Appendix A
Hyperspheres

Abstract The mathematical tool to evaluate the Hilbert space averages and variances that play a crucial role throughout this book is the averaging of polynomial functions over hyperspheres. In this appendix we explain the corresponding mathematics in some detail and thus provide this tool.

A.1 Averaging of Functions over Hyperspheres

As explained in Sects. 7.3 and 8.1 the accessible regions (ARs) we are considering essentially confine the real and imaginary parts of the amplitudes $\psi_i$ (which define the state with respect to a given basis) to hyperspheres of given radii. Thus the Hilbert space averages we have to do are eventually averages over such hyperspheres. As stated below (see Appendix B.1) the respective functions are not too complicated. They simply consist of products of powers of the coordinates (real and imaginary parts). The largest products we will have to consider contain four factors. Hence in this section we abstractly compute the averages of such functions over hyperspheres of given dimension $d$ and given radius $R$.

We start by considering the integral of the above function over a hypersphere which we denote by $Z(R, d, l, k, u, v)$ as follows:

$$Z(R, d, l, k, u, v) \equiv \int_{-\infty}^{\infty} \prod_{i=1}^{d} dx_i \delta\left(\sqrt{\sum_{j=1}^{d} x_j^2} - R\right) x_l^i x_k^j x_u^a x_v^b.$$  (A.1)

The arguments of $Z$ denote radius, dimension, and the respective powers of the four coordinate factors. Note that $Z(R, d, 0, 0, 0, 0)$ simply is the “surface area” of the hypersphere. Thus the evaluation of (A.1) will eventually suffice to compute not only the interval but the wanted average over the hypersphere, cf. (A.10). Since it is not obvious how to evaluate $Z$ in a straightforward manner, we instead analyze a function $G$ which is related to $Z$ in the following way:

DOI 10.1007/978-3-540-70510-9_BM2 © Springer-Verlag Berlin Heidelberg 2009
\[ \int_0^\infty dR \, e^{-R^2} Z(R, d, l, k, u, v) \]  

(A.2)

\[ = \int_0^\infty dR \int_{-\infty}^\infty dx_i \delta\left(\sqrt{\sum_{j} x_j^2} - R\right) e^{-\frac{d}{m} \sum x_m^2} x_a^l x_b^k x_g^u x_h^v }  

= \int_{-\infty}^\infty dx_i e^{-\frac{d}{m} \sum x_m^2} x_a^l x_b^k x_g^u x_h^v }  

\equiv G(d, l, k, u, v).

Here we first replaced \( R^2 \) in the exponent of the integrand by \( \sum_{j} x_j^2 \) which is in order since the \( \delta \)-function requires them to be equal anyway. Then we performed the integration over \( R \) which simply makes the \( \delta \)-function vanish. The remaining integral factorizes; it may be evaluated using any standard textbook on basic integrals:

\[
G(d, l, k, u, v) = \begin{cases} 
\pi^\frac{d+2}{2} \Gamma\left(\frac{l+1}{2}\right) \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{u+1}{2}\right) \Gamma\left(\frac{v+1}{2}\right) & \text{for } * \\
0 & \text{for } **
\end{cases}
\]

\[
* (l, k, u, v) \text{ all even, } \quad ** (l, k, u, v) \text{ any of them odd.}
\]  

(A.3)

To further pursue the evaluation of \( Z \) we now perform a coordinate transformation \( x_i \rightarrow y_i \) according to

\[ x_i = R \, y_i. \]  

(A.4)

Applying this transformation we may “extract” the \( R \)-dependence from \( Z \):

\[ Z(R, d, l, k, u, v) \]  

(A.5)

\[ = R^{d+l+k+u+v} \int_{-\infty}^\infty \prod_{i=1}^d dy_i \delta\left(R\sqrt{\sum_{j} y_j^2} - 1\right) y_a^l y_b^k y_g^u y_h^v \]

\[ = R^{d+l+k+u+v-1} \int_{-\infty}^\infty \prod_{i=1}^d dy_i \delta\left(\sum_{j} y_j^2 - 1\right) y_a^l y_b^k y_g^u y_h^v \]

\[ \equiv z(d, l, k, u, v). \]

Here we simply made use of the fact that prefactors in the arguments of \( \delta \)-functions may be (inversely) pulled out of the arguments. The function defined by the underbrace, \( z \), is essentially the wanted function \( Z \), only with the \( R \)-dependence factored out. Using this formulation we may write the above function \( G \) from (A.2) in an yet other form:
\[ \int_0^\infty dR \ e^{-R^2} R^{d+l+k+u+v-1} z(d, l, k, u, v) = G(d, l, k, u, v). \]  
(A.6)

Again, the function defined by the underbrace, \( F \), may be evaluated using any standard textbook on basic integrals:

\[ F(R, d, l, k, u, v) = \frac{\Gamma\left(\frac{d+l+k+u+v}{2}\right)}{2}. \]  
(A.7)

Now, eventually plugging (A.5) and (A.6) together yields

\[ Z(R, d, l, k, u, v) = \frac{G(d, l, k, u, v)}{F(R, d, l, k, u, v)} R^{d+l+k+u+v-1}. \]  
(A.8)

The different functions needed to write out (A.8) explicitly may be read off from (A.3) and (A.7), thus we find

\[ Z(R, d, l, k, u, v) = \begin{cases} 
2\pi & \frac{\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{d+l+k+u+v}{2}\right)} R^{d+l+k+u+v-1} \text{ for } * \\
0 & \text{for } ** 
\end{cases} \]

\[ * \ (l, k, u, v) \text{ all even, } * * \ (l, k, u, v) \text{ any of them odd.} \]  
(A.9)

As mentioned above we are eventually not primarily interested in integrals but in averages over hyperspheres. However, since those are linked in a rather obvious way we may define the average of a function of the above type over the above hypersphere by

\[ E(R, d, l, k, u, v) \equiv \frac{Z(R, d, l, k, u, v)}{Z(R, d, 0, 0, 0, 0)}. \]  
(A.10)

Note that the averages of all functions containing odd powers of any coordinate vanish. As turns out below, cf. Appendices B.1, B.2, apart from this general statement, we only need the concrete values of three of those averages. They all may be computed from (A.10), (A.9), thus we simply list them here

\[ E(R, d, 2, 0, 0, 0) = \frac{R^2}{d}, \]  
(A.11)

\[ E(R, d, 2, 2, 0, 0) = \frac{R^4}{d^2 + 2d}, \]  
(A.12)

\[ E(R, d, 4, 0, 0, 0) = \frac{3R^4}{d^2 + 2d}. \]  
(A.13)
Appendix B
General Hilbert Space Averages and Variances

Abstract  In this appendix we concretely calculate the Hilbert space averages and variances of some quantities which are imperative for the general reasoning given in Chap. 8. To calculate these averages we exploit the averages of polynomial functions over hyperspheres in high-dimensional spaces as given in Appendix A.

B.1 Hilbert Space Averages of Polynomial Terms

To account for the Hilbert space averages and variances of the physically relevant quantities (observables, etc.) discussed in Sects. 8.2 and 8.3 we eventually need to evaluate Hilbert space averages of products of different \( \psi_{\alpha i} \) or their complex conjugates. Thus, schematically these products look like \( \psi_{\alpha i} \psi^{\ast}_{\beta j} \psi_{\gamma k} \ldots \). With \( \psi_{\alpha i} = \eta_{\alpha i} + i \xi_{\alpha i} \) the averages of those \( \psi \)-products may be expanded into linear combinations of averages over hyperspheres of polynomial functions of their coordinates \( \eta_{\alpha i}, \xi_{\alpha i} \). Thus, identifying \( \eta_{\alpha i}, \xi_{\alpha i} \) with \( x_i, x_j, \ldots \) we may use the results from Sect. A.1. (The largest product we need to consider contains four \( \psi \)-factors; thus the general results on the integration of polynomial functions over hyperspheres discussed in Sect. A.1 suffice.) Two features facilitate the evaluation of those averages significantly: (i) according to (7.24) the average of such a \( \psi \)-product decomposes into a product of averages of \( \psi \)-products, the factors of which correspond to the same subspace \( \alpha, \beta, \ldots \). (Those products may possibly consist of a single \( \psi \)-factor.) (ii) The averages of \( \psi \)-products corresponding to some individual subspace will be zero at any rate unless the \( \psi \)-factors are pairwise equal. Otherwise the corresponding averages over hyperspheres will be averages of polynomial functions in which at least one coordinate appears in an odd power. According to (A.10) those averages are all zero. As a consequence of these two features the averages over all original \( \psi \)-products are zero at any rate unless the \( \psi \)-factors are pairwise equal. Equal here means equal with respect to the subspace \( \alpha, \beta, \ldots \) as well as with respect to the individual level within the subspace, i.e., \( i, j, \ldots \).

In the following we evaluate the specific \( \psi \)-product averages that we eventually need. Based on these general considerations the average of a single \( \psi \) may directly
be given as
\[ \langle \psi_{\alpha i} \rangle = 0. \] (B.1)

Furthermore we may immediately write
\[ \langle \psi^*_\alpha \psi_{\beta j} \rangle = \delta_{\alpha \beta} \delta_{ij} \langle |\psi_{\alpha i}|^2 \rangle. \] (B.2)

In order to evaluate \( \langle |\psi_{\alpha i}|^2 \rangle \) we now explicitly exploit (A.11):
\[ \langle |\psi_{\alpha i}|^2 \rangle = \langle \eta^2_{\alpha i} + \xi^2_{\alpha i} \rangle = 2E(W_{\alpha}, 2N_{\alpha}, 2, 0, 0, 0) = \frac{W_{\alpha}}{N_{\alpha}}. \] (B.3)

In the same fashion we now evaluate the following collection of Hilbert space averages which suffices to compute the Hilbert space averages and variances of expectation values of observables which are needed in Chaps. 8 and 10. The respective Hilbert space averages and variances are then concretely evaluated on the basis of the below list in Sect. B.2:

\[ \langle |\psi_{\alpha i}|^2 \rangle = \langle \eta^2_{\alpha i} - \xi^2_{\alpha i} + 2i\eta_{\alpha i}\xi_{\alpha i} \rangle = 2iE(W_{\alpha}, 2N_{\alpha}, 1, 1, 0, 0) = 0, \] (B.4)

\[ \langle |\psi_{\alpha i}|^2 |\psi_{\alpha j}|^2 \rangle = \langle (\eta^2_{\alpha i} + \xi^2_{\alpha i})(\eta^2_{\alpha j} + \xi^2_{\alpha j}) \rangle \\
= 4E(W_{\alpha}, 2N_{\alpha}, 2, 2, 0, 0) \\
= \frac{1}{N_{\alpha}(N_{\alpha} + 1)}, \] (B.5)

\[ \langle |\psi_{\alpha i}|^4 \rangle = \langle \eta^4_{\alpha i} + \xi^4_{\alpha i} + 2\eta^2_{\alpha i}\xi^2_{\alpha i} \rangle \\
= 2(E(W_{\alpha}, 2N_{\alpha}, 4, 0, 0, 0) + E(W_{\alpha}, 2N_{\alpha}, 2, 2, 0, 0)) \\
= \frac{2}{N_{\alpha}(N_{\alpha} + 1)}, \] (B.6)

\[ \langle (\psi^*_{\alpha i})^2 \psi^2_{\alpha j} \rangle = \langle \eta^2_{\alpha i}\eta^2_{\alpha j} - \eta^2_{\alpha i}\xi^2_{\alpha j} - \eta^2_{\alpha i}\eta^2_{\alpha j} + \xi^2_{\alpha i}\xi^2_{\alpha j} + \\
+ i(2\eta_{\alpha i}\xi_{\alpha i}(\xi^2_{\alpha j} - \eta^2_{\alpha j}) + 2\eta_{\alpha j}\xi_{\alpha j}(\eta^2_{\alpha i} - \xi^2_{\alpha i})) \rangle \\
= 0. \] (B.7)
B.2 Hilbert Space Variance of an Observable

In the following we derive the result quoted in (8.18). Expanded according to subspaces the corresponding Hilbert space variance reads

\[ \langle \hat{A} \rangle^2 = \sum_{\alpha \beta \alpha' \beta'} \left[ \langle \psi_{\alpha} | \hat{A} | \psi_{\beta} \rangle \langle \psi_{\alpha'} | \hat{A} | \psi_{\beta'} \rangle \right]. \]  

(B.8)

As explained in Sect. B.1 such a Hilbert space average can only be non-zero if the full indices of the corresponding \( \psi \)'s are either pairwise equal or all equal. This can only occur if, regardless of the “intra-subspace index,” the corresponding subspaces are either pairwise equal or all equal. Thus, the sum decomposes into four different terms:

\[ \langle \hat{A} \rangle^2 = P_1 + P_2 + P_3 + P_4, \]  

(B.9)

with

\[ P_1 : \alpha = \beta = \alpha' = \beta', \]  

(B.10)

\[ P_2 : \alpha = \alpha', \quad \beta = \beta', \]  

\[ P_3 : \alpha = \beta, \quad \alpha' = \beta', \]  

\[ P_4 : \alpha = \beta', \quad \beta = \alpha'. \]  

(B.11)

In the following we discuss those terms separately. The first term reads

\[ P_1 = \sum_{\alpha} \left[ \langle \psi_{\alpha} | \hat{A} | \psi_{\alpha} \rangle \right]^2 = \sum_{ijkl} A_{ij} A_{kl} \left[ \psi_{\alpha i}^* \psi_{\alpha j} \psi_{\alpha k} \psi_{\alpha l} \right]. \]  

(B.12)

Again as explained in Sect. B.1 not only the subspaces but also the “intra-subspace indices” have to be either pairwise equal or all equal for the Hilbert space average to be non-zero at all. Thus, similar to the scheme described in (B.10) one may single out the addends that contribute to \( P_1 \). These are found to be

\[ P_1 = \sum_{\alpha i j} \left[ \langle \psi_{\alpha i} |^2 \langle \psi_{\alpha j} |^2 \right] (A_{\alpha i, \alpha i} A_{\alpha j, \alpha j} + A_{\alpha i, \alpha j} A_{\alpha j, \alpha i}) 

+ \sum_{\alpha i j} \left[ \langle \psi_{\alpha i}^* \rangle^2 \langle \psi_{\alpha j} \rangle^2 \right] A_{\alpha i, \alpha j}^2 + \sum_{\alpha i} A_{\alpha i, \alpha i}^2 \left[ \langle \psi_{\alpha i} |^4 \right], \]  

(B.13)

where we again used the notation introduced in (8.11). Exploiting (B.5), (B.6), and (B.7) we find
\[ P_1 = \frac{W^2_\alpha}{N_\alpha(N_\alpha + 1)} \left( \sum_{aij} (A_{ai,ai} A_{aj,aj} + A_{ai,aj} A_{aj,ai}) + 2 \sum_{ai} A^2_{ai,ai} \right). \] (B.14)

The terms that are left out in the first sum due to \( i \neq j \) are exactly the terms in the second sum. Thus one may simply simultaneously drop \( i \neq j \) and the second sum. Without \( i \neq j \) the sum may be performed in a straightforward manner. The result may be written as

\[ P_1 = \sum_{\alpha} \frac{W^2_\alpha}{N_\alpha(N_\alpha + 1)} \left( \text{Tr} \{ \hat{A}^2_{\alpha\alpha} \} + \left( \text{Tr} \{ \hat{A}_{\alpha\alpha} \} \right)^2 \right). \] (B.15)

The second term yields

\[ P_2 = \sum_{\alpha\beta} \left[ \langle \psi_{\alpha} | \hat{A} | \psi_{\beta} \rangle \right]^2 \]

\[ = \sum_{\alpha\beta} \sum_{ijkl} A_{ai,\beta j} \psi_{ai}^* \psi_{ak}^* A_{ak,\beta l} \psi_{bj} \psi_{bl} \]

\[ = \sum_{\alpha\beta} \sum_{ijkl} A_{ai,\beta j} A_{ak,\beta l} \left[ \psi_{ai}^* \psi_{ak}^* \psi_{bj} \psi_{bl} \right]. \] (B.16)

Since \( \alpha \neq \beta \) this term can only be non-zero for \( i = k \) and \( j = l \) finding

\[ P_2 = \sum_{\alpha\beta} \sum_{ij} A^2_{ai,\beta j} \left[ \langle \psi_{ai}^* \rangle^2 \langle \psi_{bj} \rangle^2 \right]. \] (B.17)

According to (7.24) this factorizes as

\[ P_2 = \sum_{\alpha\beta} \sum_{ij} A^2_{ai,\beta j} \left[ \langle \psi_{ai}^* \rangle^2 \langle \psi_{bj} \rangle^2 \right] = 0, \] (B.18)

which is zero according to (B.4). The third term may also be factorized and evaluated on the basis of (8.9) as follows:
For the last term we find

\[
\begin{align*}
P_4 &= \sum_{\alpha \beta} \sum_{i j k l} [A_{\alpha i, \beta j} A_{\beta k, \alpha l} \psi_{\alpha i}^* \psi_{\alpha l} \psi_{\beta j} \psi_{\beta k}] \\
&= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} W_{\alpha} W_{\beta} N_{\alpha} N_{\beta} \text{Tr} \{ \hat{A}_{\alpha \beta} \hat{A}_{\alpha \beta}^\dagger \} .
\end{align*}
\]  

(B.20)

Again, since \( \alpha \neq \beta \) this term can only be non-zero for \( i = l \) and \( j = k \). The remaining addends read and factorize as follows:

\[
\begin{align*}
P_4 &= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} \sum_{ij} A_{\alpha i, \beta j} [\psi_{\alpha i} |^2 [\psi_{\beta j} |^2] \\
&= \sum_{\alpha \beta} W_{\alpha} W_{\beta} N_{\alpha} N_{\beta} \text{Tr} \{ \hat{A}_{\alpha \beta} \hat{A}_{\alpha \beta}^\dagger \} .
\end{align*}
\]  

(B.21)

where we have used (B.3). Plugging all those contributions together we find for the first term of the variance

\[
[\langle \hat{A} \rangle^2] = \sum_{\alpha \beta} W_{\alpha} W_{\beta} N_{\alpha} N_{\beta} \left( \text{Tr} \{ \hat{A}_{\alpha \beta} \hat{A}_{\alpha \beta}^\dagger \} + \text{Tr} \{ \hat{A}_{\alpha \alpha} \} \text{Tr} \{ \hat{A}_{\beta \beta} \} \right) .
\]  

(B.22)

This result is quoted in (8.18).
Appendix C
Special Hilbert Space Averages and Variances

Abstract  In this part of the appendix various Hilbert space averages and variances are evaluated which are needed in the context of typicality of reduced local states in composite quantum systems, especially in Chaps. 10 and 11.

C.1 General Distances Between States

First we derive the relation between the mean-squared distance of states from some average state and the Hilbert space variances of certain observables as expressed in (10.13). Starting from (10.12) we have to compute first the trace of the average of the squared density operator

\[
\text{Tr} \left\{ \left[ \rho^2 \right] \right\} = \text{Tr} \left\{ \left[ \sum_{AC} \sum_{ac} (X^{Aa,Cc} - i Y^{Aa,Cc}) |A, a\rangle \langle C, c| \right. \right.

\left. \left. \sum_{A'C'} \sum_{a'c'} (X^{A'a',C'c'} - i Y^{A'a',C'c'}) |A', a'\rangle \langle C', c'| \right] \right\}, \quad (C.1)
\]

which is after exchanging trace and average and performing the trace operation

\[
\text{Tr} \left\{ \left[ \rho^2 \right] \right\} = \left[ \sum_{AC} \sum_{ac} (X^{Aa,Cc} - i Y^{Aa,Cc}) (X^{Cc,Aa} - i Y^{Cc,Aa}) \right]. \quad (C.2)
\]

The term in the second bracket is just the complex conjugate of the first one, and thus we find

\[
\text{Tr} \left\{ \left[ \rho^2 \right] \right\} = \sum_{AC} \sum_{ac} \left( \|X^{Aa,Cc}\|^2 + \|Y^{Aa,Cc}\|^2 \right). \quad (C.3)
\]

The trace over the squared Hilbert space average (HA) of the density operator, using (10.10) and (10.11) yields
\[ \text{Tr} \left\{ \hat{\omega}^2 \right\} = \text{Tr} \left\{ \sum_{A'C'} \sum_{a'c'} \left( [X^{A'a',C'}] - i[Y^{A'a',C'}] \right) |A', a'\rangle \langle C', c'| \right\} \]  \[ \text{(C.4)} \]

which reduces to

\[ \text{Tr} \left\{ \hat{\omega}^2 \right\} = \sum_{AC} \sum_{ac} \left( [X^{Aa,Cc}] - i[Y^{Aa,Cc}] \right) \left( [X^{Cc,Aa}] - i[Y^{Cc,Aa}] \right) \]  \[ \text{(C.5)} \]

where we used again that the second bracket is the complex conjugate of the first one. Plugging this into (10.12) and exploiting furthermore (10.10) and (10.11) yields

\[ D^2(\hat{\rho}, \hat{\omega}) = \sum_{AC} \sum_{ac} \left( \Delta^2_H(X^{Aa,Cc}) + \Delta^2_H(Y^{Aa,Cc}) \right) \]  \[ \text{(C.6)} \]

This result is the one that is referred to below (10.13).

### C.2 Microcanonical Hilbert Space Averages

Here we compute the HA which is required for the result quoted in (10.17): The average of expectation value of the operator \( \hat{Y}^{Aa,Cc} \) can be computed in the very same way as the average of expectation value of the operator \( \hat{X}^{Aa,Cc} \). The latter computation is given in Sect. 10.2.1. Starting with the operator

\[ \hat{Y}^{Aa,Cc} \]  \[ \text{(A.13)} \]

we find the trace

\[ \text{Tr} \left\{ \hat{Y}^{Aa,Cc} \right\} = 0 \]  \[ \text{(C.8)} \]

and thus its Hilbert space average of \( \hat{Y}^{Aa,Cc} \) vanishes, \( \left[ \{ \hat{Y}^{Aa,Cc} \} \right] = 0 \). Thus exploiting (10.10) and (10.16) we find for the Hilbert space average of a density matrix element
\[ \rho^{Aa,Cc} = \sum_B \frac{W_{AB}}{N_A} \delta_{AC} \delta_{ac} \cdot \]  

\[ (C.9) \]

This result is quoted in (10.17).

**C.3 Microcanonical Hilbert Space Variances**

Here we are going to compute the Hilbert space variances that are quoted in (10.20) and (10.21). The pertinent accessible region (AR) is given by (9.15). The general formula for the Hilbert space variance of the expectation value of a Hermitian operator is given in (8.19). Using this, we first need the operator

\[
\hat{X}^{Aa,Cc}_{AB,CD} = \frac{1}{2} \hat{\Pi}_A \hat{\Pi}_B \left( |A, a\rangle \langle C, c| + |C, c\rangle \langle A, a| \otimes \hat{1} \right) \hat{\Pi}_C \hat{\Pi}_D
\]

\[
= \frac{1}{2} \delta_{BD} \delta_{A'C} |A, a\rangle \langle C, c| + \delta_{A'C} \delta_{A'C} |C, c\rangle \langle A, a| \otimes \hat{\Pi}_B . \]  

\[ (C.10) \]

The adjoint operator is given by

\[
(\hat{X}^{Aa,Cc}_{AB,CD})^\dagger = \frac{1}{2} \delta_{BD} \delta_{A'C} |C, a\rangle \langle A, c| + \delta_{A'C} \delta_{A'C} |A, c\rangle \langle C, a| \otimes \hat{\Pi}_B . \]  

\[ (C.11) \]

The product of (C.10) and (C.11) is

\[
\hat{X}^{Aa,Cc}_{AB,CD} (\hat{X}^{Aa,Cc}_{AB,CD})^\dagger = \frac{1}{4} \delta_{BD} \delta_{A'A} \delta_{C'C} |A, a\rangle \langle A, a| + \delta_{A'C} \delta_{A'C} |C, c\rangle \langle C, c| \otimes \hat{\Pi}_B , \]  

\[ (C.12) \]

and its trace

\[
\text{Tr} \left\{ \hat{X}^{Aa,Cc}_{AB,CD} (\hat{X}^{Aa,Cc}_{AB,CD})^\dagger \right\} = \frac{1}{4} \delta_{BD} \delta_{A'A} \delta_{C'C} + \delta_{A'C} \delta_{A'C} N_B . \]  

\[ (C.13) \]

Plugging this trace into (8.19), realizing that \( N_{AB} = N_A N_B \) and using (10.15) for the second term, we find
\[
\Delta^2_H(\{\hat{X}^{Aa,Cc}\}) = \frac{1}{4} \sum_{A'B'C'D} W_{A'B} W_{C'D} N_{A'B}(N_{C'D} + \delta_{A'C} \delta_{BD}) \delta_{BD}(\delta_{A'A} \delta_{C'C} + \delta_{A'C} \delta_{C'A}) N_B
\]
\[
- \delta_{AC} \delta_{ac} \sum_B \frac{W_{AB}^2 N_B^2}{N_{AB}(N_{AB} + 1)}
\]
\[
= \frac{1}{4} \sum_B \frac{W_{AB} W_{CB}}{N_B(N_{AB} + \delta_{AC})} + \frac{1}{4} \sum_B \frac{W_{AB} W_{CB}}{N_B(N_{AB} N_C + N_C \delta_{AC})}
\]
\[
- \delta_{AC} \delta_{ac} \sum_B \frac{W_{AB}^2}{N_A(N_A N_B + 1)}.
\] (C.14)

Because of the delta function \(\delta_{AC}\) we may change the dimension \(N_C\) in the denominator of the second term in front of the delta function to a dimension \(N_A\). Thus, the first two terms are equivalent and the variance simplifies to

\[
\Delta^2_H(\{\hat{X}^{Aa,Cc}\}) = \frac{1}{2} \sum_B \frac{W_{AB} W_{CB}}{N_B(N_{AB} + \delta_{AC})} - \delta_{AC} \delta_{ac} \sum_B \frac{W_{AB}^2}{N_A(N_A N_B + 1)}.
\] (C.15)

This result is quoted in (10.20). Now we do the same analysis for the corresponding \(\hat{Y}\)-operator

\[
\hat{Y}_{A'B'C'D}^{Aa,Cc} = \frac{i}{2} \hat{\Pi}_A \hat{\Pi}_B \left( |A, a\rangle \langle C, c| - |C, c\rangle \langle A, a| \right) \otimes 1 \right) \hat{\Pi}_C \hat{\Pi}_D
\]
\[
= \frac{i}{2} \delta_{BD}(\delta_{A'A} \delta_{C'C}|A, a\rangle \langle C, c| - \delta_{AC} \delta_{C'A}|A, a\rangle \langle C, c| \otimes \hat{\Pi}_B.
\] (C.16)

and its adjoint reads

\[
(\hat{Y}_{A'B,C'D}^{Aa,Cc})^\dagger = -\frac{i}{2} \delta_{BD}(\delta_{A'A} \delta_{C'C}|A, a\rangle \langle C, c| - \delta_{AC} \delta_{C'A}|A, a\rangle \langle C, c| \otimes \hat{\Pi}_B.
\] (C.17)

The product of (11.13) and (11.14) yields

\[
\hat{Y}_{A'B,C'D}^{Aa,Cc} (\hat{Y}_{A'B,C'D}^{Aa,Cc})^\dagger = 1/4 \delta_{BD}(\delta_{A'A} \delta_{C'C}|A, a\rangle \langle C, c| + \delta_{AC} \delta_{C'A}|C, c\rangle \langle C, c| \otimes \hat{\Pi}_B.
\] (C.18)
and its trace is

\[
\text{Tr} \left\{ \hat{Y}^{Aa,Cc}_{A'B,C'D}(\hat{Y}^{Aa,Cc}_{A'B,C'D})^\dagger \right\} = 1/4 \delta_{BD} (\delta_{A'C} \delta_{C'B} + \delta_{A'C} \delta_{B'C}) N_B . \tag{C.19}
\]

Since this result is equivalent to (C.13) the first term of the variance is identical to the first two terms of (C.14). However, because the trace of \( \hat{Y}^{Aa,Cc} \) vanishes according to (C.8) the second term does not exist here. The Hilbert space variance thus reads

\[
\Delta^2_H(\langle \hat{Y}^{Aa,Cc} \rangle) = \frac{1}{2} \sum_B W_{AB} W_{CB} N_A (N_C N_B + \delta_{AC}) . \tag{C.20}
\]

This result is quoted in (10.21).

### C.4 Energy Exchange Hilbert Space Averages

Here we derive the result quoted in (10.27). The pertinent AR is given by (9.19). Again, we use the general result (8.12) to compute the Hilbert space average. Therefore we need the operator

\[
\hat{X}^{Aa,Cc}_{EE} = \frac{1}{2} \hat{\Pi}_E (|A,a\rangle \langle C,c| + |C,c\rangle \langle A,a|) \hat{\Pi}_E
\]

\[
= \frac{1}{2} \sum_{A'B} \hat{\Pi}_{A'} \hat{\Pi}_B M_{E,A'B} (|A,a\rangle \langle C,c| + |C,c\rangle \langle A,a|) \sum_{C'D} \hat{\Pi}_{C'} \hat{\Pi}_D M_{E,C'D}
\]

\[
= \frac{1}{2} \sum_{A'B'C'} M_{E,A'B} M_{E,C'D} \hat{\Pi}_{A'} (|A,a\rangle \langle C,c| + |C,c\rangle \langle A,a|) \hat{\Pi}_{C'} \hat{\Pi}_B . \tag{C.21}
\]

Realizing that \( M_{E,A'B} M_{E,C'D} = M_{E,A'B} \delta_{A'C'} \) we get

\[
\hat{X}^{Aa,Cc}_{EE} = \frac{1}{2} \sum_{A'B} M_{E,A'B} \hat{\Pi}_{A'} (|A,a\rangle \langle C,c| + |C,c\rangle \langle A,a|) \hat{\Pi}_{B} = \frac{1}{2} \sum_B M_{E,AB} \delta_{AC} (|A,a\rangle \langle C,c| + |C,c\rangle \langle A,a|) \hat{\Pi}_B . \tag{C.22}
\]

The trace over this operator yields

\[
\text{Tr} \left\{ \hat{X}^{Aa,Cc}_{EE} \right\} = \sum_B M_{E,AB} \delta_{AC} \delta_{ac} N_B , \tag{C.23}
\]
and thus the average reads
\[
\langle \hat{X}^{Aa,Cc}\rangle = \sum_E \frac{W_E}{N_E} \text{Tr} \left\{ \hat{X}^{Aa,Cc}_{EE} \right\} = \sum_E \frac{W_E}{N_E} M_{E,AB} \delta_{AC} \delta_{ac} N_B .
\] (C.24)

Again the average of \( \hat{Y}^{Aa,Cc} \) vanishes, and finally, the Hilbert space average of the density matrix element is
\[
\rho^{Aa,Cc} = \sum_E \frac{W_E}{N_E} M_{E,AB} \delta_{AC} \delta_{ac} N_B .
\] (C.25)

This result is quoted in (10.27).

### C.5 Energy Exchange Hilbert Space Variance

Here we derive the result quoted in (10.30) on the basis of the AR as given in (9.19). Again we use the general formula for the Hilbert space variance of the expectation value of a Hermitian operator as given in (8.19). Using this formula we first need the operator
\[
\hat{X}^{Aa,Cc}_{EE'}
\]
\[
= \frac{1}{2} \hat{\Pi}_E \left( |A, a\rangle \langle C, c| + |C, c\rangle \langle A, a| \right) \hat{\Pi}_E' \\
= \frac{1}{2} \sum_{A'B} \hat{\Pi}_{A'} \hat{\Pi}_B M_{E,AB} \left( |A, a\rangle \langle C, c| + |C, c\rangle \langle A, a| \right) \sum_{C'D} \hat{\Pi}_{C'} \hat{\Pi}_D M_{E',C'D} \\
= \frac{1}{2} \sum_{A'BC'} \left( \delta_{A'B} \delta_{C'C}|A, a\rangle \langle C, c| + \delta_{A'C} \delta_{AC'}|C, a\rangle \langle A, a| \right) M_{E,AB} M_{E',C'B} \hat{\Pi}_B \\
= \frac{1}{2} \sum_B \left( M_{E,AB} M_{E',CB} |A, a\rangle \langle C, c| + M_{E,CB} M_{E',AB} |C, a\rangle \langle A, a| \right) \hat{\Pi}_B .
\] (C.26)

The adjoint operator reads
\[
(\hat{X}^{Aa,Cc}_{EE'})^\dagger
\]
\[
= \frac{1}{2} \sum_B \left( M_{E,AB} M_{E',CB} |C, a\rangle \langle A, c| + M_{E,CB} M_{E',AB} |A, a\rangle \langle C, c| \right) \hat{\Pi}_B .
\] (C.27)
and the product yields
\[ \hat{X}_{EE'}^{Aa,Cc} \hat{X}_{EE'}^{Aa,Cc} = \frac{1}{4} \sum_B \left( M_{E,AB} M_{E',CB} |A, a\rangle \langle A, a| + M_{E,AB} M_{E',CB} |C, c\rangle \langle C, c| \right) \hat{\Pi}_B , \]  
where we have used the identity \( M_{E,AB} M_{E,AD} = M_{E,AB} \delta_{BD} \). The trace over this operator yields
\[ \text{Tr} \left\{ \hat{X}_{EE'}^{Aa,Cc} \hat{X}_{EE'}^{Aa,Cc} \right\} = \frac{1}{4} \sum_B \left( M_{E,AB} M_{E',CB} + M_{E,AB} M_{E',AB} \right) N_B , \]
and thus, we finally find for the variance
\[ \Delta^2_H(\langle \hat{X}_{EE'}^{Aa,Cc} \rangle) = \sum_{EE'} \sum_B \left( \sum_{E'} \sum_{B} \frac{W_E W_E N_B}{N_E (N_E + \delta_{EE'})} \right) M_{E,AB} M_{E',CB} \]
\[ - \delta_{AC} \delta_{ac} \sum_{E'} \sum_{B} \sum_{E} \frac{W_E^2 N_B^2}{N_E^2 (N_E + 1)} M_{E,AB} . \]  

As in the microcanonical situation the Hilbert space variance of the expectation value \( \langle \hat{Y}_{Aa,Cc} \rangle \) is similar to \( \Delta^2_H(\langle \hat{X}_{Aa,Cc} \rangle) \). However, since the trace of \( \hat{Y}_{Aa,Cc} \) vanishes the second term is zero again. Using (10.13) mean-squared distance yields
\[ D^2(\hat{\rho}, \hat{\omega}) = \sum_{AC} \sum_{ac} \left( \sum_{EE'} \sum_{B} \frac{W_E W_E N_B}{N_E (N_E + \delta_{EE'})} M_{E,AB} M_{E',CB} \right) \]
\[ - \delta_{AC} \delta_{ac} \sum_{E'} \sum_{B} \sum_{E} \frac{W_E^2 N_B^2}{N_E^2 (N_E + 1)} M_{E,AB} \]
\[ = \sum_{EE'} \sum_{ABC} \frac{W_E W_E N_B N_A N_C}{N_E (N_E + \delta_{EE'})} M_{E,AB} M_{E',CB} \]
\[ - \sum_{E} \sum_{AB} \frac{W_E^2 N_B^2 N_A}{N_E^2 (N_E + 1)} M_{E,AB} . \]  
This result is quoted in (10.30).
Appendix D
Power of a Function

Abstract In this appendix we show that the $k$th power of any function with a global maximum will essentially be a Gaussian.

For this purpose we consider a function $f(x)$ with a global maximum at $x = 0$. Because of the positivity of $f(x) > 0$ we can rewrite the function

$$f(x) = e^{g(x)}, \quad \text{with} \quad g(x) = \ln f(x). \quad (D.1)$$

Since the logarithm is a monotonous function, we consider instead of $f(x)$ the expansion of the function $g(x)$ around the global maximum $x = 0$,

$$g(x) = \sum_i C_i x^i = C_0 - C_2 x^2 + C_3 x^3 + \cdots, \quad (D.2)$$

with $C_2 > 0$ and thus

$$f(x) = e^{C_0} e^{-C_2 x^2} e^{C_3 x^3} \cdots, \quad (D.3)$$

with some constants $C_i$. Since multiplying the function with itself will amplify the maximum in the center, we can truncate the decomposition in this way. Multiplying the function $k$ times with itself we get

$$\left(f(x)\right)^k = e^{kC_0} e^{-kC_2 x^2} e^{kC_3 x^3} \cdots. \quad (D.4)$$

The value of $x$, for which the quadratic part will have reduced the function to half maximum, i.e., for which

$$\exp(-kC_2 x_h^2) = 1/2, \quad (D.5)$$
is

\[ x_h = \pm \sqrt[3]{ \frac{\ln 2}{kC_2} } \]. \quad (D.6)

Evaluating the third-order part of \((f(x))^k\) at \(x_h\) then yields

\[ \exp(kC_3x_h^3) = \exp\left( \frac{C_3 \left( \frac{\ln 2}{C_3} \right)^{3/2}}{\sqrt{k}} \right), \quad (D.7) \]

which tends to 1 if \(k\) approaches infinity. For the relevant region, in which the function is peaked, we can thus approximate

\[ \left( f(x) \right)^k \approx e^{kC_0}e^{-kC_2x^2}, \quad (D.8) \]

which is essentially a Gaussian.
Appendix E
Local Temperature Conditions for a Spin Chain$^1$

Abstract In this appendix we present the technical details of the application of the local temperature conditions to a spin chain.

The entire chain with periodic boundary conditions may be diagonalized via successive Jordan–Wigner, Fourier, and Bogoliubov transformations [4]. The relevant energy scale is introduced via the thermal expectation value (without the ground state energy)

$$E = \frac{NN^G}{2\pi} \int_{-\pi}^{\pi} \frac{\omega_k}{\exp(\beta \omega_k) + 1} ,$$  \hspace{1cm} (E.1)

where

$$\omega_k = 2\Delta E \sqrt{(1 - K \cos k)^2} ,$$  \hspace{1cm} (E.2)

with $K = \lambda/\Delta E$. The ground state energy $E_0$ is given by

$$E_0 = -\frac{NN^G}{2\pi} \int_{-\pi}^{\pi} \frac{\omega_k}{2} .$$  \hspace{1cm} (E.3)

Since $N^G \gg 1$, the sums over all modes have been replaced by integrals.

If one partitions the chain into $N^G$ groups of $N$ subsystems each, the groups may also be diagonalized via a Jordan–Wigner and a Fourier transformation [4] and the energy $E_a$ reads

$$E_a = 2\Delta E \sum_{v=1}^{N^G} \sum_k (1 - K \cos(k)) \left(n_k^v(v) - \frac{1}{2}\right) ,$$  \hspace{1cm} (E.4)

---

$^1$ Based on [1–4] Hartmann et al.
where \( k = \pi l / (N + 1) \) \((l = 1, \ldots, N)\) and \( n_k^a(\nu) \) is the fermionic occupation number of mode \( k \) of group number \( \nu \) in the state \(|a\rangle\). It can take on the values 0 and 1.

For the model at hand one has \( \epsilon_a = 0 \) for all states \(|a\rangle\), while the squared variance \( \Delta_a^2 \) reads

\[
\Delta_a^2 = \sum_{\nu=1}^{N^G} \Delta_{\nu}^2 , \quad (E.5)
\]

with

\[
\Delta_{\nu}^2 = \frac{\Delta E^2 K^2}{2} - \frac{8 \Delta E^2 K^2}{(N + 1)^2} \sum_k \sin^2(k) \left( n_k^a(\nu) - \frac{1}{2} \right) \sum_p \sin^2(p) \left( n_p^a(\nu + 1) - \frac{1}{2} \right) . \quad (E.6)
\]

We now turn to analyze conditions (23.16) and (23.18). According to (E.6), \( \Delta_{\nu}^2 \) cannot be expressed in terms of \( E_{\nu-1} \) and \( E_{\nu} \). We therefore approximate (23.16) and (23.18) by simpler expressions.

Let us first analyze condition (23.16). Since it cannot be checked for every state \(|a\rangle\), we make the following approximations. For the present model with \(|K| < 1\) all occupation numbers \( n_k^a(\nu) \) are zero in the ground state and thus \( \Delta_a^2 = 0 \) as well as \( E_a - E_0 = 0 \). Therefore (23.16) cannot hold for this state. However, if one occupation number is changed from 0 to 1, \( \Delta_a^2 \) changes at most by \( 4 \Delta E^2 K^2 / (N + 1) \) and \( E_a \) changes at least by \( 2 \Delta E (1 - |K|) \). Therefore (23.16) will hold for all states except the ground state if

\[
N > 2 \Delta E \beta \frac{K^2}{1 - |K|} . \quad (E.7)
\]

If \(|K| > 1\), occupation numbers of modes with \( \cos(k) < 1/|K| \) are 0 in the ground state and occupation numbers of modes with \( \cos(k) > 1/|K| \) are equal to 1. \( \Delta_a^2 \) for the ground state then is \( \lfloor \Delta_a^2 \rfloor_{gs} \approx \lfloor \Delta_a^2 \rfloor_{\text{max}} / 2 \) (in this entire chapter, \( \lfloor x \rfloor_{\text{min}} \) and \( \lfloor x \rfloor_{\text{max}} \) denote the minimal and maximal values \( x \) can take on). We therefore approximate (23.16) with the stronger condition

\[
\frac{E_{\text{min}} - E_0}{N^G} > \beta \frac{\lfloor \Delta_a^2 \rfloor_{\text{max}}}{N^G} , \quad (E.8)
\]

which implies that (23.16) holds for all states \(|a\rangle\) in the energy range \([E_{\text{min}}, E_{\text{max}}]\) (see (23.20) and (23.21)). Equation (E.8) can be rewritten as a condition on the group size \( N \)
\[ N > \beta \left( \frac{\delta^2_a}_{\text{max}} \right) e_{\text{min}} - e_0 , \]  
(E.9)

where \( e_{\text{min}} = E_{\text{min}} / (NN^G) \), \( e_0 = E_0 / (NN^G) \), and \( \delta^2_a = \Delta^2_a / N^G \).

We now turn to analyze condition (23.18). Equation (E.6) shows that the \( \Delta^2_a \) do not contain terms that are proportional to \( E_a \). One thus has to determine when the \( \Delta^2_a \) are approximately constant, which is the case if

\[ \beta \left( \Delta^2_a \right)_{\text{max}} - \left( \Delta^2_a \right)_{\text{min}} \ll [E_a]_{\text{max}} - [E_a]_{\text{min}} . \]  
(E.10)

As a direct consequence, we get \(|c_1| \ll 1\) which means that temperature is intensive.

Defining the quantity \( e_a = E_a / (NN^G) \), we can rewrite (E.10) as a condition on \( N \),

\[ N \geq \frac{\beta}{2 \varepsilon} \left( \frac{\delta^2_a}_{\text{max}} - \frac{\delta^2_a}_{\text{min}} \right) , \]  
(E.11)

where the accuracy parameter \( \varepsilon \ll 1 \) is equal to the ratio of the left-hand side and the right-hand side of (E.10).

Since (E.10) does not take into account the energy range (23.20), its application needs some further discussion.

If the occupation number of one mode of a group is changed, say from \( n_a^k(\nu) = 0 \) to \( n_a^k(\nu) = 1 \), the corresponding \( \Delta^2_a \) differ at most by \( 4 \Delta E^2 K^2 / (n + 1) \). On the other hand, \( \left( \Delta^2_a \right)_{\text{max}} - \left( \Delta^2_a \right)_{\text{min}} = N^G \Delta E^2 K^2 \). The state with the maximal \( \Delta^2_a \) and the state with the minimal \( \Delta^2_a \) thus differ in nearly all occupation numbers and, therefore, their difference in energy is close to \( [E_a]_{\text{max}} - [E_a]_{\text{min}} \). On the other hand, states with similar energies \( E_a \) also have a similar \( \Delta^2_a \). Hence the \( \Delta^2_a \) only change quasi-continuously with energy and (E.10) ensures that the \( \Delta^2_a \) are approximately constant even locally, i.e., on any part of the possible energy range.

To compute the required group size \( N_{\text{min}} \), we need to know the maximal and minimal values \( E_a \) and \( \Delta^2_a \) can take on. For \( E_a \) they are given by

\[ \left\{ \begin{array}{c} [E_a]_{\text{max}} \\ [E_a]_{\text{min}} \end{array} \right\} = \left\{ \begin{array}{c} + \\ - \end{array} \right\} N^G N \Delta E , \]  
(E.12)

for \(|K| \leq 1\), and by

\[ \left\{ \begin{array}{c} [E_a]_{\text{max}} \\ [E_a]_{\text{min}} \end{array} \right\} = \left\{ \begin{array}{c} + \\ - \end{array} \right\} N^G N \Delta E \frac{2}{\pi} \left[ \sqrt{K^2 - 1} + \arcsin \left( \frac{1}{|K|} \right) \right] , \]  
(E.13)

for \(|K| > 1\), where the sum over all modes \( k \) has been approximated by an integral. The maximal and minimal values of \( \Delta^2_a \) are given by
\begin{equation}
\begin{bmatrix}
\Delta^2_{a,\text{max}} \\
\Delta^2_{a,\text{min}}
\end{bmatrix} =
\begin{bmatrix}
\mathcal{N}^G & K^2 \Delta E^2 \\
0 & \end{bmatrix}.
\tag{E.14}
\end{equation}

Plugging these results into (E.11) as well as (E.1) and (E.3) (taking into account (23.20) and (23.21)) into (E.9) for $|K| > 1$ and using (E.7) for $|K| < 1$, the minimal number of systems per group can be calculated.

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