 - \frac{11}{32} \mu^2 \Delta^4 \right. \]

\[ - \frac{1}{4} \mu^2 A^4 - \frac{1}{8} \left( 1 + \frac{3}{4} \Delta^2 \right)^2 \right\} W \]

\[ + \frac{\Omega^2}{16(\Omega^2 - \omega^2)} \left\{ (\Omega + 3\omega) \mu^2 A^2 \Delta^2 + \frac{\Omega}{2} \mu^2 A^4 \right\} W \]

\[ + \frac{\Omega^2}{16(\Omega^2 - 9\omega^2)} \mu^2 A^2 \Delta^2 W \]

\[ + \frac{\lambda G \Omega}{4} \left[ S(\Omega) \left( \frac{\partial^2}{\partial A^2 \partial^2} \right) + \frac{v^2 \Omega^2}{\omega^2} S(\omega) \left( \frac{\partial^2}{\partial \Delta^2} + \frac{1}{\Delta^2} \frac{\partial^2}{\partial \psi^2} \right) \right] W. \]
We note that the drift coefficients for the phases become infinite at the synchro-betatron resonances

\[ \begin{align*}
\Omega &= 2\omega, \\
\Omega &= \pm 3\omega, \\
3\Omega &= \pm \omega.
\end{align*} \] (49)

It is clear that higher-order resonances would show up if our expansion were carried out to higher orders. We shall assume that the machine is operated well away from the resonances (49). Otherwise our fundamental assumptions break down.

Assuming a uniform distribution in the phases we reduce Eq. (48) to an equation for

\[ W(\Lambda, \Delta) = \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi W(\Lambda, \Delta, \phi, \psi). \] (50)

The value of \( S(\omega) \) is found in the same way as that of \( S(\Omega) \) in Part I:

\[ \nu \lambda \Theta S(\omega) = \frac{4\omega^2}{\sigma_X^2 A W} D_W^2, \] (51)

where \( \sigma_X \) is the natural spread of betatron amplitudes at the wiggler (i.e. the zero current limit when the wiggler is not excited) and \( D_W \) is the horizontal dispersion at the wiggler.

Thus the Fokker-Planck equation finally reduces to

\[ \begin{align*}
\frac{2}{\Lambda \Omega} \dot{W} &= \frac{3}{\Lambda \Delta} \left[ \left( 1 - \frac{1}{4} \frac{\mu \Delta^2}{2} \right) \Delta W + \frac{2\sigma_E^2}{a E_0} \cdot \frac{W}{A} \right] \\
&- \frac{3}{\Delta \Lambda} \left[ \left( 1 + \frac{3}{a} \frac{\mu \Delta^2}{4} - \frac{1}{2} \right) \Delta W + \frac{\sigma_X^2}{a D_W^2} \cdot \frac{W}{\Lambda} \right] \\
&+ \frac{2\sigma_E^2}{a E_0} \frac{\sigma^2}{3A^2} W + \frac{\sigma_X^2}{a D_W^2} \frac{\sigma^2}{3A^2} W.
\end{align*} \] (52)

The "potential conditions" are not satisfied so that in equilibrium the particle current is not zero. There is always a rotational flow of particle density in \( (\Lambda, \Delta) \) space, as well as the flows along the \( \phi \) and \( \psi \) directions in the full phase space.
4. THE QUASI-EQUILIBRIUM DISTRIBUTION

Equating the right-hand side of Eq. (52) to zero gives a partial differential equation which is not as easily solved as Eq. (65) of Part I. Indeed, it is clear that no normalisable stationary solution exists because of the anti-damping at large betatron amplitudes. We may, however, hope that a quasi-equilibrium condition exists, in which the bulk of the particles is contained within the region of damping at small betatron amplitudes.

In order to calculate such a solution we make the statistical assumption that, in the drift coefficient for $A$, we may replace $\Delta^2$ by its average value and vice versa. This is justified by the fact that the betatron frequency is usually much larger than the synchrotron frequency, so that $A$ and $\Delta$ may, in an excellent approximation, be regarded as independent stochastic processes. This allows us to factorise $W$ into

$$W(A,\Delta) = W_1(A)W_2(\Delta)$$

where $W_1$ and $W_2$ satisfy

$$\frac{\partial}{\partial A} \left[ 1 - \frac{\mu A^2}{4} - \frac{\mu (\Delta^2)}{2} \right] A W_1 + \frac{2\sigma_E^2}{aE^2} \frac{W_1}{A} = \frac{2\sigma_E^2}{aE^2} \frac{\partial^2 W_1}{\partial A^2}$$

and

$$\frac{\partial}{\partial \Delta} \left[ 1 + \frac{3}{a} - \frac{\mu \Delta^2}{4} - \frac{\mu (\Delta^2)}{2} \right] \Delta W_2 + \frac{\sigma_x^2}{aD_w^2} \frac{W_2}{\Delta} = \frac{\sigma_x^2}{aD_w^2} \frac{\partial^2 W_2}{\partial \Delta^2}.$$  

The integration is now simple and gives

$$W_1(A) = N^{-1} \frac{\sqrt{2\pi} A}{2} \exp \left[ \frac{aE^2}{4\sigma_E^2} \right] \int \left[ 1 - \frac{\mu}{2} (\Delta^2) - \frac{\mu A^2}{8} \right],$$

$$W_2(\Delta) = M^{-1} \frac{\sqrt{2\pi} \Delta^2}{2} \exp \left[ - \frac{aD_w^2}{2\sigma_x^2} \Delta^2 \right] \left[ 1 + \frac{3}{a} - \frac{\mu}{2} (\Delta^2) - \frac{\mu}{8} \Delta^2 \right],$$

where $N$ and $M$ are suitable constants. Equation (56) is of the same form as Eq. (67) of Part I if we make the replacements

$$R + R' = \frac{a}{(2b)^{\sqrt{2}}(\sigma_E/E_x)} \left[ 1 - \frac{\mu}{2} (\Delta^2) \right] = R \left[ 1 - \frac{2(\Delta^2)}{A^2} \right]$$

$$A^2 + \Delta^2 = \frac{4}{\mu} \left[ 1 - \frac{\mu}{2} (\Delta^2) \right].$$
To find $\langle \Delta^2 \rangle$ we need to replace the unbounded solution (57) by its restriction to a finite volume of betatron phase space which must include the emittance of the beam. This is possible provided $a$ is sufficiently small that we may write, as a rough approximation:

$$W_0^{(0)}(\Delta) = \mathcal{M}^{-1} \frac{\sqrt{\Delta}}{2} \exp \left[ -\frac{D_w^2}{2\sigma_x^2} (3 + a)\Delta^2 \right] .$$  \hspace{1cm} (60)

This would be a Gaussian distribution with a spread rather reduced compared to the natural one. The particle distribution will be adequately contained if

$$\left\langle \frac{\mathcal{H}_0^2(\Delta^2 + 2a^2)}{A_0^2} \right \rangle^{(0)} = \frac{1}{A_0^2} \left\langle \frac{(\Delta^2)^2}{(\Delta^2 + 2a^2)} \right \rangle^{(0)} \leq 1 + \frac{3}{a} .$$  \hspace{1cm} (61)

Putting back the value of $a$ found in Part I and the "ideal" value of $\langle \Delta^2 \rangle$ gives the condition

$$\frac{\left\langle \Delta^2 \right \rangle^{(0)}}{A_0^2} \leq 1 + \frac{3}{0.192} - 2(1.5) = 13.6 .$$  \hspace{1cm} (62)

The first approximation to $\langle \Delta^2 \rangle$, from (60) is

$$\langle \Delta^2 \rangle^{(0)} = \frac{\sigma_x^2}{(3 + a)D_w^2} .$$  \hspace{1cm} (63)

In LEF, with the design horizontal emittance $\sigma_x^2/\beta_H = 5.4 \times 10^{-8}$ m, $\sqrt{\beta_H} = 6.0$ m$^{1/2}$, $D_w = 1.2$ m at the wiggler\textsuperscript{6)}, $a = 0.192$, and $b = 3.67 \times 10^4$, we find

$$\frac{\sigma_x^2}{D_w^2} = 1.9 \times 10^{-6} ,$$

$$A_0^2 = 2.1 \times 10^{-8} ,$$

$$\frac{\left\langle \Delta^2 \right \rangle^{(0)}}{A_0^2} = 0.028 ,$$

and the condition (62) is amply satisfied.

Equation (60) is, however, a poor approximation. For purposes of calculation, a better one is obtained by expanding Eq. (57):

$$W_0^{(1)}(\Delta) = \mathcal{M}^{-1} \frac{\Delta}{A_0} \exp \left[ -\frac{D_w^2}{2\sigma_x^2} (3 + a)\Delta^2 \right] \left[ 1 + \frac{aD_w^2}{\sigma_x^2} \Delta^2 \left( \frac{\left\langle \Delta^2 \right \rangle}{A_0^2} + \frac{\Delta^2}{4A_0^2} \right) \right] ,$$  \hspace{1cm} (65)
where the normalisation constant is

$$ M = \frac{\sigma_X^2}{A_0 D_\nu^2} \left[ 1 + \frac{2a}{3 + a} \frac{\langle A^2 \rangle}{A_0^2} + \frac{2a}{3 + a} \frac{\sigma_X^2}{D_\nu^2 A_0^2} \right] . $$  

(66)

This gives a larger value for the r.m.s. betatron amplitude:

$$ \frac{\langle \Delta^2 \rangle^{(1)}}{A_0^2} = \frac{28 \left[ 9 + 6(1 + 2\chi + \theta) a + (1 + 2\chi) a^2 \right]}{(3 + a) \left[ 9 + 2 (3 + 3\chi + \theta) a + (1 + 2\chi) a^2 \right]} , $$  

(67)

where

$$ \theta = \frac{\sigma_X^2}{D_\nu^2 A_0^2} , \quad \chi = \frac{\langle A^2 \rangle}{A_0^2} . $$  

(68)

In fact, Eq. (67) is really a transcendental equation, to be solved for $\langle \Delta^2 \rangle$, since $R'$ depends on $\langle \Delta^2 \rangle/A_0^2$ through Eq. (58), and

$$ \chi = \frac{\langle A^2 \rangle}{A_0^2} = 1 + \frac{1}{\pi \sqrt{2} R' \omega (-iR')} $$  

(69)

[see Eq. (86) of Part I]. However if $R'$ lies in the region of weak dependence of Eq. (69) this presents little difficulty (see Fig. 4 of Part I). For example, if we impose the condition $R' = 0.5409$ and take the parameters already quoted for LEP Version 11, we quickly find a solution with

$$ a = 0.192 , \quad \frac{\langle \Delta^2 \rangle^{(1)}}{A_0^2} = 0.0490 , \quad b = 3.67 \times 10^4 . $$  

(70)

Everything fits and we are sure of the stability of the distribution in both planes. In fact $a$ and $b$ change only slightly.

Equation (70) actually implies a narrowing of the transverse distribution. We find the ratio of the spread with the non-linear wiggler to the natural spread to be

$$ \left[ \frac{\langle \Delta^2 \rangle^{(1)}_{D^2 \nu}}{\sigma_X^2} \right]^{\frac{1}{4}} = \left( \frac{0.0490}{0.192} \right)^{\frac{1}{4}} = 74\% . $$  

(71)

In passing, we note that in the linear damping limit (see Part I, Section 5) the approximate distribution (65) does not go over into the Gaussian because, as the limit is approached, it passes into a region of invalidity of Eq. (62). Formally, however, the unbounded distribution (57) does have the required limiting property.
Figure 1 shows the optimum line-current density in amperes as a function of the longitudinal coordinate in cm for the parameters of LEP version II, Phase I \(^6\). The dotted line shows the natural bunch shape without the wiggler. The peak current is reduced from 1 kA to about 250 A and a satisfactory lifetime of 10 h is maintained.

In Figs. 2 and 3, we provide a graphical illustration of the above argument. Defining scaled synchrotron and betatron amplitudes
\[
x = \mu \Delta^* / 2 , \quad y = \mu \Delta^* / 2 ,
\]
we can combine Eqs. (34) and (35) into
\[
\frac{dy}{dx} = \frac{-y(1 + 3/a - 2x^2 - 2y^2)}{x(1 - x^2 - 2y^2)} .
\]
This defines the phase trajectories of the deterministic system.

Except for special values of \(a\), this equation cannot be integrated analytically \(^7\). The dash-dotted curves in Fig. 2 show the results of a numerical integration with \(a = 0.192\). The phase space divides into two regions. All particles starting out in one region eventually fetch up at the stable limit cycle \((1,0)\), while all particles starting out in the other are repelled by the unstable limit cycle \((0, \sqrt{1+3a})\) and approach asymptotically \((0,0)\). [In fact \((1,0)\) is a stable node of Poincaré index +1 while \((0, \sqrt{1+3a})\) is a saddle of index -1.]

Let us recall that these smoothly varying quantities are only thus as a result of the averaging procedure. The original amplitudes \(A\) and \(\Delta\) do not, even in the deterministic system, have trajectories which are predictable without knowledge of the phases \(\phi\) and \(\psi\). Each smooth trajectory in Fig. 1 will break up into many rapidly gyrating curves which will only in some average sense follow the smooth curve.

On top of this comes the stochastic disturbance due to the quantum excitation. We have argued that longitudinal and transverse stability are composable because a solution of the Fokker-Planck equation exists which is exponentially small in a sufficiently broad region straddling the border of the acceptance of the machine. In Fig. 3, we have superposed a contour plot of the function
\[
\log \left[ \frac{M}{\chi y} W_1(A_0 x) W_2(A_0 y) \right] ,
\]
with \(W_1\) and \(W_2\) given in Eqs. (56) and (57). Only a band of contours between \(-200\) and zero are shown, across which the phase-space density decreases by a factor \(e^{-200}\).
It is clear from this figure that if the particles are originally inside the
region \(0 < x \leq 2.0, 0 < y \leq 1.0\) then they stay there for a very long time. \(W_2\)
does not regain its value at the origin until \(y \approx 5\), a very large value.

Finally, it can be shown that replacement of \(\Delta^2\) and \(A^2\) by their average
values, in Eqs. (34) and (35), respectively, does not significantly alter the
trajectories of Fig. 1 in the region where the particles are confined.

5. QUANTUM LIFETIME

Having established that the transverse distribution is approximately
Gaussian, it is not necessary to derive a cumbersome formula for the transverse
quantum lifetime \(\tau^X_q\), although this could be done easily enough if high accuracy
were required. Instead we may simply adapt the standard formula for Gaussian
distributions\(^8,9\), assuming that the aperture limit is at \(n\) times the natural \(\sigma_x\).

\[ \tau^X_q \approx \frac{2\tau}{n^2} \frac{\sigma_x^2}{\langle \Delta^2 \rangle (1)} \exp \left[ \frac{n^2\sigma_x^2}{\langle \Delta^2 \rangle (1)} \right]. \]  \hspace{1cm} (74)

Equation (71) shows that this will be somewhat longer than \(\tau^X_q\) without the dipole-
octupole wiggler, although the correction factor in Eq. (65) may make this a
little too optimistic.

Equation (75) of Part I, with the replacements (58) and (59), gives \(\tau^E_q\). The
total lifetime is given by

\[ \frac{1}{\tau_q} = \frac{1}{\tau^E_q} + \frac{1}{\tau^X_q} + \ldots, \]  \hspace{1cm} (75)

where the reciprocals of the finite lifetimes due to other effects should be added
to the right-hand side.

6. CHOICE OF WIGGLER STRENGTHS

To choose the wiggler strengths required to achieve a given bunch shape and
longitudinal quantum lifetime, one should proceed as follows:

1) From Fig. 5 of Part I, choose the values of \(R'\) and \(a^2\) which give the required
bunch shape and lifetime. If the requirement is for a given peak current,
then \(R'\) should be chosen first, with the help of Fig. 4 of Part I.

2) Given now \(R'\) and \(a^2\), \(a\) and \(b\) have to be calculated. In most practical cases,
Eqs. (88) and (89) of Part I will give rather accurate values. If greater
accuracy is required, then Eq. (58) must be solved simultaneously with

\[ \bar{a}^2 = a^2 \left( \frac{\sigma_e}{\bar{E}_0} \right)^2 \frac{b}{4a} \left[ 1 - \frac{2a}{b} \langle \Delta^4 \rangle \right]^{-1}. \]
and Eq. (67). Iteration, starting with the values given in Eqs. (88) and (89) of Part I will quickly converge (cf. the example of LEP Version I given above).

3) Check that the distribution in betatron amplitudes is contained within the unstable limit cycle, i.e. verify condition (62).

4) The relation of \( a \) and \( b \) to the physical parameters of the wiggler magnets is given by the formulae in Ref. 2 and the present Section 2. The derivation of these formulae assumed uniform field properties along the length of the wigglers. Determining exactly what wiggler is required is now a problem of detailed magnet design.

In general, the choice of \( R \) and \( \delta^2 \) given in Section 8 of Part I is recommended as likely to produce the best amelioration of collective effects at high current. In practice, however, a given wiggler will run out of influence as the beam is accelerated above a certain energy, but some compromise will still be useful with other values of \( R' \).

7. TOPOLOGICAL ASPECTS

In most of our discussion we have talked about positive \( a \) and \( b \), so that a stable limit cycle exists in synchrotron phase space. The unstable limit cycle in betatron phase space does not present a serious problem because of the happy numerical values of the parameters of large e\( ^+e^- \) machines. It is interesting to consider what may happen in the whole range of real values of \( a \) and \( b \), the "control space" of the wiggler.

Three lines in the \((a,b)\) plane are significant (the "deterministic bifurcation set")

\[
b = 0, \quad a = 0, \quad a = -3. \tag{76}
\]

With only a dipole-quadrupole wiggler excited, the point representing the machine has to stay on the segment \( b = 0, -3 < a < 0 \) for stability in both planes. With the dipole-octupole wiggler excited, it has to move into the upper half-plane for stability. Whenever one of the lines (76) is crossed, a bifurcation of the stable phase angle occurs. With Minorsky's\(^{10}\) convention, we may write

\[(S,SU) \rightarrow (US,SU)\]

for the bifurcation which occurs as \( a \) increases through zero for positive \( b \). (US denotes, for example, an Unstable point surrounded by a Stable limit cycle.) The first symbol in brackets gives the stability structure surrounding the origin in synchrotron phase space and the second applies to betatron phase space.

Figure 4 shows the stability properties in the whole control space. The trajectories of the deterministic system are shown schematically. The qualitative structure of the equilibrium distribution for the fluctuating system may be
inferred by associating a peak with a stable point, a crater with a stable limit

cycle, etc.\textsuperscript{11}). Several different dissipative structures in four dimensions are
generated.

It has usually been assumed that all storage rings are located exactly on the
segment $b = 0$, $-3 < a < 0$, which is part of the bifurcation set. Clearly, this
is never exactly true because of machine imperfections.

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Fig. 1 Bunch shapes showing the instantaneous current in amperes for LEP Version II in acceleration mode, with and without the dipole-octupole wiggler. The bunch shape parameter $R_{rms}$ is 0.5409.
Fig. 2  Averaged trajectories of the deterministic system
Fig. 3 Same as Fig. 2, with contours of the logarithm of the distribution function superposed
Fig. 4 Stability structures and bifurcation set in the control space of dipole-quadrupole and dipole-octupole wigglers.