Exercises

Chapter 1: Elementary Newtonian Mechanics

1.1 Under the assumption that the orbital angular momentum $l = \mathbf{r} \times \mathbf{p}$ of a particle is conserved show that its motion takes place in a plane spanned by $\mathbf{r}_0$, the initial position, and $\mathbf{p}_0$, the initial momentum. Which of the orbits of Fig. 1 are possible in this case? ($O$ denotes the origin of the coordinate system.)

1.2 In the plane of motion of Exercise 1.1 introduce polar coordinates $\{r(t), \varphi(t)\}$. Calculate the line element $(ds)^2 = (dx)^2 + (dy)^2$, as well as $v^2 = \dot{x}^2 + \dot{y}^2$ and $l^2$, in the polar coordinates. Express the kinetic energy in terms of $\dot{r}$ and $l^2$.

1.3 For the description of motions in $\mathbb{R}^3$ one may use Cartesian coordinates $\mathbf{r}(t) = \{x(t), y(t), z(t)\}$, or spherical coordinates $\{r(t), \theta(t), \varphi(t)\}$. Calculate the infinitesimal line element $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ in spherical coordinates. Use this result to derive the square of the velocity $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ in these coordinates.
1.4 Let $\hat{x}$, $\hat{y}$, $\hat{z}$ be Cartesian unit vectors. They then fulfill $\hat{e}_x^2 = \hat{e}_y^2 = \hat{e}_z^2 = 1$, $\hat{e}_x \cdot \hat{e}_y = \hat{e}_x \cdot \hat{e}_z = \hat{e}_y \cdot \hat{e}_z = 0$, $\hat{e}_z = \hat{e}_x \times \hat{e}_y$ (plus cyclic permutations). Introduce three, mutually orthogonal unit vectors $\hat{e}_r$, $\hat{e}_\phi$, $\hat{e}_\theta$ as indicated in Fig. 2. Determine $\hat{e}_r$ and $\hat{e}_\phi$ from the geometry of this figure. Confirm that $\hat{e}_r \cdot \hat{e}_\phi = 0$. Assume $\hat{e}_\theta = \alpha \hat{e}_x + \beta \hat{e}_y + \gamma \hat{e}_z$ and determine the coefficients $\alpha$, $\beta$, $\gamma$ such that $\hat{e}_\theta^2 = 1$, $\hat{e}_\theta \cdot \hat{e}_r = 0 = \hat{e}_\theta \cdot \hat{e}_\phi$. Calculate $v = \dot{r} = d(r \hat{e}_r)/dt$ in this basis as well as $v^2$.

1.5 A particle is assumed to move according to $r(t) = v^0 t$ with $v^0 = \{0, v, 0\}$, with respect to the inertial system $K$. Sketch the same motion as seen from another reference frame $K'$, which is rotated about the $z$-axis of $K$ by an angle $\Phi$,

$$x' = x \cos \Phi + y \sin \Phi, \quad y' = -x \sin \Phi + y \cos \Phi, \quad z' = z,$$

for the cases $\Phi = \omega$ and $\Phi = \omega t$, were $\omega$ is a constant.

1.6 A particle of mass $m$ is subject to a central force $F = F(r)r/r$. Show that the angular momentum $l = m r \times \dot{r}$ is conserved (i.e. its magnitude and direction) and that the orbit lies in a plane perpendicular to $l$.

1.7 (i) In an $N$-particle system that is subject to internal forces only, the potentials $V_{ik}$ depend only on the vector differences $r_{ik} = r_i - r_k$, but not on the individual vectors $r_i$. Which quantities are conserved in this system?

(ii) If $V_{ik}$ depends only on the modulus $|r_{ik}|$ the force acts along the straight line joining $i$ to $k$. There is one more integral of the motion.

1.8 Sketch the one-dimensional potential

$$U(q) = -5q e^{-q} + q^{-4} + 2/q \quad \text{for} \quad q \geq 0$$

and the corresponding phase portraits for a particle of mass $m = 1$ as a function of energy and initial position $q_0$. In particular, find and discuss the two points of equilibrium. Why are the phase portraits symmetric with respect to the abscissa?
1.9 Study two identical pendula of length $l$ and mass $m$, coupled by a harmonic spring, the spring being inactive when both pendulums are at rest. For small deviations from the vertical the energy reads

$$E = \frac{1}{2m}(x_2^2 + x_4^2) + \frac{1}{2}m\omega_0^2(x_1^2 + x_3^2) + \frac{1}{2}m\omega_1^2(x_1 - x_3)^2$$

with $x_2 = m\dot{x}_1$, $x_4 = m\dot{x}_3$. Identify the individual terms of this equation. Derive from it the equations of motion in phase space,

$$\frac{dx}{dt} = Mx .$$

The transformation

$$x \rightarrow u = Ax \text{ with } A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

decouples these equations. Write the equations obtained in this way in dimensionless form and solve them.

1.10 The one-dimensional harmonic oscillator satisfies the differential equation

$$m\ddot{x}(t) = -\lambda x(t) , \quad (1.1)$$

with $m$ the inertial mass, $\lambda$ a positive constant, and $x(t)$ the deviation from equilibrium. Equivalently, $(1.1)$ can be written as

$$\ddot{x} + \omega^2 x = 0 , \quad \omega^2 \overset{\text{def}}{=} \frac{\lambda}{m} . \quad (1.2)$$

Solve the differential equation $(1.2)$ by means of $x(t) = a \cos(\mu t) + b \sin(\mu t)$ for the initial condition

$$x(0) = x_0 \quad \text{and} \quad p(0) = m\dot{x}(0) = p_0 . \quad (1.3)$$

Let $x(t)$ be the abscissa and $p(t)$ the ordinate of a Cartesian coordinate system. Draw the graph of the solution with $\omega = 0.8$ that goes through the point $(x_0 = 1, p_0 = 0)$.

1.11 Adding a weak friction force to the system of Exercise 1.10 yields the equation of motion

$$\ddot{x} + \kappa \dot{x} + \omega^2 x = 0 .$$

“Weak” means that $\kappa < 2\omega$. Solve the differential equation by means of

$$x(t) = e^{at}[x_0 \cos(\omega t) + \left(p_0/m\omega\right) \sin(\omega t)] .$$

Draw the graph $(x(t), p(t))$ of the solution with $\omega = 0.8$ which goes through $(x_0 = 1, p_0 = 0)$. 
1.12 A mass point of mass $m$ moves in the piecewise constant potential (see Fig. 3)

$$U = \begin{cases} U_1 & \text{for } x < 0 \\ U_2 & \text{for } x > 0. \end{cases}$$

In crossing from the domain $x < 0$, where its velocity was $v_1$, to the domain $x > 0$, it changes its velocity (modulus and direction). Express $U_2$ in terms of the quantities $U_1, v_1, \alpha_1, \text{and } \alpha_2$. What is the relation of $\alpha_1$ to $\alpha_2$ when (i) $U_1 < U_2$ and (ii) $U_1 > U_2$? Work out the relationship to the law of refraction of geometrical optics.

*Hint:* Make use of the principle of energy conservation and show that one component of the momentum remains unchanged in crossing from $x < 0$ to $x > 0$.

1.13 In a system of three mass points $m_1, m_2, m_3$ let $S_{12}$ be the center-of-mass of 1 and 2 and $S$ the center-of-mass of the whole system. Express the coordinates $r_1, r_2, r_3$ in terms of $r_s, s_a, \text{and } s_b$, as defined in Fig. 4. Calculate the total kinetic energy in terms of the new coordinates and interpret the result. Write the total angular momentum in terms of the new coordinates and show that $\sum_i l_i = l_s + l_a + l_b$, where $l_s$ is the angular momentum of the center-of-mass and $l_a$ and $l_b$ are relative angular momenta. By considering a Galilei transformation $r' = r + \omega t + a$, $t' = t + s$ show that $l_s$ depends on the choice of the origin, while $l_a$ and $l_b$ do not.
1.14 *Geometric similarity.* Let the potential $U(r)$ be a homogeneous function of degree $\alpha$ in the coordinates $(x, y, z)$, i.e. $U(\lambda r) = \lambda^\alpha U(r)$.

(i) Show by making the replacements $r \rightarrow \lambda r$ and $t \rightarrow \mu t$, and choosing $\mu = \lambda^{1-\alpha/2}$, that the energy is modified by a factor $\lambda^\alpha$ and that the equation of motion remains unchanged.

The consequence is that the equation of motion admits solutions that are geometrically similar, i.e. the time differences $(\Delta t)_a$ and $(\Delta t)_b$ of points that correspond to each other on geometrically similar orbits (a) and (b) and the corresponding linear dimensions $L_a$ and $L_b$ are related by

$$\frac{(\Delta t)_b}{(\Delta t)_a} = \left(\frac{L_b}{L_a}\right)^{1-\alpha/2}.$$

(ii) What are the consequences of this relationship for

– the period of harmonic oscillation?
– the relation between time and height of free fall in the neighborhood of the earth’s surface?
– the relation between the periods and the semimajor axes of planetary ellipses?

(iii) What is the relation of the energies of two geometrically similar orbits for

– the harmonic oscillation?
– the Kepler problem?

1.15 *The Kepler problem.* (i) Show that the differential equation for $\Phi(r)$, in the case of finite orbits, has the following form:

$$\frac{d\Phi}{dr} = \frac{1}{r} \sqrt{\frac{r pr_A}{(r - r_p)(r_A - r)}}, \quad (1.4)$$

where $r_p$ and $r_A$ denote the perihelion and the aphelion, respectively. Calculate $r_p$ and $r_A$ and integrate (1.4) with the boundary condition $\Phi(r_p) = 0$.

(ii) Change the potential to $U(r) = (-A/r) + (B/r^2)$ with $|B| \ll l^2/2\mu$. Determine the new perihelion $r_p'$ and the new aphelion $r_A'$ and write the differential equation for $\Phi(r)$ in a form analogous to (1.4). Integrate this equation as in (i) and determine two successive perihelion positions for $B > 0$ and for $B < 0$.

Hint:

$$\frac{d}{dx} \arccos\left(\frac{\alpha + \beta}{\sqrt{x^2(1 - \beta^2) - 2\alpha\beta x - \alpha^2}}\right) = \frac{\alpha}{x} \frac{1}{\sqrt{x^2(1 - \beta^2) - 2\alpha\beta x - \alpha^2}}.$$

1.16 The most general solution of the Kepler problem reads, in terms of polar coordinates $r$ and $\Phi$,
\[ r(\Phi) = \frac{p}{1 + \varepsilon \cos(\phi - \phi_0)}. \]

The parameters are given by
\[ p = \frac{l^2}{A \mu}, \quad (A = Gm_1m_2), \]
\[ \varepsilon = \sqrt{1 + \frac{2E l^2}{\mu A^2}}, \quad \left( \mu = \frac{m_1m_2}{m_1 + m_2} \right). \]

What values of the energy are possible if the angular momentum is given? Calculate the semimajor axis of the earth’s orbit under the assumption \( m_{\text{Sun}} \gg m_{\text{Earth}}; \)
\[ G = 6.672 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}, \]
\[ m_S = 1.989 \times 10^{30} \text{ kg}, \]
\[ m_E = 5.97 \times 10^{24} \text{ kg}. \]

Calculate the semimajor axis of the ellipse along which the sun moves about the center-of-mass of the sun and the earth and compare the result to the solar radius (\( 6.96 \times 10^8 \text{ m} \)).

1.17 Determine the interaction of two electric dipoles \( p_1 \) and \( p_2 \) (example for noncentral potential force).

Hints: Calculate the potential of a single dipole \( p_1 \), making use of the following approximation. The dipole consists of two charges \( \pm e_1 \) at a distance \( d_1 \). Let \( e_1 \) tend to infinity and \( |d_1| \) to zero, in such a way that their product \( p_1 = d_1 e_1 \) stays constant. Then calculate the potential energy of a finite dipole \( p_2 \) in the field of the first and perform the same limit \( e_2 \to \infty, |d_2| \to 0 \), with \( p_2 = d_2 e_2 \) constant, as above. Calculate the forces that act on the two dipoles.

Answer:
\[ W(1, 2) = (p_1 \cdot p_2) / r^3 - 3(p_1 \cdot r)(p_2 \cdot r) / r^5, \]
\[ F = -\nabla_1 W = \left[ 3(p_1 \cdot p_2)/r^5 \right. \]
\[ -15(p_1 \cdot r)(p_2 \cdot r)/r^7 \] \[ \left. +3[p_1(p_2 \cdot r) + p_2(p_1 \cdot r)]/r^5 = -F_{12} \right. . \]

1.18 Let the motion of a point mass be governed by the law
\[ \dot{v} = v \times a, \quad a = \text{const}. \]

Show that \( \dot{r} \cdot a = v(0) \cdot a \) holds for all \( t \) and reduce (1.5) to an inhomogeneous differential equation of the form \( \ddot{r} + \omega^2 r = f(t) \). Solve this equation by means
of the substitution \( r_{\text{inhom}}(t) = ct + d \). Express the integration constants in terms of the initial values \( r(0) \) and \( v(0) \). Describe the curve \( r(t) = r_{\text{hom}}(t) + r_{\text{inhom}}(t) \).

**Hint:**

\[
a_1 \times (a_2 \times a_3) = a_2(a_1 \cdot a_3) - a_3(a_1 \cdot a_2).
\]

1.19 An iron ball falls vertically onto a horizontal plane from which it is reflected. At every bounce it loses the \( n \)th fraction of its kinetic energy. Discuss the orbit \( x = x(t) \) of the bouncing ball and derive the relation between \( x_{\text{max}} \) and \( t_{\text{max}} \).

**Hint:** Study the orbit between two successive bounces and sum over previous times.

1.20 Consider the following transformations of the coordinate system:

\[
\{t, r\} \rightarrow \{t, r\}, \quad \{t, r\} \rightarrow \{t, -r\}, \quad \{t, r\} \rightarrow \{-t, r\},
\]

as well as the transformation \( P \cdot T \) that is generated by performing first \( T \) and then \( P \). Write these transformations in the form of matrices that act on the four-component vector \((t, r)\). Show that \( \{E, P, T, PT\} \) form a group.

1.21 Let the potential \( U(r) \) of a two-body system be \( C^2 \) (twice continuously differentiable). For fixed relative angular momentum, under which additional condition on \( U(r) \) are there circular orbits? Let \( E_0 \) be the energy of such an orbit. Discuss the motion for \( E = E_0 + \varepsilon \) for small positive \( \varepsilon \). Study the special cases \( U(r) = r^n \) and \( U(r) = \lambda/r \).

1.22 Following the methods explained in Sect. 1.26 show the following.

(i) In the northern hemisphere a falling object experiences a *southward* deviation of second order (in addition to the first-order eastward deviation).

(ii) A stone thrown vertically upward falls down west of its point of departure, the deviation being four times the eastward deviation of the falling stone.

1.23 Let a two-body system be subject to the potential

\[
U(r) = -\frac{\alpha}{r^2}
\]

in the relative coordinate \( r \), with positive \( \alpha \). Calculate the scattering orbits \( r(\Phi) \). For fixed angular momentum what are the values of \( \alpha \) for which the particle makes one (two) revolutions about the center of force? Follow and discuss an orbit that collapses to \( r = 0 \).

1.24 A pointlike comet of mass \( m \) moves in the gravitational field of a sun with mass \( M \) and radius \( R \). What is the total cross section for the comet to crash on the sun?
1.25 Solve the equations of motion for the example of Sect. 1.21.2 (Lorentz force with constant fields) for the case

\[ B = B \hat{e}_z, \quad E = E \hat{e}_z. \]

1.26 Kepler problem and Hodograph: Let \( p_x \) and \( p_y \) be the components of the momentum in the plane of motion of the Kepler problem. Show: In momentum space, spanned by \( (p_x, p_y) \), all bound orbits are circles. Give the position and the radius of these circles. The curve described by the tip of the velocity, or momentum, vector is called hodograph.

Chapter 2: The Principles of Canonical Mechanics

2.1 The energy \( E(q, p) \) is an integral of a finite, one-dimensional, periodic motion. Why is the portrait symmetric with respect to the \( q \)-axis? The surface enclosed by the periodic orbit is

\[ F(E) = \oint p \, dq = 2 \int_{q_{\text{min}}}^{q_{\text{max}}} p \, dq. \]

Show that the change of \( F(E) \) with \( E \) equals the period \( T \) of the orbit, \( T = \frac{dF(E)}{dE} \). Calculate \( F \) and \( T \) for the example

\[ E(q, p) = \frac{p^2}{2m} + \frac{m \omega^2 q^2}{2}. \]

2.2 A weight glides without friction along a plane inclined by the angle \( \alpha \) with respect to the horizontal. Study this system by means of d’Alembert’s principle.

2.3 A ball rolls without friction on the inside of a circular annulus. The annulus is put upright in the earth’s gravitational field. Use d’Alembert’s principle to derive the equation of motion and discuss its solutions.

2.4 A mass point \( m \) that can only move along a straight line is tied to the point \( A \) by means of a spring. The distance of \( A \) to the straight line is \( l \) (cf. Fig. 5). Calculate (approximately) the frequency of oscillation of the mass point.

2.5 Two equal masses \( m \) are connected by means of a (massless) spring with spring constant \( x \). They move without friction along a rail, their distance being \( l \) when the spring is inactive. Calculate the deviations \( x_1(t) \) and \( x_2(t) \) from the equilibrium positions, for the following initial conditions:

\[
\begin{align*}
x_1(0) &= 0, \quad \dot{x}_1(0) = v_0, \\
x_2(0) &= l, \quad \dot{x}_2(0) = 0.
\end{align*}
\]

2.6 Given a function \( F(x_1, \ldots, x_f) \) that is homogeneous and of degree \( N \) in its \( f \) variables, show that
2.7 If in the integral
\[ I[y] = \int_{x_1}^{x_2} dx \ f(y, y') \]
\( f \) does not depend explicitly on \( x \), show that
\[ y' \frac{\partial f}{\partial y'} - f(y, y') = \text{const} \]
Apply this result to \( L(q, q) = T - U \) and identify the constant. \( T \) is assumed to be a homogeneous quadratic form in \( \dot{q} \).

2.8 Solve the following two problems (whose solutions are well known) by means of variational calculus:
(i) the shortest connection between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the Euclidean plane;
(ii) the shape of a homogeneous, fine-grained chain suspended at its end points \((x_1, y_1)\) and \((x_2, y_2)\) in the gravitational field.

*Hints:* Make use of the result of Exercise 2.7. The equilibrium shape of the chain is determined by the lowest position of its center of mass. The line element is given by
\[ ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2} \ dx . \]

2.9 Two coupled pendula can be described by means of the Lagrangian function
\[ L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} m \omega_0^2 (x_1^2 + x_2^2) - \frac{1}{4} m (\omega_1^2 - \omega_0^2) (x_1 - x_2)^2 . \]

(i) Show that the Lagrangian function
\[ L' = \frac{1}{2} m (\dot{x}_1 - i \omega_0 x_1)^2 + \frac{1}{2} m (\dot{x}_2 - i \omega_0 x_2)^2 - \frac{1}{4} m (\omega_1^2 - \omega_0^2) (x_1 - x_2)^2 \]
leads to the same equations of motion. Why is this so?
(ii) Show that transforming to the eigenmodes of the system leaves the Lagrange equations form invariant.

2.10 The force acting on a body in three-dimensional space is assumed to be axially symmetric with respect to the $z$-axis. Show that

(i) its potential has the form $U = U(r,z)$, where $\{r, \varphi, z\}$ are cylindrical coordinates,

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z;$$

(ii) the force always lies in a plane containing the $z$-axis.

2.11 With respect to an inertial system $K_0$ the Lagrangian function of a particle is

$$L_0 = \frac{1}{2} m \dot{x}_0^2 - U(x_0).$$

The frame of reference $K$ has the same origin as $K_0$ but rotates about the latter with constant angular velocity $\omega$. Show that the Lagrangian with respect to $K$ reads

$$L = m \dot{x}^2 + m \dot{x} \cdot (\omega \times x) + \frac{m}{2} (\omega \times x)^2 - U(x).$$

Derive the equations of motion of Sect. 1.25 from this.

2.12 A planar pendulum is suspended such that its point of suspension glides without friction along a horizontal axis. Construct the kinetic and potential energies and a Lagrangian function for this problem.

2.13 A pearl of mass $m$ glides (without friction) along a planar curve $s = s(\Phi)$ put up vertically. $s$ is the length of arc and $\Phi$ the angle between the tangent to the curve and the horizontal line (see Fig. 6).

(i) Derive the equation for $s(t)$ for harmonic oscillations.
(ii) What is the relation between $s(t)$ and $\Phi(t)$? Discuss this relation and the motion that follows from it. What happens in the limit where $s$ can reach its maximal amplitude?
(iii) From the explicit solution calculate the force of constraint and the effective force that acts on the pearl.
2.14 Geometrical interpretation of the Legendre transformation. Given \( f(x) \) with \( f''(x) > 0 \). Construct \((\mathcal{L}f)(x) = xf'(x) - f(x) = xz - f(x) \equiv F(x, z)\), where \( z = f'(x) \). The inverse \( x = x(z) \) of the latter exists and so does the Legendre transform of \( f(x) \), which is \( zx(z) - f(x(z)) = \mathcal{L}f(z) = \Phi(z) \).

(i) Comparing the graphs of the functions \( y = f(x) \) and \( y = zx \) (for fixed \( z \)) one sees with \( \frac{\partial F(x, z)}{\partial x} = 0 \) that \( x = x(z) \) is the point where the vertical distance between the two graphs is maximal (see Fig. 7).

(ii) Take the Legendre transform of \( \Phi(z) \), i.e. \((\mathcal{L}\Phi)(z) = z\Phi'(z) - \Phi(z) = zx - \Phi(z) \equiv G(z, x) \) with \( \Phi'(z) = x \). Identify the straight line \( y = G(z, x) \) for fixed \( z \) and with \( x = x(z) \) and show that one has \( G(z, x) = f(x) \). Sketch the picture that one obtains if one keeps \( x = x_0 \) fixed and varies \( z \).

2.15  

(i) Let

\[
L(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = T - U \quad \text{with}
\]

\[
T = \sum_{i,k=1}^{2} c_{ik} \dot{q}_i \dot{q}_k + \sum_{k=1}^{2} b_k \dot{q}_k + a.
\]

Under what condition can one construct \( H(q, p, t) \) and what are \( p_1, p_2, \) and \( H \)? Confirm that the Legendre transform of \( H \) is again \( L \) and that

\[
\det \left( \frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_i} \right) \det \left( \frac{\partial^2 H}{\partial p_n \partial p_m} \right) = 1.
\]

\textbf{Hint}: Take \( d_{11} = 2c_{11}, d_{12} = d_{21} = c_{12} + c_{21}, d_{22} = 2c_{22}, \pi_i = p_i - b_i \).

(ii) Assume now that \( L = L(x_1 \equiv \dot{q}_1, x_2 \equiv \dot{q}_2, q_1, q_2, t) \equiv L(x_1, x_2, u) \) with \( u \equiv (q_1, q_2, t) \) to be an arbitrary Lagrangian function. We expect the momenta \( p_i = p_i(x_1, x_2, u) \) derived from \( L \) to be independent functions of \( x_1 \) and \( x_2 \), i.e. that there is no function \( F(p_1(x_1, x_2, u), p_2(x_1, x_2, u)) \) that vanishes identically.
Show that, if \( p_1 \) and \( p_2 \) were dependent, the determinant of the second derivatives of \( L \) with respect to the \( x_i \) would vanish.

**Hint:** Consider \( \frac{dF}{dx_1} \) and \( \frac{dF}{dx_2} \).

**2.16** A particle of mass \( m \) is described by the Lagrangian function

\[
L = \frac{1}{2}m(x^2 + y^2 + z^2) + \frac{\omega}{2} l_3 ,
\]

where \( l_3 \) is the \( z \)-component of angular momentum and \( \omega \) is a constant frequency. Find the equations of motion, write them in terms of the complex variable \( x + iy \) and of \( z \), and solve them. Construct the Hamiltonian function and find the *kinematical* and *canonical* momenta. Show that the particle has only kinetic energy and that the latter is conserved.

**2.17** **Invariance under time translations and Noether’s theorem.** The theorem of E. Noether can be applied to the case of translations in time by means of the following trick. Make \( t \) a coordinate-like variable by parametrizing both \( q \) and \( t \) by 

\[
q = q(\tau), \quad t = t(\tau)
\]

and by defining the following Lagrangian function:

\[
\bar{L} \left( q, t, \frac{dq}{d\tau}, \frac{dt}{d\tau} \right) \overset{\text{def}}{=} L \left( q, \frac{1}{dt/d\tau} \frac{dq}{d\tau}, t \right) \frac{dt}{d\tau} .
\]

(i) Show that Hamilton’s variational principle applied to \( \bar{L} \) yields the same equations of motion as for \( L \).

(ii) Assume \( L \) to be invariant under time translations

\[
h^s(q, t) = (q, t + s) .
\]

Apply Noether’s theorem to \( \bar{L} \) and find the constant of the motion corresponding to the invariance (2.1).

**2.18** A mass point is scattered elastically on a sphere with center \( P \) and radius \( R \) (see Fig. 8). Show that the physically possible orbit has *maximal* length.

**Hints:** Show first that the angles \( \alpha \) and \( \beta \) must be equal and construct the action integral. Show that any other path \( AB' \Omega \) would be shorter than for those points.

![Fig. 8.](image-url)
where the sum of the distances to $A$ and $\Omega$ is constant and equal to the length of the physical orbit.

2.19 (i) Show that canonical transformations leave the physical dimension of the product $p_i q_i$ unchanged, i.e. $[P_i Q_i] = [p_i q_i]$. Let $\Phi$ be the generating function for a canonical transformation. Show that

$$[p_i q_i] = [P_k Q_k] = [\Phi] = [H \cdot t],$$

where $H$ is the Hamiltonian function and $t$ the time.

(ii) In the Hamiltonian function $H = p^2/2m + m\omega^2 q^2/2$ of the harmonic oscillator introduce the variables

$$x_1 \overset{\text{def}}{=} \omega \sqrt{mq}, \quad x_2 \overset{\text{def}}{=} p/\sqrt{m}, \quad \tau \overset{\text{def}}{=} \omega t,$$

thus obtaining $H = (x_1^2 + x_2^2)/2$. What is the generating function $\hat{\Phi}(x_1, y_1)$ for the canonical transformation $x \to y$ that corresponds to the function $\Phi(q, Q) = (m\omega q^2/2) \cot Q$? Calculate the matrix $M_{ik} = \partial x_i / \partial y_k$ and confirm $\det M = 1$ and $M^T J M = J$.

2.20 The group $Sp_{2f}$ is particularly simple for $f = 1$, i.e. in two dimensions.

(i) Show that every matrix

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is symplectic if and only if $a_{11} a_{22} - a_{12} a_{21} = 1$.

(ii) Therefore, the orthogonal matrices

$$O = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

and the symmetric matrices

$$S = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \quad \text{with} \quad xz - y^2 = 1$$

belong to $Sp_{2f}$. Show that every $M \in Sp_{2f}$ can be written as a product

$$M = S \cdot O$$

of a symmetric matrix $S$ with determinant 1 and an orthogonal matrix $O$.

2.21 (i) Evaluate the following Poisson brackets for a single particle:

$$\{l_i, r_k\}, \quad \{l_i, p_k\}, \quad \{l_i, r\}, \quad \{l_i, p^2\}.$$
(ii) If the Hamiltonian function in its natural form \( H = T + U \) is invariant under rotations, what quantities can \( U \) depend on?

2.22 Making use of the Poisson brackets show that for the system \( H = T + U(r) \) with \( U(r) = \gamma/r \) and \( \gamma \) a constant, the vector \( A = p \times l + x m \gamma/r \) is an integral of the motion (Lenz’ vector or Hermann–Bernoulli–Laplace vector).

2.23 The motion of a particle of mass \( m \) is described by

\[
H = \frac{1}{2m} \left( p^2 + p_2^2 \right) + m \alpha q_1, \quad \alpha = \text{const}.
\]

Construct the solution of the equations of motion for the initial conditions

\[
q_1(0) = x_0, \quad q_2(0) = y_0, \quad p_1(0) = px, \quad p_2(0) = py,
\]

making use of Poisson brackets.

2.24 For a three-body system with masses \( m_i \), coordinates \( r_i \), and momenta \( p_i \) introduce the following coordinates (Jacobian coordinates\(^2\)):

\[
\begin{align*}
\varphi_1 &\equiv r_2 - r_1 \\
\varphi_2 &\equiv r_3 - \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \\
\varphi_3 &\equiv \frac{m_1 r_1 + m_2 r_2 + m_3 r_3}{m_1 + m_2 + m_3} \\
\pi_1 &\equiv \frac{m_1 p_2 - m_2 p_1}{m_1 + m_2}, \\
p_2 &\equiv \frac{(m_1 + m_2) p_3 - m_3 (p_1 + p_2)}{m_1 + m_2 + m_3}, \\
\pi_3 &\equiv p_1 + p_2 + p_3.
\end{align*}
\]

(i) What is the physical interpretation of the momenta \( \pi_1, \pi_2, \pi_3 \)?
(ii) How would you define such coordinates for four or more particles?
(iii) Show in at least two (equivalent) ways that the transformation

\[
\{r_1, r_2, r_3, p_1, p_2, p_3\} \rightarrow \{\varphi_1, \varphi_2, \varphi_3, \pi_1, \pi_2, \pi_3\}
\]

is canonical.

2.25 Given a Lagrangian function \( L \) for which \( \partial L/\partial t = 0 \), study only those variations of the orbits \( q_k(t, \alpha) \) which belong to a fixed energy \( E = \sum_k \dot{q}_k (\partial L/\partial \dot{q}_k) - L \)

and whose end points are kept fixed irrespective of the time \((t_2 - t_1)\) that the system needs to move from the initial to the end point, i.e.

\[
q_k(t, \alpha) \quad \text{with} \quad \begin{cases} 
q_k(t_1(\alpha), \alpha) = q_k^{(1)} \\
q_k(t_2(\alpha), \alpha) = q_k^{(2)} 
\end{cases} \quad \text{for all } \alpha .
\]  

(2.1)

Thus, initial and final times are also varied, \(t_i = t_i(\alpha)\).

(i) Calculate the variation of \(I(\alpha)\),

\[
\delta I = \left. \frac{dI(\alpha)}{d\alpha} \right|_{\alpha=0} \, d\alpha = \int_{t_1(\alpha)}^{t_2(\alpha)} L(q_k(t, \alpha), \dot{q}_k(t, \alpha)) \, dt .
\]

(2.2)

(ii) Show that the variational principle

\[
\delta K = 0 \quad \text{with} \quad K \overset{\text{def}}{=} \int_{t_1}^{t_2} (L + E) \, dt
\]

together with the prescriptions (2.1) is equivalent to the Lagrange equations (the \textit{Principle of Euler and Maupertuis}).

2.26 The kinetic energy

\[
T = \sum_{i, k=1}^{f} q_{ik} \dot{q}_i \dot{q}_k = \frac{1}{2} (L + E)
\]

is assumed to be a positive symmetric quadratic form in the \(\dot{q}_i\). The orbit in the space spanned by the \(q_k\) is described by the length of arc \(s\) such that \(T = (ds/dt)^2\). With \(E = T + U\) the integral \(K\) of Exercise 2.25 can be replaced with an integral over \(s\). Show that the integral principle obtained in this way is equivalent to Fermat’s principle of geometric optics,

\[
\delta \int_{x_1}^{x_2} n(x, \nu) \, ds = 0
\]

\((n: \text{index of refraction}, \nu: \text{frequency})\).

2.27 Let \(H = p^2/2 + U(q)\), where the potential is such that it has a local minimum at \(q_0\). Thus, in an interval \(q_1 < q_0 < q_2\) the potential forms a potential well. Sketch a potential with this property and show that there is an interval \(U(q_0) < E \leq E_{\text{max}}\) where there are periodic orbits. Consider the characteristic equation of Hamilton and Jacobi (2.154). If \(S(q, E)\) is a complete integral then \(t - t_0 = \partial S/\partial E\). Take the integral

\[
I(E) \overset{\text{def}}{=} \frac{1}{2\pi} \oint_{\Gamma_E} p \, dq
\]

over the periodic orbit \(\Gamma_E\) with energy \(E\) (this is the surface enclosed by \(\Gamma_E\)). Write \(I(E)\) as an integral over time and show that

\[
\frac{dI}{dE} = \frac{T(E)}{2\pi} .
\]
2.28 In Exercise 2.27 replace \( S(q, E) \) by \( \bar{S}(q, I) \) with \( I = I(E) \) as defined there. \( \bar{S} \) generates the canonical transformation \( (q, p, H) \rightarrow (\theta, I, \bar{H} = E(\Omega)) \). What are the canonical equations in the new variables? Can they be integrated?

2.29 Let \( H^0 = p^2/2 + q^2/2 \). Calculate the integral \( I(E) \) defined in Exercise 2.27. Solve the characteristic equation of Hamilton and Jacobi (2.154) and write the solution as \( \bar{S}(q, I) \). Then \( \theta = \partial \bar{S}/\partial I \). Show that \( (q, p) \) and \( (\theta, I) \) are related by the canonical transformation (2.95) of Sect. 2.24 (ii).

2.30 We assume that the Lagrangian of a mechanical system with one degree of freedom does not depend explicitly on time. In Hamilton’s variational principle we make a smooth change of the end points \( q^a \) and \( q^b \), as well as of the running time \( t = t_2 - t_1 \), in the sense that the solution \( \varphi(t) \) for the values \( (q^a, q^b, t) \) and the solution \( \phi(s, t) \) which belongs to the values \( (q^a, q^b, t') \) are related in a smooth manner: \( \varphi(t) \mapsto \phi(s, t) \) such that \( \phi(s, t) \) is differentiable in \( s \) and \( \phi(s = 0, t) = \varphi(t) \).

Show that the corresponding change of the action integral \( I_0 \) into which the physical solution is inserted (this function is called Hamilton’s principal function), is given by the following expression

\[
\delta I_0 = -E \delta t + p^b \delta q^b - p^a \delta q^a .
\]

2.31 The vector \( A \) that is introduced in Exercise 2.22 lies in the plane perpendicular to \( \ell \). Calculate \( |A| \) as a function of the energy. When does this vanish? Let \( \phi \) denote the angle between \( x \) (orbit vector) and \( A \). Calculate \( x \cdot A \) and show that this yields the orbit’s equation in the form \( r = r(\phi) \). Determine the modulus and the direction of \( A \), calculate the cross product \( \ell \times A \) and from there the quantity

\[
\left( p - \frac{1}{\ell^2} \ell \times A \right)^2 .
\]

This calculation yields an alternative solution of Exercise 1.26.

Chapter 3: The Mechanics of Rigid Bodies

3.1 Let two systems of reference \( K \) and \( \tilde{K} \) be fixed in the center of mass of a rigid body, the axes of the former being fixed in \textit{space}, those of the latter fixed in the body. If \( J \) is the inertia tensor with respect to \( K \) and \( \tilde{J} \) the one as calculated in \( \tilde{K} \), show that (i) \( J \) and \( \tilde{J} \) have the same eigenvalues. (Use the characteristic polynomial.)

(ii) \( \tilde{K} \) is now assumed to be a system of principal axes of inertia. What is the form of \( \tilde{J} \)? Calculate \( J \) for the case of rotation of the body about the 3-axis.

3.2 Two particles with masses \( m_1 \) and \( m_2 \) are held by a rigid but massless straight connection with length \( l \). What are the principal axes and what are the moments of inertia?
3.3 The inertia tensor of a rigid body is found to have the form

\[
I_{ik} = \begin{pmatrix}
I_{11} & I_{12} & 0 \\
I_{21} & I_{22} & 0 \\
0 & 0 & I_{33}
\end{pmatrix}, \quad I_{21} = I_{12}.
\]

Determine the three moments of inertia and consider the following special cases.
(i) \( I_{11} = I_{22} = A, \ I_{12} = B. \) Can \( I_{33} \) be arbitrary?

Fig. 9.

(ii) \( I_{11} = A, \ I_{22} = 4A, \ I_{12} = 2A. \) What can you say about \( I_{33} \)? What is the shape of the body in this example?

3.4 Construct the Lagrangian function for general, force-free motion of a conical top (height \( h \), mass \( M \), radius of base circle \( R \)). What are the equations of motion? Are there integrals of the motion and what is their physical interpretation?

3.5 Calculate the moments of inertia of a torus filled homogeneously with mass. Its main radius is \( R \); the radius of its section is \( r \).

3.6 Calculate the moment of inertia \( I_3 \) for two arrangements of four balls, two heavy (radius \( R \), mass \( M \)) and two light (radius \( r \), mass \( m \)) with homogeneous mass density, as shown in Fig. 9. As a model of a dancer’s pirouette compare the angular velocity for the two arrangements, with \( L_3 \) fixed and equal in the two cases.

3.7 (i) Let the boundary of a homogeneous body be defined by the formula (in spherical coordinates)

\[
R(\theta) = R_0(1 + \alpha \cos \theta),
\]

i.e. \( \varrho(r, \theta, \Phi) = \varrho_0 = \text{const} \) for \( r \leq R(\theta) \) and all \( \theta \) and \( \Phi \), and \( \varrho(r, \theta, \Phi) = 0 \) for \( r > R(\theta) \). If \( M \) is the total mass, calculate \( \varrho_0 \) and the moments of inertia.

(ii) Perform the same calculation for a homogeneous body whose shape is given by
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\[ R(\theta) = R_0 (1 + \beta Y_{20}(\theta)) \]

with \( Y_{20}(\theta) = \sqrt{5/16\pi} (3 \cos^2 \theta - 1) \) being the spherical harmonic with \( l = 2, m = 0 \). In both examples sketch the body.

![Fig. 10.](image)

3.8 Determine the moments of inertia of a rigid body whose inertia tensor with respect to a system of reference \( K_1 \) (fixed in the body) is given by

\[
J = \begin{pmatrix}
9 & 1 & -\sqrt{3} \\
8 & 4 & 8 \\
1 & 3 & -\sqrt{3} \\
-\sqrt{3} & 2 & 4 \\
8 & -\sqrt{3} & 11 \\
4 & 8 & 8
\end{pmatrix}.
\]

Can one indicate the relative position of the principal inertia system \( K_0 \) relative to \( K_1 \) ?

3.9 A ball with radius \( a \) is filled homogeneously with mass such that the density is \( \varrho_0 \). The total mass is \( M \).

(i) Write the mass density \( \varrho \) with respect to a body-fixed system centered in the center of mass and express \( \varrho_0 \) in terms of \( M \). Let the ball rotate about a point \( P \) on its surface (see Fig. 10).

(ii) What is the same density function \( \varrho(r, t) \) as seen from a space-fixed system centered on \( P \)?

(iii) Give the inertia tensor in the body-fixed system of (i). What is the moment of inertia for rotation about a tangent to the ball in \( P \)?

*Hint:* Use the step function \( \Theta(x) = 1 \) for \( x \geq 0 \), \( \Theta(x) = 0 \) for \( x < 0 \).

3.10 A homogeneous circular cylinder with length \( h \), radius \( r \), and mass \( m \) rolls along an inclined plane in the earth’s gravitational field.

(i) Construct the full kinetic energy of the cylinder and find the moment of inertia relevant to the described motion.

(ii) Construct the Lagrangian function and solve the equation of motion.
3.11 Manifold of motions of the rigid body. A rotation $R \in \text{SO}(3)$ can be determined by a unit vector $\hat{\phi}$ (the direction about which the rotation takes place) and an angle $\varphi$.

(i) Why is the interval $0 \leq \varphi \leq \pi$ sufficient for describing every rotation?

(ii) Show that the parameter space $(\hat{\phi}, \varphi)$ fills the interior of a sphere with radius $\pi$ in $\mathbb{R}^3$. This ball is denoted by $D^3$. Confirm that antipodal points on the ball’s surface represent the same rotation.

(iii) There are two types of closed orbit in $D^3$, namely those which can be contracted to a point and those which connect two antipodal points. Show by means of a sketch that every closed curve can be reduced by continuous deformation to either the former or the latter type.

3.12 Calculate the Poisson brackets (3.92–95).

Chapter 4: Relativistic Mechanics

4.1 (i) A neutral $\pi$ meson ($\pi^0$) has constant velocity $v_0$ along the $x^3$-direction. Write its energy-momentum vector. Construct the special Lorentz transformation that leads to the particle’s rest system.

(ii) The particle decays isotropically into two photons, i.e. with respect to its rest system the two photons are emitted in all directions with equal probability. Study their decay distribution in the laboratory system.

4.2 The decay $\pi \rightarrow \mu + \nu$ (cf. Example (i) of Sect. 4.9.2) is isotropic in the pion’s rest system. Show that above a certain fixed energy of the pion in the laboratory system there is a maximal angle beyond which no muons are emitted. Calculate that energy and the maximal emission angle as a function of $m_\pi$ and $m_\mu$ (see Fig. 11). Where do muons go in the laboratory system that in the pion’s rest system were emitted forward, backward, or transversely with respect to the pion’s velocity in the laboratory?

Fig. 11.

4.3 Consider a two-body reaction $A + B \rightarrow A + B$ for which the relative velocity of $A$ (the projectile) and $B$ (the target) is not small compared to the speed of light.
Examples are
\[ e^- + e^+ \rightarrow e^- + e^-, \quad \nu + e \rightarrow e + \nu, \quad p + p \rightarrow p + p. \]

Denoting the four-momenta before and after the scattering by \( q_A, q_B \) and \( q_A', q_B' \) the following quantities are Lorentz scalars, i.e. they have the same values in every system of reference,
\[ s \overset{\text{def}}{=} c^2 (q_A + q_B)^2, \quad t \overset{\text{def}}{=} c^2 (q_A - q_A')^2. \]

Conservation of energy and momentum requires \( q_A' + q_B' = q_A + q_B \). Furthermore, we have \( q_A^2 = q_A'^2 = (m_A c^2)^2, q_B^2 = q_B'^2 = (m_B c^2)^2 \).

(i) Express \( s \) and \( t \) in terms of the energies and momenta of the particles in the center-of-mass frame. Denoting the modulus of the 3-momentum by \( q^* \) and the scattering angle by \( \theta^* \), write \( s \) and \( t \) in terms of these variables.

(ii) Define \( u = c^2 (q_A - q_B')^2 \) and show that
\[ s + t + u = 2 \left( m_A^2 + m_B^2 \right) c^4. \]

4.4 Calculate the variables \( s \) and \( t \) (as defined in Exercise 4.3) in the laboratory system, i.e. in that system where \( B \) is at rest before the scattering. What is the relation between the scattering angle \( \theta \) in the laboratory system and \( \theta^* \) in the center-of-mass frame? Compare to the nonrelativistic expression (1.80).

4.5 In its rest system the electron’s spin is described by the 4-vector \( s^\alpha = (0, s) \). What is the form of this vector in a frame where the electron has the momentum \( p \)? Calculate the scalar product \( (s \cdot p) = s^\alpha p_\alpha \).

4.6 Show that
(i) every lightlike vector \( z (z^2 = 0) \) can be brought to the form \((1,1,0,0)\) by means of Lorentz transformations;
(ii) every spacelike vector can be transformed to the form \((0, z^1, 0, 0)\), where \( z^1 = \sqrt{-z^2} \).

Indicate the necessary transformations in both cases.

4.7 If \( J_i \) and \( K_j \) denote the generators of rotations and boosts, respectively (cf. Sect. 4.5.2 (iii)) define
\[ A_p \overset{\text{def}}{=} \frac{1}{2} \left( J_p + iK_p \right), \quad B_q \overset{\text{def}}{=} \frac{1}{2} \left( J_q - iK_q \right), \quad p, q = 1, 2, 3. \]

Making use of the commutation rules (4.59) calculate \([A_p, A_q], [B_p, B_q], [A_p, B_q]\) and compare to (4.59).

4.8 Study the behavior of \( J_i \) and \( K_j \) with respect to space inversion, i.e. determine \( P^* J_i P^{-1}, P K_j P^{-1} \).
4.9 In quantum theory one prefers to use the quantities
\[ \hat{J}_i \overset{\text{def}}{=} iJ_i, \quad \hat{K}_j \overset{\text{def}}{=} -iK_j. \]

What are the commutators (4.59) for these matrices? Show that the matrices \( \hat{J}_i \)
are Hermitian, i.e. that \( (\hat{J}_i^\dagger)^* = \hat{J}_i \).

4.10 A muon decays predominantly into an electron and two nearly massless neutrinos, \( \mu^- \rightarrow e^- + \nu_1 + \nu_2 \). If the muon is at rest, show that the electron assumes its maximal momentum whenever the neutrinos are emitted parallel to each other. Calculate the maximal and minimal energies of the electron as functions of \( m_\mu \) and \( m_e \).

Answer:
\[ E_{\text{max}} = \frac{m_\mu^2 + m_e^2 c^2}{2m_\mu}, \quad E_{\text{min}} = m_e c^2. \]

Draw the corresponding momenta in the two cases.

4.11 A particle of mass \( M \) is assumed to decay into three particles (1,2,3) with masses \( m_1, m_2, m_3 \). Determine the maximal energy of particle 1 in the rest system of the decaying particle as follows. Set
\[ p_1 = -f(x)\hat{n}, \quad p_2 = x f(x)\hat{n}, \quad p_3 = (1-x) f(x)\hat{n}, \]
where \( \hat{n} \) is a unit vector and \( x \) is a number between 0 and 1. Find the maximum of \( f(x) \) from the principle of energy conservation.

Examples:
(i) \( \mu^- \rightarrow e^- + \nu_1 + \nu_2 \) (cf. Exercise 4.10),
(ii) Neutron decay: \( n \rightarrow p + e + \nu \).

What is the maximal energy of the electron? What is the value of \( \beta = |v|/c \) for the electron? \( m_n - m_p = 2.53m_e, \quad m_p = 1836m_e \).

4.12 Pions \( \pi^+, \pi^- \) have the mean lifetime \( \tau \approx 2.6 \times 10^{-8} \text{ s} \) and decay predominantly into a muon and a neutrino. Over what distance can they fly, on average, before decaying if their momentum is \( p_\pi = x \cdot m_\pi c \) with \( x = 1, 10, \text{ or } 1000? \) \( (m_\pi \approx 140 \text{ MeV}/c^2 = 2.50 \times 10^{-28} \text{ kg}) \).

4.13 The free neutron is unstable. Its mean lifetime is \( \tau \approx 900 \text{ s} \). How far can a neutron fly on average if its energy is \( E = 10^{-2} m_n c^2 \) or \( E = 10^{14} m_n c^2 \)?

4.14 Show that a free electron cannot radiate a single photon, i.e. the process
\[ e \rightarrow e + \gamma \]
cannot take place because of energy and momentum conservation.
4.15 The following transformation

\[ I : x^\mu \mapsto \bar{x}^\mu = \frac{R^2}{x^2} x^\mu \]

implies the relation \( \sqrt{x^2} \sqrt{\bar{x}^2} = R^2 \). This is an obvious generalization of the well-known inversion at the circle of radius \( R \), \( r \cdot \bar{r} = R^2 \). Show that the sequence of transformations: inversion \( I \) of \( x^\mu \), translation \( T \) of the image by the vector \( R^2 e^\mu \), and another inversion of the result, i.e.,

\[ x' = (I \circ T \circ I) x \]

is precisely the special conformal transformation (4.102).

4.16 Consider the following Lagrangian

\[ L = \frac{1}{2} m \left( \dot{\psi} q^2 - c_0^2 \left( \frac{\psi - 1}{\psi} \right)^2 \right) \equiv L(\dot{q}, \psi) \]

which contains the additional, dimensionless, degree of freedom \( \psi \). The parameter \( c_0 \) has the physical dimension of a velocity. Show: The extremum of the action integral yields a theory obeying special relativity for which \( c_0 \) is the maximal velocity, in other words, one obtains the Lagrangian (4.97) with the velocity of light \( c \) replaced by \( c_0 \). Consider the limit \( c_0 \to \infty \).

Chapter 5: Geometric Aspects of Mechanics

5.1 Let \( k \omega \) be an exterior \( k \)-form, \( l \omega \) an exterior \( l \)-form. Show that their exterior product is symmetric if \( k \) and/or \( l \) are even and antisymmetric if both are odd, i.e.

\[ k \omega \wedge l \omega = (-1)^{k \cdot l} l \omega \wedge k \omega . \]

5.2 Let \( x_1, x_2, x_3 \) be local coordinates in the Euclidean space \( \mathbb{R}^3 \), \( ds^2 = E_1 \, dx_1^2 + E_2 \, dx_2^2 + E_3 \, dx_3^2 \) the square of the line element, and \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) unit vectors along the coordinate directions. What is the value of \( dx_i(\hat{e}_j) \), i.e. of the action of the one-form \( dx_i \) on the unit vector \( \hat{e}_j \)?

5.3 Let \( a = \sum_{i=1}^3 a_i(x) \hat{e}_i \) be a vector field with \( a_i(x) \) smooth functions on \( M \). To every such vector field we associate a one-form \( \omega_a \) and a two-form \( \dot{\omega}_a \) such that

\[ \frac{1}{2} \omega_a(\xi) = (a \cdot \xi), \quad \frac{1}{2} \dot{\omega}_a(\xi, \eta) = (a \cdot (\xi \times \eta)). \]
Show that
\[ \frac{1}{\omega_a} = \sum_{i=1}^{3} a_i(x) \sqrt{E_i} \, dx_i , \]
\[ 2 \omega_a = a_i(x) \sqrt{E_2 E_3} \, dx_2 \wedge dx_3 + \text{cyclic permutations} , \]

5.4 Making use of the results of Exercise 5.3 determine the components of \( \nabla f \) in the basis \( \{ \hat{e}_1, \hat{e}_2, \hat{e}_3 \} \)

Answer:
\[ \nabla f = \sum_{i=1}^{3} \frac{1}{\sqrt{E_i}} \frac{\partial f}{\partial x^i} \hat{e}_i . \]

5.5 Determine the functions \( E_i \) for the case of Cartesian, cylindrical, and spherical coordinates. In each case give the components of \( \nabla f \).

5.6 To the force \( \mathbf{F} = (F_1, F_2) \) in the plane we associate the one-form \( \omega = F_1 \, dx_1 + F_2 \, dx_2 \). When we apply \( \omega \) onto a displacement vector, \( \omega(\xi) \) is the work done by the force. What is the dual \( \ast \omega \) of the form \( \omega \)? What is its interpretation?

5.7 The Hodge star operator assigns to every \( k \)-form \( \omega \) the \( (n-k) \)-form \( \ast \omega \). Show that
\[ \ast(\ast \omega) = (-1)^{k(n-k)} \omega . \]

5.8 Let \( \mathbf{E} = (E_1, E_2, E_3) \) and \( \mathbf{B} = (B_1, B_2, B_3) \) be electric and magnetic fields that in general depend on \( x \) and \( t \). We assign the following exterior forms to them:
\[ \varphi \equiv \sum_{i=1}^{3} E_i \, dx^i , \]
\[ \omega \equiv B_1 \, dx^2 \wedge dx^3 + B_2 \, dx^3 \wedge dx^1 + B_3 \, dx^1 \wedge dx^2 . \]
Write the homogeneous Maxwell equation \( \text{curl} \, \mathbf{E} + \frac{\mathbf{B}}{c} = 0 \) as an equation between the forms \( \varphi \) and \( \omega \).

5.9 If \( \text{d} \) denotes the exterior derivatives and \( \ast \) the Hodge star operator, the codifferential \( \delta \) is defined by
\[ \delta \equiv \ast \text{d} \ast . \]
Show that \( \Delta \equiv \text{d} \circ \delta + \delta \circ \text{d} \), when applied to functions, is the Laplacian operator
\[ \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x^i} . \]
5.10 Let
\[ k \omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k}(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \]
be an exterior \( k \)-form over a vector space \( W \). Let \( F : V \to W \) be a smooth mapping of the vector space \( V \) onto \( W \). Show that the pull-back \( F^*(k \omega \wedge \ell \omega) \) of the exterior product of two such forms is equal to the exterior product of the pull-back of the individual forms \( (F^*k \omega) \wedge (F^*\ell \omega) \).

5.11 With the same assumptions as in Exercise 5.10 show that the exterior derivative and the pull-back commute,
\[ d(F^*\omega) = F^*(d\omega). \]

5.12 Let \( x \) and \( y \) be Cartesian coordinates in \( \mathbb{R}^2 \), \( V = y \partial_x \) and \( W = x \partial_y \) two vector fields on \( \mathbb{R}^2 \). Calculate the Lie bracket \([V, W]\). Sketch the vector fields \( V \), \( W \), and \([V, W]\) along circles about the origin.

5.13 Prove the follow assertions.
(i) The set of all tangent vectors to the smooth manifold \( M \) at the point \( p \in M \) form a real vector space, denoted by \( T_pM \), whose dimension is \( n = \dim M \).
(ii) If \( M = \mathbb{R}^n \), \( T_pM \) is isomorphic to that space.

5.14 The canonical two-form for a system with two degrees of freedom reads
\[ \omega = \sum_{i=1}^2 dq^i \wedge dp_i. \]
Calculate \( \omega \wedge \omega \) and confirm that this product is proportional to the oriented volume element in phase space.

5.15 Let \( H^{(1)} = p^2/2 + (1 - \cos q) \) and \( H^{(2)} = p^2/2 + q(q^2 - 3)/6 \) be the Hamiltonian functions for two systems with one degree of freedom. Construct the corresponding Hamiltonian vector fields and sketch them along some of the solution curves.

5.16 Let \( H = H^0 + H' \) with \( H^0 = (p^2 + q^2)/2 \) and \( H' = \epsilon q^3/3 \). Construct the Hamiltonian vector fields \( X_{H^0} \) and \( X_H \) and calculate \( \omega(X_H, X_{H^0}) \).

5.17 Let \( L \) and \( L' \) be two Lagrangian functions on \( TQ \) for which \( \Phi_L \) and \( \Phi_{L'} \) are regular. The corresponding vector fields and canonical two-forms are \( X_E, X_{E'}, \omega_L, \) and \( \omega_{L'} \). Show that each of the following assertions implies the other:
(i) \( L' = L + \alpha \), where \( \alpha : TQ \to \mathbb{R} \) is a closed one-form, i.e. \( d\alpha = 0 \);
(ii) \( X_E = X_{E'} \) and \( \omega_L = \omega_{L'} \).
Show that in local coordinates this is the result obtained in Sect. 2.10.
Chapter 6: Stability and Chaos

6.1 Study the two-dimensional linear system \( \dot{y} = Ay \), where \( A \) has one of the Jordan normal forms

(i) \( A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \), (ii) \( A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \), (iii) \( A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \).

In all three cases determine the characteristic exponents and the flow (6.13) with \( s = 0 \). Suppose the system is obtained by linearizing a dynamical system in the neighborhood of an equilibrium position. (i) corresponds to the situations shown in Figs. 6.2a–c. Draw the analogous pictures for (ii) for \( a = 0, b > 0 \) and \( a < 0, b > 0 \), and for (iii) with \( \lambda < 0 \).

6.2 The variables \( \alpha \) and \( \beta \) on the torus \( T^2 = S^1 \times S^1 \) define the dynamical system

\[ \dot{\alpha} = a/2\pi, \quad \dot{\beta} = b/2\pi, \quad 0 \leq \alpha, \beta \leq 1, \]

where \( a \) and \( b \) are real constants. Cutting the torus at \( (\alpha = 1, \beta) \) and at \( (\alpha, \beta = 1) \) yields a square of length 1. Draw the solutions with initial condition \( (\alpha_0, \beta_0) \) in this square for \( b/a \) rational and irrational.

6.3 Show that in an autonomous Hamiltonian system with one degree of freedom (and hence two-dimensional phase space) neighboring trajectories can diverge at most linearly with increasing time as long as one keeps clear from saddle points. 

Hint: Make use of the characteristics equation (2.154) of Hamilton and Jacobi.

6.4 Study the system

\[ \dot{q}_1 = -\mu q_1 - \lambda q_2 + q_1 q_2 \]
\[ \dot{q}_2 = \lambda q_1 - \mu q_2 + \left( q_1^2 - q_2^2 \right)/2, \]

where \( 0 \leq \mu \ll 1 \) is a damping term and \( \lambda \) with \( |\lambda| \ll 1 \) is a detuning parameter. Show that if \( \mu = 0 \) the system is Hamiltonian. Find a Hamiltonian function for this case. Draw the projection of its phase portraits for \( \lambda > 0 \) onto the \( (q_1, q_2) \)-plane and determine the position and the nature of the critical points.

Show that the picture obtained above is structurally unstable when \( \mu \) is chosen to be different from zero and positive, by studying the change of the critical points for \( \mu \neq 0 \).

6.5 Given the Hamiltonian function on \( \mathbb{R}^4 \)

\[ H(q_1, q_2, p_1, p_2) = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + \frac{1}{2} \left( q_1^2 + q_2^2 \right) + \frac{1}{3} \left( q_1^3 + q_2^3 \right) \]

show that this system possesses two independent integrals of the motion and sketch the structure of its flow.
6.6 Study the flow of the equations of motion \( p = \dot{q}, \dot{p} = q - q^3 - p \) and determine the position and the nature of its critical points. Two of these are attractors. Determine their basin of attraction by means of the Liapunov function \( V = p^2/2 - q^2/2 + q^4/4 \).

6.7 Dynamical systems of the type
\[
\dot{x} = -\partial U/\partial x \equiv -U_x
\]
are called gradient flows. They are quite different from the flows of Hamiltonian systems. Making use of a Liapunov function show that if \( U \) has an isolated minimum at \( x_0 \), then \( x_0 \) is an asymptotically stable equilibrium position. Study the example
\[
\dot{x}_1 = -2x_1(x_1 - 1)(2x_1 - 1), \quad \dot{x}_2 = -2x_2.
\]

6.8 Consider the equations of motion
\[
\dot{q} = p, \quad \dot{p} = \frac{1}{2}(1 - q^2)
\]
of a system with \( f = 1 \). Sketch the phase portrait of typical solutions with given energy. Study its critical points.

6.9 By numerical integration find the solutions of the Van der Pool equation (6.36) for initial conditions close to \((0,0)\) and for various values of \( \varepsilon \) in the interval \( 0 < \varepsilon \leq 0.4 \). Draw \( q(t) \) as a function of time, as in Fig. 6.7. Use the result to find out empirically at what rate the orbit approaches the attractor.

6.10 Choose the straight line \( p = q \) as the transverse section for the system (6.36), Fig. 6.6. Determine numerically the points of intersection of the orbit with initial condition \((0.01,0)\) with that line and plot the result as a function of time.

6.11 The system in \( \mathbb{R}^2 \)
\[
\dot{x}_1 = x_1, \quad \dot{x}_2 = -x_2 + x_1^2
\]
has a critical point in \( x_1 = 0 = x_2 \). Show that for the linearized system the line \( x_1 = 0 \) is a stable submanifold and the line \( x_2 = 0 \) an unstable one. Find the corresponding manifolds for the exact system by integrating the latter.

6.12 Study the mapping \( x_{i+1} = f(x_i) \) with \( f(x) = 1 - 2x^2 \). Substitute \( u = (4\pi)\arcsin\sqrt{(x + 1)/2} \) and show that there are no stable fixed points. Calculate numerically 50,000 iterations of this mapping for various initial values \( x_1 \neq 0 \) and plot the histogram of the points that land in one of the intervals \([n/100, (n+1)/100] \) with \( n = -100, -99, \ldots, +99 \). Follow the development of two close initial values \( x_1, x'_1 \), and verify that they diverge in the course of the iteration. (For a discussion see Collet, Eckmann 1990.)
6.13 Study the flow of Roessler’s model

\[
\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + xz - cz
\]

for \( a = b = 0.2, c = 5.7 \) by numerical integration. The graphs of \( x, y, z \) as functions of time and their projections onto the \((x, y)\)-plane and the \((x, \dot{x})\)-plane are particularly interesting. Consider the Poincaré mapping for the transverse section \( y + z = 0 \). As \( \dot{x} = 0 \), \( x \) has an extremum on the section. Plot the value of the extremum \( x_{i+1} \) as a function of the previous extremum \( x_i \) (see also Bergé, Pomeau, Vidal 1984 and references therein).

6.14 Although this is more than an exercise, the reader is strongly encouraged to study the system known as Hénon’s attractor. It provides a good illustration of chaotic behavior and extreme sensitivity to initial conditions (see also, Bergé, Pomeau, Vidal 1984, Sect. 3.2 and Devaney 1989, Sect. 2.6, Exercise 10).

6.15 Show that

\[
\sum_{\sigma=1}^{n} \exp \left[ \frac{2\pi i}{n} \sigma m \right] = n\delta_{m0}, \quad (m = 0, \ldots, n - 1). 
\]

Use this result to prove (6.63), (6.65), and (6.66).

6.16 Show that by a linear substitution \( y = \alpha x + \beta \) the system (6.67) can be transformed to \( y_{i+1} = 1 - \gamma y_i^2 \). Determine \( \gamma \) in terms of \( \mu \) and show that \( y \) lies in the interval \((-1, 1]\) and \( \gamma \) in \((0, 2]\) (cf. also Exercise 6.12 above). Making use of this transformed equation derive the values of the first bifurcation points (6.68) and (6.70).
Solution of Exercises

Cross-references to a specific section or equation in the main text of the book are marked with a capital M preceding the number of that section or equation. For instance, Sect. M3.7 refers to Chap. 3, Sect. 7, of the main text, while (M4.100) refers to eq. (4.100) in Chap. 4. Cross references within this set of solutions should be fairly obvious.

Chapter 1: Elementary Newtonian Mechanics

1.1 The time derivative of angular momentum is \( \dot{l} = \dot{r} \times p + r \times \dot{p} = m \dot{r} \times \dot{r} + r \times F \). By assumption this is zero which implies that the force \( F \) must be proportional to \( r \), \( F = \alpha r \), \( \alpha \in \mathbb{R} \). If we decompose the velocity into a component along \( r \) and a component perpendicular to it, then \( F \) will change only the former. Therefore, the motion takes place in a spatially fixed plane perpendicular to the angular momentum \( l = m r(t) \times \dot{r}(t) = m r_0 \times v_0 \), itself a constant. Motion along (a), (b), (e), and (f) is possible. Motion along (c) is not possible because \( l \) would vanish at the turning point but would be different from zero before and after passing through that point.

Fig. 1.
Thus one finds \( l \) of the mixed terms cancel so that \((ds)^2 = (dx)^2 + (dy)^2\) the mixed terms cancel so that \((ds)^2 = (dr)^2 + (d\theta)^2\). Thus, the velocity is \( v^2 = \dot{r}^2 + r^2 \dot{\phi}^2 \). As neither \( r \) nor \( v \) have a \( z \)-component, the \( x \)- and \( y \)-components of \( l = mr \times v \) vanish. The \( z \)-component is

\[
l_z = m(xv_z - yv_x)
\]

\[
= mr(\dot{r} \sin \varphi \cos \phi + r \dot{\phi} \cos^2 \phi - \dot{r} \cos \varphi \sin \phi + r \dot{\phi} \sin^2 \phi)
\]

\[
= mr^2 \dot{\phi}.
\]

Thus one finds

\[
v = \dot{r}^2 + \frac{l^2}{m^2 r^2} \quad \text{and} \quad T = \frac{1}{2} mr^2 + \frac{l^2}{2mr^2}.
\]

If \( l \) is constant this means that the product \( r^2 \dot{\phi} = \text{const.} \), thus correlating the angular velocity \( \dot{\phi} \) with the radial distance, cf. the examples (a), (b), (e), and (f), of Exercise 1.1. A motion of type (d) could only be possible if, on approaching \( O \), \( \phi \) were to go to infinity in such a way that the product \( r^2 \dot{\phi} \) stays finite. But then the shape of the orbit would be different, see Exercise 1.23.

\[1.3\] In analogy to the solution of the previous exercise one finds \((ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2\). Thus, \( v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \).

\[1.4\] Having solved Exercise 1.3 one first reads off \( \hat{e}_r \) from Fig. 2: \( \hat{e}_r = \hat{e}_x \sin \theta \cos \phi + \hat{e}_y \sin \theta \sin \phi + \hat{e}_z \cos \theta \). At the point with azimuth \( \varphi \), \( \hat{e}_\varphi \) is tangent to a great circle, see Fig. 3. Hence, \( \hat{e}_\varphi = -\hat{e}_x \sin \varphi + \hat{e}_y \cos \varphi \) (check the special cases \( \varphi = 0 \) and \( \pi/2 \)). One verifies that

\[
\hat{e}_r \cdot \hat{e}_\varphi = -\sin \theta \cos \phi \sin \varphi \hat{e}_x \cdot \hat{e}_x + \sin \theta \sin \varphi \cos \phi \hat{e}_y \cdot \hat{e}_y = 0 .
\]

Starting from the given ansatz for \( \hat{e}_\theta \) the coefficients \( \alpha, \beta, \gamma \) are determined from the equations

\[
\hat{e}_\theta \cdot \hat{e}_x = \alpha \sin \theta \cos \phi + \beta \sin \theta \sin \phi + \gamma \cos \theta = 0 ,
\]

\[
\hat{e}_\theta \cdot \hat{e}_y = -\alpha \sin \phi + \beta \cos \phi = 0 ,
\]

keeping in mind that \( \hat{e}_\theta \) has norm 1, i.e. that \( \alpha^2 + \beta^2 + \gamma^2 = 1 \). Furthermore, from Fig. 2 and for \( \theta = 0 \), \( \varphi = 0 \) one has \( \hat{e}_\theta = \hat{e}_x \), for \( \theta = 0 \), \( \varphi = \pi/2 \) one has \( \hat{e}_\theta = \hat{e}_y \), while for \( \theta = \pi/2 \) one has always \( \hat{e}_\theta = -\hat{e}_z \). The solution of the above equation which meets these conditions, reads

\[
\alpha = \cos \theta \cos \phi , \quad \beta = \cos \theta \sin \phi , \quad \gamma = -\sin \theta .
\]

In this basis we find
1.5  With respect to the frame $K$, $\mathbf{r}(t) = v t \hat{e}_y$, i.e., $x(t) = 0 = z(t)$ and $y(t) = v t$. In the rotating frame

\begin{align*}
\dot{x}' &= \dot{x} \cos \phi + \dot{y} \sin \phi + \dot{\phi}(-x \sin \phi + y \cos \phi) \\
\dot{y}' &= -\dot{x} \sin \phi + \dot{y} \cos \phi - \dot{\phi}(x \cos \phi + y \sin \phi) \\
\dot{z}' &= \dot{z} = 0.
\end{align*}

In the first case, $\phi = \omega = \text{const.}$, the particle moves uniformly along a straight line with velocity $v' = (v \sin \omega, v \cos \omega, 0)$. In the second case, $\phi = \omega t, \dot{x}' = v \sin \omega t + \omega v t \cos \omega t, \dot{y}' = v \cos \omega t - \omega v t \sin \omega t$. Integrating over time, $x'(t) = v t \sin \omega t, y'(t) = v t \cos \omega t$, and $z'(t) = 0$. The apparent motion as seen by an observer in the accelerated frame $K'$, is sketched in Fig. 4.

1.6  The equation of motion of the particle reads

$$m \ddot{\mathbf{r}} = \mathbf{F} = f(r) \frac{\mathbf{r}}{r}.$$ 

Take the time derivative of the angular momentum, $\dot{\mathbf{l}} = m \ddot{\mathbf{r}} \times \mathbf{r} + m \mathbf{r} \times \ddot{\mathbf{r}}$. The first term is always zero. The second term vanishes because, by the equation of motion, the acceleration is proportional to $r$. Hence, $\dot{\mathbf{l}} = 0$, which means that the magnitude and the direction of the angular momentum are conserved. As $\mathbf{l}$ is perpendicular to $r$ and the velocity $\dot{r}$ this proves the assertion.
1.7 (i) By Newton’s third law the forces between two bodies fulfill \( F_{ik} = -F_{ki} \) or \(-\nabla_i V_{ik}(r_i, r_k) = \nabla_k V_{ik}(r_i, r_k)\). Hence, \( V \) can only depend on \((r_i - r_k)\). Constants of the motion are: total momentum \( P \), energy \( E \); furthermore, we have for the center-of-mass motion

\[
r_S(t) - \frac{P}{Mt} = r_S(0) = \text{const.}
\]

(ii) When \( V_{ij} \) depends only on the modulus \(|r_i - r_k|\), we have

\[
F_{ji} = -\nabla_i V_{ij}(|r_i - r_k|) = -V'_{ij}(|r_i - r_k|) \nabla_i (r_i - r_k)
= -V'_{ij}(|r_i - r_k|) \frac{r_i - r_k}{|r_i - r_k|}.
\]

In this case the total angular momentum is another constant of the motion.

1.8 For \( q \to 0 \) the potential goes to infinity like \( 1/q^4 \), while for \( q \to \infty \) it tends to zero. Between these points it has two extrema as sketched in Fig. 5. As the energy \( E = p^2/2 + U(q) \) is conserved, the phase portraits are given by \( p = [2(E - U(q))]^{1/2} \). The figure shows a few examples. The minimum at \( q = 2 \) is a stable equilibrium point, the maximum just beyond \( q = 6 \) is unstable. The orbits with \( E \approx 0.2603 \) are separatrices. The phase portraits are symmetric with respect to reflection in the \( q \)-axis because \((q, p = +\sqrt{\ldots}) \) and \((q, p = -\sqrt{\ldots}) \) belong to the same portrait.

1.9 The term \((x_2^2 + x_4^2)/(2m)\) is the total kinetic energy while \( U(x_1, x_3) = m(\omega_0^2(x_1^2 + x_3^2) + \omega_1^2(x_1 - x_3)^2)/2 \) is the potential energy. The forces acting on pendula 1 and 2 are, respectively, \(-\partial U/\partial x_1\), and \(-\partial U/\partial x_3\). Thus, the equations of motion are

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix} =
\begin{pmatrix}
0 & 1/m & 0 & 0 \\
-m(\omega_0^2 + \omega_1^2) & 0 & m\omega_1^2 & 0 \\
0 & 0 & 0 & 1/m \\
m\omega_1^2 & 0 & -m(\omega_0^2 + \omega_1^2) & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix},
\]
or, for short, \( \dot{x} = Mx \). The transformation as given above

\[
\begin{align*}
  u_1 &= \frac{1}{\sqrt{2}} (x_1 + x_3), & u_2 &= \frac{1}{\sqrt{2}} (x_2 + x_4), \\
  u_3 &= \frac{1}{\sqrt{2}} (x_1 - x_3), & u_4 &= \frac{1}{\sqrt{2}} (x_2 - x_4)
\end{align*}
\]

leads to sums and differences of the original coordinates and momenta. We note that the matrix \( M \) has the structure

\[
M = \begin{pmatrix} B & C \\ C & B \end{pmatrix}
\]

where \( B \) and \( C \) are \( 2 \times 2 \) matrices. Furthermore the transformation \( A \) is invertible and, in fact, the inverse equals \( A \). Thus

\[
\frac{du}{dt} = AMA^{-1}u \quad \text{with} \quad A^{-1} = A.
\]

It is useful to note that one can do the calculations in terms of the \( 2 \times 2 \) submatrices as if these were (possibly noncommuting) numbers. For example,

\[
AM^{-1} = AMA = \begin{pmatrix} B + C \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ B - C \end{pmatrix}
\]

with

\[
B + C = \begin{pmatrix} 0 & 1/m \\ -m\omega_0^2 & 0 \end{pmatrix} \quad \text{and}
\]

\[
B - C = \begin{pmatrix} 0 & 1/m \\ -m(\omega_0^2 + 2\omega_1^2) & 0 \end{pmatrix}.
\]
This system now separates into two independent oscillators that can be solved in the usual manner. The first has frequency \( \omega^{(1)} = \omega_0 \) (the two pendula perform parallel, in-phase oscillations); the second has frequency \( \omega^{(2)} = (\omega_0^2 + 2\omega_1^2)^{1/2} \) (the pendula swing in antiphase). The general solution is

\[
    u_1 = a_1 \cos(\omega^{(1)} t + \varphi_1), \quad u_3 = a_2 \cos(\omega^{(2)} t + \varphi_2).
\]

As an example, consider the initial configuration

\[
    x_1(0) = a, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad x_4(0) = 0,
\]

which means that, initially, pendulum 1 is at maximal elongation with vanishing velocity while pendulum 2 is at rest. The initial configuration is realized by taking

\[
    a_2 = a_1 = a\sqrt{2}, \quad \varphi_1 = \varphi_2 = 0.\]

This gives

\[
    x_1(t) = a \cos \left( \frac{\omega^{(1)} + \omega^{(2)}}{2} t \right) \cos \left( \frac{\omega^{(2)} - \omega^{(1)}}{2} t \right) = a \cos \Omega t \cos \omega t,
\]

\[
    x_3(t) = a \sin \left( \frac{\omega^{(1)} + \omega^{(2)}}{2} t \right) \sin \left( \frac{\omega^{(2)} - \omega^{(1)}}{2} t \right) = a \sin \Omega t \sin \omega t,
\]

where \( \Omega := (\omega^{(1)} + \omega^{(2)})/2, \omega := (\omega^{(2)} - \omega^{(1)})/2. \) If \( \Omega/\omega = p/q \) with \( p, q \in \mathbb{Z} \) and \( p > q \), hence rational, the system returns to its initial configuration after time \( t = 2\pi p/\Omega = 2\pi q/\omega. \) For earlier times one has \( t = \pi p/(2\Omega): x_1 = 0, x_3 = a \) (pendulum 1 at rest, pendulum 2 has maximal elongation); \( t = \pi p/\Omega: x_1 = -a, x_3 = 0; t = 3\pi p/(2\Omega): x_1 = 0, x_3 = -a. \) The oscillation moves back and forth between pendulum 1 and pendulum 2. If \( \Omega/\omega \) is not rational, the system will come close, at a later time, to the initial configuration but will never assume it exactly (cf. Exercise 6.2). In the example considered here, this will happen if \( \Omega t \approx 2\pi n \) and \( \omega t \approx 2\pi m \) (with \( m, n \in \mathbb{Z}, \) i.e. if \( \Omega/\omega \) can be approximated by the ratio of two integers. It may happen that these integers are large so that the “return time” becomes very large.

1.10 As the differential equation is linear, the two terms are solutions precisely when \( \mu = \omega; \) \( a \) and \( b \) are integration constants which are fixed by the initial condition as follows

\[
    x(t) = a \cos \omega t + b \sin \omega t,
\]

\[
    p(t) = -am \omega \sin \omega t + mb \omega \cos \omega t.
\]

\( x(0) = x_0 \) gives \( a = x_0, \) \( p(0) = p_0 \) gives \( b = p_0/(m\omega). \) The solution with \( \omega = 0.8, x_0 = 1, \) \( p_0 = 0 \) reads \( x(t) = \cos(0.8t). \)

1.11 From the ansatz one has

\[
    \dot{x}(t) = \alpha x(t) + e^{at} (-\dot{\omega} x_0 \sin \dot{\omega} t + p_0/m \cos \dot{\omega} t)
\]

\[
    \ddot{x}(t) = \alpha^2 x(t) + 2\alpha e^{at} (-\dot{\omega} x_0 \sin \dot{\omega} t + p_0/m \cos \dot{\omega} t)
\]

\[
    -e^{at} \dot{\omega}^2 (x_0 \cos \dot{\omega} t + p_0/m \dot{\omega} \sin \dot{\omega} t)
\]

\[
    = -\alpha^2 x + 2\alpha \dot{x} - \dot{\omega}^2 x.
\]
Inserting and comparing coefficients one finds
\[ \alpha = -\frac{\kappa}{2}, \quad \tilde{\omega} = \sqrt{\omega^2 - \alpha^2} = \sqrt{\omega^2 - \kappa^2/4}. \]

The special solution \( x(t) = e^{-\kappa t/2} \cos(\sqrt{\omega^2 - \kappa^2/4} t) \), approaches the origin in a spiraling motion as \( t \to \infty \).

1.12 Energy conservation formulated for the two domains yields
\[ \frac{m}{2}v_1^2 + U_1 = E = \frac{m}{2}v_2^2 + U_2. \]

As the potential energy \( U \) depends on \( x \) only there can be no force perpendicular to the \( x \)-axis. Therefore, the component of the momentum along the direction perpendicular to that axis cannot change in going from \( x < 0 \) to \( x > 0 \): \( v_{1\perp} = v_{2\perp} \).

The law of conservation of energy hence reads
\[ \frac{m}{2}v_{1\perp}^2 + U_1 = \frac{m}{2}v_{2\perp}^2 + U_2, \quad \text{or} \quad \frac{m}{2}v_{1\parallel}^2 + U_1 = \frac{m}{2}v_{2\parallel}^2 + U_2. \]

from which follows
\[ \sin^2 \alpha_1 = \frac{v_{1\perp}^2}{v_1^2}, \quad \sin^2 \alpha_2 = \frac{v_{2\perp}^2}{v_2^2}, \quad \text{directly yielding} \]
\[ \frac{\sin \alpha_1}{\sin \alpha_2} = \frac{|v_2|}{|v_1|}. \]

For \( U_1 < U_2 \) we find \( |v_1| > |v_2| \), hence \( \alpha_1 < \alpha_2 \). For \( U_1 < U_2 \) all inequalities are reversed.

1.13 Let \( M = m_1 + m_2 + m_3 \) be the total mass and \( m_{12} = m_1 + m_2 \). From the figure one sees that \( r_2 + s_a = r_1, s_{12} + s_b = r_3 \), where \( s_{12} \) is the center-of-mass coordinate of particles 1 and 2. Solving for \( r_1, r_2, r_3 \) we find
\[ r_1 = r_S + \frac{m_3}{M}s_b, \quad r_2 = r_S - \frac{m_3}{M}s_b - \frac{m_1}{m_{12}}s_a, \quad r_3 = r_S + \frac{m_{12}}{M}s_b. \]

Inserting these into the kinetic energy all mixed terms cancel. The result contains only terms quadratic in \( \dot{r}_S, \dot{s}_a, \dot{s}_b \)
\[ T = \frac{1}{2}M\dot{r}_S^2 + \frac{1}{2}\mu_a\dot{s}_a^2 + \frac{1}{2}\mu_b\dot{s}_b^2 \quad \text{with} \quad \mu_a = \frac{m_1 m_2}{m_{12}}, \quad \mu_b = \frac{m_{12} m_3}{M}. \]
Let $S$ be the kinetic energy of the center-of-mass motion, $\mu_a$ is the reduced mass of the subsystem consisting of particles 1 and 2. $\mu_b$ is the reduced mass of the subsystem consisting of particle 3 and the center-of-mass $S_{12}$ of particles 1 and 2, $T_b$ is the kinetic energy of the relative motion of particle 3 and $S_{12}$.

In an analogous way, the angular momentum is found to be

$$L = \sum_i l_i = M r_S \times \dot{r}_S + \frac{\mu_a s_a \times \dot{s}_a}{l_a} + \frac{\mu_b s_b \times \dot{s}_b}{l_b},$$

all mixed terms having cancelled.

By a special (and proper) Galilei transformation, $r_S \to r'_S = r_S + wt + a$, $\dot{r}_S \to \dot{r}'_S = \dot{r}_S + w$, $s_a \to s_a$, $s_b \to s_b$ and, hence,

$$l'_S = l_S + M(a \times (\dot{r}_S + w) + (r_S - t\dot{r}_S) \times w),$$

while $l'_a = l_a$, $l'_b = l_b$ remain unchanged.

1.14 (i) With $U(\lambda r) = \lambda^\alpha U(r)$ and $r' = \lambda r$ the forces from $\tilde{U}(r') := U(\lambda r)$ and from $U(r)$, respectively, differ by the factor $\lambda^{\alpha-1}$. Indeed

$$F' = -\nabla_{r'} \tilde{U} = -\frac{1}{\lambda} \nabla_r \tilde{U} = -\lambda^{\alpha-1} \nabla_r U = \lambda^{\alpha-1} F.$$

Integrating $F' \cdot dr'$ over a path in $r'$ space and comparing with the corresponding integral over $F \cdot dr$, the work done in the two cases differs by the factor $\lambda^\alpha$. Changing $t$ to $t' = \lambda^{1-\alpha/2} t$,

$$\left( \frac{dr'}{dt'} \right)^2 = \lambda^{2} \lambda^{\alpha-2} \left( \frac{dr}{dt} \right)^2,$$

which means that the kinetic energy

$$T = \frac{1}{2} m \left( \frac{dr'}{dt'} \right)^2.$$
differs from the original one by the same factor \( \lambda^{\alpha} \). Thus, this holds for the total energy, too, \( E' = \lambda^{\alpha} E \). The indicated relation between time differences and linear dimensions of geometrically similar orbits follows.

(ii) For harmonic oscillation the assumption holds with \( \alpha = 2 \). The ratio of the periods of two geometrically similar orbits is \( T_a / T_b = 1 \), independently of the linear dimensions.

In the homogeneous gravitational field \( U(z) = mgz \) and, hence, \( \alpha = 1 \). Times of free fall and initial height \( H \) are related by \( T \propto H^{1/2} \).

In the case of the Kepler problem \( U = -A/r \) and, hence, \( \alpha = -1 \). Two geometrically similar ellipses with semimajor axes \( a_a \) and \( b_b \) have circumference \( U_a \) and \( U_b \), respectively, such that \( U_a / U_b = a_a / a_b \). Therefore the ratio of the periods \( T_a \) and \( T_b \) is \( T_a / T_b = (U_a / U_b)^{3/2} \) from which follows \( (T_a / T_b)^2 = (a_a / a_b)^3 \), Kepler’s third law.

(iii) The general relation is \( E_a / E_b = (L_a / L_b)^{\alpha} \). If \( A_i \) denotes the amplitude of harmonic oscillation, \( E_a / E_b = A_a^2 / A_b^2 \). In the case of Kepler motion \( E_a / E_b = a_b / a_a \): the energy is inversely proportional to the semimajor axis.

1.15 (i) From the equations of Sect. M1.24

\[
\begin{align*}
  r_P &= \frac{p}{1 + \epsilon} = -\frac{A}{2E} \frac{1 - \epsilon^2}{1 + \epsilon} = -\frac{A}{2E} (1 - \epsilon) ; \\
  r_A &= -\frac{A}{2E} (1 + \epsilon) .
\end{align*}
\]

From these we calculate

\[
\begin{align*}
  r_P + r_A &= -\frac{A}{E} , \\
  r_P \cdot r_A &= \frac{A^2}{4E^2} (1 - \epsilon^2) = \frac{l^2}{-2\mu E} .
\end{align*}
\]

Inserting this into the differential equation we obtain

\[
\frac{d\phi}{dr} = \frac{l}{r^2 \sqrt{2\mu \left( E + \frac{A}{r} - \frac{l^2}{2\mu r^2} \right)}} .
\]

This is precisely eq. (M1.67) with \( U_{\text{eff}} = -A/r + l^2 / 2\mu r^2 \). Integration of eq. (1.4) with the boundary condition as indicated implies

\[
\phi(r) - \phi(r_P) = \int_{r_P}^{r} \frac{1}{r} \left( \frac{r_P r_A}{(r - r_P)(r_A - r)} \right)^{1/2} dr .
\]

We make use of the indicated formula with

\[
\alpha = 2 \frac{r_A r_P}{r_A - r_P} , \quad \beta = -\frac{r_A + r_P}{r_A - r_P} ,
\]

and obtain
\[ \phi(r) = \arccos \frac{2r_A r_P - (r_A + r_P)r}{(r_A - r_P)r}. \]

(ii) There are two possibilities for solving this equation: (a) the new equations are obtained by replacing \( l^2 \) with \( \bar{l}^2 = l^2 + 2\mu B. \) For the remainder, the solution is exactly the same as for the Kepler problem. If \( B > 0 (B < 0), \) then \( \bar{l} > l (\bar{l} < l), \) i.e., in the case of repulsion (attraction) the orbit becomes larger (smaller). (b) With \( U(r) = U_0(r) + B/r^2, \) \( U_0(r) = -A/r, \) the differential equation for \( \phi(r) \) is written in the same form as above

\[ \frac{d\phi}{dr} = \sqrt{r_A r_P} \frac{\sqrt{r'_{A'} r'_{P'}} - r (r'_{A'} + r'_{P'})}{r (r'_{A'} - r'_{P'})}. \]

where \( r'_{P'}, r'_{A'} \) denote perihelion and aphelion, respectively, for the perturbed potential. They are obtained from the formula

\[ (r'_{P'} - r_{P}) (r'_{A} - r) + B/E = (r'_{P'} - r_{A}) (r'_{A'} - r). \]

Multiplying the differential equation by \( ((r'_{P'} r'_{A}) / (r_{P'} r_{A}))^{1/2} \) and integrating as before

\[ \phi(r) = \frac{r_{P} r_{A}}{r'_{P} r'_{A}} \arccos \frac{2r_A r_P - r (r'_{A} + r'_{P})}{r (r'_{A} - r'_{P})}. \]

From this solution follows \( r(\phi) = 2r_{P'} r_{A}' / [r_{P'} + r_{A}' + (r'_{A} - r'_{P}) \cos \sqrt{r'_{P} r'_{A}} \phi]. \)

The first passage through perihelion is set to \( \phi_{P1} = 0. \) The second is \( \phi_{P2} = 2\pi ((r_{P} r_{A}) / (r_{P'} r'_{A}))^{1/2} = 2\pi l / \sqrt{l^2 + 2\mu B} \approx 2\pi (1 - \mu B / l^2). \) The perihelion precession is \( (\phi_{P2} - 2\pi). \) It is independent of the energy \( E. \) For \( B > 0 \) (additional repulsion) the motion lags behind, and for \( B < 0 \) (additional attraction) the motion advances as compared to the Kepler case.

1.16 For fixed \( l, \) the energy must fulfill \( E \geq -\mu A^2 / (2l^2). \) The lower limit is assumed for circular orbits with radius \( r_0 = l^2 / \mu A. \) The semimajor axis (in relative motion) follows from Kepler’s third law \( a^3 = G_N (m_E + m_S) T^2 / (4\pi^2). \) This gives \( a = 1.495 \times 10^{11} \text{ m (} T = 1 \text{ y = 3.1536} \times 10^7 \text{ s). This is approximately equal to} a_E, \) the semimajor axis of the earth in the center-of-mass system. The sun moves on an ellipse with semimajor axis

\[ a_S = \frac{m_E}{m_E + m_S} a \approx 449 \text{ km}. \]

This is far within the sun’s radius \( R_S \approx 7 \times 10^5 \text{ km}. \)

1.17 We arrange the two dipoles as sketched in Fig. 7. The potential created by the first dipole at a point situated at \( r \) is

\[ \Phi_1 = e_1 \left( \frac{1}{|r - d_1|} - \frac{1}{|r|} \right) \approx e_1 \left( \frac{1}{r} + \frac{r \cdot d_1}{r^3} - \frac{1}{r} \right) = \frac{r \cdot (e_1 d_1)}{r^3}. \]
Here, we have expanded
\[ \frac{1}{|r - d_1|} = \frac{1}{\sqrt{r^2 + d_1^2 - 2r \cdot d_1}} \]
up to the term linear in \( d_1 \). In the limit we obtain \( \Phi_1 = r \cdot p_1 / r^3 \). The potential energy of the second dipole in the field of the first reads
\[
W = e_1(\Phi_1(r + d_2) - \Phi_1(r)) = e_2 \left( \frac{p_1 \cdot (r + d_2)}{|r + d_2|^3} - \frac{p_1 \cdot r}{r^3} \right).
\]
Expanding again up to terms linear in \( d_2 \)
\[
W \approx e_2 \left( \frac{p_1 \cdot r}{r^3} \left( 1 - 3 \frac{r \cdot d_2}{r^2} \right) + \frac{p_1 \cdot d_2}{r^3} - \frac{p_1 \cdot r}{r^3} \right).
\]
Finally, taking the limit \( e_2 \to \infty, d_2 \to 0 \), with \( e_2 d_2 = p_2 \) finite, this yields
\[
W(1, 2) = \frac{p_1 \cdot p_2}{r^3} - 3 \frac{(p_1 \cdot r)(p_2 \cdot r)}{r^5}.
\]
From this expression one calculates the components of \( F_{21} = -\nabla_1 W = -F_{12} \), making use of relations such as
\[
\frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1} \frac{\partial}{\partial r} = \frac{x_1 - x_2}{r} \frac{\partial}{\partial r}, \text{ etc.}
\]
So, for example
\[
\frac{\partial W(1, 2)}{\partial x_1} = -(p_1 \cdot p_2) \frac{3}{r^4} \frac{x_1 - x_2}{r} - \frac{3}{r^5} \left( p_1^* (p_2 \cdot r) \right)
\]
\[
+ (p_1 \cdot r) p_2^* + (p_1 \cdot r)(p_2 \cdot r) \frac{15}{r^6} \frac{x_1 - x_2}{r}.
\]
1.18 Take the time derivative of \( \dot{r} \cdot a \),
\[
\frac{d}{dt} \dot{r} \cdot a = \ddot{r} \cdot a = \dot{v} \cdot a = (v \times a) \cdot a = 0.
\]
Thus, \( \dot{r} \cdot a \) is constant in time and the indicated relation holds for all times. Taking the time derivative of (5) and inserting \( \dot{v} \), we find \( \ddot{v} = \dot{v} \times a = (v \times a) \times a = \)

\(-a^2 v + (v \cdot a) a\). The second term is constant as shown above. Thus, integrating this equation over \(t\) from 0 to \(t\), we have \(\ddot{r}(t) - \ddot{r}(0) = -\omega^2 (r(t) - r(0)) + (v(0) \cdot a) at\), where \(\omega^2 := a^2\). By eq. (5) \(\ddot{r}(0) = v(0) \times a\), so that we may write

\[
\ddot{r}(t) + \omega^2 r(t) = (v(0) \cdot a) at + v(0) \times a + \omega^2 r(0)
\]

This is the desired form, the general solution of the homogeneous differential equations is

\[
r_{\text{hom}}(t) = c_1 \sin \omega t + c_2 \cos \omega t
\]

With the given ansatz for a special solution of the inhomogeneous equation the constants are found to be

\[
c_1 = \frac{1}{\omega^2} \left( a^2 v(0) - (v(0) \cdot a) a \right) = \frac{1}{\omega^2} (a \times (v(0) \times a))
\]
\[
c_2 = -\frac{1}{\omega^2} v(0) \times a
\]
\[
c = \frac{1}{\omega^2} (v(0) \cdot a) a
\]
\[
d = \frac{1}{\omega^2} v(0) \times a + r(0)
\]

The solution therefore reads

\[
r(t) = \frac{1}{\omega^2} a \times (v(0) \times a) \sin \omega t + \frac{1}{\omega^2} (v(0) \cdot a) at
\]
\[
+ \frac{1}{\omega^2} v(0) \times a (1 - \cos \omega t) + r(0)
\]

It represents a helix winding around the vector \(a\).

1.19 The ball falls from initial height \(h_0\). It hits the plane for the first time at \(t_1 = \sqrt{2h_0/g}\), the velocity then being \(u_1 = -\sqrt{2h_0g} = -gt_1\). Furthermore, with \(\alpha := \sqrt{(n - 1)/n}\)

\[
v_i = -\alpha u_i , \quad u_{i+1} = -v_i , \quad t_{i+1} - t_i = \frac{2v_i}{g} .
\]

The first two equations give \(v_1 = \alpha g t_1\) and \(v_i = \alpha^i g t_1\). The third equation yields

\[
t_i^0 - t_i = \frac{v_i}{g} = t_{i+1} - t_i^0 \quad \text{and} \quad t_{i+1}^0 - t_{i+1} = \frac{v_{i+1}}{g} ,
\]

and, from there, \(t_{i+1}^0 - t_i^0 = (v_{i+1} + v_i)/g = t_1(\alpha + 1)\alpha^i\). With \(t_0^0 = 0\) we have at once

\[
t_i^0 = t_1(1 + \alpha) \sum_{v=0}^{i-1} \alpha^v .
\]

From \(h_i = v_i^2/(2g)\), finally, \(h_i = \alpha^{2i} h_0\).
1.20 The answer is contained in the following table giving the products of the elements

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1.21 Let \( R \) and \( E_0 \) denote the radius and the energy of a circular orbit, respectively. The differential equation for the radial motion reads

\[
\frac{dr}{dt} = \sqrt{\frac{2}{\mu} \sqrt{E_0 - U_{\text{eff}}(r)}} \quad U_{\text{eff}}(r) = U(r) + \frac{l^2}{2\mu r}.
\]

From this follows \( E_0 = U_{\text{eff}}(R) \), \( U'_{\text{eff}}|_{r=R} = 0 \), \( U''_{\text{eff}}|_{r=R} > 0 \) or, for \( U(r) \),

\[
U'(R) = \frac{l^2}{\mu R^3} \quad \text{and} \quad U''(R) > -\frac{3l^2}{\mu R^4}.
\]

If \( E = E_0 + \epsilon \),

\[
\frac{dr}{dt} = \sqrt{\frac{2}{\mu} \sqrt{\epsilon - \frac{1}{2} (r-R)^2 U''_{\text{eff}}(R)}}.
\]

Setting \( \kappa := U''_{\text{eff}}(R) \) we obtain, choosing \( \zeta = r' - R \),

\[
t - t_0 = \sqrt{\frac{\mu}{\kappa}} \int_{r_0-R}^{r-R} \frac{d\zeta}{\sqrt{2\epsilon/\kappa - \zeta^2}} = \sqrt{\frac{\mu}{\kappa}} \arcsin \left( \frac{(r-R)\sqrt{\kappa}}{2\epsilon} \right) .
\]

Solving for \( r - R \) yields

\[
r - R = \sqrt{\frac{2\epsilon}{\kappa}} \sin \sqrt{\frac{\kappa}{\mu}} (t - t_0).
\]

Thus, the radial distance oscillates around the value \( R \). More specifically, one finds

(i) \( U(r) = r^n \), \( U'(r) = nr^{n-1} \), \( U''(r) = n(n-1)r^{n-2} \). This yields the equation

\[
nR^{n-1} = \frac{l^2}{\mu R^3} \Rightarrow R = \left( \frac{l^2}{\mu n} \right)^{\frac{1}{n-1}},
\]

\[
\kappa = n(n-1)R^{n-2} + \frac{3l^2}{\mu R^3} > 0 \Leftrightarrow \frac{n(n-1)R^{n+2}}{l^2/(\mu n)} + \frac{3l^2}{\mu} = \frac{(n + 2)l^2}{\mu} > 0.
\]

(ii) \( U(r) = \lambda/r \), \( U'(r) = -\lambda/r^2 \), \( U''(r) = 2\lambda/r^3 \). From this \( R = -l^2/(\mu\lambda) \),

\( \kappa = -\lambda/R^3 \). This is greater than zero of \( \lambda \) is negative.
1.22 (i) The eastward deviation follows from the formula given in Sect. M1.26, \[ \Delta \approx \left( \frac{2\sqrt{2}}{3} \right) g^{-1/2} H^{3/2} \omega \cos \varphi \]. With \( \omega = 2\pi/(1 \text{ day}) = 7.27 \times 10^{-5} \text{ s}^{-1} \) and \( g = 9.81 \text{ ms}^{-2} \) one finds \( \Delta \approx 2.2 \text{ cm} \).

(ii) We proceed as in Sect. M1.26 (b) and determine the eastward deviation \( u \) from the linearized ansatz \( r(t) = r(0)(t) + \omega u(t) \), inserting here the unperturbed solution, \( r(0)(t) = gt(T - \frac{1}{2} t) \hat{e}_v \). This gives \( (d^2/dt^2)u(t) \approx 2g \cos \varphi(T - t) \hat{e}_v \). Integrating twice,

\[ u(t) = \frac{1}{3} g \cos \varphi(t^3 - 3Tt^2) \hat{e}_0 . \]

The stone returns to the surface of the earth after time \( t = 2T \). The eastward deviation is found to be negative, \( \Delta \approx -\frac{4}{3} g \omega \cos \varphi T^3 \), which means, in reality, that it is a westward deviation. Its magnitude is four times larger than in case (i).

(iii) Denote the eastward deviation by \( u \) as before (directed from west to east), the southward deviation by \( s \) (directed from north to south). A local, earth-bound, coordinate system is given by \( (\hat{e}_1, \hat{e}_0, \hat{e}_v) \), \( \hat{e}_1 \) defining the direction N–S, \( \hat{e}_0 \) and \( \hat{e}_v \) being defined as in Sect. M1.26 (b). Thus, \( u = u \hat{e}_0, s = s \hat{e}_1 \). The equation of motion (M1.74'), together with \( \omega = \omega(-\cos \varphi, 0, \sin \varphi) \), implies

\[ \ddot{s} = 2\omega^2 \sin \varphi \dot{u} . \]

Inserting the approximate solution \( u \approx \frac{1}{3} gt^3 \cos \varphi \) and integrating over time twice, one obtains

\[ s(t) = \frac{1}{6} \omega^2 g \sin \varphi \cos \varphi t^4 . \]

1.23 For \( E > 0 \) all orbits are scattering orbits. If \( l^2 > 2\mu \alpha \),

\[ \phi - \phi_0 = \frac{l}{\sqrt{2\mu E}} \int_{r_0}^{r} \frac{dr'}{\sqrt{r'^2 - (l^2 - 2\mu \alpha)/(2\mu E)}} \]

\[ = r_p^{(0)} \int_{r_0}^{r} \frac{dr'}{\sqrt{r'^2 - r_p^{(0)}^2}} , \tag{1} \]

where \( \mu \) is the reduced mass, \( r_p = \sqrt{(l^2 - 2\mu \alpha)/(2\mu E)} \) the perihelion and \( r_p^{(0)} = l/\sqrt{2\mu E} \). The particle is assumed to come from infinity, traveling parallel to the \( x \)-axis. Then the solution is \( \phi(r) = l/\sqrt{l^2 - 2\mu \alpha} \arcsin(r_p/r) \). If \( \alpha = 0 \), the corresponding solution is \( \phi^{(0)}(r) = \arcsin(r_p^{(0)}/r) \); the particle moves along a straight line parallel to the \( x \)-axis, at the distance \( r_p^{(0)} \). For \( \alpha \neq 0 \)

\[ \phi(r = r_p) = \frac{l}{\sqrt{l^2 - 2\mu \alpha}} \frac{\pi}{2} \]

that is, after the scattering and asymptotically, the particle moves in the direction \( l/\sqrt{l^2 - 2\mu \alpha} \pi \). Before that it travels around the center of force \( n \) times if the condition...
\[
\frac{l}{\sqrt{l^2 - 2\mu \alpha}} \left( \arcsin \frac{r_P}{\infty} - \arcsin \frac{r_P}{r_P} \right) = \frac{r_P^{(0)}}{r_P} \left( \pi - \frac{\pi}{2} \right) > n\pi
\]
is fulfilled. The number
\[
n = \left[ \frac{r_P^{(0)}}{2r_P} \right]
\]
is independent of energy \( E \).

In the case \( l^2 < 2\mu \alpha \) eq. (1) can also be integrated. With the same initial condition one obtains
\[
\phi(r) = \frac{r_P^{(0)}}{b} \ln \frac{b + \sqrt{b^2 + r^2}}{r},
\]
where \( b = ((2\mu \alpha - l^2)/(2\mu E))^{1/2} \). The particle travels around the force center on a spiral-like orbit, towards the center. As the radial distance tends to zero, the angular velocity increases in such a way as to respect Kepler’s second law (M1.22).

1.24 Let the comet and the sun approach each other with energy \( E \). Long before the collision the relative momentum has the magnitude \( q = \sqrt{2\mu E} \), with \( \mu \) the reduced mass, the angular momentum has the magnitude \( l = qb \). The comet crashes when the perihelion \( r_P \) of its hyperbola is smaller or equal \( R \), i.e., when \( b \leq b_{\text{max}} \) with \( b_{\text{max}} \) following from the condition \( r_P = R \), viz.
\[
\frac{p}{1 + \varepsilon} = R \quad \text{with} \quad \frac{l^2}{A\mu} = \frac{q^2 b^2}{A\mu}, \quad \varepsilon = \sqrt{1 + \frac{2E q^2 b^2}{\mu A^2}}
\]
and \( A = GmM \). One finds \( b_{\text{max}} = \sqrt{1 + A/(ER)} \) and, hence,
\[
\sigma = \int_0^{b_{\text{max}}} 2\pi b \, db = \pi R^2 \left( 1 + \frac{A}{ER} \right).
\]
For \( A = 0 \) this is the area of the sun seen by the comet. With increasing gravitational attraction \( (A > 0) \) this surface increases by the ratio (potential energy at the sun’s edge)/(energy of relative motion).

1.25 As explained in Sect. M1.21.2 the equation of motion reads
\[
\dot{x} = Ax + b,
\]
with \( A \) as given in eq. (M1.50), and
\[
A = \begin{bmatrix} 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m \\ 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad b = e \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_x \\ E_y \\ E_z \end{bmatrix}.
\]
The last of the six equations is integrated immediately, giving $x_6 = eEt + C_1$. Inserting this into the third and integrating yields

$$x_3 = z = \frac{eE_z}{2m} t^2 + C_1 t + C_2.$$ 

The initial conditions $z(0) = z^{(0)}$, $\dot{z}(0) = v_z^{(0)}$ give $C_2 = z^{(0)}$, $C_1 = v_z^{(0)}$. The remaining equations are coupled equations. Taking the time derivative of the fourth and replacing $\dot{x}_5$ by the right-hand side of the fifth gives

$$\ddot{x}_4 = -K^2 x_4 + eKE_y,$$ 

which is integrated to $x_4 = C_3 \sin Kt + C_4 \cos Kt + eE_y/K$. Making use of the fifth equation once more yields $x_3 = C_3 \cos Kt - C_4 \sin Kt + C_5$. Also the fourth equation yields the condition $C_5 = -eE_x/K$. These two expressions are inserted into the first and second equations so that these can be integrated yielding

$$x_1 = -\frac{C_3}{Km} \cos Kt + \frac{C_4}{Km} \sin Kt + \frac{e}{Km} E_y t + C_6,$$

$$x_2 = +\frac{C_3}{Km} \sin Kt + \frac{C_4}{Km} \cos Kt - \frac{e}{Km} E_x t + C_7.$$ 

Upon insertion of the initial conditions $x(0) = x^{(0)}$, $y(0) = y^{(0)}$, $\dot{x}(0) = v_x^{(0)}$, $\dot{y}(0) = v_y^{(0)}$ we finally obtain

$$C_3 = mv_y^{(0)} + \frac{e}{K} E_x,$$ 

$$C_4 = mv_x^{(0)} - \frac{e}{K} E_y,$$ 

$$C_6 = x^{(0)} + \frac{v_y^{(0)}}{K} + \frac{e}{mK^2} E_x,$$ 

$$C_7 = y^{(0)} - \frac{v_x^{(0)}}{K} + \frac{e}{mK^2} E_y.$$ 

If the electric field points along the $z$-direction, $E = E\hat{e}_z$, then the motion is the superposition of a uniformly accelerated motion along the $z$-direction and a circular motion in the $(x, y)$-plane. That is to say the particle runs along a spiral.

**1.26** Using Cartesian coordinates in the plane of the motion and allowing for an arbitrary initial position of the perihelion, the solution eq. (M1.21) reads

$$x(t) = \frac{p}{1+\varepsilon \cos(\phi - \phi_0)} \cos(\phi - \phi_0),$$

$$y(t) = \frac{p}{1+\varepsilon \cos(\phi - \phi_0)} \sin(\phi - \phi_0).$$ 

Differentiate these formulae with respect to the time variable and replace the derivative $\dot{\phi}$ by $\ell/(\mu r^2)$, by means of eq. (M1.19a). Inserting $p = \ell^2/(A\mu)$ one obtains

$$p_x(t) = \mu \dot{x} = -\frac{A\mu}{\ell} \sin(\phi - \phi_0),$$

$$p_y(t) = \mu \dot{y} = \frac{A\mu}{\ell} \{\cos(\phi - \phi_0) + \varepsilon\}.$$ 

This equation describes a circle with radius $A\mu/\ell$ whose center has the coordinates

$$\left(0, \varepsilon \frac{A\mu}{\ell}\right) = \left(0, \sqrt{(A\mu/\ell)^2 + 2\mu E}\right).$$ 

See also Exercise 2.31 below.
Chapter 2: The Principles of Canonical Mechanics

2.1 We take the derivative of $F(E)$ with respect to $E$

$$
\frac{dF}{dE} = 2 \frac{d}{dE} \int_{q_{\text{min}}(E)}^{q_{\text{max}}(E)} \sqrt{2m(E-U(q))} \, dq = 2 \int_{q_{\text{min}}(E)}^{q_{\text{max}}(E)} \frac{m}{\sqrt{2m(E-U(q))}} \, dq 
$$

$$
+ 2 \sqrt{2m(E-U(q_{\text{max}}))} \bigg|_{q_{\text{max}}(E)}^{q_{\text{min}}(E)}
$$

To find $T$ we must calculate the time integral over one period. In doing so we note that

$$
m \frac{dq}{dt} = p = \sqrt{2m(E-U(q))}, \quad \text{and hence,}
$$

$$
dt = \frac{mdq}{\sqrt{2m(E-U(q))}}.
$$

Therefore,

$$
T = 2 \int_{q_{\text{min}}(E)}^{q_{\text{max}}(E)} \frac{m}{\sqrt{2m(E-U(q))}} \, dq.
$$

This, however, is precisely the expression calculated above. For the example of the oscillator with $q = q_0 \sin \omega t$, $p = m \omega q_0 \cos \omega t$, one finds $F = m \omega \pi q_0^2 = (2\pi/\omega)E$ and $T = 2\pi/\omega$.

2.2 Choose the plane as sketched in Fig. 8. D’Alembert’s principle $(F - \dot{p}) \cdot \delta r = 0$, with $F = -mg \hat{e}_3$, admits virtual displacements along the line of intersection of the inclined plane and the (1,3)-plane as well as along the 2-axis. Denoting the two independent variables by $q_1, q_2$, this means that $\delta r = \delta q_1 \hat{e}_\alpha + \delta q_2 \hat{e}_2$ with

---

Fig. 8.

Fig. 9.
\[ \hat{e}_\alpha = \hat{e}_1 \cos \alpha - \hat{e}_3 \sin \alpha. \] Inserting this yields the equations of motion \( \ddot{q}_1 = g \sin \alpha, \) \( \ddot{q}_2 = 0 \) whose solutions read

\[ q_1(t) = \frac{(g \sin \alpha) t^2}{2} + v_1 t + a_1, \quad q_2(t) = v_2 t + a_2. \]

2.3 Choose the (1,3)-plane to coincide with the plane of the annulus and take its center to be the origin. Choosing the unit vectors \( \hat{t} \) and \( \hat{n} \) as shown in Fig. 9, viz.

\[ \hat{t} = \hat{e}_1 \cos \phi + \hat{e}_3 \sin \phi, \quad \hat{n} = \hat{e}_1 \sin \phi - \hat{e}_3 \cos \phi \]

we find

\[ \delta r = \hat{t} R \delta \phi, \quad \dot{r} = R \dot{\phi} \hat{t}, \quad \ddot{r} = R \ddot{\phi} \hat{t} - R \dot{\phi}^2 \hat{n}, \]

the force acting on the system being \( F = -mg \hat{e}_3. \)

D’Alembert’s principle \( (F - p) \cdot \delta r = 0 \) yields the equation of motion \( \ddot{\phi} + g \sin \phi/R = 0. \) This is the equation of motion of the planar pendulum that was studied in Sect. M1.17.

2.4 Let \( d_0 \) be the length of the spring in its rest state and let \( \kappa \) be the string constant. When the mass point is at the position \( x \) the length of the string is \( d = \sqrt{x^2 + l^2}. \) The corresponding potential energy is

\[ U(x) = \frac{1}{2} \kappa (d - d_0)^2. \]

For \( d_0 \leq l \) the only stable equilibrium position is \( x = 0. \) For \( d_0 > l, \) \( x = 0 \) is unstable, while the points \( x = \pm \sqrt{d_0^2 - l^2} \) are stable equilibrium positions.

As an example we study here the case \( d_0 \leq l. \) Expanding \( U(x) \) around \( x = 0, \)

\[ U(x) \approx \frac{1}{2} \kappa \left( l - d_0 + \frac{x^2}{2l} - \frac{x^4}{8l^3} \right) \approx \frac{1}{2} \kappa \left( (l - d_0)^2 + \frac{l - d_0}{l} x^2 + \frac{d_0}{4l^3} x^4 \right). \]

From this expression we would conclude that the frequency of oscillation is approximately
\[ \omega = \sqrt{\frac{\kappa}{m}} \frac{l - d_0}{l}. \]

However, this does not hold for all values of \( d_0 \). For \( d_0 = l \) the quadratic term vanishes, and \( x^4 \) is the leading order. In the other extreme, \( d_0 = 0 \), we have \( U(x) = \kappa(x^2 + l^2)/2 \), i.e. a purely harmonic potential (the constant terms in the potential are irrelevant). Thus, the approximation is acceptable only when \( d_0 \) is small compared to \( l \).

### 2.5 A suitable Lagrangian function for this system reads

\[
L = \frac{1}{2} m \left( \dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{1}{2} \kappa (x_1 - x_2)^2 .
\]

Introduce the following coordinates: \( u_1 := x_1 + x_2 \), \( u_2 := x_1 - x_2 \). Except for a factor 1/2 these are the center-of-mass and relative coordinates, respectively. The Lagrangian becomes \( L = m(u_1^2 + u_2^2)/4 - \kappa u_2^2/2 \). The equations of motion that follow from it are

\[
\ddot{u}_1 = 0 , \quad m \ddot{u}_2 + 2\kappa u_2 = 0 .
\]

The solutions are \( u_1(t) = C_1 t + C_2 \), \( u_2(t) = C_3 \sin \omega t + C_4 \cos \omega t \), with \( \omega = \sqrt{2\kappa/m} \).

It is not difficult to rewrite the initial conditions in the new coordinates, viz.

\[
\begin{align*}
 u_1(0) &= +l , & \dot{u}_1(0) &= v_0 \\
 u_2(0) &= -l , & \dot{u}_2(0) &= v_0 .
\end{align*}
\]

The constants are determined from these so that the final solution is

\[
\begin{align*}
 x_1(t) &= \frac{v_0}{2} \left( t + \frac{1}{\omega} \sin \omega t \right) - \frac{l}{2} (1 - \cos \omega t) \\
 x_2(t) &= \frac{v_0}{2} \left( t - \frac{1}{\omega} \sin \omega t \right) + \frac{l}{2} (1 + \cos \omega t).
\end{align*}
\]

### 2.6 By hypothesis \( F(\lambda x_1, \ldots, \lambda x_n) = \lambda^N F(x_1, \ldots, x_N) \). We take the first derivative of this equation with respect to \( \lambda \) and set \( \lambda = 1 \). The left-hand side is

\[
\frac{d}{d\lambda} \left. F(\lambda x_1, \ldots, \lambda x_n) \right|_{\lambda=1} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{d(\lambda x_i)}{d\lambda} \bigg|_{\lambda=1} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} x_i .
\]

The same operation on the right-hand side gives \( NF \).

### 2.7 In the general case the Euler-Lagrange equation reads

\[
\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y'} .
\]
Multiply this equation by \( y' \) and add the term \( y'' \partial f / \partial y' \) on both sides. The right-hand side is combined to

\[
y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} = \frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right).
\]

If \( f \) does not depend explicitly on \( x \) then the left-hand side is \( df(y, y')/dx \). The whole equation can be integrated directly and yields the desired relation. Applying this result to \( L(\dot{q}, q) = T(\dot{q}) - U(q) \) gives

\[
\sum_i \dot{q}_i \frac{\partial T(\dot{q})}{\partial \dot{q}_i} - T + U = \text{const}.
\]

If \( T \) is a homogeneous, quadratic form in \( \dot{q} \) the solution to Exercise 2.6 tells us that the first term equals \( 2T \). Therefore, the constant is the energy \( E = T + U \).

2.8  (i) We must minimize the arc length

\[
L = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx
\]
i.e. we must choose \( f(y, y') = \sqrt{1 + y'^2}. \) Applying the result of the preceding exercise we obtain

\[
y' \frac{\sqrt{y}}{\sqrt{1 + y'^2}} - \sqrt{1 + y'^2} = \text{const},
\]
or \( y' = \text{const} \). Thus, \( y = ax + b \). Inserting the boundary conditions \( y(x_1) = y_1 \), \( y(x_2) = y_2 \) gives

\[
y(x) = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) + y_1.
\]

(ii) The position of the center of mass is determined by the equation

\[
Mr_S = \int r \, dm,
\]

\( M \) denoting the mass of the chain, \( dm \) the mass element. If \( \lambda \) is the mass per unit length, \( dm = \lambda \, ds \). As the \( x \)-coordinate of the center of mass is irrelevant, the problem is to find the shape for which its \( y \)-coordinate is lowest. Thus we have to minimize the functional

\[
\int y \, ds = \int_{x_1}^{x_2} y \sqrt{1 + y'^2} \, dx.
\]
The result of the preceding exercise leads to
\[
\frac{yy'^2}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} = -\frac{\sqrt{y}}{\sqrt{1 + y'^2}} = C.
\]
This equation can be solved for \( y' \),
\[
y' = \sqrt{C y^2 - 1}.
\]
This is a separable differential equation whose general solution is
\[
y(x) = \frac{1}{\sqrt{C}} \cosh \left( \sqrt{C} x + C' \right).
\]
The constants \( C \) and \( C' \), finally, must be chosen such that the boundary conditions \( y(x_1) = y_1, \ y(x_2) = x_2 \) are fulfilled.

2.9 (i) In either case the equations of motion read
\[
\ddot{x}_1 = -m\omega_0^2 x_1 - \frac{1}{2} m(\omega_1^2 - \omega_0^2)(x_1 - x_2)
\]
\[
\ddot{x}_2 = -m\omega_0^2 x_2 + \frac{1}{2} m(\omega_1^2 - \omega_0^2)(x_1 - x_2).
\]
The reason for this result becomes clear when we calculate the difference \( L' - L \):
\[
L' - L = -i\omega_0 m (x_1 \dot{x}_1 + x_2 \dot{x}_2) = -\frac{i}{2} \omega_0 m \frac{d}{dt} (x_1^2 + x_2^2).
\]
The two Lagrangian functions differ by the total time derivative of a function which depends on the coordinates only. By the general considerations of Sects. M2.9 and M2.10 such an addition does not alter the equations of motion.

(ii) The transformation to eigenmodes reads
\[
z_1 = \frac{1}{\sqrt{2}} (x_1 + x_2), \quad z_2 = \frac{1}{\sqrt{2}} (x_1 - x_2).
\]
This transformation
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]
is one-to-one. Both \( F \) and \( F^{-1} \) are differentiable. Thus, \( F \), being a diffeomorphism, leaves the Lagrange equations invariant.

2.10 The axial symmetry of the force suggests the use of cylindrical coordinates. In these coordinates the force must not have a component along the unit vector \( \hat{e}_\phi \). Furthermore, since
\[ \nabla U(r, \varphi, z) = \frac{\partial U}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial U}{\partial \varphi} \hat{e}_\varphi + \frac{\partial U}{\partial z} \hat{e}_z, \]

\( U \) must not depend on \( \varphi \). The unit vectors \( \hat{e}_r \) and \( \hat{e}_z \) span a plane that contains the \( z \)-axis.

2.11 By a (passive) infinitesimal rotation we have

\[ x \approx x_0 - (\hat{\varphi} \times x_0) \varepsilon \quad \text{or} \quad x_0 \approx x + (\hat{\varphi} \times x) \varepsilon. \]

Here \( \hat{\varphi} \) is the direction about which the rotation takes place, \( \varepsilon \) is the angle of rotation, so that \( \hat{\varphi} \varepsilon = \omega dt \). Thus \( x_0 = \dot{x} + (\omega \times x) \), the dot denoting the time derivative in the system of reference that one considers. Inserting this into the kinetic energy one finds

\[ T = m \left( \dot{x}^2 + 2 \dot{x} \cdot (\omega \times x) + (\omega \times x)^2 \right) / 2. \]

Meanwhile \( U(x_0) \) becomes \( U(x) = \tilde{U}(R^{-1}(t)x) \). We calculate

\[ \frac{\partial L}{\partial \dot{x}_i} = m \ddot{x}_i + m(\omega \times x)_i \]

\[ \frac{\partial L}{\partial x_i} = -\frac{\partial \tilde{U}}{\partial x_i} + m(\dot{x}_i \times \omega) + m((\omega \times x) \times \omega)_i. \]

This leads to the equation of motion

\[ m \ddot{x} = -\nabla U - 2m(\omega \times \dot{x}) - m\omega \times (\omega \times x) - m(\dot{\omega} \times x). \]

2.12 Let the coordinates of the point of suspension be \((x_A, 0)\), \( \varphi \) the angle between the pendulum and the vertical, with \(-\pi \leq \varphi \leq \pi \). The coordinates of the mass point are

\[ x = x_A + l \sin \varphi, \quad y = -l \cos \varphi, \]

\( m \) denoting the mass, \( l \) the length of the pendulum. Inserting these into

\[ L = \frac{m}{2} (\dot{x}_A^2 + l^2 \dot{\varphi}^2) - mg(y + l) \]

gives the answer

\[ L = \frac{m}{2} (\dot{x}_A^2 + l^2 \dot{\varphi}^2 + 2l \cos \varphi \dot{x}_A \dot{\varphi}) + mgl(\cos \varphi - 1). \]

2.13 (i) If the oscillation is to be harmonic \( s(t) \) must obey the following equation

\[ \ddot{s} + \kappa^2 s = 0 \Rightarrow s(t) = s_0 \sin \kappa t. \]

(ii) The Lagrangian function reads
Fig. 11.

\[ L = \frac{m}{2} s^2 - U \]

where the potential energy is given by (see Fig. 11)

\[ U = mgy = mg \int_0^s \sin \phi \, ds. \]

The Euler-Lagrange equation reads \( m\ddot{s} + mg \sin \phi = 0 \). Inserting the above relation for \( s(t) \) we obtain the equation \( s_0 \kappa^2 \sin \kappa t = g \sin \phi \).

Since the absolute value of the sine function is always smaller than, or equal to, 1 one has

\[ \lambda := \frac{s_0 \kappa^2}{g} \leq 1. \]

Thus we obtain the equation \( \phi(t) = \arcsin(\lambda \sin \kappa t) \) whose derivatives are

\[ \dot{\phi} = \frac{\lambda \kappa \cos \kappa t}{\sqrt{1 - \lambda^2 \sin^2 \kappa t}}, \quad \ddot{\phi} = \frac{-\lambda \kappa^2 (1 - \lambda^2) \sin \kappa t}{(1 - \lambda^2 \sin^2 \kappa t)^{3/2}}. \]

In the limit \( \lambda \to 1, \phi \) goes to zero and \( \dot{\phi} \) goes to \( \kappa \), except if \( \kappa t = (2n + 1)/2\pi \) where they are singular.

(iii) The force of constraint is the one perpendicular to the orbit. It is

\[ Z(\phi) = mg \cos \phi \begin{pmatrix} -\sin \phi \\ \cos \phi \end{pmatrix}. \]

The effective force is then

\[ E = -mg \begin{pmatrix} 0 \\ 1 \end{pmatrix} + Z(\phi) = -mg \sin \phi \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}. \]

2.14  (i) The condition \( \partial F(x, z)/\partial x = 0 \) implies \( z - \partial f/\partial z = 0 \), i.e., \( z = f'(x) \). Therefore, \( x = x(z) \) is the point where the vertical distance between \( y = zx \) (with \( z \) fixed) and \( y = f(x) \) is largest (see Fig. 12).

(ii) The figure shows that \( (L\Phi)(z) = zx - \Phi(z) \equiv G(x, z) \), \( z \) fixed, is tangent to \( f(x) \) at the point \( x = x(z) \) (the derivative being \( z \)).

Keeping \( x = x_0 \) fixed and varying \( z \) yields the picture shown in Fig. 13. For fixed \( z \) \( y = G(x, z) \) is the tangent to \( f(x) \) in \( x(z) \). \( G(x_0, z) \) is the ordinate of the
intersection point of that tangent with the straight line \( x = x_0 \). The maximum is at \( x_0 = x(z) \), i.e. \( z(x_0) = f''(x)|_{x=x_0} \). As \( f'' > 0 \) all tangents are below the curve. The envelope of this set of straight lines is the curve \( y = f(x) \).

**2.15** (i) In a first step we determine the canonically conjugate momenta

\[
p_1 = \frac{\partial L}{\partial \dot{q}_1} = 2c_{11}\dot{q}_1 + (c_{12} + c_{21})\dot{q}_2 + b_1
\]

\[
p_2 = \frac{\partial L}{\partial \dot{q}_2} = (c_{12} + c_{21})\dot{q}_1 + 2c_{22}\dot{q}_2 + b_2 .
\]

Using the given abbreviations this can be written in the form

\[
\pi_1 = d_{11}\dot{q}_1 + d_{12}\dot{q}_2 , \quad \pi_2 = d_{21}\dot{q}_1 + d_{22}\dot{q}_2 .
\]

For this to be solvable in terms of \( \dot{q}_i \) the determinant

\[
D := d_{11}d_{22} - d_{12}d_{21} = \det \left( \frac{\partial L}{\partial \dot{q}_i \partial \dot{q}_k} \right) \neq 0
\]

must be different from zero. The \( \dot{q}_i \) can then be expressed in terms of the \( \pi_i \):

\[
\dot{q}_1 = \frac{1}{D} (d_{22}\pi_1 - d_{12}\pi_2) , \quad \dot{q}_2 = \frac{1}{D} (-d_{21}\pi_1 - d_{11}\pi_2) .
\]

We construct the Hamiltonian function and obtain

\[
H = p_1\dot{q}_1 + p_2\dot{q}_2 - L = \frac{1}{D} \left( c_{22}\pi_1^2 - (c_{12} + c_{21})\pi_1\pi_2 + c_{11}\pi_2^2 \right) - a + U .
\]

The above determinant is found to be
\begin{align*}
\det \left( \frac{\partial H}{\partial p_i \partial p_k} \right) &= \det \left( \frac{\partial^2 H}{\partial \pi_i \partial \pi_k} \right) \\
&= \frac{1}{D^2} \left| \begin{array}{cc}
d_{22} & (d_{12} + d_{21})/2 \\
d_{11} & (d_{11} + d_{22})/2 \\
\end{array} \right| = \frac{1}{D} .
\end{align*}

The inverse transformation, the construction of \( L \) from \( H \), proceeds along the same lines.

(ii) Assume that there exists a function

\[ F(p_1(x_1, x_2, u), p_2(x_1, x_2, u)) \]

which vanishes identically in the domain of definition of the \( x_i, u \) fixed. Take the derivatives

\[ 0 = \frac{dF}{dx_1} = \frac{\partial F}{\partial p_1} \frac{\partial p_1}{\partial x_1} + \frac{\partial F}{\partial p_2} \frac{\partial p_2}{\partial x_1} \]

\[ 0 = \frac{dF}{dx_2} = \frac{\partial F}{\partial p_1} \frac{\partial p_1}{\partial x_2} + \frac{\partial F}{\partial p_2} \frac{\partial p_2}{\partial x_2} \]

By assumption the partial derivatives of \( F \) with respect to \( p_i \) do not vanish (otherwise the system of equations would be trivial). Therefore, the determinant

\[ D = \det \left( \frac{\partial p_1}{\partial x_1} \frac{\partial p_2}{\partial x_2} \right) = \det \left( \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_k} \right) \]

must be different from zero. This proves the assertion.

2.16 We introduce the complex variable \( w := x + iy \). Then

\[ x = (w + w^*)/2 , \quad y = -i(w - w^*)/2 , \quad \dot{x}^2 + \dot{y}^2 = \dot{w}\dot{w}^* . \]

\( l_3 \) is calculated to be

\[ l_3 = m(x\dot{y} - y\dot{x}) = \frac{m}{2i} (\dot{w}w^* - w\dot{w}^*) . \]

Expressed in the new coordinates the Lagrangian function reads

\[ L = \frac{m}{2} (\dot{w}\dot{w}^* + \dot{z}^2) - \frac{im\omega}{4} (\dot{w}w^* - w\dot{w}^*) . \]

The equations of motion are

\[ \frac{m}{2} \ddot{w}^* - \frac{im\omega}{4} w^* = \frac{im\omega}{4} \dot{w}^* , \quad m\ddot{z} = 0 . \]

The first of these is written in terms of the variable \( u := \dot{w}^* \). It becomes \( \dot{u} = i\omega u \), its solution being \( u = e^{i\omega t} . \) \( w^* \) is the time integral of this function, viz.
Solution of Exercises

\[ w^* = -\frac{i}{\omega} e^{i\omega t} + C , \]

where \( C \) is a complex constant. Take the complex conjugate

\[ w = \frac{i}{\omega} e^{-i\omega t} + C^* , \]

from which follow the solutions for \( x \) and \( y \)

\[ x = \frac{1}{\omega} \sin \omega t + C_1 , \quad y = \frac{1}{\omega} \cos \omega t + C_2 , \]

where \( C_1 = \Re\{C\}, \quad C_2 = -\Im\{C\} \).

The solution for the \( z \) coordinate is simple: \( z = C_3 t + C_4 \), i.e., uniform motion along a straight line. The canonically conjugate momenta are

\[ p_x = m \dot{x} - \frac{m}{2} \omega y , \quad p_y = m \dot{y} + \frac{m}{2} \omega x , \quad p_z = m \dot{z} , \]

while the kinetic momenta are given by \( p_{\text{kin}} = m \dot{x} \). In order to construct the Hamiltonian function the velocities are expressed in terms of the canonical momenta

\[ \dot{x} = \frac{1}{m} p_x + \frac{\omega}{2} y , \quad \dot{y} = \frac{1}{m} p_y - \frac{\omega}{2} x , \quad \dot{z} = \frac{1}{m} p_z . \]

Then \( H \) is found to be

\[ H = p \cdot \dot{x} - L = \frac{1}{2m} p_{\text{kin}}^2 . \]

2.17 (i) Hamilton’s variational principle, when applied to \( \bar{L} \), requires

\[ \bar{I} := \int_{\tau_1}^{\tau_2} \bar{L} d\tau \]

to be an extremum. Now, since

\[ \int_{\tau_1}^{\tau_2} \bar{L} d\tau = \int_{t_1}^{t_2} L dt \quad \text{with} \quad t_i = t(\tau_i) , \quad i = 1, 2 , \]

the action integral \( \bar{I} \) is extremal if and only if the Lagrange equations that follow from \( L \) are fulfilled.

(ii) We define \( q = (q_1, \ldots, q_f) \), \( t = q_{f+1} \). From Noether’s theorem the quantity

\[ I = \sum_{i=1}^{f+1} \frac{\partial L}{\partial q_i} d\tau \frac{d}{ds} h^s(q_1, \ldots, q_{f+1})|_{s=0} \]
is an integral of the motion provided $\bar{L}$ is invariant under $(q_1, \ldots, q_{f+1}) \rightarrow h^s(q_1, \ldots, q_{f+1})$, i.e., in the case considered here, under $(q_1, \ldots, q_{f+1}) \rightarrow (q_1, \ldots, q_{f+1} + s)$. Here

$$\left. \frac{dh^s}{ds} \right|_{s=0} = (0, \ldots, 0, 1)$$

and

$$\frac{\partial \bar{L}}{\partial \dot{q}_{f+1}} = \frac{\partial \bar{L}}{\partial (dt/d\tau)} = L + \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \left( -\frac{1}{(dt/d\tau)^2} \right) \frac{dq_i}{d\tau} \frac{dt}{d\tau} = L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{dq_i}{dt}.$$

The integral of the motion is

$$I = L - \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{dq_i}{dt}.$$

Except for a sign this is the expression for the energy.

2.18 The points for which the sum of their distances to $A$ and to $\Omega$ is constant lie on the ellipsoid with foci $A$, $\Omega$, semi-major axis $\sqrt{R^2 + a^2}$, and semi-minor axis $R$. The reflecting sphere lies inside that ellipsoid and is tangent to it in $B$. Thus, any other path than the one shown in Fig. 14 would be shorter than the one through $B$ for which $\alpha = \beta$.

2.19 (i) As usual we set $x_\alpha = (q_1, \ldots, q_f; p_1, \ldots, p_f)$, and $y_\beta = (Q_1, \ldots, Q_f; P_1, \ldots, P_f)$, as well as $M_{\alpha\beta} = \partial x_\alpha / \partial y_\beta$. We have

$$M^T J M = J \quad \text{and} \quad J = \begin{pmatrix} 0_{f \times f} & I_{f \times f} \\ -I_{f \times f} & 0_{f \times f} \end{pmatrix}.$$ (1)

The equation always relates $\partial P_k / \partial p_i$ to $\partial q_i / \partial Q_k$, $\partial Q_j / \partial p_t$ to $\partial q_l / \partial P_j$ etc. From this follows that $[P_k \cdot Q_k] = [p_j \cdot q_j]$. Let $\Phi(x, y)$ be generating function of the
canonical transformation. As $\tilde{H} = H + \partial \Phi / \partial t$, the function $\Phi$ has the dimension of the product $H \cdot t$. The assertion then follows from the canonical equations.

(ii) With the canonical transformation $\Phi$ and using $\tau := \omega t$, $H$ goes over into $\tilde{H} = H + \partial \Phi / \partial \tau$. Hence $[\Phi] = [H] = [x_1 x_2] = [\omega] [pq]$. The new generalized coordinate $y_1 = Q$ has no dimension. As $y_1 y_2$ has the same dimension as $x_1 x_2$, $y_2$ must have the dimension of $H$, or $\tilde{H}$, that is, $y_2$ must equal $\omega P$. Therefore,

$$\dot{\Phi}(x_1, y_1) = \frac{1}{2} x_1^2 \cot y_1 .$$

From this one calculates

$$x_2 = \frac{\partial \dot{\Phi}}{\partial x_1} = x_1 \cot y_1 , \quad y_2 = -\frac{\partial \dot{\Phi}}{\partial y_1} = \frac{x_1^2}{2 \sin^2 y_1}$$

or,

$$x_1 = \sqrt{2 y_2} \sin y_1 , \quad x_2 = \sqrt{2 y_2} \cos y_1 .$$

Using these formulas one finds

$$M_{\alpha \beta} = \frac{\partial x_\alpha}{\partial y_\beta} = \begin{pmatrix} (2 y_2)^{1/2} \cos y_1 & (2 y_2)^{-1/2} \sin y_1 \\ -(2 y_2)^{1/2} \sin y_1 & (2 y_2)^{-1/2} \cos y_1 \end{pmatrix} .$$

One easily verifies the conditions $\det M = 1$ and $M^T JM = J$.

2.20 (i) For $f = 1$ the condition $\det M = 1$ is necessary and sufficient because, quite generally,

$$M^T JM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (a_{11} a_{22} - a_{12} a_{21}) = J \det M .$$

(ii) We calculate $S \cdot O$, set it equal to $M$, and obtain the equations

$$x \cos \alpha - y \sin \alpha = a_{11} \quad \text{(1)}$$
$$x \sin \alpha + y \cos \alpha = a_{12} \quad \text{(2)}$$
$$y \cos \alpha - z \sin \alpha = a_{21} \quad \text{(3)}$$
$$y \sin \alpha + z \cos \alpha = a_{22} \quad \text{(4)}$$

From the combination $((2) - (3))/((1) + (4))$ of the equations

$$\tan \alpha = \frac{a_{12} - a_{21}}{a_{11} + a_{22}} .$$

This allows us to calculate $\sin \alpha$ and $\cos \alpha$, so that the subsystems ((1), (2)) and ((3), (4)) can be solved for $x$, $y$, and $z$. One finds $x = a_{11} \cos \alpha + a_{12} \sin \alpha$, $z = a_{22} \cos \alpha - a_{21} \sin \alpha$, $y^2 = xz - 1$.

There is a special case, however, that must be studied separately: This is when $a_{11} + a_{22} = 0$. If $a_{12} \neq a_{21}$ we take the reciprocal of the above relation
\[ \cot \alpha = \frac{a_{11} + a_{22}}{a_{12} - a_{21}}. \]

If, however, \( a_{12} = a_{21} \) the matrix \( \mathbf{M} \) is symmetric and \( \mathbf{O} \) can be taken to be the unit matrix, i.e., \( \alpha = 0 \).

2.21 (i) Using the product rule one has \( \{fg, h\} = f\{g, h\} + g\{f, h\} \). Hence

\[ \{l_i, r_k\} = \{\varepsilon_{imn} r_m p_n, r_k\} = \varepsilon_{imn} r_m \{p_n, r_k\} + \varepsilon_{imn} p_n \{r_m, r_k\} \]

\[ \{r_m, r_k\} = \varepsilon_{imn} r_m \delta_{nk} = \varepsilon_{imk} r_m \]

and, in a similar fashion, \( \{l_i, p_k\} = \varepsilon_{ikm} p_m \). In calculating the third Poisson bracket we note that

\[ \{l_i, r\} = \{\varepsilon_{imn} r_m p_n, r\} = \varepsilon_{imn} r_m \{p_n, r\} + \varepsilon_{imn} p_n \{r_m, r\} \]

\[ = \varepsilon_{imn} r_m \frac{\partial r}{\partial r_n} = \varepsilon_{imn} r_m r_n \frac{1}{r} = 0. \]

Finally, we have

\[ \{l_i, p^2\} = \{\varepsilon_{imn} r_m p_n, p_k p_k\} = \varepsilon_{imn} r_m \{p_n, p_k p_k\} + \varepsilon_{imn} p_n \{r_m, p_k p_k\} \]

\[ = -2\varepsilon_{imn} p_n p_k \delta_{mk} = -\varepsilon_{imn} p_n p_m = 0. \]

(ii) \( U \) can only depend on \( r \).

2.22 The vector \( \mathbf{A} \) is a constant of the motion precisely when the Poisson bracket of each of its components with the Hamiltonian function vanishes. Therefore, we calculate

\[ \{H, A_k\} = \left\{ \frac{1}{2m} \mathbf{p}^2 + \frac{\gamma}{r}, \varepsilon_{klm} p_l l_m + \frac{m\gamma}{r} r_k \right\} \]

\[ = \frac{1}{2m} \varepsilon_{klm} \left\{ \mathbf{p}^2, p_l l_m \right\} + \gamma \varepsilon_{klm} \{1/r, p_l l_m\} \]

\[ + \frac{\gamma}{2} \left\{ \mathbf{p}^2, r_k/r \right\} + m\gamma^2 \{1/r, r_k/r\}. \]

The fourth bracket vanishes. The first three are calculated as follows

\[ \left\{ \mathbf{p}^2, p_l l_m \right\} = \left\{ \mathbf{p}^2, p_l \right\} l_m + \left\{ \mathbf{p}^2, l_m \right\} p_l = 0 \]

\[ \{1/r, p_l l_m\} = \{1/r, p_l\} l_m + \{1/r, l_m\} p_l = r_l/r^3 l_m \]

\[ \left\{ \mathbf{p}^2, r_k/r \right\} = 1/r \left\{ \mathbf{p}^2, r_k \right\} + r_k \left\{ \mathbf{p}^2, 1/r \right\} = 2p_k/r - 2r_k \mathbf{p} \cdot \mathbf{x}/r^3. \]

Inserting these results we obtain

\[ \{H, A_k\} = \gamma \varepsilon_{klm} r_l/r^3 l_m + \gamma \left( p_k/r - r_k \mathbf{p} \cdot \mathbf{x}/r^3 \right) = 0. \]

This vector is often called Lenz vector or, in the German literature, Lenz-Runge vector, although apparently neither H.F.E. Lenz nor C. Runge claimed priority
for it. Its discovery is due to Jakob Hermann (published in Giornale dei Letterati d’Italia, vol. 2 (1710) p. 447). The conservation of this vector was also known to Joh. I Bernoulli and to P.-S. de Laplace, see H. Goldstein, Am. J. Phys. 44 (1976) No. 11. very much in the spirit of linear algebra.

2.23  Calculation of the Poisson brackets yields differential equations which are solved taking proper account of the initial conditions as follows:

\[
\begin{align*}
\dot{p}_1 &= \{H, p_1\} = -m\alpha \Rightarrow p_1 = -m\alpha t + p_x , \\
\dot{p}_2 &= \{H, p_2\} = 0 \Rightarrow p_2 = p_y , \\
\dot{q}_1 &= \{H, q_1\} = \frac{1}{m} p_1 \Rightarrow q_1 = -\frac{1}{2} \alpha t^2 + \frac{p_x}{m} t + x_0 , \\
\dot{q}_2 &= \{H, q_2\} = \frac{1}{m} p_2 \Rightarrow q_2 = \frac{p_y}{m} t + y_0 .
\end{align*}
\]

2.24  (i) Let \( \mu_1 = \frac{m_1 m_2}{m_1 + m_2} \) and \( \mu_2 = \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3} \) be the reduced masses of the two two-body systems \((1, 2)\) and \(((\text{center-of-mass of } 1 \text{ and } 2), 3)\), respectively. Then \( \pi_1 = \mu_1 \dot{q}_1 \) and \( \pi_2 = \mu_2 \dot{q}_2 \). This explains the meaning of these two momenta. The momentum \( \pi_3 \) is the center-of-mass momentum.

(ii) We define

\[ M_j := \sum_{i=1}^{j} m_i , \]

i.e., \( M_j \) is the total mass of particles \( 1, 2, \ldots, j \). We can then write

\[
\begin{align*}
q_j &= r_{j+1} - \frac{1}{M_j} \sum_{i=1}^{j} m_i r_i , \quad j = 1, \ldots, N - 1 \\
q_N &= \frac{1}{M_N} \sum_{i=1}^{N} m_i r_i , \\
\pi_j &= \frac{1}{M_{j+1}} \left( M_j p_{j+1} - m_{j+1} \sum_{i=1}^{j} p_i \right) , \quad j = 1, \ldots, N - 1 \\
\pi_N &= \sum_{i=1}^{N} p_i .
\end{align*}
\]

(iii) We choose the following possibilities:

a) As the Poisson bracket of \( r_i \) and \( p_k \) ist \( \{p_k, r_i\} = \mathbb{I}_{3 \times 3} \delta_{ik} \) we must also have \( \{\pi_k, q_j\} = \mathbb{I}_{3 \times 3} \delta_{ik} \). We use this suggestive short hand notation for \( \{(p_k)_m, (q_i)_n\} = \delta_{ik} \delta_{mn} \) with \( (\cdot)_m \) denoting the \( m \)th Cartesian coordinate. Using the former brackets one calculates the latter brackets from the defining formulas. For instance, with \( m_{12} := m_1 + m_2, M := m_1 + m_2 + m_3, \)
\{\pi_1, \varrho_1\} = \left( \frac{m_1}{m_{12}} + \frac{m_2}{m_{12}} \right) \mathbb{I} = \mathbb{I},

\{\pi_2, \varrho_1\} = \left( \frac{m_3}{M} - \frac{m_3}{M} \right) \mathbb{I} = 0, \quad \text{etc.}

b) In the 18-dimensional phase space introduce the variables

\( x = (r_1, r_2, r_3, p_1, p_2, p_3) \) and \( y = (\varrho_1, \varrho_2, \varrho_3, \pi_1, \pi_2, \pi_3) \),
calculate the matrix \( M_{\alpha\beta} := \frac{\partial y_\alpha}{\partial x_\beta} \) and verify that this matrix is symplectic, i.e. that it satisfies eq. (M2.113). This calculation can be simplified by noting that \( M \) has the form

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]
such that

\[
M^T J M = \begin{pmatrix}
0 & A^T B \\
-B^T A & 0
\end{pmatrix}
\]

Thus, it suffices to verify that \( A^T B = \mathbb{I}_{9 \times 9} \). One finds

\[
A = \begin{pmatrix}
-\mathbb{I} & \mathbb{I} & 0 \\
-m_1/m_{12} \mathbb{I} - m_2/m_{12} \mathbb{I} & \mathbb{I} & 0 \\
m_1/M \mathbb{I} & m_2/M \mathbb{I} & m_3/M \mathbb{I}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
-m_2/m_{12} \mathbb{I} & m_1/m_{12} \mathbb{I} & 0 \\
-m_3/M \mathbb{I} & -m_3/M \mathbb{I} & m_2/M \mathbb{I} \\
\mathbb{I} & \mathbb{I} & \mathbb{I}
\end{pmatrix},
\]

the entries being themselves \( 3 \times 3 \) matrices. In a next step one calculates \((A^T B)_{ik} = \sum_l A_{il} B_{lk}\). For instance, one finds

\[
(A^T B)_{11} = \frac{m_2}{M_{12}} + \frac{m_1 m_3}{m_{12} M} + \frac{m_1}{M} = 1, \quad \text{etc.}
\]

and verifies, eventually, that \( A^T B = \mathbb{I}_{9 \times 9} \).

2.25 (i) In the situation described in the exercise the variation of \( I(\alpha) \) is

\[
\delta I = \left. \frac{dI(\alpha)}{d\alpha} \right|_{\alpha=0} \, d\alpha
\]

\[
= L \left( q_k (t_2(0), 0), \dot{q}_k (t_2(0), 0) \right) \left. \frac{dt_2(\alpha)}{d\alpha} \right|_{\alpha=0} \, d\alpha
\]

\[
- L \left( q_k (t_1(0), 0), \dot{q}_k (t_1(0), 0) \right) \left. \frac{dt_1(\alpha)}{d\alpha} \right|_{\alpha=0} \, d\alpha
\]

\[
+ \int_{t_1(0)}^{t_2(0)} \left( \sum_k \frac{\partial L}{\partial q_k} \frac{\partial q_k(t, \alpha)}{\partial \alpha} \bigg|_{\alpha=0} \, d\alpha + \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k(t, \alpha)}{\partial \alpha} \bigg|_{\alpha=0} \, d\alpha \right) dt.
\]

We define, as usual
\[
\frac{\partial q_k}{\partial \alpha} \bigg|_0 \, d\alpha = \delta q_k \quad \text{and} \quad \frac{\partial \dot{q}_k}{\partial \alpha} \bigg|_0 \, d\alpha = \delta \dot{q}_k = \frac{d}{dt} \delta q_k ,
\]

and, in addition, \( dt_i(\alpha)/d\alpha|_0 d\alpha = \delta t_i \), \( i = 1, 2 \). The time derivative \( d\delta q_k/dt \), by partial integration, is shifted onto \( \partial L/\partial \dot{q}_k \). Here, however, the terms at the boundaries do not vanish because the variations \( \delta t_i \) do not vanish. One has

\[
\int_{t_1(0)}^{t_2(0)} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k \, dt = \left[ \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1(0)}^{t_2(0)} - \int_{t_1(0)}^{t_2(0)} \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt .
\]

The end points are kept fixed which means that

\[
\frac{dq_k(t_i(\alpha),\alpha)}{d\alpha} \bigg|_{\alpha=0} = 0 , \quad i = 1, 2 .
\]

Taking the derivative with respect to \( \alpha \) this implies

\[
\frac{dq_k(t_i(\alpha),\alpha)}{d\alpha} \bigg|_{\alpha=0} = \frac{\partial q_k}{\partial t} \bigg|_{t=t_i} \frac{dt_i(\alpha)}{d\alpha} \bigg|_{\alpha=0} + \frac{\partial q_k}{\partial \alpha} \bigg|_{t=t_i,\alpha=0} \, d\alpha \\
\equiv \dot{q}_k(t_i) \delta t_i + \delta q_k |_{t=t_i} = 0 .
\]

Inserting this into \( \delta I \) one obtains the result

\[
\delta I = \left[ \left( L - \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right) \delta t_i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \, dt \sum_k \left( \frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k .
\]

(ii) One calculates \( \delta K \) in exactly the way, viz.

\[
\delta K = K \int_{t_1}^{t_2} (L + E) \, dt \\
= \left[ \left( L - \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k \right) \delta t_i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \, dt \sum_k \left( \frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k + \left[ E \delta t_i \right]_{t_1}^{t_2} = 0 .
\]

Now, by assumption \( E = \sum_k \dot{q}_k (\partial L/\partial \dot{q}_k) \) is constant. As a consequence the first and third terms of the equation cancel. As the variations \( \delta q_k \) are independent one finds indeed the implication

\[
\delta K \overset{!}{=} 0 \Leftrightarrow \frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = 0 , \quad k = 1, \ldots, f .
\]

2.26 We write

\[
T = \sum_i g_{ik} \dot{q}_i \dot{q}_k = \left( \frac{ds}{dt} \right)^2 = \frac{1}{2} (L + E) = E - U
\]
and obtain $T \, dt = (ds/dt)ds = \sqrt{E - U} \, ds$. The principle of Euler and Maupertuis, $\delta K = 0$, requires

$$\delta \int_{q_1}^{q_2} \sqrt{E - U} \, ds = 0 .$$

On the other hand, Fermat’s principle states the following: A light pulse traverses the path $ds$ in the time $dt = (n(x, \nu)/c)ds$. The path it chooses is such that the integral $\int dt$ is an extremum, i.e. that $\delta \int n(x, \nu)ds = 0$. The analogy is established if we associate with the particle an “index of refraction” which is given by the dimensionless quantity $((E - U)/mc^2)^{1/2}$; (see also Exercise 1.12).

2.27 For $U(q_0) < E \leq E_{\text{max}}$ the points of intersection $q_1$ and $q_2$ of the curves $y = U(q)$ and $y = E$ are turning points, $q(t)$ oscillates periodically between $q_1$ and $q_2$, cf. Fig. 15. Write the characteristic equation of Hamilton and Jacobi (M2.154) as

$$H \left( q, \frac{\partial S(q, P)}{\partial q} \right) = E .$$

(1)

We know that the transformed momentum obeys the differential equation $\dot{P} = 0$, i.e., that $P = \alpha = \text{const}$. We are free to choose this constant to be the energy, $P = E$. Taking the derivative of eq. (1) with respect to $P = E$,

$$\frac{\partial H}{\partial p} \frac{\partial^2 S}{\partial q \, \partial P} = 1 .$$

If $\partial H/\partial p \neq 0$ (this holds locally if $E$ is larger than $U(q_0)$), then $(\partial^2 S)/(\partial q \, \partial P) \neq 0$. Thus, the equation $Q = \partial S(q, P)/\partial P$ can be solved locally for $q = q(Q, P)$. This yields

![Fig. 15.](image-url)
\[ H\left(q(Q, P), \frac{\partial S}{\partial q}(q(Q, P), P)\right) \equiv \tilde{H}(Q, P) = E \equiv P. \]

From this we conclude
\[
\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = 1, \quad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \Rightarrow Q = t - t_0 = \frac{\partial S}{\partial E}.
\]

The integral \( I(E) \) becomes
\[
I(E) = \frac{1}{2\pi} \oint_{\Gamma_E} p \, dq = \frac{1}{2\pi} \int_{t_0}^{t_0 + T(E)} p \cdot \dot{q} \, dt
\]
so that \( dI(E)/dE = T(E)/(2\pi) \equiv \omega(E) \), in agreement with Exercise 2.1.

2.28 The function \( \bar{S}(q, I) \), with \( I \) as in Exercise 2.27, generates the transformation from \( (q, p) \) to the action and angle variables \( (\theta, I) \),
\[
p = \frac{\partial \bar{S}(q, I)}{\partial q}, \quad \theta = \frac{\partial \bar{S}(q, I)}{\partial I}, \quad \text{with} \quad \tilde{H} = E(I).
\]

We then have \( \dot{\theta} = \partial E/\partial I = \text{const.} \), \( \dot{I} = 0 \), which are integrated to \( \theta(t) = (\partial E/\partial I)t + \theta_0 \), \( I = \text{const.} \). Call the circular frequency \( \omega(E) := \partial E/\partial I \) so that \( \theta(t) = \omega t + \theta_0 \), \( I = \text{const.} \).

2.29 We calculate the integral \( I(E) \) of Exercise 2.27 for the case \( H = p^2/2 + q^2/2 \): With \( p = \sqrt{2E - q^2} \)
\[
I(E) = \frac{1}{2\pi} \oint_{\Gamma_E} p \, dq = \frac{1}{2\pi} \int_{\Gamma_E} \sqrt{2E - q^2} \, dq
\]
\[
= \frac{1}{\pi} \int_{-A}^{A} \sqrt{A^2 - q^2} \, dq, \quad (A = \sqrt{2E}).
\]

Using
\[
\int_{-A}^{A} \sqrt{A^2 - x^2} \, dx = \frac{\pi A^2}{2}
\]
one finds \( I(E) = A^2/2 = E \), i.e., \( H = I \). The characteristic equation (M2.154) reads in the present example
\[
\frac{1}{2} \left( \frac{\partial S}{\partial q} \right)^2 + \frac{1}{2} q^2 = E.
\]

Its solution can be written as an indefinite integral \( S = \int \sqrt{2E - q^2} \, dq' \), or \( \bar{S}(q, I) = \int \sqrt{2I - q'^2} \, dq' \). The angle variable follows from this
\[
\theta = \frac{\partial \tilde{S}}{\partial I} = \int \frac{1}{\sqrt{2I - q'^2}} \, dq' = \arcsin \frac{q}{\sqrt{2I}},
\]
giving \(q = \sqrt{2I} \sin \theta\). In a similar fashion one calculates
\[
p = \frac{\partial \tilde{S}}{\partial q} = \sqrt{2I - q^2} = \sqrt{2I} \cos \theta.
\]
These are identical with the formulas that follow from the canonical transformation \(\Phi(q, Q) = (q^2/2) \cot Q\), cf. eq. (M2.95).

2.30 Following Exercise 2.17 we take the time variable \(t\) to be a generalized coordinate \(t = q_{f+1}\) and introduce in its place a new variable \(\tau\) such that
\[
\bar{L}\left(q, \tau, \frac{dq}{d\tau}, \frac{dt}{d\tau}\right) := L\left(q, \frac{1}{(dt/d\tau)} \frac{dq}{d\tau}\right) \frac{dt}{d\tau}.
\]
By assumption \(f = 1\), i.e., \(q_1 = q\) and \(q_2 = q_{f+1} = t\). The action integral (the principal function) with the boundary conditions modified accordingly, reads
\[
I_0^s = \int_{\tau_1}^{\tau_2} d\tau \bar{L}(\phi_1(s, \tau), \phi_{f+1}(s, \tau), \phi'_1(s, \tau), \phi'_{f+1}(s, \tau)),
\]
the prime denoting the derivative with respect to \(\tau\). We now take the derivative of \(I_0^s\) with respect to \(s\), at \(s = 0\),
\[
\left. \frac{d}{ds} I_0^s \right|_{s=0} = \int_{\tau_1}^{\tau_2} d\tau \times \left\{ \frac{\partial \bar{L}}{\partial \phi_1} \frac{d\phi_1}{ds} + \frac{\partial \bar{L}}{\partial \phi'_1} \frac{d\phi'_1}{ds} + \frac{\partial \bar{L}}{\partial \phi_{f+1}} \frac{d\phi_{f+1}}{ds} + \frac{\partial \bar{L}}{\partial \phi'_{f+1}} \frac{d\phi'_{f+1}}{ds} \right\}. \tag{1}
\]
Replacing \(\partial \bar{L}/\partial \phi_1\) in the first term by \((d/d\tau)(\partial \bar{L}/\partial \phi'_1)\), through the equations of motion, the first two terms of the curly brackets in eq.(1) can be combined to a total derivative with respect to \(\tau\). The integral over \(\tau\) can be rewritten as an integral over \(t\), so that the first two terms give the contribution
\[
\int_{\tau_1}^{\tau_2} dt \left( \frac{d\bar{L}}{d\phi_1} \frac{d\phi_1}{ds} \right) = p^b \frac{dq^b}{ds} - p^s \frac{dq^s}{ds}.
\]
(Note that here the dot means the derivative with respect to \(t\), as before.) In the third term we replace \(\partial \bar{L}/\partial \phi_{f+1}\) by \((d/d\tau)(\partial \bar{L}/\partial \phi'_{f+1})\), (again by virtue of the equation of motion). In this term and in the last term of the curly brackets in eq.(1) we make use of the solution to Exercise 2.17, viz.
\[
\frac{\partial \bar{L}}{\partial (\partial \phi_{f+1}/\partial \tau)} = \frac{\partial \bar{L}}{\partial (\partial t/\partial \tau)} = L - \frac{\partial L}{\partial \phi_1} \frac{d\phi_1}{dt}.
\]
so that these terms can also be combined to a total derivative
\[
\frac{d}{d\tau} \left( \left( L - \frac{\partial L}{\partial \dot{\phi}_1} \right) \frac{d\phi_{f+1}}{ds} \right).
\]
(2)

As above, in doing the integral one replaces \(\tau\) by the variable \(t\). The inner bracket in eq. (2) is the energy (to within the sign) so that we obtain the difference of the term \(-E(d\phi_{f+1}/ds)\) at the two boundary points, that is, \((-E)\) times the derivative of the time \(t = t_2 - t_1\) by \(s\). Summing up one indeed obtains the result of the assertion. Finally, the generalization to \(f > 1\) is obvious.

2.31 Start from the Hamiltonian function \(H = \frac{p^2}{2m} + U(r)\) with \(U(r) = \gamma/r\), and from \(A = p \times \ell + mU(r)x\). Clearly, \(A \cdot \ell = 0\), the vector \(A\) is perpendicular to \(\ell\) and, hence, lies in the plane of the orbit. Making use of the formula \(x \cdot (p \times \ell) = \ell \cdot (x \times p) = \ell^2\) one calculates (with \(\ell = |\ell|\))
\[
A^2 = (p \times \ell)^2 + 2mU(r)x \cdot (p \times \ell) + m^2\gamma^2
= \ell^2(p^2 + 2mU(r)) + m^2\gamma^2
= m^2\gamma^2 + 2m\ell^2H = m^2\gamma^2 + 2m\ell^2E.
\]

This vanishes only if the energy \(E\) and hence also \(\gamma\) are negative. In the case of the Kepler problem \(\gamma = -Gm_1m_2 \equiv -A\) (notation as in Sect. 1.7.2), \(m \equiv \mu\) is the reduced mass. The vector \(A\) vanishes for \(E = -\mu A^2/(2\ell^2)\) which is the case of the circular orbit.

Calculate the scalar product \(x \cdot A = x \cdot (p \times \ell) - \mu Ar = \ell^2 - \mu Ar\), set this equal to \(r|A|\cos\phi\) to obtain
\[
r(\phi) = \ell^2/|A| \cos\phi + \mu A
= \ell^2/(\mu A)/1 + \sqrt{1 + 2E\ell^2/(\mu A^2)} \cos\phi \equiv p/1 + \epsilon \cos\phi.
\]

This is the solution given in Sect. 1.21, with \(\phi_0 = 0\). One concludes that \(A\) points along the 1-axis in the orbital plane, from the center of force to the perihelion, its modulus being \(|A| = \epsilon \mu A\).

The cross product is \(\ell \times A = \ell^2p - (\mu A/r)x \times \ell\) from which one finds
\[
\left( p - 1/\ell^2 \ell \times A \right)^2 = \mu^2A^2/\ell^2.
\]

Noting that \(A = \epsilon \mu A\hat{e}_1\) and \(\ell \times A = \ell \epsilon \mu A\hat{e}_3 \times \hat{e}_1 = \ell \epsilon \mu A\hat{e}_2\), and decomposing \(p = p_1\hat{e}_1 + p_2\hat{e}_2\) one obtains
\[
p_1^2 + (p_2 - \epsilon \mu A/\ell)^2 = (\mu A)^2/\ell^2.
\]

This is the equation of the hodograph of Exercise 1.26.
Chapter 3: The Mechanics of Rigid Bodies

3.1 (i) As $K$ and $\vec{K}$ differ by a time-dependent rotation, $J$ is related to $\vec{J}$ by $J = R(t)\vec{J}^{-1}(t)$, with $R(t)$ the rotation matrix that describes the relative rotation of the two coordinate systems. The characteristic polynomial of $J$ is invariant under similarity transformations. Indeed, by the multiplication law for determinants,

$$\det |J - \lambda \mathbb{I}| = \det |R(t)\vec{J}^{-1}(t) - \lambda \mathbb{I}|$$

$$= \det |R(t)(J - \lambda \mathbb{I})R^{-1}(t)|$$

$$= \det |\vec{J} - \lambda \mathbb{I}| .$$

The characteristic polynomials of $J$ and $\vec{J}$ are the same. Hence, their eigenvalues are pairwise equal.

(ii) If $\vec{K}$ is a principal-axes system, $\vec{J}$ has the form

$$\vec{J} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} .$$

A rotation about the 3-axis reads

$$R(t) = \begin{pmatrix} \cos \phi(t) & \sin \phi(t) & 0 \\ -\sin \phi(t) & \cos \phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

This allows us to compute $J$ with the result

$$J = \begin{pmatrix} I_1 \cos^2 \phi + I_2 \sin^2 \phi & (I_2 - I_1) \sin \phi \cos \phi & 0 \\ (I_2 - I_1) \sin \phi \cos \phi & I_1 \sin^2 \phi + I_2 \cos^2 \phi & 0 \\ 0 & 0 & I_3 \end{pmatrix} .$$

3.2 The straight line connecting the two atoms is a principal axis. The remaining axes are chosen perpendicular to the first and perpendicular to each other, as sketched in Fig. 16. Using the notation of that figure we have $m_1a_1 = m_2a_2$, $a_1 + a_2 = l$, and therefore

$$a_1 = \frac{m_2}{m_1 + m_2} l, \quad a_2 = \frac{m_1}{m_1 + m_2} l ,$$

with $\mu$ denoting the reduced mass,

$$I_1 = I_2 = m_1a_1^2 + m_2a_2^2 = \frac{m_1m_2}{m_1 + m_2} l^2 = \mu l^2 ,$$

$$I_3 = \mu l^2 .$$
3.3 The moments of inertia are determined from the equation

$$\det(J - \lambda I) = \begin{vmatrix} I_{11} - \lambda & I_{12} & 0 \\ I_{21} & I_{22} - \lambda & 0 \\ 0 & 0 & I_{33} - \lambda \end{vmatrix} = 0,$$

whose solutions are

$$I_{1,2} = \frac{I_{11} + I_{22}}{2} \pm \sqrt{\frac{(I_{11} - I_{22})^2}{4} + I_{12}I_{21}}, \quad I_3 = I_{33}.$$

(i) $I_{1,2} = A \pm B$. Thus it follows that $B \leq A$ and $A + B \geq 0$. Since $I_1 + I_2 \geq I_3$ we also have $I_3 \leq 2A$, i.e., $A \geq 0$.

(ii) $I_1 = 5A$, $I_2 = 0$. From $I_1 + I_2 \geq I_3$ and $I_2 + I_3 \geq I_1$ follows $I_3 = 5A$. The body is axially symmetric with respect to the 2-axis.

3.4 The motion being free we choose a principal-axes system attached to the center-of-mass, letting the 3-axis coincide with the symmetry axis. The moments of inertia are easily calculated, the result being

$$I_1 = I_2 = \frac{3}{20} M \left(R^2 + \frac{1}{4} h^2\right), \quad I_3 = \frac{3}{10} M R^2.$$

A Lagrangian function is

$$L = T_{\text{rot}} = \frac{1}{2} \sum_{i=1}^{3} I_i \tilde{\omega}_i^2,$$

where $\tilde{\omega}_i$ are the components of the angular velocity in the body fixed system. They are related to the Eulerian angles and their time derivatives by the formulae (M3.82),

$$\tilde{\omega}_1 = \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi$$

$$\tilde{\omega}_2 = -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi$$

$$\tilde{\omega}_3 = \dot{\phi} \cos \theta + \dot{\psi}.$$

Inserting these into $L$ and noting that $I_1 = I_2$, we have
The Lagrangian function is

\[ L(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = \frac{1}{2} I_1 \left( \dot{\phi}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right)^2 . \]

The variables \( \phi \) and \( \psi \) are cyclic, hence

\[ p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \cos \theta , \]

\[ p_{\psi} = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) , \]

are conserved. Furthermore, the energy \( E = T_{\text{rot}} = L \) is conserved. Note that \( p_{\phi} = I_1 (\bar{\omega}_1 \sin \psi + \bar{\omega}_2 \cos \psi) \sin \theta + I_3 \bar{\omega}_3 \cos \theta \). This is the scalar product \( L \cdot \hat{e}_3 \) of the angular momentum and the unit vector in the 3-direction of the laboratory system, i.e. \( p_{\phi} = L_3 \). Regarding \( p_{\psi} \) we have \( p_{\psi} = I_3 \bar{\omega}_3 = \bar{L}_3 \). The equations of motion read

\[ \frac{d}{dt} p_{\phi} = \frac{d}{dt} \left( I_1 (\bar{\omega}_1 \sin \psi + \bar{\omega}_2 \cos \psi) \sin \theta + I_3 \bar{\omega}_3 \cos \theta \right) = 0 \quad (1) \]

\[ \frac{d}{dt} p_{\psi} = I_3 \frac{d}{dt} \left( \dot{\psi} + \dot{\phi} \cos \theta \right) = 0 \quad (2) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = I_1 \ddot{\theta} - I_1 \dot{\phi}^2 \sin \theta \cos \theta + I_3 \left( \dot{\psi} + \dot{\phi} \cos \theta \right) \dot{\phi} \sin \theta = 0 . \quad (3) \]

From the first of these \( \dot{\phi} = (L_3 - \bar{L}_3 \cos \theta)/(I_1 \sin^2 \theta) \). Inserting this into the Lagrangian function gives

\[ L = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2 I_1 \sin^2 \theta} (L_3 - \bar{L}_3 \cos \theta)^2 + \frac{1}{2 I_3} \bar{L}_3^2 = E = \text{const.} \quad (4) \]

If, on the other hand, \( \dot{\phi} \) is inserted into the third equation of motion (3), one obtains

\[ I_1 \ddot{\theta} - \frac{\cos \theta}{I_1 \sin^2 \theta} (L_3 - \bar{L}_3 \cos \theta)^2 + \frac{1}{I_1 \sin \theta} \bar{L}_3 \left( L_3 - \bar{L}_3 \cos \theta \right) = 0 , \]

which is nothing but the time derivative of eq. (4).

3.5 We choose the 3-axis to be the symmetry axis of the torus. Let \( (r', \phi) \) be polar coordinates in a section of the torus and \( \psi \) be the azimuth in the plane of the torus as sketched in Fig. 17. The coordinates \( (r', \psi, \phi) \) are related to the cartesian coordinates by

\[ x_1 = (R + r' \cos \phi) \cos \psi , \quad x_2 = (R + r' \cos \phi) \sin \psi , \quad x_3 = r' \sin \phi . \]

The Jacobian is

\[ \frac{\partial (x_1, x_2, x_3)}{\partial (r', \psi, \phi)} = r' (R + r' \cos \phi) . \]

The volume of the torus is calculated to be

\[ V = \int_0^R dr' \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi r' (R + r' \cos \phi) = 2\pi^2 r^2 R , \]
so that the mass density is \( \rho_0 = M / (2\pi r^2 R) \). Thus

\[
I_3 = \int d^3 x \rho_0 \left( x_1^2 + x_2^2 \right) = \rho_0 \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^r dr' r'(R + r' \cos \phi)^3 \\
= M \left( R^2 + \frac{3}{4} r^2 \right)
\]

\[
I_1 = \int d^3 x \rho_0 \left( x_1^2 + x_3^2 \right) = \rho_0 \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^r dr' r'(R + r' \cos \phi) \\
\cdot \left( (R + r' \cos \phi)^2 \sin^2 \psi + r'^2 \sin^2 \phi \right) = \frac{1}{2} M \left( R^2 + \frac{5}{4} r^2 \right).
\]

![Fig. 17.](image)

![Fig. 18.](image)

3.6 In the first position we have \( I_3^{(a)} = 2(2/5)(MR^2 + mr^2) \) and \( \omega_3^{(a)} = L_3 / I_3^{(a)} \). In the second position the contribution of the two smaller balls is calculated by means of Steiner’s theorem, \( I_i' = I_i + m(a^2 - a_i^2) \), now with \( I_3 = 2mr^2 / 5 \), \( a = \pm(b/2)\hat{e}_1 \):

\[
I_3^{(b)} = 2 \left( (2/5)(MR^2 + mr^2) + mb^2 / 4 \right).
\]
It follows that \( \omega(b) = L_3/I_3(b) \), and \( \omega_3(a)/\omega_3(b) = 1 + mb^2/(2I_3(a)) \). One rotates faster if the arms are close to one’s body than if they are stretched out. Making use of Steiner’s theorem once more we finally calculate \( I_1 = I_2 \). For the two arrangements one obtains the result

\[
I_1^{(a)} = I_3(a) + \frac{1}{2} a^2 \left( M + \frac{1}{9} m \right), \quad I_1^{(b)} = \frac{4}{5} \left( MR^2 + mr^2 \right) + \frac{5}{9} Ma^2.
\]

3.7 The relation between density and mass reads

\[
M = \int \varrho(r) \, d^3r = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \int_0^\infty r^2 \, dr \varrho(r, \theta, \phi).
\]

In our case, where \( \varrho \) depends on \( \theta \) only, this means

\[
M = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \int_0^{R(\theta)} r^2 \, dr \varrho_0 \frac{2\pi}{3} \varrho_0 \int_0^\pi \sin \theta \, d\theta \, R(\theta)^3.
\]

The moments of inertia \( I_3, I_1 = I_2 \) are calculated from the formulas

\[
I_3 = \int d^3x \varrho r^2 (1 - \cos^2 \theta), \quad I_1 + I_2 + I_3 = 2 \int d^3x \varrho r^2,
\]

(i) Integration gives the results

\[
M = \frac{4\pi}{3} \varrho_0 R_0^3 (1 + \alpha)^2, \quad \text{i.e.,} \quad \varrho_0 = \frac{3}{4\pi} \frac{M}{R_0^3 (1 + \alpha^2)}.
\]

\[
I_1 = I_2 = \frac{2MR_0^2}{5 (1 + \alpha^2)} \left\{ 1 + 4\alpha^2 + \frac{9}{7} \alpha^4 \right\}, \quad I_3 = \frac{2MR_0^2}{5 (1 + \alpha^2)} \left\{ 1 + 2\alpha^2 + \frac{3}{7} \alpha^4 \right\}.
\]

(ii) Substituting \( z = \cos \theta \) the integrals are easily evaluated. With the abbreviation \( \gamma := \sqrt{5/16\pi} \beta \) we obtain

\[
M = \frac{4\pi}{3} \varrho_0 R_0^3 \left( \frac{16}{35} \gamma^3 + \frac{12}{5} \gamma^2 + 1 \right),
\]

that is

\[
\varrho_0 = \frac{3}{4\pi} \frac{M}{R_0^3} \left( \frac{16}{35} \gamma^3 + \frac{12}{5} \gamma^2 + 1 \right)^{-1}.
\]

\[
I_1 = I_2 = \frac{2MR_0^2}{5} \cdot \frac{1 + \gamma + 64\gamma^2/7 + 8\gamma^3 + 688\gamma^4/77 + 2512\gamma^5/1001}{1 + 12\gamma^2/5 + 16\gamma^3/35},
\]

\[
I_3 = \frac{2MR_0^2}{5} \cdot \frac{1 - 2\gamma + 40\gamma^2/7 - 16\gamma^3/7 + 208\gamma^4/77 - 32\gamma^5/1001}{1 + 12\gamma^2/5 + 16\gamma^3/35}.
\]
3.8 The (principal) moments of inertia are the eigenvalues of the given tensor and, hence, are the roots of the characteristic polynomial \( \det(\lambda \mathbb{I} - J) \). Calculating this determinant we are led to the cubic equation \( \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \). Its solutions are \( \lambda_1 = \lambda_2 = 1, \lambda_3 = 2 \). The inertia tensor in diagonal form reads

\[
\overset{\circ}{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
\]

We write \( J = R \overset{\circ}{J} R^T \) and decompose the rotation matrix according to \( R(\psi, \theta, \phi) = R_3(\psi)R_2(\theta)R_3(\phi) \). As the factor \( R_3(\phi) \) leaves \( J \) invariant we can choose \( \phi = 0 \). With

\[
R_2(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \quad \text{and} \quad R_3(\psi) = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

we calculate

\[
R \overset{\circ}{J} R^T = \mathbb{I} + \begin{bmatrix} \cos^2 \psi \sin^2 \theta & -\sin \psi \cos \psi \sin^2 \theta & -\cos \psi \cos \theta \sin \theta \\ -\sin \psi \cos \psi \sin^2 \theta & \sin^2 \psi \sin^2 \theta & \sin \psi \cos \theta \sin \theta \\ -\cos \psi \cos \theta \sin \theta & \sin \psi \cos \theta \sin \theta & \cos^2 \theta \end{bmatrix}.
\]

If this is set equal to \( J \) as given, we find \( \cos^2 \theta = 3/8 \), and, with the following choice of signs for \( \theta : \cos \theta = \sqrt{3}/(2\sqrt{2}) \) and \( \sin \theta = \sqrt{5}/(2\sqrt{2}) \), the result is \( \cos \psi = 1/\sqrt{5}, \sin \psi = -\sqrt{4/5} \).

3.9 (i) \( \varrho(r) = \varrho_0(a - |r|) \). The total mass is equal to the volume integral of \( \varrho(r) \), viz.

\[
M = \frac{4\pi}{3} a^3 \varrho_0 \Rightarrow \varrho_0 = \frac{3M}{4\pi a^3}.
\]

(ii) We choose the \( 3 \)-axis to be the axis of rotation. Let the coordinate in the body fixed system be \( (x, y, z) \), the coordinates in the space fixed system are

\[
x' = x \cos \omega t - (y + a) \sin \omega t, \quad y' = x \sin \omega t + (y + a) \cos \omega t, \quad z' = z.
\]

![Fig. 19](image-url)
Inverting these equations we have

\[
x = x' \cos \omega t + y' \sin \omega t, \quad y = -x' \sin \omega t + y' \cos \omega t - a,
\]

whence

\[
x^2 + y^2 = x'^2 + y'^2 + a^2 + 2a(x' \sin \omega t - y' \cos \omega t).
\]

From this we get

\[
\varrho(r', t) = \varrho_0 \theta \left( a - \sqrt{r'^2 + a^2 + 2a(x' \sin \omega t - y' \cos \omega t)} \right).
\]

(iii) In the case of a homogeneous sphere the inertia tensor is diagonal, all three moments of inertia are equal, \( I_1 = I_2 = I_3 \equiv I \). Hence,

\[
3I = I_1 + I_2 + I_3 = 2 \int d^3r \varrho(r)r^2 = \frac{6M a^2}{5}.
\]

Making use of Steiner’s theorem (M3.23) we find

\[
I'_3 = I_3 + M \left( a^2 \delta_{33} - a_3^2 \right) = \frac{7M a^2}{5}.
\]

3.10 (i) The volume of the cylinder is \( V = \pi r^2 h \), hence the mass density is \( \varrho_0 = m / (\pi r^2 h) \). The moment of inertia relevant for rotations about the symmetry axis is best calculated using cylindrical coordinates,

\[
I_3 = \varrho_0 \int_0^{2\pi} d\phi \int_0^h dz \int_0^r \varrho^3 d\varrho = \frac{1}{2} mr^2.
\]

Call \( q(t) \) the projection of the center-of-mass’ orbit onto the inclined plane. When the center-of-mass moves by an amount \( dq \), the cylinder rotates by an angle \( d\phi = dq/r \). Therefore, the total kinetic energy is

\[
T = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} I_3 \frac{\dot{q}^2}{r^2} = \frac{3}{4} m \dot{q}^2.
\]

(ii) A Lagrangian function is

\[
L = T - U = 3m \dot{q}^2 / 4 - mg(q_0 - q) \sin \alpha,
\]

where \( q_0 \) is the length of the inclined plane and \( \alpha \) its angle of inclination. The equation of motion reads \( 3m \ddot{q} / 2 = mg \sin \alpha \), the general solution being \( q(t) = q(0) + v(0)t + (g \sin \alpha) t^2 / 3 \).

3.11 (i) The rotation \( R(\phi \cdot \hat{\phi}) \) is a right-handed rotation by an angle \( \phi \) about the direction \( \hat{\phi} \), with \( 0 \leq \phi \leq \pi \). Any desired position is reached by means of rotations about \( \hat{\phi} \) and \( -\hat{\phi} \).
(ii) With \( \hat{\phi} \) an arbitrary direction in \( \mathbb{R} \), and with \( \phi \) between 0 and \( \pi \), the parameter space \( (\phi, \hat{\phi}) \) is the ball \( D^3 \) (surface and interior of the unit sphere in \( \mathbb{R}^3 \)). Every point \( p \in D^3 \) represents a rotation, the direction \( \hat{\phi} \) being given by the polar coordinates of \( p \), and the angle \( \phi \) being given by its distance from center. Note, however, that \( A : (\hat{\phi}, \phi = \pi) \) and \( B : (-\hat{\phi}, \phi = \pi) \) represent the same rotation.

(iii) There are two types of closed curves in \( D^3 \): curves of the type of \( C_1 \) as shown in Fig. 20, which can be contracted, by a continuous deformation, to a point, and curves such as \( C_2 \), which do not have this property. \( C_2 \) connects the antipodes \( A \) and \( B \). As these points represent the same rotation, \( C_2 \) is a closed curve. Any continuous deformation of \( C_2 \) which shifts \( A \) to \( A' \) also shifts \( B \) to \( B' \), the antipodal point of \( A' \).

While \( C_1 \) contains no jumps between antipodes, \( C_2 \) contains one such jump. One easily convinces oneself, by means of a drawing, that any closed curve with an even number of antipodal jumps can be deformed continuously into \( C_1 \) or, equivalently, into a point.

Take the example of a closed curve with two such jumps as shown in Fig. 21. One can let \( A_1 \) move to \( B_2 \) in such a way that the arc \( B_1 A_2 \) goes to zero, the sections \( A_1 B_1 \) and \( A_2 B_2 \) become equal and opposite so that the curve \( A_1 B_2 \) becomes like \( C_1 \) in Fig. 20. In a similar fashion one shows that all closed curves with an odd number of jumps can be continuously deformed into \( C_2 \). (One says that the two types of curves form homotopy classes.)
3.12 We do the calculation for the example of (M3.93). Equations (M3.89) yield expressions for the components of angular momentum in the body fixed system.

Making use of the relation \( \{ p_i, f(q_j) \} = \delta_{ij} f'(q_j) \) which follows from the definition of the Poisson brackets, we calculate readily

\[
\{ \bar{L}_1, \bar{L}_2 \} = \left\{ \begin{array}{c} p_\phi \frac{\sin \psi}{\sin \theta} - p_\psi \sin \psi \cot \theta + p_\theta \cos \psi, \\
p_\phi \frac{\cos \psi}{\sin \theta} - p_\psi \cos \psi \cot \theta - p_\theta \sin \psi \end{array} \right\}
\]

\[
= p_\theta \frac{\cos \theta}{\sin^2 \theta} \left( -\{ \sin \psi, p_\psi \cos \psi \} - \{ p_\psi \sin \psi, \cos \psi \} \right)
\]

\[
+ p_\phi \left( -\sin^2 \psi \left\{ \frac{1}{\sin \theta}, p_\theta \right\} + \cos^2 \psi \left\{ p_\theta, \frac{1}{\sin \theta} \right\} \right)
\]

\[
+ \cot^2 \left( p_\psi \sin \psi, p_\psi \cos \psi \right) + \left( \sin \theta \left\{ p_\psi \cos \theta, p_\theta \sin \psi \right\} \right)
\]

\[
- \cos \psi \left\{ p_\theta \cos \psi, p_\psi \cot \theta \right\} \right) \right) \right)
\]

\[
= + p_\phi \frac{\cos \theta}{\sin^2 \theta} - p_\phi \frac{\cos \theta}{\sin^2 \theta} - p_\psi \cot^2 \theta + p_\phi \frac{1}{\sin^2 \theta}
\]

\[
= p_\psi = \bar{L}_3.
\]

Chapter 4: Relativistic Mechanics

4.1 (i) Let the neutral pion fly in the 3-direction with velocity \( v = v_0 \hat{e}_3 \). The full energy–momentum vector of the pion is

\[
q = \left( \frac{1}{c} E_q, q \right) = (\gamma_0 m_\pi c, \gamma_0 m_\pi v) = \gamma_0 m_\pi c \left( 1, \beta_0 \hat{e}_3 \right),
\]

where \( \beta_0 = v_0/c, \gamma_0 = (1 - \beta_0^2)^{-1/2} \). The special Lorentz transformation which takes us to the rest system of the pion, is

\[
\mathbf{L}_{-v} = \left( \begin{array}{ccc} \gamma_0 & 0 & -\gamma_0 \beta_0 \\ 0 & 1 & 0 \\ -\gamma_0 \beta_0 & 0 & \gamma_0 \end{array} \right).
\]

Indeed, \( \mathbf{L}_{-v} q = q^* = (m_\pi c, 0) \).

(ii) In the rest system (Fig. 22, left-hand side) the two photons have the four-momenta \( k^*_i = (E^*_i/c, k^*_i) \), \( i = 1, 2 \). Conservation of energy and momentum requires \( q^* = k^*_1 + k^*_2 \), i.e. \( E^*_1 + E^*_2 = m_\pi c^2 \) and \( k^*_1 + k^*_2 = 0 \). As photons are massless, \( E^*_i = |k^*_i| c \), and, as \( k^*_1 = -k^*_2 \), one has \( E^*_1 = E^*_2 \). Denote the absolute value of the spatial momenta by \( \kappa^* \). Then \( |k^*_1| = |k^*_2| = \kappa^* = m_\pi c/2 \).
In the rest system the decay is isotropic. In the laboratory system only the direction $\hat{e}_3$ of the pion’s momentum is singled out. Therefore, in this system the decay distribution is symmetric with respect to the 3-axis. We first study the situation in the (1, 3)-plane and then obtain the complete answer by rotation about the 3-axis. We have $(k_1^*)_3 = \kappa\cos\theta^* = -(k_2^*)_3$, $(k_1^*)_1 = \kappa\sin\theta^* = -(k_2^*)_1$, while the 2-components vanish. In the laboratory system we have $k_i = L\gamma_0 k_i^*$, viz.

$$\frac{1}{2} E_1 = \gamma_0 \kappa^* (1 + \beta_0 \cos\theta^*) , \quad \frac{1}{c} E_2 = \gamma_0 \kappa^* (1 - \beta_0 \cos\theta^*) ,$$

$$(k_1)_1 = (k_1^*)_1 = \kappa\sin\theta^* , \quad (k_2)_1 = (k_2^*)_1 = -\kappa\sin\theta^* ,$$

$$(k_1)_3 = \gamma_0 \kappa^* (\beta_0 + \cos\theta^*) , \quad (k_2)_3 = \gamma_0 \kappa^* (\beta_0 - \cos\theta^*) ,$$

$$(k_1)_2 = 0 = (k_2)_2 .$$

In the laboratory system we then find

$$\tan\theta_1 = \frac{(k_1)_1}{(k_1)_3} = \frac{\sin\theta^*}{\gamma_0 (\beta_0 + \cos\theta^*)} ; \quad \tan\theta_2 = \frac{-\sin\theta^*}{\gamma_0 (\beta_0 - \cos\theta^*)} .$$

Examples:

a) $\theta^* = 0$ (forward emission of one photon, backward emission of the other): From the formulas above one finds $E_1 = m_\pi c^2 \gamma_0 (1 + \beta_0)/2$, $E_2 = m_\pi c^2 \gamma_0 (1 - \beta_0)/2$, $k_1 = m_\pi c\gamma_0 (\beta_0 + 1)\hat{e}_3/2$, $k_2 = m_\pi c\gamma_0 (\beta_0 - 1)\hat{e}_3/2$, and, as $\beta_0 \leq 1$, $\theta_1 = 0$, $\theta_2 = \pi$.

b) $\theta^* = \pi/2$ (transverse emission): In this case $E_1 = E_2 = m_\pi c^2 \gamma_0/2$, $k_1 = m_\pi c(\hat{e}_1 + \gamma_0 \beta_0\hat{e}_3)/2$, $k_2 = m_\pi c(-\hat{e}_1 + \gamma_0 \beta_0\hat{e}_3)/2$, $\tan\theta_1 = \tan\theta_2 = 1/(\gamma_0 \beta_0)$.

c) $\theta^* = \pi/4$ and $\beta_0 = 1/\sqrt{2}$, i.e., $\gamma_0 = \sqrt{2}$: In this case one finds $E_1 = 3m_\pi c^2 \gamma_0/4$, $E_2 = m_\pi c^2 \gamma_0/4$, $k_1 = m_\pi c(\hat{e}_1 - 2\sqrt{2}\hat{e}_3)/2\sqrt{2}$, $k_2 = -m_\pi c\hat{e}_1/(2\sqrt{2})$, $\theta_1 = \arctan(1/(2\sqrt{2})) \approx 0.108\pi$, $\theta_2 = \pi/2$.

In the rest system the decay distribution is isotropic, which means that the differential probability $d\Gamma$ for $k_1^*$ to lie in the interval $d\Omega^* = \sin\theta^* d\theta^* d\phi^*$ is independent of $\theta^*$ and of $\phi^*$. (To see this enclose the decaying pion by a unit
sphere. If one considers a large number of decays then photon 1 will hit every surface element \( d\Omega^* \) on that sphere with equal probability.) Thus,

\[
d\Gamma = \Gamma_0 d\Omega^* \quad \text{with} \quad \Gamma_0 = \text{const}.
\]

In the laboratory system the analogous distribution is no longer isotropic. It is distorted in the direction of flight but is still axially symmetric about that axis. We have

\[
\frac{1}{\Gamma_0} \frac{d\Gamma}{d\Omega} = \frac{|d\Omega^*|}{d\Omega} \quad \text{where} \quad \frac{d\Omega^*}{d\Omega} = \frac{\sin \theta^* \, d\theta^*}{\sin \theta \, d\theta}.
\]

The factor \( \sin \theta^*/\sin \theta \) is calculated from the above formula for \( \tan \theta_1 \), viz.

\[
\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sin \theta^*}{\gamma_0 (\beta_0 + \cos \theta^*)},
\]

and the derivative \( d\theta/d\theta^* \) is obtained from \( \theta = \arctan(\sin \theta^*/\gamma_0 (\beta_0 + \cos \theta^*)) \) by making use of the relation \( \gamma_0^2 \beta_0^2 = \gamma_0^2 - 1 \). One finds

\[
\frac{d\Omega^*}{d\Omega} = \gamma_0^2 \left(1 + \beta_0 \cos \theta^*\right)^2.
\]

The cosine of \( \theta^* \) is expressed in terms of the corresponding laboratory angle by the formula

\[
\cos \theta^* = \frac{\cos \theta - \beta_0}{1 - \beta_0 \cos \theta}.
\]

The shift of the angular distribution is well illustrated by the graph of the function

\[
F(\theta) := \frac{d\Omega^*}{d\Omega} = \gamma_0 \left(1 + \beta_0 \frac{\cos \theta - \beta_0}{1 - \beta_0 \cos \theta}\right)^2
\]

for different values of \( \beta_0 \). Quite generally we have \( dF/d\theta|_{\theta=0} = 0 \). For \( \beta_0 \to 1 \) the value \( F(0) = (1 + \beta_0)/(1 - \beta_0) \) tends to infinity. For small argument \( \theta = \varepsilon \ll 1 \), on the other hand,

\[
F(\varepsilon) \approx \frac{1 + \beta_0}{1 - \beta_0} \left(1 - \frac{\varepsilon^2}{1 - \beta_0}\right),
\]

which means that for \( \varepsilon^2 \approx (1 - \beta_0) \) \( F \) becomes very small. Therefore, when \( \beta_0 \to 1 \) the function \( F(\theta) \) falls off very quickly with increasing \( \theta \). Figure 23 shows the examples \( \beta_0 = 0, \beta_0 = 1/\sqrt{2} \), and \( \beta_0 = 11/13 \).

4.2 Let the energy-momentum four-vectors of \( \pi, \mu \), and \( \nu \) be \( q, p \), and \( k \), respectively. We have always \( q = p + k \). In the pion’s rest system

\[
q = (m_\pi c, q = 0), \quad p = \left(\frac{1}{c} E_p^*, p^*\right), \quad k = \left(\frac{1}{c} E_k^*, -p^*\right).
\]
If \( \kappa^* := |\mathbf{p}^*| \) denotes the magnitude of the momentum of the muon and of the neutrino, then

\[
E^*_k = \kappa^* c = \frac{m^2_\pi - m^2_\mu}{2m_\pi} c^2, \quad E^*_p = \sqrt{(\kappa^* c)^2 + (m_\mu c)^2} = \frac{m^2_\pi + m^2_\mu}{2m_\pi} c^2.
\]

In the laboratory system the situation is as follows: The pion has velocity \( \mathbf{v}_0 = v_0 \hat{e}_3 \) and, therefore,

\[
q = (E_q/c, q) = (\gamma_0 m_\pi c, \gamma_0 m_\pi v_0) = \gamma_0 m_\pi c (1, \beta_0 \hat{e}_3)
\]

with \( \beta_0 = v_0/c, \ \gamma_0 = 1/\sqrt{1 - \beta^2_0} \). It is sufficient to study the kinematics in the \((1, 3)\)-plane. The transformation from the pion’s rest system to the laboratory system yields

\[
\frac{1}{c} E_p = \gamma_0 \left( \frac{1}{c} E^*_p + \beta_0 p^*_3 \right) = \gamma_0 \left( \frac{1}{c} E^*_p + \beta_0 \kappa^* \cos \theta^* \right),
\]

\[
p^1 = p^{*1},
\]

\[
p^2 = p^{*2} = 0,
\]

\[
p^3 = \gamma_0 \left( \frac{1}{c} \beta_0 E^*_0 + p^*_3 \right) = \gamma_0 \left( \frac{1}{c} \beta_0 E^*_p + \kappa^* \cos \theta^* \right)
\]

and, therefore, the relation between the angles of emission \( \theta^* \) and \( \theta \) is (cf. Fig. 24)
\[ \tan \theta = \frac{p^1}{p^3} = \frac{\kappa^* \sin \theta^*}{\gamma_0 (\beta_0 E^*_p/c + \kappa^* \cos \theta^*)} , \]  

(1)

or

\[ \tan \theta = \frac{\left( m^2_\pi - m^2_\mu \right) \sin \theta^*}{\gamma_0 \left( \beta_0 \left( m^2_\pi + m^2_\mu \right) + \left( m^2_\pi - m^2_\mu \right) \cos \theta^* \right)} . \]  

(2)

Making use of \( \beta^* := c \kappa^*/E^*_p = (m^2_\pi - m^2_\mu)/(m^2_\pi + m^2_\mu) \) (this is the beta factor of the muon in the rest system of the pion), eq. (1) is rewritten

\[ \tan \theta = \frac{\beta^* \sin \theta^*}{\gamma_0 (\beta_0 + \beta^* \cos \theta^*)} . \]  

(3)

There exists a maximal angle \( \theta \) if the muons which are emitted backwards in the pion’s rest system (\( \theta^* = \pi \)), have momenta

\[ p^3 = \gamma_0 E^*_p / c (\beta_0 + \beta^* \cos \theta^*) = \gamma_0 E^*_p / c (\beta_0 - \beta^*) > 0 , \]

i.e., if \( \beta_0 > \beta^* \). The magnitude of the maximal angle is obtained from the condition \( d \tan \theta / d\theta^* = 0 \) which gives \( \cos \beta^* = -\beta^*/\beta_0 \) and, finally,

\[ \tan \theta_{\text{max}} = \frac{\beta^* \sqrt{\beta_0^2 - \beta^{*2}}}{\gamma_0 (\beta_0^2 - \beta^{*2})} = \frac{\beta^* \sqrt{1 - \beta_0^2}}{\gamma_0 \sqrt{\beta_0^2 - \beta^{*2}}} = \frac{\beta^*}{\sqrt{\beta_0^2 - \beta^{*2}}} . \]  

(4)

4.3 The variables \( s \) and \( t \) are the squared norms of four-vectors and are thus invariant under Lorentz transformations. The same is true for \( u = c^2 (q_A - q_B')^2 \). For our calculations it is convenient to choose units such that \( c = 1 \). It is not difficult to re-insert the constant \( c \) in the final results. (This is important if we wish to expand in terms of \( v/c \).) To reconstruct those factors one must keep in mind that terms like (mass times \( c^2 \)) and (momentum times \( c \)) have the physical dimension of energy.

Conservation of energy and momentum means that the four equations

\[ q_A + q_B = q'_A + q'_B \]  

(1)
must be satisfied. This means that the variables \( s, t, u \) can each be expressed in two different ways (now setting \( c = 1 \)):

\[
\begin{align*}
    s &= (q_A + q_B)^2 = (q'_A + q'_B)^2 \\
    t &= (q_A - q'_A)^2 = (q'_B - q_B)^2 \\
    u &= (q_A - q'_B)^2 = (q'_A - q_B)^2.
\end{align*}
\]

(i) In the center-of-mass system we have

\[
\begin{align*}
    q_A &= (E^*_A, q^*_A) , \quad q_B = (E^*_B, -q^*_B) , \\
    q'_A &= (E'^*_A, q'^*_A) , \quad q'_B = (E'^*_B, -q'^*_B) ,
\end{align*}
\]

where \( E^*_A = \sqrt{m_A^2 + (q^*_A)^2} \) etc., with \( q^*_A = |q^*_A| \). Like in the nonrelativistic case energy conservation requires the magnitudes of the three-momenta in the center-of-mass system to be equal. However, the simple nonrelativistic formula

\[
(q^*)_{\text{n.r.}} = \frac{m_B}{m_A + m_B} |q^*_A| 
\]

that follows from eq. (M1.79a) no longer holds. This is so because neither the nonrelativistic energy

\[
T_r = \frac{m_A + m_B}{2m_A m_B} (q^*)_{\text{n.r.}}^2
\]

nor the quantity

\[
\frac{(q_A + q_B)^2}{2(m_A + m_B)}
\]

are conserved. We have

\[
\begin{align*}
    s &= (E^*_A + E^*_B)^2 = m_A^2 + m_B^2 + 2(q^*_A)^2 \\
    &\quad + 2\sqrt{(q^*_A)^2 + m_A^2} ((q^*_A)^2 + m_B^2) \cdot \\
    &\quad (q^*_A)^2 + m_B^2 + m_B^2)
\end{align*}
\]

Thus, \( s \) is the square of the total energy in the center-of-mass system. Reintroducing the velocity of light,

\[
s = m_A c^4 + m_B c^4 + 2(q^*_A)^2 c^2 + 2\sqrt{(q^*_A)^2 c^2 + m_A^2 c^4} \left( (q^*_A)^2 c^2 + m_B^2 c^4 \right)
\]

In a first step we check that \( s \), when expanded in terms of \( 1/c \), gives the correct nonrelativistic kinetic energy \( T_r \) of relative motion (except for the rest masses, of course)

\[
s \approx \left( m_A c^2 + m_B c^2 \right)^2 \left( 1 + \frac{1}{m_A m_B} (q^*_A)^2 / c^2 + O\left( \frac{(q^*_A)^4}{m^4 c^4} \right) \right),
\]
and, thus,
\[ \sqrt{s} \approx m_A c^2 + m_B c^2 + \frac{m_A + m_B}{2m_A m_B} (q^*)^2 + O\left(\frac{(q^* c)^4}{(mc^2)^4}\right). \]

The magnitude of the center-of-mass momentum is obtained from (6)
\[ q^*(s) = \frac{1}{2\sqrt{s}} \sqrt{s - (m_A + m_B)^2} \left(s - (m_A - m_B)^2\right). \tag{7} \]

Clearly, the reaction can take place only if \( s \) is at least equal to the square of the sum of the rest energies,
\[ s \geq s_0 := (m_A + m_B)^2 \geq \left(m_A c^2 + m_B c^2\right)^2. \]

\( s_0 \) is called the threshold of the reaction. For \( s = s_0 \) the momentum vanishes, which means that at threshold the kinetic energy of relative motion vanishes.

The variable \( t \) is expressed in terms of \( q^* \) and the scattering angle \( \theta^* \) as follows:
\[ t = (q_A - q'_A)^2 = q_A^2 + q_A'^2 - 2q_A \cdot q_A' = 2m_A^2 - 2E^*_A E_A'^* + 2q^* \cdot q'^* . \]

As the magnitudes of \( q^* \) and of \( q'^* \) are equal, \( E^*_A = E_B^* \). Therefore,
\[ t = -2(q^*)^2(1 - \cos \theta^*). \tag{8} \]

Except for the sign, \( t \) is the square of the momentum transfer \((q^* - q'^*)\) in the center-of-mass system. For fixed \( s \geq s_0 \), \( t \) varies as follows
\[ -4(q^*)^2 \leq t \leq 0 . \]

**Examples:**

a) \( e^- + e^- \rightarrow e^- + e^- \)
\[ s \geq s_0 = 4 \left(m_e c^2\right)^2, \quad -(s - s_0) \leq t \leq 0 . \]

b) \( \nu + e^- \rightarrow e^- + \nu \)
\[ s \geq s_0 = \left(m_e c^2\right)^2, \quad -\frac{1}{s} (s - s_0)^2 \leq t \leq 0 . \]

(ii) Calculating \( s + t + u \) from the formulas (2)–(4) and making use of (1), we find \( s + t + u = 2(m_A^2 + m_B^2)c^4 \). More generally, for the reaction \( A + B \rightarrow C + D \), one finds
\[ s + t + u = \left(m_A^2 + m_B^2 + m_C^2 + m_D^2\right)c^4 . \]
4.4 In the laboratory system

\[ q_A = (E_A \cdot q_A), \quad q_B = (m_B c^2, 0), \]
\[ q'_A = (E'_A \cdot q'_A), \quad q'_B = (E'_B \cdot q'_B), \]

(1)

The scattering angle \( \theta \) is the angle between the three-vectors \( q_A \) and \( q'_A \). From eq. (3) of the solution to Exercise 4.3 above (using \( c = 1 \)),

\[ t = q_A^2 + q_A^2 - 2q_Aq'_A = 2m_A^2 - 2E_AE'_A + 2|q_A||q'_A| \cos \theta . \]  

(2)

Equation (8) of Exercise 4.3 above gives an alternative expression for \( t \). The aim is now to express the laboratory quantities \( E_A, E'_A, |q_A|, |q'_A| \) in terms of the invariants \( s \) and \( t \). Using (1) \( s \) is found to be, in the laboratory system,

\[ s = m_A^2 + m_B^2 + 2E_Am_B, \]  

(3)

From this, using \( q_A^2 = E_A^2 - m_A^2 \),

\[ |q_A| = \frac{1}{2m_B} \sqrt{(s - (m_A + m_B)^2)} \]  

(4)

with \( q^* \) as given by eq. (7) of Exercise 4.3 above. We now calculate \( t = (q_B - q'_B)^2 \) in the laboratory system and find \( E'_B = (2m_B^2 - t)/(2m_B) \) and from this \( E'_A = E_A + m_B - E'_B \) to be equal to

\[ E'_A = \frac{1}{2m_B} \left( s + t - m_A^2 - m_B^2 \right) = E_A + \frac{t}{2m_B}, \]  

(5)

and, eventually, from \( q'_A^2 = E'_A^2 - m_A^2 \)

\[ |q'_A| = \frac{1}{2m_B} \sqrt{(s + t - (m_A + m_B)^2)} \]  

(6)

From (2)

\[ \cos \theta = \left( E_AE'_A - m_A^2 + \frac{t}{2} \right) \frac{1}{|q_A||q'_A|} \]  

This is used to calculate \( \sin \theta \) and \( \tan \theta \), replacing all quantities which are not invariants by the expressions (3)–(6). With the abbreviations \( \Sigma := (m_A + m_B)^2 \) and \( \Delta := (m_A - m_B)^2 \) we find

\[ \tan \theta = \frac{2m_B \sqrt{-t \left( s + (s - \Sigma)(s - \Delta) \right)}}{(s - \Sigma)(s - \Delta) + t \left( s - m_A^2 + m_B^2 \right)} \]  

Finally, \( \cos \theta^* \) and \( \sin \theta^* \) can also be expressed in terms of \( s \) and \( t \), starting from eqs. (8) and (7) of Exercise 4.3 above,
\[
\cos \theta^* = \frac{2st + (s - \Sigma)(s - \Delta)}{(s - \Sigma)(s - \Delta)}, \\
\sin \theta^* = 2\sqrt{s} \frac{\sqrt{-t(st + (s - \Sigma)(s - \Delta))}}{(s - \Sigma)(s - \Delta)}.
\]

Replace the square root in the numerator of \(\tan \theta\) by \(\sin \theta^*\) and insert \(t\) in the denominator, as a function of \(\cos \theta^*\), to obtain the final result

\[
\tan \theta = \frac{2m_B \sqrt{s}}{s - m_A^2 + m_B^2} \frac{\sin \theta^*}{\cos \theta^* + \frac{s + m_A^2 - m_B^2}{s - m_A^2 + m_B^2}}.
\]

For \(s \approx (m_A + m_B)^2\) one recovers the nonrelativistic result (M1.80). The case of two equal masses is particularly interesting. With \(m_A = m_B \equiv m\)

\[
\tan \theta = \frac{2m}{\sqrt{s}} \tan \frac{\theta^*}{2}.
\]

As \(\sqrt{s} \geq 2m\) the scattering angle \(\theta\) is always smaller than in the nonrelativistic situation.

4.5 If we wish to go from the rest system of a particle to another system where its four-momentum is \(p = (E/c, p)\), we have to apply a special Lorentz transformation \(L(v)\) with \(v\) related to \(p\) by \(p = m\gamma v\). Solving for \(v\),

\[
v = \frac{pc}{\sqrt{p^2 + m^2c^2}} = \frac{pc^2}{E}.
\]

Insertion into eq. (M4.41) and application to the vector \((0, s)\) gives

\[
s = L(v)(0, s) = \left(\frac{\gamma}{c} s \cdot v, s + \frac{\gamma^2}{c^2(1 + \gamma)} v \cdot sv\right).
\]

As \(s_\alpha p^\alpha\) is a Lorentz scalar, and hence is independent of the frame of reference that one uses, this quantity may be evaluated most simply in the rest system. It is found to vanish there and, hence, in any frame of reference.

4.6 In either case the coordinate system can be chosen such that the \(y-\) and \(z-\)components of the four-vector vanish and the \(x-\)component is positive, i.e., \(z = (z^0, z^1, 0, 0)\) with \(z^1 > 0\). If \(z^0\) is smaller than zero we apply the time reversal operation (M4.30) so that, from here on, we assume \(z^0 > 0\), without loss of generality.

(i) A light-like vector has \(z^2 = 0\) and, hence, \(z^0 = z^1\). We apply a boost with parameter \(\lambda\) along the \(x-\)direction, cf. (M4.39). In order to obtain the desired form of the four-vector we must have

\[
z^0 \cosh \lambda - z^0 \sinh \lambda = 1 \quad \text{or} \quad z^0 e^{-\lambda} = 1,
\]
from which follows $\lambda = \ln z^0$.

(ii) For a space-like vector $z^2 = (z^0)^2 - (z^1)^2 < 0$, i.e., $0 < z^0 < z^1$. Applying a boost with parameter $\lambda$, it is transformed to

$$\left( z^0 \cosh \lambda - z^1 \sinh \lambda, \ z^0 \cosh \lambda - z^1 \sinh \lambda, 0, 0 \right).$$

For the time component to vanish, one must have $\tanh \lambda = z^0 / z^1$. Calculating $\sinh \lambda$ and $\cosh \lambda$ from this yields the assertion $z^1 = \sqrt{-z^2}$.

4.7 The commutation relations (M4.59) can be summarized as follows, making use of the Levi-Civita symbol:

$$[J_p, J_q] = \varepsilon_{pqr} J_r,$$
$$[K_p, K_q] = -\varepsilon_{pqr} J_r,$$
$$[J_p, K_q] = \varepsilon_{pqr} K_r.$$

From this one obtains

$$[A_p, A_q] = \varepsilon_{pqr} A_r,$$
$$[B_p, B_q] = \varepsilon_{pqr} B_r,$$
$$[A_p, B_q] = 0.$$

4.8 The explicit calculation gives

$$P J_i P^{-1} = J_i, \quad P K_j P^{-1} = -K_j.$$

This corresponds to the fact that space inversion does not alter the sense of rotation but reverses the direction of motion.

4.9 The commutators (M4.59) read in this basis

$$[\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk} \hat{J}_k,$$
$$[\hat{J}_i, \hat{K}_j] = i\varepsilon_{ijk} \hat{K}_k,$$
$$[\hat{K}_i, \hat{K}_j] = -i\varepsilon_{ijk} \hat{J}_k.$$

The matrix $J_i$ and $K_j$ being real and skew-symmetric, we have for instance

$$\left( \hat{J}_i^T \right)^* = -i\hat{J}_i = iJ_i = \hat{J}_i.$$

4.10 This exercise is a special case of Exercise 4.11. The result is obtained from there by taking $m_2 = 0 = m_3$.

4.11 Energy conservation implies (again taking $c = 1$)
\[ M = E_1 + E_2 + E_3 = \sqrt{m_1^2 + f^2} + \sqrt{m_2^2 + x^2 f^2} + \sqrt{m_3^2 + (1-x)^2 f^2} \]
\[ = M(x f(x)) . \]

The maximum of \( f(x) \) is found from the equation
\[
0 = \frac{df}{dx} = -\frac{\partial M/\partial x}{\partial M/\partial f} = -\frac{f E_1 - (1-x) E_2}{E_2 E_3 + x^2 E_1 E_3 + (1-x)^2 E_1 E_2} ,
\]
or \( x E_3 = (1-x) E_2 \). Squaring this equation gives \( x^2 (m_3^2 + (1-x)^2 f^2) = (1-x)^2 (m_2^2 + x^2 f^2) \), and from this the condition
\[ x = \frac{m_2}{m_2 + m_3} . \]

Taking into account the condition
\[ E_3 = \frac{1-x}{x} E_2 = \frac{m_3}{m_2} E_2 , \]
one obtains
\[ M - E_1 = \frac{m_2 + m_3}{m_2} E_2 . \]

The square of this yields
\[ M^2 - 2ME_1 + m_1^2 + f^2 = \frac{(m_2 + m_3)^2}{m_2^2} \left( m_2^2 + \frac{m_2^2}{(m_2 + m_3)^2} f^2 \right) \]
and from this
\[ (E_1)_{\text{max}} = \frac{1}{2M} \left( M^2 + m_1^2 - (m_2 + m_3)^2 \right) . \]

Examples:
(i) \( \mu \to e + v_1 + v_2 : m_2 = m_3 = 0, M = m_\mu, m_1 = m_e \). Thus
\[ (E_e)_{\text{max}} = \frac{1}{2m_\mu} \left( m_\mu^2 + m_e^2 \right) c^2 . \]

With \( m_\mu/m_e \approx 206.8 \) one finds \( (E_e)_{\text{max}} \approx 104.4 m_e c^2 \).

(ii) \( n \to p + e + v ; M = m_n, m_1 = m_e, m_2 = m_p, m_3 = 0 \). Therefore one obtains
\[ (E_e)_{\text{max}} = \frac{1}{2m_n} \left( m_n^2 + m_e^2 - m_p^2 \right) c^2 = \frac{1}{2m_n} \left( (2m_n - \Delta) \Delta + m_e^2 \right) c^2 , \]
where \( \Delta := m_n - m_p \). Inserting numerical values yields \( (E_e)_{\text{max}} \approx 2.528 m_e c^2 \).
Thus \( \gamma_{\text{max}} = 2.528 \) and \( \beta_{\text{max}} = \sqrt{\gamma_{\text{max}}^2 - 1}/\gamma_{\text{max}} = 0.918 \). The electron is highly relativistic at the maximal energy.
4.12 The apparent lifetime $\tau^{(v)}$ in the laboratory system is related to the real lifetime $\tau^{(0)}$ by $\tau^{(v)} = \gamma \tau^{(0)}$. During this time the particle, on average, travels a distance

\[ L = v \tau^{(v)} = \beta \gamma \tau^{(0)} c . \]

Now, the product $\beta \gamma$ equals $|p|/(mc^2)$, cf. eq. (M4.83), so that for $|p| = x mc$ there follows the relation

\[ L = x \tau^{(0)} c . \]

For pions, for instance, one has $\tau^{(0)}_\pi c \approx 780 \text{ cm}$.

4.13 From the results of the preceding exercise we obtain $\tau^{(0)}_n c \approx 2.7 \times 10^{13} \text{ cm}$. For $E = 10^{-2} m_n c^2$ one has $x = \sqrt{\gamma^2 - 1} = 0.142$, while for $E = 10^{14} m_n c^2$ one has $x \approx 10^{14}$.

4.14 Let $p_1$, $p_2$ be the energy–momentum four-vectors of the incoming and outgoing electron, respectively, and $k$ that of the photon. Energy and momentum conservation in the reaction $e \rightarrow e + \gamma$ requires $p_1 = p_2 + k$. Squaring this relation and making use of

\[ p_1^2 = m_e c^2 = p_2^2 , \quad k^2 = 0 , \]

one deduces $p_2 \cdot k = 0$. As $k$ is a light-like four-vector this relation can only hold if $p_2$ is light-like, too, i.e., if $p_2^2 = 0$. This is in contradiction with the outgoing electron being on its mass shell, $p_2^2 = m_e^2$. Hence, the reaction cannot take place.

4.15 The first inversion leads from $x^\mu$ to $(R^2/x^2)x^\mu$, the translation that follows leads to $R^2(x^\mu/x^2 + c^\mu)$, and the second inversion, finally, to

\[ x'^\mu = \frac{R^4 \left( x^\mu + x^2 c^\mu \right)}{x^2 R^4 \left( x + x^2 c \right)^2} \cdot \frac{x^\mu + x^2 c^\mu}{1 + 2 (c \cdot x) + c^2 x^2} . \]

The inversion $J$ leaves invariant the two halves of the time-like hyperboloid $x^2 = R^2$, but interchanges those of the space-like hyperboloid $x^2 = -R^2$. The image of the light-cone by the inversion is at infinity. The light-cone as a whole stays invariant under the combined transformation $J \circ \mathcal{T} \circ J$.

4.16 As $L$ does not depend on $q$, the equation of motion for this variable reads

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = m \frac{d}{dt} (\dot{\psi} \dot{q}) = 0 . \]

In turn, $L$ is independent of $\dot{\psi}$. The condition for the action integral to be extremal leads to the following equation for $\psi$,
\[
\frac{\partial L}{\partial \psi} = \frac{1}{2} m \left( \dot{\mathbf{r}}^2 - c_0^2 \frac{\psi^2}{\psi^2} - 1 \right) = 0.
\]

The solutions of this equations are
\[
\psi_1 = \frac{c_0}{\sqrt{c_0^2 - \dot{\mathbf{r}}^2}}, \quad \psi_2 = -\frac{c_0}{\sqrt{c_0^2 - \dot{\mathbf{r}}^2}}.
\]

Insertion of \(\psi_1\) into the Lagrangian function yields
\[
L (\dot{\mathbf{r}}, \psi = \psi_1) = \frac{1}{2} m \left( -2c_0\sqrt{c_0^2 - \dot{\mathbf{r}}^2} + 2c_0^2 \right) = -mc_0^2\sqrt{1 - \dot{\mathbf{r}}^2/c_0^2} + mc_0^2.
\]

This is nothing but eq. (M4.97), with \(c\) replaced by \(c_0\), to which the constant energy \(mc_0^2\) is added.

If we let \(c_0\) go to infinity, \(\psi_1\) trends to 1 and the Lagrangian function becomes \(L_{nr} = m\dot{\mathbf{r}}^2/2\), well-known from nonrelativistic motion. One verifies easily that (1) is the correct equation of motion in either case.

The second solution \(\psi_2\) must be excluded. Obviously, the additional term
\[
\frac{1}{2} m (\psi - 1) \left( \dot{\mathbf{r}}^2 - c_0^2 \frac{\psi - 1}{\psi} \right).
\]

which is added to the Lagrangian function \(L_{nr}\) takes care of the requirement that the velocity \(\dot{\mathbf{r}}\) should not exceed the value \(c_0\).

**Chapter 5: Geometric Aspects of Mechanics**

5.1 We make use of the decomposition (M5.52) for \(\omega\) and \(\omega\)
\[
\omega \wedge \omega \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} \sum_{j_1 < \ldots < j_l} \omega_{j_1 \ldots j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.
\]

The analogous decomposition of \(\omega \wedge \omega\) (\(k\) and \(l\) interchanged) is obtained from this by shifting first \(dx^{j_1}\), then \(dx^{j_2}\), and so forth up to \(dx^{j_l}\), across the product \(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}\) from the right to the left. Each one of these operations gives rise to a factor \((-)^{k}\), so that one obtains the total factor \((-)^{k+l}\).

5.2 We calculate
\[
ds^2 (\hat{e}_i, \hat{e}_j) = \sum_{k=1}^{3} E_k dx^k(\hat{e}_i) dx^k(\hat{e}_j) = \sum_{k=1}^{3} E_k a_i^k a_j^k,
\]

where we have set \(a_i^k := dx^k(\hat{e}_i)\). As \(ds^2(\hat{e}_i, \hat{e}_k) = \delta_{ik}\), we must have \(a_i^k = b_i^k / \sqrt{E_k}\), where \(\{b_i^k\}\) is an orthogonal matrix. This matrix must be orthogonal because the coordinate axes were chosen orthogonal. Therefore \(dx^k(\hat{e}_i) = \delta_i^k / \sqrt{E_k}\).
Consider

1. \[ \omega_a = \sum_{i=1}^{3} \omega_i(x) \, dx^i, \]

2. \[ \hat{\omega}_a = b_1(x) \, dx^2 \wedge dx^3 + \text{cyclic permutations} \]

whose coefficients, \( \omega_i(x) \) and \( b_1(x) \), are to be determined.

a) We calculate

1. \[ \hat{\omega}_a(\xi) = \sum_i \omega_i(x) \, dx^i(\xi) = \sum_i \omega_i(x) \, dx^i \left( \sum_k \xi^k \hat{e}_k \right) = \sum_i \omega_i(x) \xi^i \frac{1}{\sqrt{E_i}}. \]

Since, on the other hand,

1. \[ \hat{\omega}_a(\xi) = \mathbf{a} \cdot \xi = \sum_i a_i(x) \xi^i \]

we deduce

1. \[ \omega_i(x) = a_i(x) \sqrt{E_i}. \]

b) We calculate

2. \[ \hat{\omega}_a(\xi, \eta) = b_1(x) (dx^2(\xi) \, dx^3(\eta) - dx^2(\eta) \, dx^3(\xi)) + \text{cycl. perms.} \]

\[ = b_1(x) (\xi^2 \eta^3 - \eta^2 \xi^3) / \sqrt{E_2 E_3} + \text{cycl. perms.}. \]

Comparing this with the scalar product of \( \mathbf{a} \) and \( \xi \times \eta \) yields

1. \[ b_1(x) = \sqrt{E_2 E_3} a_1(x) \quad (\text{cyclic permutations}). \]

5.4 Denote by \((\nabla f)_i\) the components of \( \nabla f \) with respect to the orthogonal basis that we consider. We then have, according to the solution of Exercise 5.3,

1. \[ \hat{\omega}_{\nabla f} = \sum_i (\nabla f)_i \sqrt{E_i} \, dx^i. \]

With \( \hat{\xi} = \sum_i \xi^i \hat{e}_i \) a unit vector, the function \( \hat{\omega}_{\nabla f}(\hat{\xi}) = \sum_i (\nabla f)_i \xi^i \) is the directional derivative of \( f \) along the direction \( \hat{\xi} \). This quantity can be calculated alternatively from the total differential

1. \[ df = \sum_i \frac{\partial f}{\partial x^i} \, dx^i \]

to be
Chapter 5: Geometric Aspects of Mechanics

\[ df(\hat{\xi}) = \sum_{i,k} \frac{\partial f}{\partial x^i} \xi^k \, dx^i \quad (\hat{e}_k) = \sum_i \frac{1}{\sqrt{E_i}} \frac{\partial f}{\partial x^i} \xi^i. \]

Comparing the two expressions yields the result

\[ (\nabla f)_i = \frac{1}{\sqrt{E_i}} \frac{\partial f}{\partial x^i}. \]

5.5 For Cartesian coordinates we have \( E_1 = E_2 = E_3 = 1 \).

For cylindrical coordinates \((\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z)\) we have \( ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \), i.e., \( E_1 = E_3 = 1, E_2 = \rho^2 \) and, therefore,

\[ \nabla f = \left( \frac{\partial f}{\partial \rho}, \frac{1}{\rho} \frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z} \right). \]

For spherical coordinates \((\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)\) we have \( ds^2 = dr^2 + r^2 d\theta^2 + r \sin^2 \theta d\phi^2 \), which means that \( E_1 = 1, E_2 = r^2, E_3 = r^2 \sin^2 \theta \) and, hence,

\[ \nabla f = \left( \frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \right). \]

5.6 The defining equation (M5.58) can be written alternatively as follows

\[ (\ast \omega)(\hat{e}_{i_1+1}, \ldots, \hat{e}_{i_n}) = \varepsilon_{i_1 \ldots i_{i_1+1} \ldots i_n} \omega(\hat{e}_{i_1}, \ldots, \hat{e}_{i_{i_1+1}}). \]

Here \( \varepsilon_{i_1 \ldots i_n} \) is the totally antisymmetric Levi-Civita symbol. It equals \(+1(-1)\) if \((i_1 \ldots i_n)\) is an even (odd) permutation of \((1, \ldots, n)\), and vanishes whenever two of its indices are equal. Thus, for \( n = 2, \ast dx^1 = dx^2, \ast dx^2 = -dx^1, \) and \( \ast \omega = F_1 dx^2 - F_2 dx^1 \). Therefore, \( \omega(\xi) = \mathbf{F} \cdot \xi \), while \( \ast \omega(\xi) = \mathbf{F} \times \xi \). If \( \xi \) is a displacement vector \( \xi = r_A - r_B \), \( \mathbf{F} \) a constant force, \( \omega(\xi) \) is the work of the force along that displacement. In turn, \( \ast \omega(\xi) \) describes the change of the external torque.

5.7 For any base \( k \)-form \( dx^{i_1} \wedge \ldots \wedge dx^{i_k} \) with \( i_1 < \cdots < i_k \)

\[ \ast(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = \varepsilon_{i_1 \ldots i_{i+1} \ldots i_n} dx^{i_{i+1}} \wedge \ldots \wedge dx^{i_n}. \]

Here we have assumed the indices on the right-hand side to be ordered, too, viz. \( i_{k+1} < \cdots < i_n \). The dual of this form is again a \( k \)-form and is given by

\[ \ast \ast(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = \varepsilon_{i_1 \ldots i_{k+1} \ldots i_n} \varepsilon_{i_{k+1} \ldots i_n j_1 \ldots j_k} dx^{j_1} \wedge \ldots \wedge dx^{j_k}. \]

All indices \( i_1 \ldots i_n \) must be different. Therefore, the set \((j_1 \ldots j_k)\) must be a permutation of \((i_1 \ldots i_k)\). If we choose the ordering \( j_1 < \cdots < j_k \), then \( j_1 = i_1, \ldots, j_k = i_k \). In the second \( \varepsilon \)-symbol interchange the group of indices \((i_1, \ldots, i_k)\) with the group \((i_{k+1}, \ldots, i_n)\). For \( i_1 \) this requires exactly \((n - k)\) exchanges of neighbors. The same holds true for \( i_2 \) up to \( i_k\). This gives \( k \) times a sign factor \((-1)^{n-k}\). As \((\varepsilon_{i_1 \ldots i_n})^2 = 1\) for all indices different, we conclude
\[ * \ast (dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = (-)^{k(n-k)} dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \]

**5.8** The exterior derivative of \( \phi \) is calculated following the rule (CD3) of Sect. M5.4.4, viz.

\[
d\phi = \left( -\frac{\partial E_1}{\partial x^2} + \frac{\partial E_2}{\partial x^1} \right) dx^1 \wedge dx^2 + \text{cyclic permutations}.
\]

This yields the result \( d\phi + \dot{\omega}/c = 0. \)

**5.9** With \( f \) a smooth function \( df = \sum (\partial f/\partial x^i) dx^i \), thus

\[
*df = \left( \frac{\partial f}{\partial x^1} \right) dx^2 \wedge dx^3 + \text{cyclic permutations},
\]

\[
d(*df) = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^1 \wedge dx^2 \wedge dx^3 + \text{cyclic permutations},
\]

and

\[
* \ast (df) = \sum_i \frac{\partial^2 f}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3.
\]

Furthermore, \( *f = f dx^1 \wedge dx^2 \wedge dx^3 \) and \( d(*f) = 0. \)

**5.10** If \( \omega \) is a \( k \)-form which is applied to \( k \) vectors \((\hat{e}_1, \ldots, \hat{e}_k)\), then, by the definition of the pull-back (special case, for vector spaces, of (M5.41), Sect. 5.4.1)

\[
F^* \omega(\hat{e}_1, \ldots, \hat{e}_k) = \omega(F(\hat{e}_1), \ldots, F(\hat{e}_k)).
\]

Then

\[
F^*(\omega \wedge \omega)(\hat{e}_1, \ldots, \hat{e}_{k+1}) = (\omega \wedge \omega)(F(\hat{e}_1), \ldots, F(\hat{e}_{k+1})) ,
\]

which, in turn, equals \((F^* \omega) \wedge (F^* \omega)\).

**5.11** This exercise is solved in close analogy to the solution of Exercise 5.10 above.

**5.12** With \( V := y \partial_x \) and \( W := x \partial_y \) we find readily \( Z := [V, W] = (y \partial_x)(x \partial_y) - (x \partial_y)(y \partial_x) = y \partial_x - x \partial_y. \)

**5.13** Let \( v_1 \) and \( v_2 \) be elements of \( T_p M \). Addition of vectors and multiplication by real numbers being defined as in (M5.20), it is clear that both \( v_1 + v_2 \) and \( av_i \) with \( a \in R \) belong to \( T_p M \), too. The dimension of \( T_p M \) is \( n = \dim M \). \( T_p M \) is a vector space. In the case \( M = \mathbb{R}^n \), \( T_p M \) isomorphic to \( M \).

**5.14** We have
\[ \omega \wedge \omega = \sum_{i=1}^{2} \sum_{j=1}^{2} dq^i \wedge dp_i \wedge dq^j \wedge dp_j = -2 dq^1 \wedge dq^2 \wedge dp_1 \wedge dp_2. \]

This is so because we interchanged \(dp_i\) and \(dq^j\), and because the terms \((i = 1, j = 2)\) and \((i = 2, j = 1)\) are equal.

5.15 \(H^{(1)} = p^2/2 + 1 - \cos q\) is the Hamiltonian function that describes the planar mathematical pendulum. The corresponding Hamiltonian vector field reads

\[ X_H^{(1)} = \frac{\partial H}{\partial p} \partial q - \frac{\partial H}{\partial q} \partial p = p \partial q - \sin q \partial p. \]

A sketch of this vector field will yield the vectors tangent to the curves of Fig. M1.10. The neighborhood of the point \((p = 0, q = \pi)\) is particularly interesting as this represents an unstable equilibrium. For \(H^{(2)} = \frac{1}{2} p^2 + \frac{1}{6} q(q^2 - 3)\)

the Hamiltonian vector field is

\[ X_H^{(2)} = p \partial q - \frac{1}{2} (q^2 - 1) \partial p. \]

This vector field has two equilibrium points, \((p = 0, q = +1)\) and \((p = 0, q = -1)\). A sketch of \(X_H^{(2)}\) will show that the former is a stable equilibrium (center), while the latter is unstable (saddle point). Linearization in the neighborhood of \(q = +1\) means that we set \(u := q - 1\) and keep up to linear terms in \(u\) only. Then \(X_H^{(2)} \approx p \partial u - u \partial p\). This is the vector field of the harmonic oscillator or, equivalently, the vector field \(X_H^{(1)}\) above, for small values of \(q\).

Linearization of the system in the neighborhood of \((p = 0, q = -1)\), in turn, means setting \(u := q + 1\) so that \(X_H^{(2)} \approx p \partial u + u \partial p\). Here the system behaves like the mathematical pendulum (described by \(X_H^{(1)}\) above) in the neighborhood of its unstable equilibrium \((p = 0, q = \pi)\) where \(\sin q = -\sin(q - \pi) \approx -(q - \pi)\).

(See also Exercise 6.8.)

5.16 One finds \(X_{H^0} = p \partial q - q \partial p\), \(X_H = p \partial q - (q + \varepsilon q^2) \partial p\), and, finally

\[ \omega(X_H, X_{H^0}) = dH(X_{H^0}) = \varepsilon pq^2 = \{H^0, H\}. \]

5.17 For the proof consult for example Sect. 3.5.18 of Abraham and Marsden (1981).
Chapter 6: Stability and Chaos

6.1 (i) $A$ is already diagonal. The flux is

$$\exp(tA) = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}.$$ 

(ii) The characteristic exponents (i.e., the eigenvalues of $A$) are $\lambda_1 = a + ib$, $\lambda_2 = a - ib$ so that in the diagonalized form the flux reads as follows: With

$$y \to u = \mathbf{U}y, \quad \overset{\circ}{A} = \mathbf{U}A\mathbf{U}^{-1} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix},$$

we have

$$u(t) = \exp(rA)u(0) = \begin{pmatrix} e^{(a+ib)t} \\ 0 \end{pmatrix} u(0).$$

For $a = 0$, $b > 0$ we find a (stable) center. For $a < 0$, $b > 0$ we find an (asymptotically stable) node.

(iii) The characteristic exponents are equal, $\lambda_1 = \lambda_2 \equiv \lambda$. For $\lambda < 0$ we again find a node.

6.2 The flux of this system is

$$\begin{pmatrix} \alpha(\tau) = \frac{a}{2\pi} \tau + \alpha_0 \pmod{1} \\ \beta(\tau) = \frac{b}{2\pi} \tau + \beta_0 \pmod{1} \end{pmatrix}.$$ 

If the ratio $b/a$ is rational, i.e., $b/a = m/n$ with $m, n \in \mathbb{Z}$, the system returns to its initial position after the time $\tau = T$ where $T$ follows from $\alpha_0 + aT/(2\pi) = \alpha_0 \pmod{1}$ and $\beta_0 + bT/(2\pi) = \beta_0 \pmod{1}$, i.e., $T = 2\pi n/a = 2\pi m/b$. We study the example $(a = 2/3, b = 1)$ with initial condition $(\alpha_0 = 1/2, \beta_0 = 0)$. This yields the results shown in Table 1 and in Fig. 25. In the figure the sections of the orbit are numbered in the order in which they appear.

<table>
<thead>
<tr>
<th>$\varrho = \frac{\tau}{2\pi}$</th>
<th>0</th>
<th>$\frac{3}{4}$</th>
<th>1</th>
<th>2</th>
<th>$\frac{9}{4}$</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{5}{6}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0</td>
<td>$\frac{3}{4}$</td>
<td>1</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>1</td>
</tr>
</tbody>
</table>

If the ratio $b/a$ is irrational then the flux will cover the torus, or the square of Fig. 25, densely. As an example one may choose $a$ “out of tune” at the value $a = 1/\sqrt{2} \approx 0.7071$, keeping $b = 1$ fixed, and plot the flux in the square. A
specific example is provided by two coupled oscillators, cf. Exercises 1.9 and 2.9. The modes of the system obey the differential equation \( \ddot{u}_i + \omega^{(i)^2} u_i = 0 \), \( i = 1, 2 \). These are rewritten in terms of action and angle variables \( I_i \) and \( \Theta_i \), respectively, by means of the canonical transformation (M2.95). They become \( \dot{I}_i = 0 \), \( \dot{\Theta}_i = \omega^{(i)} \), \( i = 1, 2 \). Embedded in the phase space which is now four-dimensional, we find two two-dimensional tori which are determined by the given values of \( I_i = I_0^i = \text{const.} \). Each of these tori carries the flux described above.

6.3 The Hamiltonian function has the form \( H = p^2/(2m) + U(q) \). The characteristic equation

\[
\frac{1}{2m} \left( \frac{\partial S_0(q, \alpha)}{\partial q} \right)^2 + U(q) = E_0
\]

is integrated by quadrature:

\[
S_0(q, \alpha) = \int_{q_0}^q \sqrt{2m \left( E_0 - U(q') \right)} \, dq'.
\]

We have \( p = m \dot{q} = \partial S_0 / \partial q = \sqrt{2m(E_0 - U(q))} \) and hence

\[
t(q) - t(q_0) = \int_{q_0}^q \frac{m}{\sqrt{2m \left( E_0 - U(q') \right)}} \, dq' = \frac{\partial S_0}{\partial E_0}.
\]

(This holds away from equilibrium positions, in case the system possesses any equilibria.) Choose \( P = \alpha = E_0 \). Then \( Q = \partial S_0 / \partial E_0 = t - t_0 \), or, alternatively, \( (P = 0, \dot{Q} = 1) \). Thus, we have achieved rectification of the Hamiltonian vector field: In the coordinates \((P, Q)\) the particle moves on the straight line \( P = E_0 \) with velocity \( \dot{Q} = 1 \).

Consider now an energy in the vicinity of \( E_0 \), \( E = \beta E_0 \), with \( \beta \) not far from 1. Let the particle travel from \( q_0 \) to a point \( q' \) in such a way that the time
The new momentum is chosen to be \( p \) and the flux tends to \( \min \) of \( U(q) \). \( p \) is a saddle point. The same is true for \( H(q, \dot{q}) \) solutions of \( E = H(q) \). For \( \dot{q} = 0 \) and \( \dot{\theta} = \beta(t - t_0) \), in the new coordinates the particle again moves along the straight line \( P = E_0 \), this time, however, with velocity \( \beta \). The particles on the orbits to be compared move apart linearly in time. If \( U(q) \) is such that in some region of phase space all orbits are periodic, one should transform to action and angle variables, \( I(E) = \) const., \( \Theta = \omega(E)t + \Theta_0 \). Also in this situation one sees that the orbits separate at most linearly in time. In the case of the oscillator, where \( \omega \) is independent of \( E \) or \( I \), their distance remains constant.

The integration described above is possible only if \( E \) is larger than the maximum of \( U(q) \). For \( E = U_{\max}(q) \) the running time goes to infinity logarithmically (cf. Sect. M1.23). The statement of this exercise does not apply when one of the trajectories is a separatrix.

**6.4** For \( \mu = 0 \) the system becomes \( \dot{q}_1 = \partial H/\partial q_2 \) and \( \dot{q}_2 = -\partial H/\partial q_1 \), with \( H = -\lambda (q_1^2 + q_2^2)/2 + (q_1 q_2^2 - q_1^3)/2 \). The critical points (where the Hamiltonian vector field vanishes) are obtained from the system of equations \(-\lambda q_2 + q_1 q_2 = 0, \lambda q_1 + (q_1^2 - q_2^3)/2 = 0\). One finds the following solutions: \( P_0 = (q_1 = 0, q_2 = 0) \), \( P_{1/2} = (q_1 = \lambda, q_2 = \pm \sqrt{3}\lambda) \), \( P_3 = (q_1 = -2\lambda, q_2 = 0) \). Linearization in the neighborhood of \( P_0 \) leads to \( \dot{q}_1 \approx -\lambda q_2, \dot{q}_2 \approx \lambda q_1 \). Thus \( P_0 \) is a center. Linearization in the neighborhood of \( P_1 \) is achieved by the transformation \( u_1 := q_1 - \lambda, u_2 := q_2 - \sqrt{3}\lambda \), whereby the differential equations become

\[
\begin{align*}
\dot{u}_1 &= \sqrt{3}\lambda u_1 + u_1 u_2 \approx \sqrt{3}\lambda u_1; \\
\dot{u}_2 &= 2\lambda u_1 - \sqrt{3}\lambda u_2 + (u_1^2 - u_2^2)/2 \approx 2\lambda u_1 - \sqrt{3}\lambda u_2.
\end{align*}
\]

The flux tends to \( P_1 \) along \( u_1 = 0 \) but tends away from it along \( u_2 = 0 \). Thus \( P_1 \) is a saddle point. The same is true for \( P_2 \) and \( P_3 \). One easily verifies that these three points belong to the same energy \( E = H(P) = -2\lambda^3/3 \) and that they are pairwise connected by separatrices. Indeed, the straight lines \( q_2 = \pm(q_1 + 2\lambda)/\sqrt{3} \) and \( q_1 = \lambda \) are curves with constant energy \( E = -2\lambda^3/3 \) and build up the triangle \( (P_1, P_2, P_3) \).

If one switches on the damping terms by means of \( 1 \gg \mu > 0 \), \( P_0 \) is still an equilibrium point because in the neighborhood of \( (q_1 = 0, q_2 = 0) \) we have

\[
\begin{pmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{pmatrix} \approx \begin{pmatrix}
-\mu & -\lambda \\
\lambda & -\mu
\end{pmatrix} \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} \equiv A \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}.
\]
From the equation \( \det(x \mathbb{I} - \mathbf{A}) = 0 \) one finds the characteristic exponents to be \( x_{1/2} = -\mu \pm i\lambda \). Thus \( P_0 \) becomes a node (a sink). The points \( P_1, P_2, \) and \( P_3 \), however, are no longer equilibrium positions. The lines which connect them are broken up.

6.5 We write \( H \) in two equivalent forms

\[
\begin{align*}
(i) \quad & H = I_1 + I_2 \quad \text{with} \quad I_1 = \left(p_1^2 + q_1^2 \right)/2 + q_1^3/3, \\
& I_2 = \left(p_2^2 + q_2^2 \right)/2 - q_2^3/3, \\
(ii) \quad & H = \left(p_1^2 + p_2^2 \right)/2 + U(q_1, q_2) \quad \text{with} \quad U = \left(q_1^2 + q_2^2 \right)/2 + \left(q_1^3 - q_2^3 \right)/3 \\
& = \left(\Sigma^2 + \Delta^2 \right)/4 + \Sigma^2 \Delta/4 + \Delta^3/12, \\
& \text{where} \quad \Sigma := q_1 + q_2, \quad \Delta := q_1 - q_2.
\end{align*}
\]

Then the equations of motion are

\[
\begin{align*}
\dot{q}_1 &= p_1, \quad \dot{q}_2 = p_2, \\
\dot{p}_1 &= -q_1 - q_1^3, \quad \dot{p}_2 = -q_2 + q_2^3.
\end{align*}
\]

The critical points of this system are \( P_0 : (q_1 = 0, q_2 = 0, p_1 = 0, p_2 = 0), \) \( P_1 : (0, 1, 0, 0), \) \( P_2 : (-1, 0, 0, 0), \) \( P_3 : (-1, 1, 0, 0). \) One easily verifies that \( dI_i/dt = 0, i = 1, 2, \) i.e., \( I_1 \) and \( I_2 \) are independent integrals of the motion. The points \( P_1 \) and \( P_2 \) lie on two equipotential lines, viz. the straight line \( q_1 - q_2 = -1, \) and the ellipse \( 3(q_1 + q_2)^2 + (q_1 - q_2)^2 + 2(q_1 - q_2) = 2 = 0. \) In either case \( U = 1/6. \) As an example and using these results, one may sketch the projection of the flux onto the plane \( (q_1, q_2). \)

6.6 The critical points of the system \( \dot{q} = p, \dot{p} = q - q^3 - p \) are \( P_0 : (q = 0, p = 0), P_1 : (1, 0), \) and \( P_2 : (-1, 0). \) Linearization around \( P_0 \) gives

\[
\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} \approx \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{A} \begin{pmatrix} q \\ p \end{pmatrix}.
\]

The eigenvalue of \( \mathbf{A} \) are \( \lambda_{1/2} = (-1 \pm \sqrt{5})/2, \) hence \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \) which means that \( P_0 \) is a saddle point. Linearizing in the neighborhood of \( P_1 \) and introducing the variables \( u := q - 1, v := p, \) the system becomes

\[
\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\]

The characteristic exponents are \( \mu_{1/2} = (-1 \pm i\sqrt{7})/2. \) The same values are found for the system linearized in the neighborhood of \( P_2. \) This means that both \( P_1 \) and \( P_2 \) are sinks.

The Liapunov function \( V(q, p) \) has the value 0 in \( P_0, \) and the value \(-1/4\) in \( P_1 \) and \( P_2. \) One easily verifies that \( P_1 \) and \( P_2 \) are minima and that \( V \) increases monotonically in a neighborhood of these points. For instance, close to \( P_1 \) take \( u := \)
q−1, v := p. Then \( \Phi_1(u, v) := V(q = u+1, p = v) + 1/4 = v^2/2 + u^2 + u^3 + u^4/4. \) Indeed, at the point \( P_1 \) we find \( \Phi_1(0, 0) = 0 \) while \( \Phi_1 \) is positive in a neighborhood of \( P_1 \).

Along solutions the function \( V(q, p) \), or, equivalently, \( \Phi_1(u, v) \), decreases monotonically. Let us check this for \( V \):

\[
\frac{dV}{dt} = \frac{dV}{dp} \dot{p} + \frac{dV}{dq} \dot{q} = \frac{\partial V}{\partial p} (q - q^3 - p) + \frac{\partial V}{\partial q} p = -p^2.
\]

In order to find out towards which of the two sinks a given initial configuration will tend, one has to calculate the two separatrices that end in \( P_0 \). They form the boundaries of the basins of \( P_1 \) and \( P_2 \) as indicated in Fig. 26 by the blank and dotted areas, respectively.

6.7 As \( x_0 \), by assumption, is an isolated minimum, a Liapunov function is chosen as follows: \( V(x) := U(x) - U(x_0) \). In a certain neighborhood \( M \) of \( x_0 \), \( V(x) \) is positive semi-definite and we have

\[
\frac{d}{dt} V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = -\sum_{i=1}^{n} \left( \frac{\partial U}{\partial x_i} \right)^2.
\]

If we follow a solution in the domain \( M \setminus \{x_0\} \), \( V(x) \) decreases, i.e., all solution curves tend “inwards”, towards \( x_0 \). Thus, this point is asymptotically stable.

In the example \( U(x_1, x_2) = x_1^2(x_1 - 1)^2 + x_2^2 \). The points \( x_0 = (0, 0) \) and \( x_0' = (1, 0) \) are isolated minima and, hence, are asymptotically stable equilibria.

6.8 This system is Hamiltonian. A Hamiltonian function is \( H = p^2/2 + q(q^2 - 3)/6 \). The phase portraits are obtained by drawing the curves \( H(q, p) = E = \ldots \)
const. The Hamiltonian vector field \( v_H = (p, (1 - q^2)/2) \) has two critical points whose nature is easily identified by linearizing in their neighborhoods. One finds

\( P_1: \ (q = -1, \ p = 0) \) and, with \( u := q + 1, \ v := p : (\dot{u} \approx v, \ \dot{v} \approx u) \). Thus, \( P_1 \) is a saddle point.

\( P_2: \ (q = 1, \ p = 0) \) and, with \( \dot{u} := q - 1, \ v := p : (\dot{u} \approx v, \ \dot{v} \approx -u) \). Thus, \( P_2 \) is a center. In the neighborhood of \( P_2 \) there will be harmonic oscillations with period \( 2\pi \) (see also Exercise 5.15).

6.9 The differential equation \( \ddot{q} = f(q, \dot{q}) \), with \( f(q, \dot{q}) = -q + (\varepsilon - q^2)\dot{q} \), is solved numerically by means of a Runge–Kutta procedure as follows

\[
q_{n+1} = q_n + h \left( \dot{q}_n + \frac{1}{6} (k_1 + k_2 + k_3) \right) + O(h^5)
\]

\[
\dot{q}_{n+1} = \dot{q}_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4),
\]

\( h \) being the integration step in the time variable, and the auxiliary quantities \( k_i \) being defined by

\[
k_1 = hf(q_n, \dot{q}_n),
\]

\[
k_2 = hf \left( q_n + \frac{h}{2} \dot{q}_n + \frac{h}{8} k_1, \dot{q}_n + \frac{1}{2} k_1 \right),
\]

\[
k_3 = hf \left( q_n + \frac{h}{2} \dot{q}_n + \frac{h}{8} k_1, \dot{q}_n + \frac{1}{2} k_2 \right),
\]

\[
k_4 = hf \left( q_n + h\dot{q}_n + \frac{h}{2} k_3, \dot{q}_n + k_3 \right).
\]

One lets the dimensionless time variable \( \tau = \omega t \) run from 0 to, say, \( 6\pi \), in steps of 0.1, or 0.05, or 0.01. This will produce pictures of the type shown in Figs. (M6.6)–(M6.8). Alternatively, one may follow the generation of these figures on the screen of a PC. One will notice that all of them tend quickly to the attractor.

6.10 The program developed in Exercise 6.9 may be used to print out, for a given initial configuration, the time \( \tau \) and the distance from the origin \( d \), each time the orbit crosses the line \( p = q \). One finds the following result:

\[
\begin{array}{c|cccccccc}
 p = q > 0 : & \tau & 5.46 & 11.87 & 18.26 & 24.54 & 30.80 & 37.13 & 43.47 \\
 & d & 0.034 & 0.121 & 0.414 & 1.018 & 1.334 & 1.375 & 1.378 \\
 p = q < 0 : & \tau & 2.25 & 8.86 & 15.07 & 21.42 & 27.66 & 33.96 & 40.30 \\
 & d & 0.018 & 0.064 & 0.227 & 0.701 & 1.238 & 1.366 & 1.378
\end{array}
\]

Plotting \( \ln d \) versus \( \tau \) shows that this function increases approximately linearly (with slope \( \approx 0.1 \)) until it has reached the attractor. Thus, the point of intersection of the orbit and the straight line \( p = q \) wanders towards the attractor at an
approximately exponential rate. One finds a similar result for orbits which approach the attractor from the outside.

6.11 The motion manifold of this system is $\mathbb{R}^2$. For the linearized system $(\dot{x}_1 = x_1, \dot{x}_2 = -x_2)$ the straight line $U_{\text{stab}} = (x_1 = 0, x_2)$ is a stable submanifold. Indeed, the velocity field points towards the equilibrium point $(0, 0)$ and the characteristic exponent is $-1$. The straight line $U_{\text{unst}} = (x_1, x_2 = 0)$, in turn, is an unstable submanifold: The velocity field points away from $(0, 0)$ and the characteristic exponent is $+1$. The full system can be transformed to

$$\dot{x}_2 - \dot{x}_2 - 2x_2 = 0, \quad x_1^2 = x_2 + \dot{x}_2,$$

whose general solution is

$$x_2(t) = a \exp(2t) + b \exp(-t), \quad x_1(t) = \sqrt{3}a \exp t,$$

or, equivalently,

$$x_2 = \frac{1}{3}x_1^2 + b\sqrt{3}a \frac{1}{x_1} \equiv \frac{1}{3}x_1^2 + \frac{c}{x_1}.$$  

Among this set of solutions the orbit with $c = 0$ goes through the point $(0, 0)$ and is tangent to $U_{\text{unst}}$ in that point. On the submanifold $V_{\text{unst}} = (x_1, x_2 = x_1^2/3)$ the velocity field moves away from $(0, 0)$.

The corresponding stable submanifold of the full system coincides with $U_{\text{stab}}$ because, with $a = 0$, $x_1(t) = 0$, $x_2(t) = b \exp(-t)$ which means that $V_{\text{stab}} = (x_1 = 0, x_2)$.

6.12 We have $x_{n+1} = 1 - 2x_n^2$ and $y_i = 4/\pi \arcsin(\sqrt{(x_i + 1)/2}) - 1$. With $-1 \leq x_i \leq 0$ also $-1 \leq y_i \leq 0$, and with $0 \leq x_i \leq 1$ also $0 \leq y_i \leq 1$. We wish to know the relation between $y_{n+1}$ and $y_n$. First, the relation $x_n \rightarrow y_{n+1}$ is $y_{n+1} = 4/\pi \arcsin(1 - x_n^2)^{1/2} - 1$. Using the addition theorem $\arcsin u + \arcsin v = \arcsin(u\sqrt{1 - v^2} + v\sqrt{1 - u^2})$ and setting $u = v = (1 + x)/2$ one shows

$$\arcsin \sqrt{1 - x^2} = 2 \arcsin \sqrt{\frac{x + 1}{2}} \quad \text{for} \quad -1 \leq x \leq 0,$$

$$\arcsin \sqrt{1 - x^2} = \pi - 2 \arcsin \sqrt{\frac{x + 1}{2}} \quad \text{for} \quad 0 \leq x \leq 1.$$  

In the first case $y_n \leq 0$ and $y_{n+1} = 1 + 2y_n$, in the second case $y_n \geq 0$ and $y_{n+1} = 1 - 2y_n$. These can be combined to $y_{n+1} = 1 - 2|y_n|$. The derivative of this iterative mapping is $\pm 2$; its magnitude is larger than 1. There are no stable fixed points.

6.15 If, for instance, $m = 1$, then $z_\sigma := \exp(i2\pi \sigma/n)$ are the roots of the equation $z^n - 1 = (z - z_1) \ldots (z - z_n) = 0$. They lie on the unit circle in the complex plane and neighboring roots are separated by the angle $2\pi/n$. Expanding the product $(z - z_1) \ldots (z - z_n) = z^n - z^n \sum_{\sigma=1}^{n} z_\sigma + \ldots$, one sees that, indeed, $\sum_{\sigma=1}^{n} z_\sigma = 0$. 


For the values \( m = 2, \ldots, n - 1 \) we renumber the roots and obtain the same result. For \( m = 0 \) and for \( m = n \), however, the sum is equal to 0. Multiplying \( \bar{x}_\sigma = \sum_{\tau=1}^{n} x_\tau \exp(-2i\pi \sigma \tau / n) / \sqrt{n} \) by \( 1/\sqrt{n} \exp(2i\pi \sigma \lambda / n) \) and summing over \( \sigma \) one obtains

\[
\frac{1}{\sqrt{n}} \sum_{\sigma=1}^{n} \bar{x}_\sigma e^{2i\pi \sigma \lambda / n} = \frac{1}{\sqrt{n}} \sum_{\tau=1}^{n} x_\tau \sum_{\sigma=1}^{n} e^{2i\pi \sigma (\tau - \lambda) / n} = \sum_{\tau=1}^{n} x_\tau \delta_{\tau \lambda} = x_\lambda .
\]

This is used to calculate

\[
g_\lambda = \frac{1}{n} \sum_{\sigma=1}^{n} x_\sigma x_{\sigma + \lambda} = \frac{1}{n} \sum_{\mu, v} \bar{x}_\mu \bar{x}_{\mu - v} \sum_{\sigma} e^{2i\pi / n (\sigma \mu + v + \lambda v)} .
\]

The orthogonality relation implies \( \mu + v = 0 \pmod{n} \), and we have used \( \bar{x}_{n-v} = \bar{x}_v \). Furthermore, \( \bar{x}_\mu \pmod{n} = \bar{x}_\mu \) and, finally, \( \bar{x}_\mu \) and \( \bar{x}_{n-\mu} \) have the same modulus. It follows that \( g_\lambda = 1/n \sum_{\mu=1}^{n} |\bar{x}_\mu|^2 \cos(2\pi \lambda \mu / n) \). The inverse of this formula \(|\bar{x}_\sigma|^2 = \sum_{\lambda=1}^{n} g_\lambda \cos(2\pi \sigma \lambda / n) \) is obtained in the same way.

6.16 If we set \( y = \alpha x + \beta \), i.e., \( x = y/\alpha - \beta/\alpha \), the relation

\[
x_{i+1} = \mu x_i (1 - x_i) = \mu \left( \frac{1}{\alpha} y_i - \frac{\beta}{\alpha} \right) \left( 1 + \frac{\beta}{\alpha} - \frac{1}{\alpha} y_i \right)
\]

will take the desired form provided \( \alpha \) and \( \beta \) are chosen such that they fulfill the equations \( \alpha + 2\beta = 0 \), \( \beta(1 - \mu(\alpha + \beta)/\alpha) = 1 \). These give \( \alpha = 4/(\mu - 2) \), \( \beta = -\alpha/2 \), and, therefore, \( \gamma = \mu(\mu - 2)/4 \). From \( 0 \leq \mu < 4 \) follows \( 0 \leq \gamma < 2 \). One then sees easily that \( y_i \in [-1, +1] \) is mapped onto \( y_{i+1} \) in the same interval. Let \( h(y, \gamma) := 1 - \gamma y^2 \). The first bifurcation occurs when \( h(y, \gamma) = y \) and \( \partial h(y, \gamma) / \partial y = 1 \), i.e., when \( y_0 = 3/4 \), \( y_0 = 2/3 \), or, correspondingly, \( \mu_0 = 3/4 \), \( x_0 = 2/3 \). Take then \( k := h \circ h \), i.e., \( k(y, \gamma) = 1 - \gamma(1 - \gamma y^2)^2 \). The second bifurcation occurs at \( \gamma_1 = 5/4 \). The corresponding value \( y_1 \) of \( y \) is calculated from the system

\[
k \left( y, \frac{5}{4} \right) = -\frac{1}{4} + \frac{25}{8} y^2 - \frac{125}{64} y^4 = y ,
\]

\[
\frac{\partial k}{\partial y} \left( y, \frac{5}{4} \right) = \frac{25}{4} y \left( 1 - \frac{5}{4} y^2 \right) = -1 .
\]

Combining these equations according to \( y \cdot (2) - 4 \cdot (1) \), one finds the quadratic equation

\[
y^2 - \frac{4}{5} y - \frac{4}{25} = 0 ,
\]

whose solutions are \( y_{1/2} = 2(1 \pm \sqrt{2})/5 \). From \( \gamma_1 = 5/4 \) we have \( \mu_1 = 1 + \sqrt{6} \), from this and from \( y_{1/2} \), finally

\[
x_{1/2} = \frac{1}{10} \left( 4 + \sqrt{6} \pm (2\sqrt{3} - \sqrt{2}) \right) = 0.8499 \quad \text{and} \quad 0.4400 .
\]
Appendix

A. Some Mathematical Notions

“Order” and “Modulo”. The symbol $O(\varepsilon^n)$ stands for terms of the order $\varepsilon^n$ and higher powers thereof that are being neglected.

Example. If in a Taylor series one wishes to take account only of the first three terms, one writes

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + O(x^3). \quad (A.1)$$

This means that the right-hand side is valid up to terms of order $x^3$ and higher.

The notation $y = x \pmod{a}$ means that $x$ and $x + na$ should be identified, $n$ being any integer. Equivalently, this means that one should add to $x$ or subtract from it the number $a$ as many times as are necessary to have $y$ fall in a given interval.

Example. Suppose two angles $\alpha$ and $\beta$ are defined in the interval $[0, 2\pi]$. The equation $\alpha = f(\beta) \pmod{2\pi}$ means that one must add to the value of the function $f(\beta)$, or subtract from it, an integer number of terms $2\pi$ such that $\alpha$ does not fall outside its interval of definition.

Mappings. A mapping $f$ that maps a set $A$ onto a set $B$ is denoted as follows:

$$f : A \to B : a \mapsto b. \quad (A.2)$$

It assigns to a given element $a \in A$ the element $b \in B$. The element $b$ is said to be the image and $a$ its preimage.

Examples. (i) The real function $\sin x$ maps the real $x$-axis onto the interval $[-1, 1]$,

$$\sin : \mathbb{R} \to [-1, 1] : x \mapsto y = \sin x.$$

(ii) The curve $\gamma : x = \cos \omega t, \ y = \sin \omega t$ in $\mathbb{R}^2$ is a mapping from the real $t$-axis onto the unit circle $S^1$ in $\mathbb{R}^2$,

$$\gamma : \mathbb{R} \to S^1 : t \mapsto (x = \cos \omega t, \ y = \sin \omega t).$$

A mapping is called surjective if $f(A) = B$, i.e. if $B$ is covered completely. The mapping is called injective if two distinct elements in $A$ also have distinct images.
in $B$, in other words, if every $b \in B$ has at most one original $a \in A$. If it has both properties it is said to be bijective. In this case every element of $A$ has exactly one image in $B$, and for every element of $B$ there is exactly one preimage in $A$. In other words the mapping is then one-to-one.

**Examples:** (i) The mapping

$$f : \mathbb{R} \to \mathbb{R} : a \mapsto b = f(a) = a^3$$

is injective. Indeed, $b_1 = f(a_1) = f(a_2) = b_2$ implies the equality $a_1 = a_2$. The mapping is also surjective: For any $b \in \mathbb{R}$ the preimage is $a = b^{1/3}$ if $b$ is positive, and $a = -b^{1/3}$ if $b$ is negative.

(ii) The mapping

$$f : \mathbb{R} \to \mathbb{R} : a \mapsto b = f(a) = a^2$$

is not injective because $a_1 = 1$ and $a_2 = -1$ have the same image.

The *composition* $f \circ g$ of two mappings $f$ and $g$ means that $g$ should be applied first and then $f$ should be applied to the result of the first mapping, viz.

If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$. \hspace{1cm} (A.3)

**Example:** Suppose $f$ and $g$ are functions over the reals. Then, with $y = g(x)$ and $z = f(y)$ we have $z = (f \circ g)(x) = f(g(x))$.

The *identical mapping* is often denoted by “id”, i.e.

$$id : A \to A : a \mapsto a.$$ 

**Special Properties of Mappings.** A mapping $f : A \to B$ is said to be *continuous at the point* $u \in A$ if for every neighborhood $V$ of its image $v = f(u) \in B$ there exists a neighborhood $U$ of $u$ such that $f(U) \subset V$. The mapping is *continuous* if this property holds at every point of $A$.

Homeomorphisms are bijective mappings $f : A \to B$ that are such that both $f$ and its inverse $f^{-1}$ are continuous.

Diffeomorphisms are differentiable, bijective mappings $f$ that are such that both $f$ and $f^{-1}$ are smooth (i.e. differentiable, $C^\infty$).

**Derivatives.** Let $f(x^1, x^2, \ldots, x^n)$ be a function over the space $\mathbb{R}^n$, $\{\hat{e}_1, \ldots, \hat{e}_n\}$, a set of orthogonal unit vectors. The *partial derivative* with respect to the variable $x^i$ is defined as follows:

$$\frac{\partial f}{\partial x^i} = \lim_{h \to 0} \frac{f(x + h\hat{e}_i) - f(x)}{h}. \hspace{1cm} (A.4)$$

Thus one takes the derivative with respect to $x^i$ while keeping all other arguments $x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n$ fixed.
Collecting all partial derivatives yields a vector field called the gradient,
\[ \nabla f = \left( \frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right). \]  

(A.5)

Since any direction \( \hat{n} \) in \( \mathbb{R}^n \) can be decomposed in terms of the basis vectors \( \hat{e}_1, \ldots, \hat{e}_n \), one can take the directional derivative of \( f \) along that direction, viz.
\[ \frac{\partial f}{\partial \hat{n}} = \sum_{i=1}^{n} \hat{n}^i \frac{\partial f}{\partial x^i} \equiv \hat{n} \cdot \nabla f. \]  

(A.6)

The total differential of the function \( f(x^1, \ldots, x^n) \) is defined as follows:
\[ df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n. \]  

(A.7)

**Examples.** (i) Let \( f(x, y) = \frac{1}{2}(x^2 + y^2) \) and let \( (x = r \cos \phi, \ y = r \sin \phi) \) with fixed \( r \) and \( 0 \leq \phi \leq 2\pi \) be a circle in \( \mathbb{R}^2 \). The normalized vector tangent to the circle at the point \( (x, y) \) is given by \( v_t = (-\sin \phi, \ \cos \phi) \). Similarly, the normalized normal vector at the same point is given by \( v_n = (\cos \phi, \ \sin \phi) \). The total differential of \( f \) is \( df = x dx + y dy \) and its directional derivative along \( v_t \) is \( v_t \cdot \nabla f = -x \sin \phi + y \cos \phi = 0 \); the directional derivative along \( v_n \) is given by \( v_n \cdot \nabla f = x \cos \phi + y \sin \phi = r \). For an arbitrary unit vector \( v = (\cos \alpha, \sin \alpha) \) we find \( v \cdot \nabla f = r(\cos \phi \cos \alpha + \sin \phi \sin \alpha) \). For fixed \( \phi \) the absolute value of this real number becomes maximal if \( \alpha = \phi \) (mod \( \pi \)). Thus, the gradient defines the direction along which the function \( f \) grows or falls fastest.

(ii) Let \( U(x, y) = xy \) be a potential in the plane \( \mathbb{R}^2 \). The curves along which the potential \( U \) is constant (they are called *equipotential lines*) are obtained by taking \( U(x, y) = c \), with \( c \) a constant real number. They are given by \( y = c/x \), i.e. by hyperbolas whose center of symmetry is the origin. Along these curves \( dU(x, y) = ydx + xdy = 0 \) because \( dy = -(c/x^2)dx = -(y/x)dx \). The gradient is given by \( \nabla U = (\partial U/\partial x, \ \partial U/\partial y) = (y, x) \) and is perpendicular to the curves \( U(x, y) = c \) at any point \( (x, y) \). It is the tangent vector field of another set of curves that obey the differential equation
\[ \frac{dy}{dx} = \frac{x}{y}. \]

These latter curves are given by \( y^2 - x^2 = a \).

**Differentiability of Functions.** The function \( f(x^1, \ldots, x^i, \ldots, x^n) \) is said to be \( C^r \) with respect to \( x^i \) if it is \( r \)-times continuously differentiable in the argument \( x^i \). A function is said to be \( C^\infty \) in some or all of its arguments if it is differentiable an infinite number of times. It is then also said to be *smooth*.

**Variables and Parameters.** Physical quantities often depend on two kinds of arguments, the *variables* and the *parameters*. This distinction is usually made on the
basis of a physical picture and, therefore, is not canonical. Generally, variables are
dynamical quantities whose time evolution one wishes to study. Parameters, on the
other hand, are given numbers that define the system under consideration. In the
example of a forced and damped oscillator, the deviation $x(t)$ from the equilib-
rium position is taken to be the dynamical variable while the spring constant, the
damping factor, and the frequency of the driving source are parameters.

**Lie Groups.** The definition assumes that the reader is familiar with the notion of
differentiable manifold, cf. Sect. 5.2. A Lie group is a finite dimensional, smooth,
manifold $G$ which in addition is a group and for which the product “·”,

$$G \times G \to G : (g, g') \mapsto g \cdot g' ,$$

as well as the transition to the inverse,

$$G \to G : g \mapsto g^{-1}$$

are smooth operations.

$G$ being a group means that it fulfills the group axioms, cf. e.g., Sect. 1.13:
There exists an associative product; $G$ contains the unit element $e$; for every $g \in G$
there exists an inverse $g^{-1} \in G$. Loosely speaking, smoothness means that the
group elements depend differentiably on parameters, which may be thought of as
angles for instance, and that group elements can be deformed in a continuous and
differentiable manner.

A simple example is the unitary group

$$U(1) = \left\{ e^{i\alpha} | \alpha \in [0, 2\pi] \right\} .$$

This is an Abelian group (i.e., a commutative group). Further examples are
provided by the rotation group $SO(3)$ in three real dimensions, and the unitary
group $SU(2)$ which are dealt with in Sects. 2.21 and 5.2.3 (iv). The Galilei group
is defined in Sect. 1.13, the Lorentz group is discussed in Sects. 4.4 and 4.5.

**B. Historical Notes**

There follow some biographical notes on scientists who made important contri-
butions to mechanics. Some of these are marked by an asterisk and are treated
in somewhat more detail, though without striving for completeness, because their
impact on the understanding and the development of mechanics was particularly
important.

* d’Alembert, Jean-Baptiste: born 17 November 1717 in Paris, died 29 October
1783 in Paris. Writer, philosopher, and mathematician. Co-founder of the *Encyc-
opédie*. Important contributions to mathematics, mathematical physics and ast-
ronomy. His principal work “Traité de dynamique” contains the principle which
bears his name.
Arnol’d, Vladimir Igorevich: 1937–, Russian mathematician.


Coriolis, Gustave-Gaspard: 1792–1843, French mathematician.


*Descartes, René (Cartesius): born 31 March 1596 in La Haye (Touraine), died 11 February 1650 in Stockholm. French philosopher, mathematician and natural philosopher. In spite of the fact that Descartes’ contributions to mechanics were not too successful (for instance, he proposed incorrect laws of collision), he contributed decisively to the development of analytic thinking without which modern natural science would not be possible. In this regard his book *Discours de la Méthode pour bien conduire sa Raison*, published in 1637, was particularly important. Also his imaginative conceptions – ether whirls carrying the planets around the sun; God having given eternal motion to the atoms relative to the atoms of the ether which span our space; the state of motion of atoms being able to change only by collisions – inspired the amateur researchers of the 17th century considerably. It was in this community where the real scholars of science found the resonance and support which they did not obtain from the scholastic and rigid attitude of the universities of their time.

*Einstein, Albert: born 14 March 1879 in Ulm (Germany), died 18 April 1955 in Princeton, N.J. (U.S.A.). German-Swiss physicist, 1940 naturalized in the U.S.A. His most important contribution to mechanics is Special Relativity which he published between 1905 and 1907. In his General Relativity, published between 1914 and 1916, he succeeded in establishing a (classical) description of gravitation as one of the fundamental interactions. While Special Relativity is based on the assumption that space-time is the flat space $\mathbb{R}^4$, General Relativity is a dynamical and geometric field theory which allows one to determine the metric on space-time from the sources, i.e., from the given distribution of masses in the universe.

*Euler, Leonhard: born 15 April 1707 in Basel (Switzerland), died 18 September 1783 in St. Petersburg (Russia). Swiss mathematician. Professor initially of physics, then of mathematics at the Academy of Sciences in St. Petersburg (from 1730 until 1741, and again from 1766 onwards), and, at the invitation of Frederick the Great, member of the Berlin Academy (1741–1766). Among his gigantic scientific work particularly relevant for mechanics: development of variational calculus; law of conservation of angular momentum as an independent principle; equations of motion for the top. He also made numerous contributions to continuum mechanics.

Fibonacci, Leonardi: ~1175–~1240. Italian mathematician who introduced the Indian-Arabic system of numbers. See also Fibonacci numbers in Sect. 6.5.

*Galilei, Galileo: born 15 February 1564 in Pisa (Italy), died 8 January 1642 in Arcetri near Florence (Italy). Italian mathematician, natural philosopher, and
philosopher, who belongs to the founding-fathers of natural sciences in the modern sense. Professor in Pisa (1589–1592), in Padua (1592–1610), mathematician and physicist at the court of the duke of Florence (1610–1633). From 1633 on under confinement to his house in Arcetri, as a consequence of the conflict with Pope Urban VIII and the Inquisition, which was caused by his defence of the Copernican, heliocentric, planetary system. Galilei made important contributions to the mechanics of simple machines and to observational astronomy. He developed the laws of free fall.

*Hamilton, William Rowan: born 4 August 1805 in Dublin (Ireland), died 2 September 1865 in Dunsink near Dublin. Irish mathematician, physicist, and astronomer. At barely 22 years of age he became professor at the university of Dublin. Important contributions to optics and to dynamics. Developed the variational principle which was cast in its later and more elegant form by C.G.J. Jacobi.

*Huygens, Christiaan: born 14 April 1629 in The Hague (Netherlands), died 8 July 1695 in The Hague. Dutch mathematician, physicist and astronomer. From 1666 until 1681 member of the Academy of Sciences in Paris. Although Huygens is not mentioned explicitly in this book he made essential contributions to mechanics: among others the correct laws for elastic, central collisions and, building on Galilei’s discoveries, the classical principle of relativity.

Jacobi, Carl Gustav Jakob: 1804–1851, German mathematician.

*Kepler, Johannes: born 27 December 1571 in Weil der Stadt (Germany), died 15 November 1630 in Regensburg (Germany). German astronomer and mathematician. Led a restless live, in part due to numerous misfortunes during the turbulent times before and during the Thirty Years War, but also for reasons to be found in his character. Of greatest importance for him was his acquaintance with the Danish astronomer Tyge (Tycho) Brahe, in the year 1600 in Prague, whose astronomical data were the basis for Kepler’s most important works. Succeeding T. Brahe, Kepler became mathematician and astrologer at the imperial court, first under emperor Rudolf II, later under Mathias. Finally, from 1628 until his death in 1630, he was astrologer of the duke of Friedland and Sagan, A. von Wallenstein. Kepler’s first two laws are contained in his Astronomia nova (1609), the third is contained in his main work Harmonices Mundi (1619). They were not generally accepted, however, until Newton’s work who gave them a new, purely mechanical foundation. Kepler’s essential achievement was to overcome the ancient opposition between celestial mechanics (where the circle was believed to be the natural inertial type of movement) on one hand and terrestrial mechanics on the other (where the straight line is the inertial motion).


Academy, succeeding d’Alembert, then from 1786 member of the French Academy of Sciences, professor at Ecole Normale, Paris, in 1795 and at Ecole Polytechnique in 1797. His major work *Mécanique Analytique* which appeared in 1788, after Newton’s *Principia* (1688) and Euler’s *Mechanica* (1736), is the third of the historically important treatises of mechanics. Of special relevance for mechanics were his completion of variational calculus which he used to derive the Euler-Lagrange equations of motion, and his contributions to celestial perturbation theory.

*Laplace, Pierre Simon de:* born 28 March 1749 in Beaumont-en-Auge (France), died 5 March 1827 in Paris. French mathematician and physicist. From 1785 on member of the Academy, he became professor for mathematics at Ecole Normale (Paris) in 1794. He must have been rather flexible because he survived four political systems without harm. Under Napoleon he was for a short time Secretary of the Interior. Besides important contributions to celestial mechanics on the basis of which the stability of the planetary system was rendered plausible, he developed potential theory, along with Gauß and Poisson. Other important publications of his deal with the physics of vibrations and with thermodynamics.

**Legendre, Adrien Marie:** 1752–1833, French mathematician.

**Leibniz, Gottfried Wilhelm:** 1646–1716, German natural philosopher and philosopher.

**Lie, Marius Sophus:** 1842–1899, Norwegian mathematician.

**Liapunov, Aleksandr Mikhailovich:** 1857–1918, Russian mathematician.

**Liouville, Joseph:** 1809–1882, French mathematician.

**Lorentz, Hendrik Antoon:** 1853–1928, Dutch physicist.

*Maupertuis, Pierre Louis Moreau de:* born 28 September 1698 in St. Malo (Brittany, France), died 27 July 1759 in Basel (Switzerland). French mathematician and natural philosopher. From 1731 paid member of the Academy of Sciences of France, in 1746 he became the first president of the Prussian *Académie Royale des Sciences et Belles Lettres*, newly founded by Frederick the Great in Berlin who called many important scientists to the academy, notably L. Euler. In 1756, seriously ill, Maupertuis returned first to France but then joined his friend Joh. II Bernoulli in Basel where he died in 1759. Along with Voltaire, Maupertuis was a supporter of Newton’s theory of gravitation which he had come to know while visiting London in 1728, and fought against Descartes’ ether whirls. Of decisive importance for the development of mechanics was his principle of least action, formulated in 1747, although his own formulation was still somewhat vague (the principle was formulated in precise form by Euler and Lagrange). A widely noticed dispute of priority started by the Swiss mathematician Samuel König who attributed the principle to Leibniz, was eventually decided in favor
of Maupertuis. This dispute alienated Maupertuis from the Prussian Academy and contributed much to his bad state of health.

**Minkowski, Hermann:** 1864–1909, German mathematician.

**Moser, Jürgen:** 1928–1999, German mathematician.

**Newton, Isaac:** born 24 December 1642 in Whoolsthorpe (Lincolnshire, England), died 20 March 1726 in Kensington (London), (both dates according to the Julian calendar which was used in England until 1752). Newton, who had studied theology at Trinity College Cambridge, learnt mathematics and natural sciences essentially by himself. His first great discoveries, differential calculus, dispersion of light, and the law of gravity which he communicated to a small circle of experts in 1669, so impressed Isaac Barrow, then holding the “Lucasian Chair” of mathematics at Trinity College, that he renounced his chair in favor of Newton. In 1696 Newton was called at the Royal Mint and became its director in 1699. The Royal Society of London elected him president in 1703. Venerated and admired as the greatest English natural philosopher, Newton was buried in Westminster Abbey.

His principal work, with regard to mechanics, is the three-volume *Philosophiae Naturalis Principia Mathematica* (1687) which he wrote at the instigation of his pupil Halley and which was edited by Halley. Until this day the Principia have been studied and completely understood only by very few people. The reason for this is that Newton’s presentation uses a highly geometrical language, divides matters into “definitions” and “axioms” which mutually complete and explain one another, following examples from antiquity, in a manner difficult to understand for us, and because he presupposes notions of scholastic and Cartesian philosophy we are normally not familiar with. Even during Newton’s lifetime it took a long time before his contemporaries learned to appreciate this difficult and comprehensive work which, in addition to Newton’s laws, deals with a wealth of problems in mechanics and celestial mechanics, and which contains Newton’s original ideas about space and time that have stimulated our thinking ever since. Newton completed a long development that began during antiquity and was initiated by astronomy, by showing that celestial mechanics is determined only by the principle of inertia and the gravitational force and, hence, that it follows the same laws as the mechanics of our everyday world. At the same time he laid the foundation for a development which to this day is not concluded.

**Noether, Emmy Amalie:** 1882–1935, German mathematician. Belongs to the great scientific personalities in the mathematics of the 20th century. Her seminal work *Invariante Variationsprobleme*, published in 1918, contains two theorems which ever since we refer to as the “Noether theorems” and which provide important keys to various parts of theoretical physics, notably mechanics and classical field theory. Barely any other mathematical publication has had such a profound and lasting impact on theoretical physics in the 20th century.

**Poincaré, Jules Henri:** born 29 April 1854 in Nantes (France), died 17 July 1912 in Paris. French mathematician, professor at Sorbonne university in Paris.
Poincaré, very broad and extraordinarily productive, made important contributions to the many-body problem in celestial mechanics for which he received a prize donated by King Oscar II of Sweden. (Originally the prize was announced for the problem of convergence of the celestial perturbation series, the famous problem of the “small denominators” that was solved much later by Kolmogorov, Arnol’d, and Moser.) Poincaré may be considered the founder of qualitative dynamics and of the modern theory of dynamical systems.

**Poisson, Siméon Denis:** 1781–1840, French mathematician.
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