THE REDUCTION OF ENERGY LOSS DUE TO SYNCHROTRON RADIATION

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The interference between the electromagnetic fields produced by two equally and oppositely charged particles rotating synchronously about the same axis with small separation is considered. Since the fields tend to disappear as the distance between the particles is reduced, it is reasoned that in this case the rate of energy loss by synchrotron radiation should also decrease. Expressions are derived for the components of the force on one particle due to the presence of the other and it is found that the component acting along the direction of motion is just sufficient, in the limit of very small separation, to compensate the rate of energy loss by synchrotron radiation. A method is proposed for producing the required electromagnetic-field cancellation by making a charged particle move close to a superconducting sheet and thus producing an oppositely charged image particle.

1. INTRODUCTION

Classical electromagnetic theory predicts that an accelerating charged particle loses energy by radiation. In the particular case of a particle moving along a circular trajectory, the radiation is known as synchrotron radiation. In present-day circulating electron accelerators a large fraction of the power input is lost in this form. To reduce this power loss, the standard solution has been to increase the radius of the electron’s orbit since, for a given particle energy, the power loss is inversely proportional to the square of the radius. It is shown in this paper that under certain special conditions the rate of energy loss by synchrotron radiation can be substantially reduced, thus perhaps leading to a reduction in the size of high-energy electron accelerators.

2. ELECTRIC AND MAGNETIC FIELDS DUE TO ACCELERATING PARTICLES

The electric and magnetic fields associated with a moving charged particle are given by the formulae

\[
E = \frac{e}{\gamma^2 R^2} \frac{(\hat{n} - \hat{\beta})}{(1 - \hat{n} \cdot \hat{\beta})^3} + eB^2 \frac{\hat{n} \times ((\hat{n} - \hat{\beta}) \times \hat{w})}{Rr} \frac{(1 - \hat{n} \cdot \hat{\beta})^3}{(1 - \hat{n} \cdot \hat{\beta})^3}
\]

\[
H = \hat{n} \times E,
\]

where \( e \) is the electric charge, \( R \) is the distance from the charge to the observation point, and \( \hat{n} \) is the unit vector in this direction, \( \gamma \) is the instantaneous radius of curvature, \( \beta = v/c \) (\( v \) is the vector velocity of the particle and \( c \) is the velocity of light), \( \hat{w} \) is the unit vector in the direction of the particle’s acceleration and \( \gamma^2 = 1/(1 - \beta^2) \).

Let us imagine a particle with charge \( e \) traveling with uniform angular velocity in a circular orbit of radius \( r \). If we consider an infinitely long cylinder of radius greater than \( r \) whose axis coincides with the perpendicular passing through the orbit center then, at any given time there will be a pattern of electric and magnetic fields on the
surface of the cylinder. If Poynting’s vector is calculated over the whole of the surface, then in unit time the quantity of energy given in Eq. (3) will be found to pass outwards through the cylindrical surface. From Eqs. (1) and (2) we note that if the sign of the electric charge of the particle is changed from \(+e\) to \(-e\), then the directions of the electric and magnetic fields are reversed and this leads to the observation that if two particles of equal and opposite charges are coincident, then, due to the mutual cancellation of the fields, there is no energy loss by radiation (of course this is obvious since two such particles would constitute an electrically neutral particle, which does not lose energy through radiation). However, we can pursue this thought experiment a little further and slightly separate the two orbits in such a way that they are synchronous but separated in the direction of the axis of rotation. On the surface of the cylinder there will be interference between the two radiation fields. As the distance between the two orbits is increased, the two radiation fields will separate and in the limit of infinite separation the rate of energy loss will be that due to two independent particles. It would seem reasonable to suppose that for some small separation distances between the two orbits, there is a noticeable reduction in energy loss from the system compared with that due to a single orbiting particle. In the next section, we shall investigate this process and we shall see that the reduction in energy loss is explainable by the fact that the particles exert accelerating forces on each other. It is clearly difficult to imagine a method of making oppositely charged particles move in virtually coincident orbits. However, the presence of a second particle is not indispensible. Imagine a charged particle moving at a distance \(d\) from the surface of a superconducting plane. The charge distribution in the surface of the superconductor will be such that the field above the plane is the same as that produced by the particle and its equal and oppositely charged image particle at distance \(d\) below the plane.

3. FORCES ACTING ON THE PARTICLES

Consider the situation shown in Fig. 1.

Two particles have synchronous orbits as defined in Sect. 2. A Cartesian coordinate system is defined such that the particles’ orbits are in the planes \(z = \pm h/2\). At time \(t = 0\), the particles have coordinates \(x = y = 0\). The \(y\) axis is directed towards the centers of the orbits and the direction of the \(x\) axis is given by the direction of motion of the particles at \(t = 0\). Particle 1 has charge \(+e\) and travels along the orbit \(A_1B_1\) of radius \(r\) with velocity \(\beta c\). Particle 2 moves directly below particle 1 along orbit \(A_2B_2\) and has charge \(-e\). Radiation emitted by particle 2 at \(A_2\) (at time \(t = 0\)) encounters particle 1 at \(B_1\). The condition for this to happen is that the time for light to travel in the straight line \(A_2B_1\) equals the time for the first particle to travel along \(A_1B_1\). Denoting by \(\alpha\) the angle moved through by particle 1, we have

\[
\frac{\alpha r}{\beta} = [(r \sin \alpha)^2 + r^2(1 - \cos \alpha)^2 + h^2]^{1/2} = R.
\]

The unit vector from \(A_2\) to \(B_1\) thus has components

\[
\frac{\beta \sin \alpha}{\alpha}, \frac{\beta (1 - \cos \alpha)}{\alpha}, \frac{\beta h}{\alpha r}.
\]
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where \( \mathbf{\beta}_1 \) is the vector velocity (divided by \( c \)) of the particle 1 at \( B \) (in the following derivations unsubscripted vector quantities correspond to properties of particle 2 at \( A_2 \), while the subscript 1 refers to particle 1 at \( B_1 \)).

Substituting Eq. (4) in Eq. (1) and recalling that particle 2 has negative charge, we find that the electric field at \( B_1 \) due to the particle 2 when it was at \( A_2 \) is given by

\[
\mathbf{E} = \frac{-e\mathbf{\beta}^2}{\gamma^2\alpha^2 r^2 (1 - \hat{n} \cdot \mathbf{\beta})^3}
\times \{ (\hat{n} - \mathbf{\beta}) + \alpha\beta\gamma^2 \hat{n} \times (\hat{n} - \mathbf{\beta}) \times \hat{w} \}.
\]

Using (7) and (2), eq. (6) can be rewritten

\[
\mathbf{F} = -\frac{e\beta^2}{\gamma^2\alpha^2 r^2 (1 - \hat{n} \cdot \mathbf{\beta})^3}
\times \{ (\hat{n} - \mathbf{\beta}) + \alpha\beta\gamma^2 \hat{n} \times (\hat{n} \times \hat{w}) - \alpha\beta\gamma^2 \hat{n}
\times (\hat{n} \times (\hat{n} \times (\hat{n} \times \hat{w}))) \}
\]

Now

\[
\mathbf{a}_1 \times (\mathbf{a}_2 \times \mathbf{a}_3) = (\mathbf{a}_1 \cdot \mathbf{a}_3)\mathbf{a}_2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_3
\]

and it can easily be shown that

\[
\mathbf{a}_1 \times (\mathbf{a}_2 \times (\mathbf{a}_3 \times (\mathbf{a}_4 \times \mathbf{a}_5))) = [(\mathbf{a}_3 \cdot \mathbf{a}_5)(\mathbf{a}_1 \cdot \mathbf{a}_4)
- (\mathbf{a}_3 \cdot \mathbf{a}_4)(\mathbf{a}_1 \cdot \mathbf{a}_5)]\mathbf{a}_2
- (\mathbf{a}_3 \cdot \mathbf{a}_5)(\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_4
+ (\mathbf{a}_3 \cdot \mathbf{a}_4)(\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_5
\]

so that Eq. (8) can be written

\[
\mathbf{F} = -\frac{e\beta^2}{\gamma^2\alpha^2 r^2 (1 - \hat{n} \cdot \mathbf{\beta})^3}
\times \{ [(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)(1 + \alpha\beta\gamma^2 \hat{n} \cdot \hat{w})
- \alpha\beta\gamma^2 \mathbf{\beta}_1 \cdot \hat{w}(1 - \hat{n} \cdot \mathbf{\beta})] \hat{n}
- (1 - \hat{n} \cdot \mathbf{\beta}_1)(1 + \alpha\beta\gamma^2 \hat{n} \cdot \hat{w})\mathbf{\beta}
- \alpha\beta\gamma^2 (1 - \hat{n} \cdot \mathbf{\beta}_1)(1 - \hat{n} \cdot \mathbf{\beta}_1)\hat{w} \}
\]

We are now in a position to find the various components of the force acting on particle 1 at \( B \). We shall find the components along the directions \( \mathbf{\beta}_1 \), toward the orbit center and perpendicular to these two directions. We denote these components by \( F_{x_1}, F_{y_1} \) and \( F_{z_1} \) respectively. Note that

\[
F_{x_1} = \frac{\mathbf{\beta}_1}{\beta} \cdot \mathbf{F},
\]

\[
F_{y_1} = \hat{w}_1 \cdot \mathbf{F} \quad \text{and} \quad F_{z_1} = \hat{k} \cdot \mathbf{F},
\]

where \( \hat{w}_1 = -\hat{i} \sin \alpha + \hat{j} \cos \alpha \); \( \hat{i}, \hat{j} \) and \( \hat{k} \) are the unit vectors in the \( x, y \) and \( z \) directions.

In evaluating the components given by Eq. (10), use will be made of the following relations

\[
\hat{n} \cdot \mathbf{\beta} = \hat{n} \cdot \mathbf{\beta}_1, \quad 1 + \alpha\beta\gamma^2 \hat{n} \cdot \hat{w} = 1 + \beta^2(1 - \cos \alpha)
= \frac{(1 - \beta^2) + \beta^2(1 - \cos \alpha)}{(1 - \beta^2)} = \gamma^2(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)
\]

and

\[
\hat{n} \cdot \hat{w}_1 = -\hat{n} \cdot \hat{w}, \quad \mathbf{\beta} \cdot \hat{w}_1 = -\mathbf{\beta}_1 \cdot \hat{w}
\]

After some manipulation we find

\[
F_{x_1} = \frac{-e\beta^2}{\alpha^2r^2(1 - \hat{n} \cdot \mathbf{\beta})^3}
\left\{ \frac{(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)}{\alpha}(\hat{n} \cdot \mathbf{\beta} - \mathbf{\beta} \cdot \mathbf{\beta}_1)
- \beta(1 - \hat{n} \cdot \mathbf{\beta}_1)\hat{w}_1 \right\}
\]

(11)

\[
F_{y_1} = \frac{-e\beta^2}{\alpha^2r^2(1 - \hat{n} \cdot \mathbf{\beta})^3}
\left\{ \hat{n} \cdot \hat{w}_1 + \frac{(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)}{\alpha} \right\}
\times (1 - \hat{n} \cdot \mathbf{\beta}_1) \hat{w} - \hat{n} \cdot \hat{w}_1 \frac{(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)^2}{\alpha}
- \frac{\mathbf{\beta} \cdot \mathbf{\beta}_1}{\beta} (1 - \hat{n} \cdot \mathbf{\beta})^2 \right\}
\]

(12)

\[
F_{z_1} = \frac{-e\beta^2}{\alpha^2(1 - \hat{n} \cdot \mathbf{\beta})^3}
\left\{ \hat{n} \cdot \hat{k} \left[ \frac{(1 - \mathbf{\beta} \cdot \mathbf{\beta}_1)^2}{\alpha} - \mathbf{\beta} \cdot \hat{w}_1 \cdot \hat{w} \right] \right\}.
\]

(13)

The expressions (11) to (13) are not very informative in their present form. We shall there-
fore attempt to find a power-series solution for small values of \( \alpha \). Although in reality the geometry of the system is defined by the separation \( h \) between the two particles, it is convenient to consider \( h \) as a function of \( \alpha \). The defining relationship is given by Eq. (4), which can be reduced to

\[
\frac{h^2}{r^2} = \frac{\alpha^2}{\beta^2 \gamma^2} + 2 \left[ \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \ldots \right].
\] (14)

As a first stage in obtaining the components in terms of \( \alpha \), note that the denominator term in Eqs. (11) to (13) can be written

\[
(1 - \hat{n} \cdot \beta)^{-3} = \left( 1 - \beta^2 \frac{\sin \alpha}{\alpha} \right)^{-3}
\]

\[
= \left( \frac{1}{\gamma^2} + \beta^2 \left( \frac{\alpha^2}{3!} - \frac{\alpha^4}{5!} + \ldots \right) \right)^{-3}
\]

\[
= \gamma^6 \left( 1 - \frac{\gamma^2 \beta^2 \alpha^2}{2} + \frac{3 \gamma^2 \beta^2}{5!} \alpha^4 - \ldots \right)
\]

Neglecting the intermediate calculations, which are relatively straightforward but tedious, we find the following expressions for the components

\[
F_{x1} = \frac{2 e^2 \beta^3 \gamma^4}{3} \left[ 1 - \beta^2 (\alpha \gamma)^2 + 0(\alpha^4) \right]
\] (15)

\[
F_{y1} = \frac{e^2 \beta^3 \gamma^2}{2 \alpha} \left( 1 - \hat{n} (\alpha \gamma)^2 + 0(\alpha^4) \right)
\] (16)

\[
F_{z^*} = -\frac{e^2 \beta^3 \gamma^2}{r^2 \alpha^3} \left( 1 - \frac{(\alpha \gamma)^2}{2} + 0(\alpha^4) \right)
\] (17)

where \( 0(\alpha^4) \) denotes quantities of order \( \alpha^4 \) and greater.

4. INTERPRETATION AND NUMERICAL RESULTS

From Eq. (15) we see that as \( \alpha \gamma \) tends to zero, the force acting along the direction of particle 1 tends to \( 2 e^2 \beta^3 \gamma^4 / 3 r^2 \), so that energy is supplied to the particle at the rate of \( 2 e^2 \beta^3 \gamma^4 c / 3 r^2 \). This is the same as the rate of energy loss by synchrotron radiation when particle 2 is absent [see Eq. (3)]. We can interpret this result as meaning that the energy lost by particle 2 is fed back into the motion of particle 1 and vice versa. In the case of small \( \alpha \), the rate of energy loss for both particles in the present system is given by \( 2 e^2 \beta^3 \gamma^4 c (\alpha \gamma)^2 / 3 r^2 \) and the ratio of energy-loss rate in this system to that in the simple one-particle system under the same conditions is \( (\alpha \beta \gamma)^2 \) \( = h^2 \gamma^4 / r^2 \) if \( \alpha \ll 1/\gamma \), see Eq. (14)]. It is interesting to calculate the values of \( h \) required to reduce radiative energy loss by various amounts. Such values are tabulated in Table 1. A radius of curvature of 1000 m has been assumed.

Equation (16) shows that the particle also experiences a force tending to direct it towards the orbit center and, from Eq. (17), we conclude that there is an attractive force between particles 1 and 2. The magnitude of these forces is very small and can be conveniently expressed in terms of the magnetic field required to produce the equivalent force on the rapidly moving particle. Considering the examples in Table 1 it can be shown that in every case the magnetic field is much smaller than 1 Gauss.

5. DISCUSSION AND CONCLUSIONS

It has been shown that, at least in principle, it is possible to find a means of substantially reducing the power loss due to synchrotron radiation. Although this note has only touched very lightly on the practicability of the scheme, it is natural to ask whether it could be applied in the design of high-energy electron accelerators. Assuming that a material can be found with suitable properties to act as a perfectly conducting sheet, the major
drawback of the scheme is that to obtain a significant reduction in radiation loss, it requires separations of fractions of microns between the particle and its image. This constraint would not be too serious were it not for the unfortunate behaviour of $F_z$, which varies with the inverse square of the distance between the particle and its image. Thus, in the scheme as it stands, there is no means of confining the particle to within a small distance from the conducting sheet. If it were possible to create an additional force repelling the particle from the surface, then the net effect could be to confine the particle close to the sheet.

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REFERENCE