SECOND ORDER CONTRIBUTIONS TO THE STRUCTURE FUNCTIONS
IN DEEP INELASTIC SCATTERING : (III) THE SINGLET CASE

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ABSTRACT

Pointlike QCD predictions for the singlet part of the structure functions are given up to next to leading order of perturbation theory. This generalises the result obtained in the first paper of this series, which deals with the non-singlet case.

An interesting by-product, here shown, is an exact and simple analytical expression for the anomalous dimension matrix to second non-trivial order in the QCD coupling constant.

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1. **INTRODUCTION**

During the last few years, deep inelastic scattering\(^1\) has attracted much interest both from the theoretical and experimental points of view, the main reason being the possibility of testing QCD through its particular pattern of scaling violations. Different approaches have been followed in comparing these predictions with the data. One of the most frequent is based on fitting the \(Q^2\) evolution of the moments of the structure functions. This procedure has several drawbacks which have been emphasized by a number of authors\(^2\),\(^3\),\(^4\),\(^5\). An alternative procedure, the pointlike reconstruction method, has been argued to be the cleanest way of testing QCD scaling violation predictions\(^2\),\(^3\). It, however, presents several theoretical difficulties which, up to now, have limited its range of applicability. As a matter of fact, one needs to handle analytically all the relevant quantities appearing in the formulae i.e., anomalous dimensions coefficient functions and the \(\delta\) function.

Pointlike reconstruction of the non-singlet part of the structure functions was first performed by Gross\(^6\) to leading order of QCD. Recently\(^7\), this method has been carried on up to next-to-leading order. However, it has remained restricted to the non-singlet part. This was due to several facts: first, the anomalous dimensions matrix was only known to leading order; second, there are a number of difficulties on the way, specific to the singlet case; last, but not least, the amount of work required to achieve the final result is several times greater than for the non-singlet case. The purpose of this work is to enlarge the method to cover the singlet part of the structure functions as well. This is done to leading and next-to-leading order in QCD.

It is well known that the O.P.E. (operator product expansion) and renormalization group allow one to write an expansion for the moments of a given structure function in the following form\(^8\):

\[
\mu_{N-2}(Q^2) = C_N(Q^2) A_N^a + \text{higher twists}
\]  

\(1.1\)

\* Summation on repeated superindices is to be understood.
In the above expression \( \mu_{N-2}(q^2) \) represents the \( N-2 \) moment of the structure function. The index \( a \) runs over the different types of operators (gluon = A, non-singlet quark = NS, singlet quark = F) and \( q^2 = -q'^2 \) will denote the transverse momentum squared at the lepton vertex.

The first term in the expansion (1.1) is known as the twist two contribution, and dominates at high \( Q^2 \) since higher twists decrease as inverse powers of \( Q^2 \). The constants \( A_N^a \) are unknown quantities, which carry all the dependence on the target nature (proton, neutron, deuterium, iron, etc). The \( C_N^a(q^2) \) satisfy a renormalization group equation and therefore can be written as:

\[
C_N^a(q^2, g_0) = C_N^b(q_0^2, \bar{g}(Q^2)) \left[ T \exp \left\{ \int_0^t dt' \gamma_N(t') \right\} \right]^{ba}
\]

(1.2)

where \( g_0 \) is the quark-gluon coupling constant defined at the reference point \( q_0^2 \). The matrix \( \gamma_N \) is the anomalous dimension matrix, \( \bar{g}(Q^2) \) is the QCD running coupling constant and \( t = \log Q^2/q_0^2 \). All the functions appearing in the right hand side of (1.2) can be calculated perturbatively in QCD. As a consequence of the asymptotic freedom of the theory, the perturbative expansion will make sense provided \( Q^2 \) and \( q_0^2 \) are large enough.

Expression (1.1) implies a similar result for the structure functions themselves, namely

\[
F(x, Q^2) = F^{(2)}(x, Q^2) + \ldots
\]

(1.3)

where the \( N-2 \) moment of \( F^{(2)} \) is given by \( C_N^a A_N^a \). Again at large enough \( Q^2 \) \( F \) and \( F^{(2)} \) can be identified. However, the actual comparison of QCD predictions with the data is done in a range of \( Q^2 \) where the systematic neglecting of twists higher than two might render the results meaningless. It has been argued that the influence of higher twists on the structure function is mostly important at \( x = 1 \) \( [(1-x)\alpha_s/Q^2] \). Therefore, provided that this region is left out of the analysis, the known twist two contribution should dominate. On the contrary, the \( N^{th} \) moment of the structure functions, for \( N \) large enough (4 or 6 may already be too much), is most sensitive to the region close to \( x = 1 \). These arguments can be given a quantitative content by using phenomenological parametrizations for the higher twist contributions. It therefore seems more favourable to work
with the structure functions themselves rather than with their moments, particularly since the latter quantities, not being directly obtained from the data, are poorly determined. To calculate the moments one has to extrapolate the structure functions to unexplored regions.

The approach which we will follow here is not the only possible way of expressing QCD predictions for the structure functions. Let us mention here the method due to Buras and Gaemers\textsuperscript{9)}, which has the great advantage of being quite simple to use. However, in its present form it only accounts for leading order QCD predictions. Besides, it seems justified for us to avoid the risks and/or the limitations of using simple parametrizations in favour of a more rigorous treatment.

Before summarizing the content of the following sections we would like to point out the relative importance of the singlet case in the whole pointlike reconstruction program. It is well known that the singlet part of the structure functions dominate at low values of \( x \). Previous analyses were thus compelled to leave out of their applicability range a region where large amount of data is available. Otherwise the singlet case was to be treated approximately\textsuperscript{10)}. With our results, the whole set of experimental data can be properly tested versus QCD predictions.

In the following section we give exact, explicit, analytical expressions for the anomalous dimension matrix elements up to next-to-leading order of QCD. This has been made possible after the work of Floratos, Ross and Sachrajda\textsuperscript{11)} who calculated all the relevant diagrams. However, their full result has not been published due to the lengthy way in which it was expressed. We have performed the single, double and triple summations appearing in their expressions, and end up with comparatively simple formulas which enable quicker numerical calculations, large \( N \) behaviour investigation and, most relevant for our purposes, the pointlike reconstruction method. Our work is therefore a necessary complement of the one by Floratos et al.

In Section 3 the strategy is presented. We give a suitable definition of the gluon distribution function. The moments of the pointlike reconstruction kernels are expanded in powers of \( \alpha \), thus getting rid of the \( T \) exponential appearing in the formulae. Our approach, though essentially equivalent to some previous ones, is presented in completely different terms.
The explicit evaluation of the kernels is done in Sections 4 and 5 for leading and next to leading order respectively. No diagonalization of the leading order anomalous dimension matrix is necessary since our procedure consists of expanding this quantity in powers of $1/N$.

In Section 6 the conclusions are presented. We include a guide for the use of our formulae in phenomenological analyses of scaling violations.

Target mass effects are not considered in this paper since their influence on the singlet part of the structure functions is small. Comparison with experiment is left for a future publication due to the length and internal unity of the present paper. The large number of published works on deep inelastic scattering has forced us to restrict ourselves to references more directly related to our work. For references of a more general character the reviews of Ref. 1) should be looked at. More specific references can be found in the first and second paper of this series 7),2).

2. THE ANOMALOUS DIMENSIONS

As shown in the Introduction, QCD predictions for the $Q^2$ evolution of the moments of the structure functions, are expressed in terms of certain quantities which can be calculated in perturbation theory. Of those quantities, the most lengthy and involved ones, by far are the anomalous dimensions of the singlet operators. Pointlike reconstruction requires the knowledge of their explicit dependence on the variable $N$ (the order number of the moment). This section is devoted to giving their analytical form up to second non-trivial order in $\alpha^\prime$.

Let us recall that the anomalous dimension of the non-singlet operators is a 2 by 2 matrix. It can be expanded in powers of $\alpha$, giving:

$$\gamma_N^{(\alpha)} = \gamma_N^{(0)} + \gamma_N^{(1)} \frac{\alpha^2}{16\pi^2} + \ldots$$

(2.1)

The leading order contribution $\gamma_N^{(0)}$ has been computed by Gross and Wilczek 12) and we recall, for completeness, its expression:

$$\gamma_N^{(0)} = 2c_F\left(\gamma_{S_1}(N) - 3 - \frac{2}{N(N+1)}\right)$$

(2.2a)

*) From now on $\bar{\alpha}$ will stand for $\bar{\alpha}^2 (Q^2)/4\pi$, $\alpha_0$ for $\bar{\alpha}^2 (Q_0^2)/4\pi$ and $\alpha$ for any one of both.
\[
\gamma_{N}^{(o) \text{FA}} = -8 T_R \frac{N^2 + N + 2}{N(N+1)(N+2)} \\
\gamma_{N}^{(o) \text{AF}} = -4 C_F \frac{N^2 + N + 2}{N(N^2 - 1)} \\
\gamma_{N}^{(o) \text{AA}} = 2 C_A \left( 4 S_1(N) - \frac{11}{3} - \frac{4}{N(N-1)} - \frac{4}{(N+1)(N+2)} \right) + \frac{8 T_R}{3} R
\] (2.2.b)

where \( T_R = \frac{1}{2} n_F \) (\( n_F \) = number of flavours), and \( C_F, C_A \) equal \( 4/3 \) and 3 respectively for the gauge group \( SU(3) \).

Floratos, Ross and Sachrajda\textsuperscript{11} have calculated the necessary diagrams to obtain the next contribution to the anomalous dimension matrix, \( \gamma_N(1) \). Their result is given in terms of some complicated auxiliary functions containing single, double and triple summations (running from 1 to N). It is possible to use these expressions in numerical computations of \( \gamma_N(1) \) for a few values of \( N \). However, their explicit behaviour with \( N \) is obscure and must be disentangled. To do so, we have performed all the summations appearing in the output of Floratos et al., kindly handed to us by the authors. We profited from the fact that all necessary intermediate formulae were already known. We refer the reader to the appendix of Ref. 7), where a complete survey of such intermediate results is given.

After a rather lengthy calculation we end up with comparatively simple expressions for the anomalous dimension matrix elements listed below:

\[
\gamma_{N}^{(o) \text{FF}} = C_F \left( 16 S_1(N) \frac{2N+1}{N^2(N+1)^2} + 16 \left( 2 S_1(N) - \frac{1}{N(N+1)} \right) \right.
\]

\[
\times \left( \frac{S_2(N) - S_2(N)}{N^2} \right) + 24 S_2(N) + 64 S_1(N) - 8 S_3(N) - \frac{1}{2} \right)
\]

\[
- 3 - 8 \left( \frac{3N^3 + N^2 - 1}{N^2(N+1)^3} - 16(-1)^N \frac{2N^2 + 2N + 1}{N^3(N+1)^3} \right) +
\]

\[
C_F C_A \left( \frac{536}{9} S_1(N) - 8 \left( 2 S_1(N) - \frac{1}{N(N+1)} \right) \right) \left( S_2(N) - S_2(N) \right)
\]
\[-\frac{88}{3} S_2(N) - 32 S(N) + 4 S_2'\left(\frac{N}{2}\right) - \frac{17}{3} - \frac{4}{9} \cdot \left[ 15N^4 + \frac{15N^2}{N^3(N+1)} \right] \]
\[+ \frac{236N^3 + 88N^2 + 3N + 18}{N^3(N+1)^3} \]
\[+ 8(-1)^N \frac{(2N^2 + 2N + 1)}{N^3(N+1)^3} \]
\[\left[ 160 S_1(N) + \frac{32}{3} S_2(N) + \frac{4}{3} + \frac{16}{9} \cdot \left[ 11N^4 + 4N^6 \right] \right] \]
\[\frac{5N^5 - 32N^4 - 514N^3 - 350N^2 - 240N - 72}{(N-1)N^3(N+1)^3(N+2)^2} \]
\[
\left[ \frac{109N^8 + 512N^7 + 879N^6 + 772N^5 + 104N^4 - 954N^3 + 278N^2 + 208N + 72}{9(N-1)^2 N^3 (N+1)^2 (N+2)^2} \right] \\
+ \frac{32}{3} C_F T_R \left\{ \frac{(s_1(N) - \frac{8}{3})}{N^2} \right\} \left\{ \frac{N^2 + N + 2}{N(N-1)} \right\} + \frac{1}{(N+1)^2} \right\};
\]

\[
\gamma_{A}^{(4)} = C_A T_R \left\{ -\frac{160}{9} S_1(N) + \frac{32}{3} + \frac{16}{9} \right\} \left\{ \frac{38N^4 + 76N^3 + 94N^2 + 56N + 12}{(N-1)N^2 (N+1)^2 (N+2)^2} \right\} \\
+ C_F T_R \left\{ \frac{8 + 16}{N(N-1)^2 N^3 (N+1)^3 (N+2)} \right\}
\]

\[
S_1(N) \left[ \frac{10N^7 - 10N^6 - 122N^5 + 182N^4 + 432N^3 + 896N^2 + 160N - 32}{(N-1)^2 N^2 (N+1)^2 (N+2)^2} \right] \\
+ S_2(N) \left[ \frac{28N^3 + 86N^2 + 22N + 207}{(N-1)N(N+1)(N+2)} \right] \left\{ -16 S_1(N) S_2(N) + 32 \tilde{S}(N) \right\} \\
+ 16 \left( S_1 - \frac{1}{(N+1)(N+2)} - \frac{1}{N(N-1)} \right) \left\{ S_2(N) - S_2 \left( \frac{N}{2} \right) \right\} \\
- \frac{64}{3} + \left\{ -\frac{468N^{10} + 3970N^9 - 11256N^8 - 11785N^7 - 4205N^6}{9(N-1)^2 N^3 (N+1)^3 (N+2)^3} \right\} \left( -\frac{4387N^5 + 5123N^4 + 2742N^3 + 7640N^2 + 7896N - 2736}{9(N-1)^2 N^3 (N+1)^3 (N+2)^3} \right) \\
+ (-1)^N \left[ \frac{-2N^7 - 7N^6 - 354N^5 - 469N^4 - 234N^3 + 8N^2 + 264N + 112}{(N-1)N^3 (N+1)^3 (N+2)^3} \right] \right\}
\]
The symbols appearing in expressions (2.2)-(2.3) are defined (as for the non-singlet case):

\[ S_q(N) = \frac{N}{2} \sum_{j=1}^{N} \frac{1}{j^2}, \quad (2.4.a) \]

\[ S(N) = \sum_{j=1}^{N} \frac{(-1)^j S_q(j)}{j^2}, \quad (2.4.b) \]

\[ S_q\left(\frac{N}{2}\right) = \frac{1-(-1)^N}{2} S_q\left(\frac{N}{2}\right) + \frac{1+(-1)^N}{2} S_q\left(\frac{N-1}{2}\right). \quad (2.4.c) \]

The non-singlet anomalous dimension is related to \( \gamma_N^{\text{FF}} \) in a simple manner. To leading order these two quantities coincide. When including next-to-leading order we have:

\[ \gamma_N^{\text{NS}}(1) = \gamma_N + 16 C_F T_R \frac{5N+32N^4+49N^3+38N^2+28N+8}{N^3(N+1)^3(N-1)(N+2)^2}, \quad (2.5) \]

where the difference is seen to vanish as \( 1/N^4 \) when \( N \) goes to infinity.

Using our final expressions (2.3) it is straightforward to investigate the asymptotic behaviour of the different matrix elements of \( \gamma_N^{(1)} \). We obtain:

\[ \gamma_N^{(1)\text{FF}} \sim \left( \frac{536}{9} - \frac{8}{3} \pi^2 \right) C_F C_A - \frac{160}{9} C_F T_R \left( \log N \right), \quad (2.6.a) \]

\[ \gamma_N^{(1)\text{AA}} \sim \left( \frac{536}{9} - \frac{8}{3} \pi^2 \right) C_A^2 - \frac{160}{9} C_A T_R \left( \log N \right), \quad (2.6.b) \]

\[ \gamma_N^{(1)\text{FA}} \sim 16 (C_F - C_A) T_R \left( \frac{\log^2 N}{N} \right), \quad (2.6.c) \]
\[
\gamma_N^{(1)} \sim 8 (c_A - c_F) c_F \frac{\log^2 N}{N}
\]

This result is in contrast with that guessed by Floratos et al. on the basis of their first computation. Especially interesting is the \( \log N \) behaviour of the diagonal elements, since it coincides with that of the leading order term thus allowing a safe manipulation of the perturbative expansion. It thus gets rid of the cautions mentioned by Peterman \(^1\) in the context of a guessed \( \log^2 N \) behaviour.

We can now display the full \( N \) dependence of \( \gamma_N(1) \) in a way which is suitable for further moment inversion. This can be done by expanding our expressions around \( N = \infty \). Use is made of the corresponding expansions of the functions defined in \((2.4)^{7,16}\). The result has the form:

\[
\gamma_N^{(a \, b)} = \sum_{P = 0}^{\infty} \sum_{\ell = 0}^{\infty} B_{P \ell} \frac{\log N}{N^\ell} + (-1)^N \sum_{\ell = 3}^{\infty} \frac{\mathcal{B}_{P \ell}}{N^\ell},
\]

(2.7)

where \( a \) and \( b \) take the values \( A \) and \( F \). The detailed list of coefficients \( B \) and \( \mathcal{B} \) are given in Appendix A.

3. POINTLIKE RECONSTRUCTION

Let us consider an arbitrary structure function \( F(x, Q^2) \). It can be split into a singlet and non-singlet piece

\[
F(x, Q^2) = F^{\text{singlet}}(x, Q^2) + F^{\text{non-singlet}}(x, Q^2),
\]

(3.1)

the moments of which satisfy an equation of the form \(^a\):

\[
\mu^{\text{NS}}_{-2}(Q^2) = \int_0^1 x^{-2} F^{\text{NS}}(x, Q^2) dx = C^{\text{NS}}(Q^2) A^{\text{NS}}_N
\]

(3.2a)

\(^a\) Only twist two terms are considered.
singlet $\mu_{N-2}(Q^2) = \int_0^1 x^{N-2} f(x, Q^2) dx = C_N^F(Q^2) A_N^{\text{singlet}}$

(3.2.5)

where $\tilde{A}(\vec{x})$ is a vector whose components are $C_N^F(\vec{x})$ and $C_N^A(\vec{x})$.

In order to perform the pointlike reconstruction of the non-singlet piece, one must get rid of the unknown constant $A_N^{\text{N.S.}}$. This is achieved by using the values of the structure function at some reference point $Q_0^2$. Inverting (3.2.a) then yields:

$$F^{\text{N.S.}}(x, Q^2) = \int_0^1 dy F^{\text{N.S.}}(y, Q_0^2) b^{\text{N.S.}}(x, y; Q^2, Q_0^2)$$

(3.3)

where the kernel $b^{\text{N.S.}}$ is given by:

$$b^{\text{N.S.}}(x, y; Q^2, Q_0^2) = \lim_{K/N \to \infty} \frac{(N+1)!}{K!} \sum_{\ell=0}^{N-K} \frac{(-1)^{N-K-\ell}}{(N-K-\ell)!} \frac{C_N^{\text{N.S.}}(Q^2)}{C_N^{\text{N.S.}}(Q_0^2)}$$

(3.4)

In the singlet case we have two unknown constants $\tilde{A}$; therefore, knowledge of the structure function at $Q_0^2$ is not enough to eliminate them. To do so, one can either use information at two reference points ($Q_0^2$ and $Q_1^2$) or else introduce a new structure function suitably related to the one under consideration. This last procedure is easier and we will follow it here. However, it can be seen that the auxiliary structure function, not being a measured quantity, is somewhat arbitrary. For physical reasons we will refer to it as the gluon distribution function and will define it in such a way that the final expressions are simple generalizations of the non-singlet case.

The final result which we are looking for has the form *):

$$F(x, Q^2) = \int_0^1 dy b(x, y; Q^2, Q_0^2) F(y, Q_0^2) + \int_0^1 dy b(x, y; Q^2, Q_0^2) G(y, Q_0^2)$$

(3.5)

*) We drop the label "singlet" for the rest of this section.
where $G$ stands for the gluon distribution function. The kernels $b^{\lambda\alpha}$ can be obtained by means of an expression of type (3.4) with $c_N^{\text{NS}}(Q^2)/c_N^{\text{NS}}(Q_o^2)$ replaced by other quantities: $\lambda_N^{\lambda}(Q^2, Q_o^2)$ and $\lambda_N^{\lambda}(Q^2, Q_o^2)$. This section is devoted to showing how these two last quantities can be suitably calculated, thus paving the way for the next two sections where the explicit calculation is performed and the result given.

We begin by substituting expression (1.2) into (3.2b), obtaining

$$
\mu_{N-2}(Q^2) = \overrightarrow{c}_N(Q_o^2, \overrightarrow{g}) M_N \overrightarrow{A}_N,
$$

(3.6)

where

$$
M_N(Q^2, Q_o^2) = \overrightarrow{T} \exp \left\{ - \int_t^\infty \overrightarrow{\gamma}_N(t') dt' \right\},
$$

(3.7)

and $\gamma_N$ is the anomalous dimension matrix of the singlet operators. In order to introduce an auxiliary structure function in the most natural way, let us think of $\overrightarrow{c}_N(Q_o^2, \overrightarrow{g})$ as the first row of a certain matrix $C_N(Q_o^2, \overrightarrow{g})$. With this in mind we can write

$$
\overrightarrow{\mu}_{N-2}(Q^2) = \overrightarrow{c}_N(Q_o^2, \overrightarrow{g}) M_N(Q^2, Q_o^2) \overrightarrow{A}_N
$$

(3.8)

which obviously implies (3.6) if the first component of $\overrightarrow{\mu}_N$ is precisely the $N$th moment of our structure function. At the same time (3.8) gives us an equivalent relation for the moments of another structure (second component of $\overrightarrow{\mu}$).

At $Q^2 = Q_o^2$ we have:

$$
\overrightarrow{\mu}_{N-2}(Q_o^2) = \overrightarrow{c}_N(Q_o^2, \overrightarrow{g}) \overrightarrow{A}_N,
$$

(3.9)

where use has been made of the fact that $M_N(Q_o^2, Q_o^2) = 1$. The last two expressions give:

$$
\overrightarrow{\mu}_N(Q^2) = \overrightarrow{c}_N(Q_o^2, \overrightarrow{g}) M_N(Q^2, Q_o^2) C_N^\dagger(Q_o^2, \overrightarrow{g}) \overrightarrow{\mu}_{N-2}(Q_o^2),
$$

(3.10)

where we have succeeded in eliminating the undesired $\overrightarrow{A}_N$ constants.
We can now make use of the arbitrariness in the definition of $C_N$ further to simplify the result. Let us fix the second row of $C_N$ by the condition that its commutator with $M_N(Q^2, Q_o^2)$ is zero, up to a given order of perturbation theory (next-to-leading in our case). We then conclude that:

$$\bar{\mu}_{N-2}(Q^2) = C_N(Q_o^2, Q^2) C_N^{-1}(Q_o^2) g_o M_N(Q, Q_o^2) \bar{\mu}_{N-2}(Q_o^2),$$  

(3.11)

whose first component gives us precisely the $N-2$ moment of the structure function under consideration.

We must now prove that $C_N$ and $M_N$ can be made to commute. Working up to next to leading order we can write:

$$C_N(Q_o^2, Q^2) = C_N^{(0)} + \frac{\alpha}{4\pi} C_N^{(4)}$$  

(3.12.a)

and

$$M_N(Q^2, Q_o^2) = M_N^{(0)}(Q, Q_o^2) + \frac{\alpha}{4\pi} M_N^{(4)}(Q, Q_o^2)$$  

(3.12.b)

where $\alpha$ stands both for $\bar{\alpha}$ and $\alpha_o = \bar{\alpha}_o^2/4\pi$. The commutation condition thus requires:

$$\left[ C_N^{(0)}, M_N^{(0)} \right] = 0$$  

(3.13.a)

$$\left[ C_N^{(4)}, M_N^{(0)} \right] = 0$$  

(3.13.b)

The first of these equations can be satisfied by choosing $C_N^{(0)}$ to be a multiple of the identity, since $C_N^{(0)FA} = 0$. The second has the solution:

$$C_N^{(4)AF} = \frac{\gamma_N^{(0)AF} C_N^{(4)FA}}{\gamma_N^{(0)FA}}$$  

(3.14.a)
\[ C^{(4)AA}_N = C^{(4)FF}_N + \left( \frac{\partial^{(0)AA}_N - \partial^{(0)FF}_N}{\partial^{(0)FA}_N} \right) C^{(4)FA}_N \]  

(3.14b)

Furthermore, the solution is unique.

We still have to face the problem of inverting (3.11). The relevant kernels shown in (3.5) can be obtained as follows:

\[ \lim_{k,N \to \infty} \frac{(N+1)!}{k!} \sum_{k=0}^{N-k} \frac{(-1)^k}{k!} \frac{e^{t\ell}}{(N-K-\ell)!} \lambda^a_k (\beta^2, \gamma^2) \]  

(3.15)

where

\[ \lambda^a_N (\beta^2, \gamma^2) = \left[ C_N (\beta^2, \gamma^2) C^{-1}_N (\beta^2, \gamma^2) M_N (\beta^2, \gamma^2) \right]^{\frac{Fa}{a}} \]  

(3.16)

and \( a = A \) and \( F \). As it is, solving (3.15)-(3.16) is a very difficult problem due to the complicated \( N \) dependence of the \( \lambda \)'s. We will now show how one can eliminate the \( T \) exponential \( M_N \) from expression (3.16).

We start by defining

\[ R_N = M^{(0)-1}_N M_N \]  

(3.17)

where \( M^{(0)}_N \) is the leading order part of \( M_N \). Changing variables in expression (3.7) we obtain:
\[ M_N = T \exp \left\{ \frac{\bar{\beta}}{g_0} \int \frac{dg'}{g'} \gamma_N(g') \right\}. \quad (3.18) \]

Now using the perturbative expansion for \( \gamma \) (2.1) and \( \beta \):
\[ \beta(g) = -\beta_0 g - \frac{\alpha_1}{4\pi} g - \frac{\alpha_2}{16\pi^2} g^2, \]

and disregarding terms of order \( \alpha^2 \) we arrive at
\[ M_N = T \exp \left\{ \frac{\bar{\beta}}{g_0} \int \frac{dg'}{g'} \left[ \gamma_N^{(2)} + \frac{\alpha'}{4\pi} \left( \gamma_N^{(4)} - \frac{\beta_1}{\beta_0} \gamma_N^{(2)} \right) \right] \right\}. \quad (3.20) \]

Up to leading order the above expression reduces to:
\[ M_N^{(o)} = \exp \left\{ -\frac{3}{32} \gamma_N^{(0)} \gamma_N^{(0)} \right\}, \quad (3.21.a) \]

with
\[ r = \frac{16}{3\beta_0} \log \left( \alpha_0 / \alpha \right). \quad (3.21.b) \]

Correspondingly, the expression of \( R_N \) is:
\[ R_N = T \exp \left\{ \int \frac{dg'}{g'} \frac{\alpha'}{4\pi \beta_0} M_N^{(o)}(r') \left( \gamma_N^{(4)} - \frac{\beta_1}{\beta_0} \gamma_N^{(2)} \right) M_N^{(o)}(r') \right\}. \quad (3.22) \]

where \( \alpha' \) and \( r' \) are given in terms of \( g' \) by the same expressions as \( \alpha' \) and \( r \) with respect to \( \bar{g} \).
To prove (3.22) notice, that $M_N^{(o)}$ satisfies the following differential equation:

$$\frac{dM_N^{(o)}}{d\bar{q}} = \frac{1}{\bar{q}\beta_0} \left[ \gamma_N^{(o)} + (\gamma_N^{(o)} - \gamma_N^{(o)} \frac{\beta_1}{\beta_0}) \frac{\alpha}{\eta} \right] M_N^{(o)}$$  (3.23)

with the initial condition $M_N^{(o)}(Q_0^2, Q_0^2) = I$. Analogously, $M_N^{(o)}(r)$ satisfies a similar equation where non-leading order terms are set to zero. When use is made of the definition of $R_N$ (3.17) one can easily verify that:

$$\frac{dR_N}{d\bar{q}} = \frac{\alpha}{\bar{q}\beta_0 \eta} M_N^{(o)}(\bar{q}) \left[ \gamma_N^{(o)} - \gamma_N^{(o)} \frac{\beta_1}{\beta_0} \right] M_N^{(o)}(\bar{q}) R_N$$  (3.24)

whose solution is given by (3.22).

The usefulness of (3.22) lies in the fact that every term in the expansion of the $T$ exponential is down by a factor of $\alpha$ with respect to the one before. Therefore, up to our level of approximation we can write:

$$R_N = I - \frac{(\alpha - \alpha_0)}{8\pi} \frac{\beta_1}{\beta_0^2} \gamma_N^{(o)} + \Delta_N$$  (3.25)

with

$$\Delta_N = - \frac{3}{32} \frac{\alpha_0}{4\pi} \int_0^\infty \exp(-\frac{3\alpha_0}{4\sigma}) \gamma_N^{(o)} M_N^{(o)}(r') \gamma_N^{(o)} M_N^{(o)}(r') dr'$$  (3.26)

The last two equations and (3.12.a) allow us to simplify (3.16) giving:

$$\lambda^a_n(Q_0^2, Q_0^2) = \left[ M_N^{(o)}(r) + \frac{(\alpha - \alpha_0)}{4\pi} \left( \frac{1}{C_N^{(o)FF}} C_N^{(1)} - \frac{1}{2} \frac{\beta_1}{\beta_0^2} \gamma_N^{(o)} \right) M_N^{(o)}(r) + M_N^{(o)}(r) \Delta_N \right] F_a$$  (3.27)

where the first term in the right-hand side represents the leading order result and the rest are second order contributions.
As previously advanced, the $T$ exponential no longer appears in (3.27). Nevertheless, inverting (3.27) [using (3.15)] is still a very difficult problem. One might face it by dropping the limit in (3.15) thus obtaining a polynomial approximation to the kernel $^{18}$). Unfortunately, this procedure is quite unstable for numerical calculations, since the coefficients of the different powers of $y$ in (3.16) are alternating in sign. Consequently, we proceed in a different way, identical to that followed in the non-singlet case $^{7})$. Our approach consists on expanding $^a_N$ in inverse powers of $N$. The transformation in (3.15) can be performed exactly analytically for each term in this expansion $^{*)}$. 

The resulting expansion for the kernels is rapidly convergent. Increasing the accuracy of the result is straightforward and one can easily achieve the desired level of approximation. For phenomenological purposes the first few terms suffice.

4. LEADING ORDER

In this section we face the problem of obtaining the pointlike reconstruction kernels to leading order in $\alpha$. Our starting point is relation (3.27), which neglecting higher orders, reduces to

$$\lambda^a_N(Q^2, Q'^2, \lambda) = \lambda^a_N (\tau)$$

(4.1)

with $\lambda^a_N$ defined in (3.21). As explained in the previous section, our procedure consists of expanding (4.1) in inverse powers of $N$ and then inverting each term separately.

We proceed by separating $\gamma^{(o)}_N$ into two pieces:

$$\gamma^{(o)}_N = -\frac{32 \alpha}{3} L_N + \epsilon_N$$

(4.2)

the first of which $L$ contains those terms that do not vanish as $N \to \infty$. It follows from (2.2) that $L_N$ is diagonal; the non-vanishing elements being:

$$L^F_N = -\log N + \frac{3}{4} + \epsilon$$

(4.3.a)

*) See the next two sections for the proof of this statement.
\[ L_N^{AA} = \frac{9}{4} \log N + \frac{33}{16} - \frac{9}{4} \zeta - \frac{\pi^2}{8} \]  

(4.3.b)

where \( \zeta \) is Euler's constant. Conversely, \( \varepsilon_N \) decreases as \( 1/N \).

We can now write \( M_N^{(o)} \) in terms of \( M_N^{(o)L} \) = \exp\{rL_N\} in the following form:

\[ M_N^{(o)} = M_N^{(o)L} T_N \]  

(4.4)

It can be shown (by a similar procedure as that leading to 3.22) that

\[ T_N = T \exp \left\{ \int_0^T \phi_N(r') \, dr' \right\} \]  

(4.5.a)

with \( \phi_N \) given by

\[ \phi_N = (M_N^{(o)L})^{-1} \varepsilon_N M_N^{(o)L} \]  

(4.5.b)

Since \( \varepsilon_N \) goes as \( 1/N \), expansion (4.5.a) can be truncated at some point dictated by the number of terms one wants to keep in the \( 1/N \) expansion.

The diagonal nature of \( L_N \) greatly simplifies the calculation. The computation of \( M_N^{(o)L} \) is straightforward, giving

\[ M_N^{(o)L} = \begin{pmatrix} K_F(r) N^{-r} & 0 \\ 0 & K_A(r) N^{-\frac{9}{4}r} \end{pmatrix} \]  

(4.6.a)

with

\[ K_F(r) = \exp \left\{ \left( \frac{3}{4} - \zeta \right) r \right\} \]

\[ K_A(r) = \exp \left\{ \left( \frac{33}{16} - \frac{9}{4} \zeta - \frac{\pi^2}{8} \right) r \right\} \]  

(4.6.b)
For $\Phi$ we obtain the result

$$
\phi_N(\tau) = \begin{pmatrix}
\epsilon_{NF}^F & \epsilon_{NF}^A \tau \delta_N \\
\epsilon_{NF}^A e^{-\tau \delta_N} & \epsilon_{NF}^A \end{pmatrix}
$$

(4.7.a)

where

$$
\delta_N = \frac{\epsilon_{NF}^A}{L^2_N - L^2_F} - \frac{5}{4} \log N + \frac{21}{16} - \frac{5}{4} \epsilon + \frac{n_F}{8}.
$$

(4.7.b)

By iteration, one can obtain the desired number of terms in the expansion of the $T$ matrix. After substitution of the $\gamma^{(o)}_N$ expansion and gathering all pieces together one ends up with:

$$
T^F_N = \sum_{j=0}^{\infty} G_j(\tau) \frac{A}{N^2} + \frac{3n_F}{20} \sum_{j=2}^{\infty} \frac{A}{N^2} \sum_{\ell=1}^{\infty} \frac{1}{(\log N - \rho)^{\ell}} \int \frac{Y^{FF}_j(\tau) + e^{\beta r - \alpha r} Z^{FF}_j(\tau)}{(\log N - \rho)^{\ell}}
$$

$$
T^A_F = \frac{3n_F}{20} \sum_{j=1}^{\infty} \frac{A}{N^2} \sum_{\ell=1}^{\infty} \frac{1}{(\log N - \rho)^{\ell}} \int \frac{Y^{FA}_j(\tau) + e^{\beta r - \alpha r} Z^{FA}_j(\tau)}{(\log N - \rho)^{\ell}}
$$

(4.8.a)

$$
T^A_F = \sum_{j=1}^{\infty} \frac{1}{N^2} \sum_{\ell=1}^{\infty} \frac{1}{(\log N - \rho)^{\ell}} \int \frac{Y^{AF}_j(\tau) + e^{\beta r - \alpha r} Z^{AF}_j(\tau)}{(\log N - \rho)^{\ell}}
$$

$$
T^{AA}_N = \sum_{j=0}^{\infty} G_j(\tau) \frac{A}{N^2} + \frac{3n_F}{20} \sum_{j=2}^{\infty} \frac{A}{N^2} \sum_{\ell=1}^{\infty} \frac{1}{(\log N - \rho)^{\ell}} \int \frac{Y^{AA}_j(\tau) + e^{\beta r - \alpha r} Z^{AA}_j(\tau)}{(\log N - \rho)^{\ell}}
$$

(4.8.b)
where

\[
\alpha' = \frac{5}{4}, \\
\beta = \frac{21}{16} - \frac{5}{4} \xi - \frac{n_f}{8}, \\
\zeta = \beta / \alpha'
\]

(4.8b)

The coefficients \( G_j^{(a)}(r) \), \( y_{j,\ell}(r) \) and \( z_{j,\ell}(r) \) are polynomials in \( r \). They are listed in Appendix B up to \( j = 4 \). Notice that \( G_j^{(F)}(r) \) are exactly the same coefficients that appear in the leading order calculation of the non-singlet piece of the structure functions\(^7\). On the other hand, \( G_j^{(A)}(r) \) play exactly the same role when computing the gluon distribution functions.

Expression (4.6) and (4.8) provide us with an expansion for \( \lambda_N^x \) which, suitably fitted into (3.15) gives rise to the corresponding kernel expansion. As a matter of fact, it can easily be seen that all we need is to invert terms of the form \( N^{-r-j}(\log N - \rho)^{-\ell} \) where \( j \) and \( \ell \) are non-negative integers. The case of \( \ell = 0 \) is already known\(^6\),\(^7\); it gives rise to kernels of the form:

\[
\frac{1}{x} b_0(z, r + j) = \frac{1}{x} \frac{\Theta(z-1)}{z^2} \frac{\log z}{\Gamma(r+j-1)},
\]

(4.9)

where \( z = y/x \). Notice that with increasing \( j \), the kernels decrease quickly as a consequence of the appearance of Euler's \( \Gamma \) function in the denominator. Furthermore, the approximation becomes worse for large values of \( z \) due to the extra \( \log z \) powers in the numerator. Since \( z \) is bounded by \( 1/x \), the number of terms one has to consider in the expansion of \( \lambda_N^x \) follows from the range in \( x \) one wishes to cover. However, due to the vanishing of the structure functions at \( x = 1 \), the accuracy obtained in the reconstruction exceeds that of the kernels themselves.

To obtain the kernel associated with an \( N \) dependence of the type \( N^{-r-j}(\log N - \rho)^{-1} \), one should proceed as follows. Notice that integrating \( N^q e^{\rho q} \) with respect to \( q \) yields the desired logarithm the denominator. Consequently, the associated kernel is related to \( b_0 \) by:
\[
\frac{1}{x} b_{-1}(z;r+j;\rho) = \frac{1}{x} e^{-(r+j)\rho} \int_{r+j}^{\infty} dq' e^{q'\rho} b_o(z; q'), \tag{4.10}
\]
and using the expression for \( b_o \) we find:

\[
b_{-1}(z;r+j;\rho) = \frac{\Theta[(z-1)\log z]}{z} \int_0^{\infty} ds \frac{(e^s\log z)^S}{\Gamma(s+r+j)}, \tag{4.11}
\]

Similarly one can prove that the kernel corresponding to \( N^{-r-j}(\log N - \rho)^{-q} \) is given by:

\[
\frac{1}{x} \epsilon_{1}(z;r+j;\rho) = \frac{1}{x} \frac{\Theta[(z-1)\log z]}{z^2} \frac{r+j-1}{(l-1)!} \int_0^{\infty} ds \frac{s^{l-1}(e^s\log z)^S}{\Gamma(s+r+j)}, \tag{4.12}
\]

The integrals appearing in the right-hand side of (4.11)-(4.12) are related to the special functions\textsuperscript{13} \( \mu(x,\beta,\alpha) \) as follows:

\[
\frac{1}{(l-1)!} \int_0^{\infty} ds \frac{s^{l-1}(e^s\log z)^S}{\Gamma(s+q)} = \left(e^q\log z\right)^{-q} \mu(e^q\log z, l-1, q-1), \tag{4.13}
\]

In Table I we give a list of the relevant kernels together with their corresponding functions of \( N \), thus collecting our results somewhat dispersed in the text and in Reference 7). We include the kernels \( b_x \) associated with \( N^{-r-j}(\log N)^q \), which will be used in the next-to-leading order calculation.

To conclude, we give the final expression for the kernels entering in (3.3) and (3.5). These expressions are:\textsuperscript{a)}

\[
b^{NS\,(0)}(x,y; q_1, q_2) = \frac{1}{x} K_F(r) \sum_{j=0}^{\infty} G^{(F)}(r)b_o(z; r+j), \tag{4.14.a}
\]

\textsuperscript{a)} The superscript \((0)\) stands for leading order contribution.
\[ b^{FF}(x, y, Q^2, Q_0^2) = b(x, y, Q_0^2) + \frac{3n_F}{20} \frac{K_F(r)}{x} \sum_{J=2}^{\infty} \sum_{J=1}^{\infty} \left\{ \right. \]
\[ \left. Y_{\ell J}^{FF}(r) b_{-\ell}(2; r+j; \rho) e^{\beta r} Z_{\ell J}^{FF}(r) b_{-\ell}(2; q; r+j; \rho) \right\} \quad (4.14.b) \]
\[ b^{FA}(x, y, Q^2, Q_0^2) = \frac{3n_F}{20} \frac{K_A(r)}{x} \sum_{J=1}^{\infty} \sum_{J=2}^{\infty} \left\{ \right. \]
\[ \left. Y_{\ell J}^{FA}(r) b_{-\ell}(2; r+j; \rho) e^{\beta r} Z_{\ell J}^{FA}(r) b_{-\ell}(2; q; r+j; \rho) \right\} \quad (4.14.c) \]

It is worthwhile to note that in the limit \( r \to 0 \) the whole scheme is consistent, since the first two kernels tend to \( \delta(x - y) \) and the last one vanishes. This fact reflects in a symmetry relation between the different coefficients, namely:
\[ Y_{\ell J}^{ab}(0) = - Z_{\ell J}^{ab}(0) \quad (4.15) \]

which can be easily checked.

As a bonus of our calculation, the \( Q^2 \) evolution of the gluon distribution function can also be obtained. The result is:
\[ G(x, Q^2) = \int_0^x b^{AF}(x, y, Q^2, Q_0^2) F^{single}(y, Q_0^2) \, dy + \int_0^x b^{AA}(x, y, Q^2, Q_0^2) G(y, Q_0^2) \, dy \quad (4.16) \]

where
\[ b^{AA}(x, y, Q^2, Q_0^2) = \frac{K_A(r)}{x} \sum_{J=0}^{\infty} G^{(A)}_J(r) b_{-\ell}(2; q; r+j; \rho) + \frac{3n_F}{20} K_A(r) \sum_{J=2}^{\infty} \sum_{J=1}^{\infty} \left\{ \right. \]
\[ \left. Y_{\ell J}^{AA}(r) b_{-\ell}(2; r+j; \rho) e^{\beta r} Z_{\ell J}^{AA}(r) b_{-\ell}(2; q; r+j; \rho) \right\} \quad (4.17.a) \]
and

\[
\begin{align*}
\tilde{b}^{\text{AF}}(0)_{(x,y;\alpha^2,\beta^2)} &= \frac{K_A(r)}{x} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \langle \gamma_{j}^{(r)} b_{\lambda}^{(2j,y^r)}_{(r)} \rangle \\
&+ e^{-\beta r} \sum_{j=1}^{\infty} \langle \gamma_{j}^{(r)} b_{\lambda}^{(2j,r+j)}_{(r)} \rangle
\end{align*}
\]

(4.17.b)

With the last expressions we close the section. The following one will extend the results to include the next order of perturbation theory.

5. **Next to Leading Order**

This section is devoted to obtaining the second order QCD contributions to the pointlike reconstruction kernels. This is justified by the findings of the non-singlet case\(^2,3\) where these contributions are non-negligible when compared with the leading order ones. In particular, they help much in determining the value of \(\Lambda\), the parameter which fixes the scale of strong interactions. When working to leading order \(\Lambda\) shifts to higher values in an effort to simulate higher order contributions. When next-to-leading order is included both its value and error is considerably reduced. The importance of the final result is obvious, since the same parameter \(\Lambda\) enters in a large variety of QCD predictions\(^14,15\). It is, however, clear that obtaining second order contributions demands many times more work than first order ones. Luckily enough, the final result does not reflect the complexity of intermediate calculations.

Let us start by expanding (3.27) in power series of \(1/N\). We make use of the expansion of \(\gamma_N^{(o)}(r)\) which was derived in the previous section. Those of \(\gamma_N^{(1)}\) and \(C_N^{(1)}\) can be readily obtained from their expressions, using the expansions of \(S_\lambda(N)\)\(^16\). Notice that the coefficient functions \(C_N^{(1)}\) depend on the structure function under consideration \((F_1, F_2\) or \(F_3\)). Restricting ourselves to \(F_2\) we have \(11,17\):
\[ C_N^{(0)FF} = 2 \]
\[ C_N^{(0)FA} = 0 \]
\[ C_N^{(1)FF} = 2C_F \left\{ 2S_1^2(N) + 3S_1(N) - 2S_2(N) - \frac{2S_2(N)}{N(N+1)} \right\} \]
\[ - 9 + \frac{3}{N} + \frac{V}{N+1} + \frac{2}{N^2} + \left( \log \frac{4n}{\pi} - \gamma \right) (4S_1(N) - 3 - \frac{2}{N(N+1)}) \]
\[ C_N^{(1)FA} = 8TR \left\{ \frac{1}{N} - \frac{1}{N^2} - \frac{6}{(N+1)} + \frac{6}{(N+2)} \right\} \]
\[ \frac{(N^2+N+2)}{N(N+1)(N+2)} \left( \log \frac{4n}{\pi} - S_1(N) - \gamma \right) \]

where \( \gamma \) is Euler's constant.

The expansion of \( \Lambda_N \) is not very difficult. It however requires a large amount of computation. For this purpose, we made use of "REDUCE", a program for analytical manipulation implemented at the IBM at CERN. The integral appearing in (3.26) can and has been performed analytically by the computer.

Finally, substituting \( \beta_0 \) and \( \beta_1 \) by their values:

\[ \beta_0 = 11 - \frac{2}{3} n_f \]
\[ \beta_1 = 102 - \frac{38}{3} n_f \]

(5.2)

and putting all terms together we arrive at:

\[ \lambda_N^{(1) FA} = K_F(r) N^{-1} \left\{ \frac{(\lambda - \alpha_0)}{4n} \right\} \sum_{p=0}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_j}{3^{p+q}} \frac{F_a}{N^j} \]
\[ N^{-\alpha r} e^{\beta qr} + \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{\chi_{pJ}^{(q)}}{N J (\log N - \rho)^P} \]

\[ \sum_{J=0}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_1)^P} e^{\beta qr} \]

\[ \sum_{J=0}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_2)^P} e^{\beta qr} \]

\[ \sum_{J=0}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_1)^P} e^{\beta qr} \]

\[ \sum_{J=0}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_2)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \rho)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_1)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_2)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \rho)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_1)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_2)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \rho)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_1)^P} e^{\beta qr} \]

\[ \sum_{J=1}^{\infty} \sum_{q=-2}^{J+1} \sum_{P=1}^{J} \frac{(q) Fa N^{-\alpha r}}{N J (\log N - \alpha_2)^P} e^{\beta qr} \]

where \( a = A F \). The quantities \( \alpha, \beta, \rho \) and \( K_\rho (r) \) were defined in the previous section. The definition of \( a_1 \) and \( a_2 \) is as follows:

\[ a_1 = e + b \]

\[ a_2 = \rho - b \]
with

\[ b = \frac{3}{2} \beta_0 \]  \hspace{1cm} (5.4.b)

The coefficients \( \xi, \bar{\xi}, \chi, \bar{\chi}, \tau, \bar{\tau}, \sigma \) and \( \bar{\sigma} \) are listed in Appendix C. Only terms up to \( 1/N^2 \) are given.

When expression (5.3) is substituted into (3.15) we obtain the second order contribution to the kernels. Notice that all the functions of \( N \) which one must invert coincide with those appearing in the non-singlet and leading order calculation (see Table I). This suggests the fact that our method will work equally well in higher orders of perturbation theory.

We conclude by giving the final expression of the pointlike reconstruction kernels up to next to leading order. This is the following:

\[
\begin{align*}
\frac{b_{NS}^{FA}}{x} (x, y; Q^2, Q_0^2) & = b_{NS}^{FA(0)} (x, y; Q^2, Q_0^2) + \frac{\lambda_{NS}}{y \eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} R_p R_q b_{p+q+1}^{F(2)} (r; r + 2r + i) \\
\frac{b_{NS}^{FA}}{x} (x, y; Q^2, Q_0^2) & = b_{NS}^{FA(0)} (x, y; Q^2, Q_0^2) + \frac{\lambda_{NS}}{y \eta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} R_p R_q b_{p+q+1}^{F(2)} (r; r + 2r + i) \\
\end{align*}
\]  \hspace{1cm} (5.5.a)
\[
\sum_{J=0}^{2} \sum_{\frac{q-2}{2}}^{J+1} \sum_{p=1}^{\infty} C_{pj}^{(q)} F_a \ e^{\beta_{gr}} (-1)^p b_{-p}^{(2; r+\alpha g+j; q_1)} + \\
k_F(r) \frac{1}{\sqrt{4\pi}} n \left[ \sum_{J=1}^{2} \sum_{\frac{q-2}{2}}^{J+1} \sum_{p=1}^{\infty} C_{pj}^{(q)} F_a \ e^{\beta_{gr}} b_{-p}^{(2; r+\alpha g+j; p)} \right] + \\
\sum_{J=1}^{2} \sum_{\frac{q-2}{2}}^{J+1} \sum_{p=1}^{\infty} C_{pj}^{(q)} F_a \ e^{\beta_{gr}} b_{-p}^{(2; r+\alpha g+j; p)} (-1)^p + \\
\sum_{J=1}^{2} \sum_{\frac{q-2}{2}}^{J+1} \sum_{p=1}^{\infty} C_{pj}^{(q)} F_a \ e^{\beta_{gr}} b_{-p}^{(2; r+\alpha g+j; q_2)} \right]
\]

(5.5.b)

where \( a = A \) and \( F \) and \( z = y/x \). The \( b^{(o)} \) kernels are shown in expressions (4.14). For completeness, the non-singlet kernel is also given. The coefficients \( C_{pj}^{NS} \) are given in Appendix C together with the ones relevant for the singlet case.

We point out, that although expressions (5.5) are lengthy, they involve no serious difficulty. All the effort required dwells in the explicit calculation of the coefficients (\( \xi, \tilde{\xi}, \chi, \tilde{\chi} \), etc.). For this reason we have evaluated the first few coefficients and displayed them in Appendix C. Many of them in fact vanish, thus simplifying numerical calculations. The number of non-zero coefficients increases quickly with the power of \( 1/N \) considered. We have stopped at \( 1/N^2 \) since the second order contributions, being smaller, do not require as much accuracy as the leading order ones, where terms up to \( 1/N^4 \) have been kept \(^*\). We think that this precision is enough for comparison with presently available data.

6. CONCLUSIONS

In this paper we have derived pointlike QCD predictions for the singlet parts of the structure functions in deep inelastic scattering. This result, together

\(^*\) See references 2) and 7) for a numerical estimation of the relative importance of each term in the expansion for the non-singlet situation.
with the non-singlet one, enables the full reconstruction of any structure function at any $Q^2$ in terms of quantities defined at $Q^2_0$. Contrary to other approaches, our algorithm is general enough to allow its continuation to arbitrary orders of perturbation theory. We have explicitly dealt with first and second order contributions.

Some intermediate results are interesting in themselves. In particular, we have obtained simple analytical expressions for the anomalous dimension matrix of the singlet operators. The treatment of the $T$ exponential appearing in the moment relation is also new. Since our procedure goes through expanding all the quantities in powers of $1/N$, no diagonalization of $\gamma^{(0)}_N$ is required.

The phenomenological implications of our formulae are not discussed. Some interesting considerations mentioned in the non-singlet calculation\cite{7} go essentially unchanged. More specific consequences are postponed to another publication, where a detailed comparison with available data will be undertaken.

Let us briefly outline the steps to be followed in using our formulas in phenomenological fits to experimental data. First, one must calculate $r$, defined in (3.21.b). If we are working to leading order we have:

$$\bar{x}(Q^2) = \frac{4n}{\beta_0 \log(Q^2/\Lambda^2)} \quad (6.1)$$

where $\beta_0$ is shown in (5.2), and $\Lambda$ is a parameter to be fixed by experiment.

When second order contributions are also considered (6.1) should be replaced by:

$$\bar{x}(Q^2) = \frac{4n}{\beta_0 \log(Q^2/\Lambda^2)} \left[ 1 - \frac{\beta_1}{\beta_0} \log \log \left( \frac{Q^2}{\Lambda^2} \right) / \log \left( \frac{Q^2}{\Lambda^2} \right) \right] \quad (6.2)$$

with $\beta$ given by (5.2).

To know the predicted value of $F(x,Q^2)$ one must perform the integrals (3.3) and (3.5). The neccessary kernels are shown in (4.14) for the leading order calculation and in (5.5) up to next-to-leading order. Enough accuracy is attained by considering the coefficients shown in Appendix B and C and setting the rest to zero.
As seen in the previous paragraph, using the set of results derived in this paper is quite straightforward. However, testing QCD by this method requires fitting experimental data to fix the structure functions at $Q_0^2$ and the value of $A$. This consumes more computer time than other methods of expressing QCD predictions. This is the price to pay for a more rigorous treatment.

ACKNOWLEDGEMENTS

Special thanks are due to D. Ross and C. Sachrajda, who gave us the results of their computation on the anomalous dimensions and solved some of the difficulties on the interpretation of their output. We also thank F.J. Yndurain for useful discussions.
Table 1

Some useful transformations under (3.15)

<table>
<thead>
<tr>
<th>$\lambda_N$</th>
<th>$b(x,y; q^2, Q^2_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^{-\alpha}$</td>
<td>$\frac{1}{x} b_0(z; \alpha) = \frac{1}{x} \frac{\Theta^{(2-1)}}{z^2} \frac{\log z}{\Gamma(\alpha)}$</td>
</tr>
<tr>
<td>$N^{-\alpha} \log N$</td>
<td>$\frac{1}{x} b_0(z; \alpha) = \frac{1}{x} b_0(z; \alpha) \left[ \gamma(\alpha) - \log \log z \right]$</td>
</tr>
<tr>
<td>$N^{-\alpha} \log N^2$</td>
<td>$\frac{1}{x} b_0(z; \alpha) = \frac{1}{x} b_0(z; \alpha) \left[ \gamma(\alpha) - \log \log z \right]^2 + \gamma'(\alpha)$</td>
</tr>
<tr>
<td>$N^{-\alpha} \log^2 N$</td>
<td>$\frac{1}{x} b_0(z; \alpha) = \frac{1}{x} b_0(z; \alpha) \left[ \gamma(\alpha) - \log \log z \right]^2 + \gamma'(\alpha)$</td>
</tr>
<tr>
<td>$N^{-\alpha} \log N$</td>
<td>$\frac{1}{x} b_0(z; \alpha) = \frac{1}{x} b_0(z; \alpha) \left[ \gamma(\alpha) - \log \log z \right]^2 + \gamma'(\alpha)$</td>
</tr>
<tr>
<td>$N^{-\alpha} (\log N^2 - \ell)$</td>
<td>$\frac{1}{x} b_0(z; \alpha; \ell) = \frac{1}{x} b_0(z; \alpha) \frac{\Gamma(\alpha)}{(\ell-1)!} \int_0^\infty \frac{e^{sq}}{s^{(\ell-1)}} \frac{\log z}{\Gamma\left(s + \alpha\right)} ds$</td>
</tr>
</tbody>
</table>
APPENDIX A

In this appendix we list the first coefficients in the expansion of $\gamma_N^{(1)}$. The general form of this expansion is:

$$
\gamma_N = \sum_{p=0}^{\infty} \sum_{\ell=0}^{\infty} B_{p\ell}^{ab} \log N + (-1)^{\ell} \sum_{\ell=3}^{\infty} \frac{B_{\ell}}{N^{\ell}}
$$

(A.1)

with $a, b \in \mathbb{F}_A$. Only terms up to (including) $N^{-4}$ will be given. Missing coefficients are zero. The set of $B$ and $\bar{B}$ are:

$$
B_{10}^{FF} = c_F c_A \left( \frac{536}{9} - \frac{8}{3} \pi^2 \right) c_F c_A - \frac{160}{9} c_F T_R = 132.946 - 11.852 \eta_f,
$$

$$
B_{00}^{FF} = \left( 4 \pi^2 - 48 \zeta(3) - 3 \right) c_F^2 + \left( \frac{536}{9} \zeta(4) - \frac{44}{9} \pi^2 \right.
$$

$$
\left. + \frac{8}{3} \eta_f + 24 \zeta(3) - \frac{17}{3} \right) c_F c_A + \left( \frac{4}{3} - \frac{160}{9} \zeta(4) + \frac{16}{9} \pi^2 \right) c_F T_R = -61.262 + 5.745 \eta_f,
$$

$$
B_{11}^{FF} = 32 c_F^2 = \frac{512}{9},
$$

$$
B_{01}^{FF} = \left( 32 \zeta(3) - 24 \right) c_F^2 + \left( \frac{532}{3} \pi^2 \right) c_F c_A - \frac{127}{9} c_F T_R = 173.977 - 13.037 \eta_f,
$$

$$
B_{12}^{FF} = -16 c_F^2 = -\frac{256}{9},
$$

$$
B_{02}^{FF} = \left( 44 - 16 \zeta(3) \right) c_F^2 + \left( \frac{14 \pi^2}{9} - \frac{2558}{27} \right) c_F c_A + \frac{712}{27} c_F T_R = -255.748 + 17.580 \eta_f,
$$

with $a, b \in \mathbb{F}_A$. Only terms up to (including) $N^{-4}$ will be given. Missing coefficients are zero. The set of $B$ and $\bar{B}$ are:
\[ B_{13}^{FF} = \frac{112}{3} c_F^2 = \frac{1792}{27}, \]
\[ B_{03}^{FF} = \frac{4}{3} (28 \xi - 53) c_F^2 + \frac{4}{3} (82 - \pi^2) c_F c_A - 32 c_F T_R = 247.376 - 21.333 n_f, \]
\[ B_{41}^{FF} = -48 c_F^2 = -\frac{256}{3}, \]
\[ B_{04}^{FF} = 4 (29 - 12 \xi) c_F^2 + \frac{1}{45} (59 \pi^2 - \frac{17633}{3}) c_F c_A - \frac{1204}{27} c_F T_R = -313.732 - 29.728 n_f, \]

The coefficients of the non-singlet anomalous dimension coincide with the ones already quoted, except for the last one being replaced by:

\[ B_{04}^{NS} = 4 (29 - 12 \xi) c_F^2 + \frac{1}{45} (59 \pi^2 - \frac{17633}{3}) c_F c_A + \frac{956}{27} c_F T_R = -313.732 + 23.605 n_f. \]

\[ \tilde{B}_3^{FA} = 8 c_A T_R = 12 n_f, \quad \tilde{B}_4^{FA} = -48 c_A T_R = -72 n_f, \]
\[ B_{21}^{FA} = -16 c_A T_R + 16 c_F T_R = -13.333 n_f, \]
\[ B_{44}^{FA} = -32 \xi c_A T_R + 32 \xi c_F T_R = -15.392 n_f, \]
\[ B_{01}^{FA} = (-16 \xi^2 + 16) \frac{C_A}{T_R} + \left(16 \xi^2 - \frac{8 \pi^2}{3} + 40\right) \frac{C_F}{T_R} = 28.678 \eta_f, \]
\[ B_{22}^{FA} = 32 \frac{C_A}{T_R} - 32 \frac{C_F}{T_R} = 26.667 \eta_f, \]
\[ B_{12}^{FA} = (64 \xi^2 - 16) \frac{C_A}{T_R} - (64 \xi^2 + 16) \frac{C_F}{T_R} = -3.882 \eta_f, \]
\[ B_{02}^{FA} = (32 \xi^2 - 16 \xi^2 - 16) \frac{C_A}{T_R} + (-32 \xi^2 - 16 \xi^2 + \frac{16}{3} \pi^2 - 64) \frac{C_F}{T_R} = -42.7 \eta_f, \]
\[ B_{23}^{FA} = -96 \frac{C_A}{T_R} + 96 \frac{C_F}{T_R} = -80 \eta_f, \]
\[ B_{13}^{FA} = (-192 \xi + \frac{488}{3}) \frac{C_A}{T_R} + (192 \xi - \frac{104}{3}) \frac{C_F}{T_R} = 128.534 \eta_f, \]
\[ B_{03}^{FA} = (-96 \xi^2 + \frac{488}{3} \xi + 44) \frac{C_A}{T_R} + (96 \xi^2 - \frac{104}{3} \xi - 16 \pi^2 + 276) \frac{C_F}{T_R} = 245.570 \eta_f, \]
\[ B_{24}^{FA} = 224 \frac{C_A}{T_R} - 224 \frac{C_F}{T_R} = 186.667 \eta_f, \]
\[ B_{14}^{FA} = \left(448 \xi - \frac{2032}{3}\right) \frac{C_A}{T_R} + \left(-448 \xi + \frac{304}{3}\right) \frac{C_F}{T_R} = -732.951 \eta_f, \]
\[ B_{04}^{FA} = \left(224 \xi^2 - \frac{2032}{3} \xi + 76\right) \frac{C_A}{T_R} + \left(-224 \xi^2 + \frac{304}{3} \xi + \frac{112}{3} \pi^2 - 684\right) \frac{C_F}{T_R} = -581.620 \eta_f, \]
\[ B_{4}^{AF} = -16 c_{F} C_{A} = -64, \quad B_{24}^{AF} = -8 c_{F}^{2} + 8 c_{F} C_{A} = 17.778, \]
\[ B_{11}^{AF} = (-16 \xi + 40) c_{F}^{2} + (16 \xi - \frac{136}{3}) c_{F} C_{A} + \frac{32}{3} c_{F} T_{R} = -89.699 + 7.111 n_{f}, \]
\[ B_{04}^{AF} = (-8 \xi^{2} + 40 \xi - \frac{4}{3} \pi^{2} - 48) c_{F}^{2} + (8 \xi^{2} - \frac{136}{3} \xi + \frac{822}{9}) c_{F} C_{A} + (\frac{32}{3} \xi - \frac{256}{9}) c_{F} T_{R} = 221.129 - 14.858 n_{f}, \]
\[ B_{22}^{AF} = -8 c_{F}^{2} + 8 c_{F} C_{A} = 17.778, \]
\[ B_{12}^{AF} = (-16 \xi + 16) c_{F}^{2} + (16 \xi - \frac{112}{3}) c_{F} C_{A} + \frac{32}{3} c_{F} T_{R} = -100.366 + 7.111 n_{f}, \]
\[ B_{02}^{AF} = (-8 \xi^{2} + 16 \xi - \frac{4}{3} \pi^{2} + 4) c_{F}^{2} + (8 \xi^{2} - \frac{112}{3} \xi + \frac{476}{9}) c_{F} C_{A} + (\frac{32}{3} \xi - \frac{112}{3}) c_{F} T_{R} = 131.416 - 4.192 n_{f}, \]
\[ B_{23}^{AF} = -24 c_{F}^{2} + 24 c_{F} C_{A} = 53.333, \]
\[ B_{13}^{AF} = (-48 \xi + \frac{436}{3}) c_{F}^{2} + (48 \xi - \frac{580}{3}) c_{F} C_{A} + 32 c_{F} T_{R} = -453.393 + 21.333 n_{f}, \]
\[ B_{03}^{AF} = (-24 \xi^{2} + \frac{436}{3} \xi - 4 \pi^{2} - \frac{3 y_{0}}{3}) c_{F}^{2} + (24 \xi^{2} - \frac{580}{3} \xi + \frac{27 y_{0}}{9}) c_{F} C_{A} + (32 \xi - \frac{920}{9}) c_{F} T_{R} = 666.637 - 55.834 n_{f}, \]
\[
B_{24}^{AF} = -8c_F^2 + 8c_Fc_A = 17.778,
\]
\[
B_{14}^{AF} = (-16 \ell - \frac{32}{3})c_F^2 + (16 \ell - \frac{356}{3})c_Fc_A + \frac{22}{3}c_FT_R = \]
\[-508.662 + 7.111 \eta_f,\]
\[
B_{04}^{AF} = (-8 \ell - \frac{32}{3} \ell - \frac{4}{3} \eta^2 + 13 \ell)c_F^2 + (8 \ell - \frac{356}{3} \ell
\]
\[-\frac{55}{3})c_Fc_A + (\frac{32}{3} \ell + \frac{56}{3})c_FT_R = -151.370 + 16.549 \eta_f,\]
\]
\[
\tilde{B}_3^{AA} = 2c_A^2 = -18, \quad \tilde{B}_4^{AA} = -23c_A^2 = -207,\]
\[
B_{10}^{AA} = -\frac{160}{9}c_A T_R + (\frac{8}{3} \eta^2 + \frac{536}{9})c_A^2 = 299.129 - 26.667 \eta_f,\]
\[
B_{00}^{AA} = (-160 \ell + \frac{32}{3})c_A T_R + 8c_F T_R + (\frac{8}{3} \ell \eta^2 + \frac{536}{9} \ell
\]
\[-24 \ell + (3 - \frac{32}{3})c_A = -278.982 - 5.333 \eta_f,\]
\]
\[
B_{24}^{AA} = -4c_A^2 = -36, \quad B_{14}^{AA} = (-8 \ell + 42) c_A^2 = 336.440,\]
\[
B_{04}^{AA} = -\frac{310}{9}c_A T_R + (-4 \ell^2 + 42 \ell + \frac{10}{3} \eta^2 - \frac{200}{9})c_A^2 = \]
\[-302.284 - 13.334 \eta_f,\]
\[
B_{12}^{AA} = 14c_A^2 = 126, \quad B_{12}^{AA} = (28 \ell - 70) c_A^2 = -484.542,\]

\[ B_{02}^{AA} = \frac{1864}{27} c_A T_R + 32 c_F T_R + \left( 14 \ell_1^2 - 70 \ell_1 + \frac{47}{9} \ell_2^2 - \frac{2189}{27} \right) \]

\[ c_A^2 = -587.461 + 124.889 \eta, \]

\[ B_{23}^{AA} = 6 c_A^2 = 54, \quad B_{13}^{AA} = (12 \ell_1 + 78) c_A = 764.339, \]

\[ B_{03}^{AA} = -\frac{608}{9} c_A T_R - 64 c_F T_R + \left( 6 \ell_1^2 + 78 \ell_1 \ell_2 - \frac{5}{3} \ell_2^2 + \frac{4917}{18} \right) c_A^2 = 523.653 - 144 \eta t \]

\[ B_{24}^{AA} = 14 c_A^2 = 116, \quad B_{14}^{AA} = (28 \ell_1 - \frac{691}{3}) c_A = -1927.542 \]

\[ B_{04}^{AA} = \frac{8156}{27} c_A T_R + 144 c_F T_R + \left( 14 \ell_1^2 - \frac{691}{3} \ell_1 + \frac{944}{45} \ell_2^2 - \frac{67178}{135} \right) c_A^2 = -3769.740 + 549.111 \eta \]

We remind the reader that \( T_R = n_F/2 \) where \( n_F \) is the number of flavours.

The values of \( c_A \) and \( c_F \) are 3 and 4/3 for the SU(3) gauge group.

The symbol \( \ell \) stands for Euler's constant and \( \ell = \sum_{j=1}^{\infty} \frac{1}{j^3} + \ldots \)
We now list the values of the polynomials in $r$ appearing in expression (4.14):

\[ Y_{21}^{FF} = r, \quad Y_{22}^{FF} = \frac{y}{5}, \quad Y_{31}^{FF} = -\frac{r^2}{2} - r, \]
\[ Y_{32}^{FF} = -\frac{r}{10} + \frac{y}{5}, \quad Y_{33}^{FF} = \frac{y}{5}, \quad Y_{41}^{FF} = \frac{r^3}{8} + \frac{13r^2}{12} + 7r, \]
\[ Y_{42}^{FF} = \left( \frac{3}{20} + \frac{3n_f}{40} \right) + \frac{35}{12} r - \frac{28}{5}, \]
\[ Y_{43}^{FF} = -6n_f - \frac{3}{25} r - \frac{45y}{75}, \quad Y_{44}^{FF} = \frac{36n_f}{125} - \frac{3}{5}, \]
\[ Z_{22}^{FF} = \frac{y}{5}, \quad Z_{32}^{FF} = -\frac{9r}{10} - \frac{y}{5}, \quad Z_{33}^{FF} = -\frac{y}{5}, \]
\[ Z_{42}^{FF} = \frac{81}{160} r^2 + \frac{93}{20} r + \frac{22}{5}, \quad Z_{43}^{FF} = \left( -\frac{3n_f}{25} + \frac{9}{10} \right) r + \frac{45y}{75}, \quad Z_{44}^{FF} = -\frac{36n_f}{125} + \frac{3}{5}, \]
\[ G_0^{(F)} = 1, \quad G_1^{(F)} = -\frac{r}{2}, \quad G_2^{(F)} = r \left( \frac{r}{8} + \frac{7}{12} \right), \]
\[ G_3^{(F)} = -r \left( \frac{r^2}{48} + \frac{7r}{24} + \frac{1}{2} \right), \]
\[ G_4^{(F)} = r \left( \frac{r^3}{384} + \frac{7r^2}{96} + \frac{121r}{288} + \frac{59}{120} \right) \]
\[ Y_{14}^{FA} = 2, \quad Y_{24}^{FA} = -r - 4, \quad Y_{22}^{FA} = -1, \]
\[
Y_{31}^{FA} = \frac{r^2}{4} + \frac{19r}{6} + 12, \quad Y_{32}^{FA} = \left(\frac{3\eta_f}{10} + \frac{1}{2}\right)r + \frac{257}{30},
\]
\[
Y_{33}^{FA} = -\frac{12}{25}\eta_f + \frac{1}{2}, \quad Y_{41}^{FA} = -\frac{r^3}{24} - \frac{13r^2}{12} - \frac{28r}{3} - 28,
\]
\[
Y_{42}^{FA} = -\left(\frac{3\eta_f}{20} + \frac{1}{8}\right)r^2 - \left(\frac{9\eta_f}{10} + \frac{73}{15}\right)r - \frac{383}{15},
\]
\[
Y_{43}^{FA} = -\left(\frac{3\eta_f}{50} + \frac{1}{4}\right)r + \frac{36\eta_f}{25} - \frac{227}{30},
\]
\[
Y_{44}^{FA} = \frac{18\eta_f}{25} - \frac{1}{4},
\]
\[
Z_{41}^{FA} = -2, \quad Z_{21}^{FA} = \frac{9r}{4} + 4, \quad Z_{22}^{FA} = 4,
\]
\[
Z_{31}^{FA} = -\frac{81r^2}{64} - \frac{111r}{8} - 12, \quad Z_{32}^{FA} = \left(\frac{3\eta_f}{10} - \frac{9}{8}\right)r - \frac{257}{30},
\]
\[
Z_{33}^{FA} = \frac{12\eta_f}{25} - \frac{1}{2},
\]
\[
Z_{41}^{FA} = \frac{243}{512}r^3 + \frac{837r^2}{64} + \frac{165r}{4} + 28,
\]
\[
Z_{42}^{FA} = \left(\frac{81}{128} - \frac{27}{80}\eta_f\right)r^2 + \left(\frac{573}{40} - \frac{9}{10}\eta_f\right)r + \frac{383}{15},
\]
\[
Z_{43}^{FA} = \left(\frac{9}{16} - \frac{21}{25}\eta_f\right)r - \frac{36}{25}\eta_f + \frac{227}{30},
\]
\[
Z_{44}^{FA} = -\frac{18\eta_f}{25} + \frac{1}{4}.
\]
\[ Y_{11}^{AF} = -\frac{2}{5}, \quad Y_{21}^{AF} = \frac{9r}{20} - \frac{2}{5}, \quad Y_{22}^{AF} = \frac{1}{5}, \]
\[ Y_{31}^{AF} = -\frac{81}{320} r^2 - \frac{57}{40} r - \frac{6}{5}, \quad Y_{32}^{AF} = (\frac{3\eta_f}{50} - \frac{9}{40}) r - \frac{167}{150}, \]
\[ Y_{33}^{AF} = \frac{12\eta_f}{125} - \frac{1}{10}, \quad Y_{41}^{AF} = \frac{2V3}{2560} r^3 + \frac{297}{160} r^2 - \frac{51}{40} r - \frac{2}{5}, \]
\[ Y_{42}^{AF} = (-\frac{27\eta_f}{400} + \frac{81}{640}) r^2 + \frac{219}{400} r + \frac{17}{30}, \]
\[ Y_{43}^{AF} = (-\frac{27}{125} \eta_f + \frac{9}{80}) r + \frac{91}{75}, \quad Y_{44}^{AF} = -\frac{18\eta_f}{125} + \frac{1}{20}, \]
\[ Z_{11}^{AF} = \frac{2}{5}, \quad Z_{21}^{AF} = -\frac{1}{5} + \frac{2}{5}, \quad Z_{22}^{AF} = -\frac{1}{5}, \]
\[ Z_{31}^{AF} = \frac{r^2}{20} + \frac{r}{30} + \frac{6}{5}, \quad Z_{32}^{AF} = \frac{3\eta_f}{50} + \frac{1}{10} r + \frac{167}{150}, \]
\[ Z_{33}^{AF} = -\frac{12\eta_f}{125} + \frac{1}{10}, \quad Z_{41}^{AF} = -\frac{r^3}{120} - \frac{r^2}{15} - \frac{17}{30} r + \frac{2}{5}, \]
\[ Z_{42}^{AF} = (-\frac{3\eta_f}{400} + \frac{1}{40}) r^2 - \frac{101}{150} r - \frac{17}{30}, \]
\[ Z_{43}^{AF} = (-\frac{3\eta_f}{250} + \frac{1}{20}) r - \frac{91}{75}, \quad Z_{44}^{AF} = \frac{18\eta_f}{125} - \frac{1}{20}, \]
\[ Y_{21}^{AA} = -r, \quad Y_{22}^{AA} = -\frac{4}{5}, \quad Y_{31}^{AA} = \frac{9r^2}{8} + r, \quad Y_{32}^{AA} = \frac{3}{5} r + \frac{4}{5}, \]
\[ Y_{33}^{AA} = \frac{4}{5}, \quad Y_{41}^{AA} = -\frac{81}{128} r^3 - \frac{93}{16} r^2 + r, \]
\[ Y_{42}^{AA} = \frac{3\eta_f}{40} - \frac{171}{160} r^2 - \frac{23}{30} r - \frac{28}{5}. \]
$Y^{AA}_{43} = \left( \frac{6\eta_f}{25} - \frac{23}{20} \right) r - \frac{454}{75}, \quad Y^{AA}_{44} = \frac{36\eta_f}{125} - \frac{3}{5},$

$Z^{AA}_{22} = \frac{4}{5}, \quad Z^{AA}_{32} = -\frac{2r}{5} - \frac{4}{5}, \quad Z^{AA}_{33} = -\frac{4}{5},$

$Z^{AA}_{42} = \frac{r^2}{10} + \frac{13}{15} r + \frac{28}{5}, \quad Z^{AA}_{43} = \left( \frac{3\eta_f}{25} + \frac{2}{5} \right) \eta_f + \frac{454}{75},$

$Z^{AA}_{44} = -\frac{36\eta_f}{125} + \frac{3}{5},$

$G^{(A)}_0 = 1, \quad G^{(A)}_1 = -\frac{9r}{8}, \quad G^{(A)}_2 = \frac{81}{128} r^2 + \frac{75}{16} r,$

$G^{(A)}_3 = -\frac{243}{1024} r^3 - \frac{675}{128} r^2 - \frac{9}{2} r,$

$G^{(A)}_4 = \frac{2187}{32768} r^4 + \frac{6075}{2048} r^3 + \frac{8217}{512} r^2 + \frac{2877}{160} r.$

In the previous expressions $r$ is given by formula (3.21b) and $\eta_f \cdot$ stands for the number of quark flavours. Notice that $G^{(F)}_3$ differs from its expression appearing in Ref. 7) by an overall minus sign which was deleted when typing that paper.
In this appendix we give the list of coefficients which are necessary to obtain the pointlike reconstruction kernels up to second non-trivial order in α. The general form of the expansion is shown in (5.5). These coefficients are in general polynomials in r. Their values up to $1/r^2$ terms are as follows:

\[
\begin{align*}
\xi_{20}^{(0)\,FF} &= 2.666, \\
\xi_{40}^{(0)\,FF} &= 17.499 - 5.333 \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (7.491 - 5.926 g), \\
\xi_{00}^{(0)\,FF} &= -14.900 + 0.922 \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (4.520 + 2.873 g), \\
\xi_{21}^{(0)\,FF} &= 1.333 r, \\
\xi_{11}^{(0)\,FF} &= -2.667 r + 2.667 + 2.667 r \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (-4.985 r + 2.962 g r + 4.267), \\
\xi_{01}^{(0)\,FF} &= 7.491 r + 20.749 + (-0.461 r - 2.667) \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (2.297 r - 1.436 g r + 13.048 - 6.519 g), \\
\xi_{22}^{(0)\,FF} &= 0.333 r^2 + 1.556 r, \\
\xi_{12}^{(0)\,FF} &= 2.187 r^2 + r (8.874 + 2.667 g) - 3.111 + (-0.667 r^2 - 3.111 r) \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (1.246 r^2 - 0.740 g r^2 + 3.683 r - 3.5 g r - 2.133), \\
\xi_{02}^{(1)\,FF} &= -0.533 g, \\
\xi_{02}^{(0)\,FF} &= -1.451 r^2 + r (-19.119 + 18.689 g + 1.778 g^2) - 8.874 + 3.2 g + (0.115 r^2 + 1.871 r - 5.333 g r + 3.111) \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (-0.574 r^2 + 0.359 r g^2 - 9.204 r + 14.906 g - \ldots)
\end{align*}
\]
\[
\begin{align*}
5.926g^2r - 19.848 - 7.743g + 7.111g^2) \\
\chi_{12}^{(0)}_{FF} = g \left[ (-6.120 + 13.347g + 1.185g^2)r + \\
5.337 + 0.711g \right] - (3.556g^2r + 1.69g - 9.6g) \frac{\beta_1}{\beta_0^2} + \\
\frac{g}{b} \left( 19.427 - 8.349 \frac{1}{b} - 14.044g + 3.924 \frac{g}{b} + \\
5.926g^2 - 3.951 \frac{g^2}{b} \right), \ \chi_{12}^{(n)FF} = g \left( 7.486 - 0.711g \right) - \\
9.6g \frac{\beta_1}{\beta_0^2} + \frac{g}{b} \left( 19.427 - 8.349 \frac{1}{b} - 14.044g + \\
3.924 \frac{g}{b} + 5.926g^2 - 3.951 \frac{g^2}{b} \right), \\
\tilde{\chi}_{12}^{(0)FF} = \frac{g}{b} \left( -23.72 + 16.698 \frac{1}{b} + 15.052g - \\
7.848 \frac{g}{b} - 7.111g^2 + 7.901 \frac{g^2}{b} \right), \ \tilde{\chi}_{12}^{(n)FF} = - \chi_{12}^{(0)FF} \\
\chi_{22}^{(0)FF} = g \left( 4.896 - 10.677g - 0.948g^2 \right) + \left( 2.844g^2 + \\
1.28g \right) \frac{\beta_1}{\beta_0^2} + \frac{g}{b} \left( -0.956 - 5.375g + 3.160g^2 \right), \\
\chi_{22}^{(n)FF} = - \chi_{22}^{(0)FF} + 3.509g, \ \tilde{\zeta}_{12}^{(0)FF} = \frac{g}{b} \left[ -2.667b^2 + \\
10.933b - 27.335 + 8.349 \frac{1}{b} - 10.667bg + \\
24.711g - 3.924 \frac{g}{b} - 5.926g^2 + 8.951 \frac{g^2}{b} \right], \\
\tilde{\zeta}_{12}^{(n)FF} = \tilde{\zeta}_{12}^{(0)FF} = \tilde{\zeta}_{12}^{(n)FF} = - \tilde{\zeta}_{12}^{(0)FF}.
\end{align*}
\]
\[ \bar{\sigma}_{12}^{(4)\text{FF}} = \frac{\Delta}{b} (-2.667b^2 + 5.6b + 13.656 + 8.349 \frac{1}{b} + 3.556gb - 9.660g - 3.924 \frac{b^2}{g} - 1.185g^2 + 3.951 \frac{b^2}{g}) \]

\[ \sigma_{12}^{(0)\text{FF}} \approx \bar{\sigma}_{12}^{(4)\text{FF}} \]

\[ \sigma_{12}^{(0)\text{FF}} \approx \sigma_{12}^{(4)\text{FF}} \]

\[ \bar{F}_{11}^{(0)\text{FA}} = -6.667g, \quad \bar{F}_{11}^{(4)\text{FA}} = 5.333g, \quad \bar{F}_{11}^{(8)\text{FA}} = 6.667g, \]

\[ \bar{F}_{04}^{(2)\text{FA}} = g(26.671 - 0.889g) - 10.667g \left( \frac{b_1}{b_0^2} + \frac{1}{b} \right) (19.942g - 11.852g^2 + 6.667gb^2), \]

\[ \bar{F}_{04}^{(4)\text{FA}} = g \left( 9.561 - 3.556g \right) + 4.444g^2 - 6.667gb, \]

\[ \bar{F}_{04}^{(8)\text{FA}} = 0.667g \gamma + 2.667g, \quad \bar{F}_{12}^{(0)\text{FA}} = -g(3.333\gamma + 13.333), \]

\[ \bar{F}_{12}^{(4)\text{FA}} = \frac{1}{b} (9.876g^3 - 9.810g^2 + 20.873g), \quad \bar{F}_{12}^{(8)\text{FA}} = \bar{F}_{12}^{(0)\text{FA}}. \]
\[ \bar{\zeta}^{(1)}_{12} = -9 \, g \, r - 16g + 5.4 \frac{g}{b}, \quad \bar{\zeta}^{(2)}_{12} = g (7.5 \, r + 13.33 \, b), \]

\[ \bar{\zeta}_{02}^{(1)} = -13.335 \, g \, r + 0.444 \, g^2 \, \gamma - 70.075 \, g + 1.778 \, g^2 + \left( 5.333 \, g \, r + 21.333 \, g \right) \frac{\beta_1}{\beta_2^2} + \frac{1}{b} \left( -3.333 \, b^2 - 13.333 \, b^2 - 9.971 \, r - 31.351 + 5.926 \, r \, g + 23.704 \, g \right), \]

\[ \bar{\zeta}_{02}^{(2)} = -10.756 \, g \, r + 4 \, g^2 \, \gamma - 69.475 \, g + 7.111 \, g^2 - (27 \, g \, r + 48) \frac{\beta_1}{\beta_2} + \frac{g}{b} \left( 50.478 \, r + 41.826 - 30 \, g \right) - 49.733 \, g), \quad \bar{\zeta}_{02}^{(\omega)} = g \left( 3.333 \, r \, b + 13.333 \, b - 5.424 \, r - 2.296 \right) + g^2 \left( -2.222 \, r - 8.889 \right), \]

\[ \bar{\zeta}_{12}^{(1)} = g \left[ -7.5 \, r \, b - 13.333 \, b + 12.204 \, r + 2.296 + 5 \, g - 8.889 \, g \right], \quad \bar{\zeta}_{12}^{(\omega)} = 6.120 \, g \, r - 13.337 \, g^2 \, \gamma - 1.185 \, g^3 + 48.482 \, g - 53.387 \, g^2 - 4.741 \, g^3 + \left( 12.444 \, g^2 \, \gamma + 1.6 \, g \, r + 49.778 \, g^2 + 6.4 \, g \right) \frac{\beta_1}{\beta_2^2} + \frac{g}{b} \left[ (8.889 \, r + 33.156) g^2 - (11.623 \, r + 16.253) g + 10.317 \, r + 86.828 \right], \]
\[ \chi_{12}^{(0)} = -13.771 \frac{g}{r} + 30.030 \frac{g^2}{r} + 2.667 \frac{g^3}{r} - 48.482 \frac{g}{r} + 53.387 \frac{g^2}{r} + 4.741 \frac{g^3}{r} - 28.925 \frac{g^2}{r} + 3.612 \frac{g^3}{r} - 49.778 \frac{g^2}{r} + 6.4 \frac{g}{r} \beta_1 \beta_2 \beta_0 \left[ (-20t - 33.156) g^2 + (26.152 r + 16.253) - 23.213 r - 86.828 \right], \]
\[ \tilde{\chi}_{12}^{(0)} = \frac{a}{b} \left[ (-4.938 r - 17.353) g^2 + g (4.905 r - 12.042) - 10.436 r - 67.146 \right], \]
\[ \tilde{\chi}_{12}^{(1)} = \frac{a}{b} \left[ g^2 (11.111 r + 17.353) + g (-11.036 r + 12.042) + 23.482 r + 67.146 \right], \]
\[ \chi_{22}^{(0)} = \frac{g}{r} \left( 6.120 - 13.346 \frac{g}{r} - 1.185 \frac{g^2}{r} + 12.444 \frac{g^2}{r} + 1.6 \frac{g}{r} \beta_1 \beta_2 \beta_0 \left( 8.889 \frac{g^2}{r} - 11.623 g + 10.317 \right), \right], \]
\[ \chi_{22}^{(1)} = - \chi_{22}^{(0)}, \quad \tilde{\chi}_{22}^{(0)} = \frac{a}{b} \left( -4.938 \frac{g^2}{r} + 4.905 \frac{g}{r} - 10.436 \right), \quad \tilde{\chi}_{22}^{(1)} = - \tilde{\chi}_{22}^{(0)} , \]
\[ \sigma_{11}^{(0)} = \frac{a}{b} \left[ (-2.962 b + 9.877) g^2 + g (8.889 b^2 + 24.148 b - 9.810) - 6.667 b^3 + 14 b^2 - 28.512 b + 83.523 \right], \quad \sigma_{11}^{(1)} = - \sigma_{11}^{(0)}, \quad \sigma_{11}^{(1)} = \sigma_{11}^{(0)} \]
\[ \sigma_{12}^{(0)FA} = \frac{g}{b} \left[ (1.481 r b + 5.926 b - 4.938 r - 17.353) g^2 \\
+ g \left( -4.444 r b^2 - 17.778 b^2 + 12.074 r b + 33.985 b + \\
4.905 r - 12.042 \right) + 3.333 r b^3 + 13.333 b^3 - 7 r b^2 - \\
11.933 b^2 - 17.069 r b - 25.316 b - 10.436 r - \\
67.146 \right] , \quad \sigma_{12}^{(0)FA} = -\sigma_{12}^{(0)FA} \]

\[ \overline{\sigma}_{12}^{(0)FA} = \frac{g}{b} \left[ (3.333 r b + 5.926 b - 11.111 r - 17.353) g^2 \\
+ g \left( -10 r b^2 - 17.778 b^2 + 27.162 r b + 33.985 b + \\
11.036 r - 12.042 \right) + 7.5 r b^3 + 13.333 b^3 - 15.75 r b^2 - \\
11.933 b^2 - 38.405 r b - 25.316 b - 23.482 r - 67.146 \right] , \]

\[ \sigma_{22}^{(0)FA} = \frac{g}{b} \left[ (1.482 b - 4.938) g^2 + g \left( -4.444 b^2 + \\
12.074 b + 4.905 \right) + 3.333 b^3 - 7 b^2 - 17.069 b - \\
10.436 \right] , \quad \overline{\sigma}_{22}^{(0)FA} = -\sigma_{22}^{(0)FA} , \quad \sigma_{22}^{(0)FA} = \sigma_{22}^{(0)FA} \]

In the above expressions we have kept the dependence on the number of flavours. This dependence is carried by the quantities \( \beta_1, \beta_0, g \) and \( b \). The first two are defined in (5.2). The definition of \( g \) and \( b \) is the following:
\[ g = \frac{3}{20} n_f \]
\[ b = \frac{3}{20} \beta_0 \]

where \( n_f \) is the number of flavours.

The coefficients entering in the expression of the non-singlet kernel are:

\[ \xi_{20}^{\text{NS}} = 2.667, \quad \xi_{521}^{\text{NS}} = -1.333 r, \quad \xi_{22}^{\text{NS}} = 0.333 r^2 + 1.556 g, \]
\[ \xi_{10}^{\text{NS}} = 17.498 - 5.333 \frac{\beta_1}{\beta_0^2} + \frac{1}{b} \left[ 9.971 - 5.926 g \right], \]
\[ \xi_{11}^{\text{NS}} = -8.749 r + 2.667 + 2.667 \frac{\beta_1}{\beta_0^2} r + \frac{1}{b} \left[ (-4.985 + 2.963 g) r + 21.333 \right], \]
\[ \xi_{12}^{\text{NS}} = 2.187 r^2 + 8.874 r - 3.111 + (0.667 r^2 - 3.111) \frac{\beta_1}{\beta_0^2} + \frac{1}{b} \left[ (3.489 - 0.741 g) r^2 + (5.520 - 3.457 g) r - 10.667 \right], \]
\[ \xi_{500}^{\text{NS}} = -14.990 + 1.843 \frac{\beta_1}{\beta_0^2} + \frac{1}{b} (-4.595 + 2.873 g), \quad \xi_{504}^{\text{NS}} = 7.495 r + 2.474 - (0.922 r + 2.667) \frac{\beta_1}{\beta_0^2} + \frac{1}{b} \left[ (2.298 + 1.437 g) r + 13.048 - 6.519 g \right], \]
\[ S_{02}^{\text{NS}} = -1.874 \tau^2 - 19.119 \tau - 11.541 + (0.230 \tau^2 + 2.408 \tau + 6.222) \frac{\rho_{\text{ref}}^2}{\rho^2} + \frac{1}{b} \left[ (-0.574 \tau^2 - 9.204 \tau - 19.181 + (0.359 \tau^2 + 4.935 \tau + 8.790) \rho) \right] \]
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