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ON SOLITONS WITH HALF INTEGRAL CHARGE

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A B S T R A C T

Can soliton states, with half integral expectation value of charge, be also eigenstates of charge -- with half integral eigenvalue ? It can be so only with a somewhat sophisticated definition of charge.

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In some very interesting papers [1,2,3,4] it has been shown that in certain models there appear states with half integral expectation value of charge in the neighbourhood of certain solitons. It has even been suggested that these states are eigenstates of charge -- with half integral eigenvalue. See in particular Goldstone and Wilczek[2], and Jackiw and Schrieffer [3] (eq.(3.20)). These ideas are quite startling, especially in connection with nonrelativistic atomic models [3], which in the usual sense have a fixed finite integral number of electrons. We believe that the states in question are eigenstates only of a 'charge' operator which is defined with some sophistication.

In this paper we consider not the nonrelativistic models, to which we will return, but the one dimensional Dirac field theory [1]. We find that the charge defined in a naive way, in a region with sharp boundaries, other than the total quantization volume, suffers large fluctuations -- as is usual in field theory. However, as is also usual in field theory, these fluctuations can be suppressed by making diffuse the boundaries of the region in question.

In contrast with Jackiw and Rebbi [1] we will work first with a finite quantization volume L, and only later take L to infinity. For simplicity we will take for the soliton function just a step function:

$$\varphi(x) = -m\frac{x}{|x|} \tag{1}$$

It will be evident that a more realistic soliton function would not change the conclusions. The one dimensional Dirac equation in the presence of an external field is

$$\left[-i\alpha \frac{d}{dx} - \beta\varphi(x)\right]\eta(x) = \varepsilon\,\eta(x) \tag{2}$$

where $\eta(x)$ is a two component spinor $\eta_i(x)$; the spinor index will often be suppressed below. We adopt the representation of Dirac matrices

$$\alpha = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \beta = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with

$$C = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the charge conjugation matrix. We impose periodic boundary conditions:

$$\eta_i(-L/2) = \eta_i(L/2)$$

Then there are the following solutions.

(1) Positive energy solutions. For each energy

$$\varepsilon_n = +\sqrt{k_n^2 + m^2}$$

with

$$k_n L = 2n\pi \qquad ; \qquad n = 1, 2, \cdots \infty$$

there are two degenerate solutions:

$$\frac{1}{\sqrt{L}}\begin{pmatrix} \cos(k_n|x| + \delta_n) \\ \\ -\sin(k_n x) \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{L}}\begin{pmatrix} \sin(k_n x) \\ \\ \cos(k_n|x| - \delta_n) \end{pmatrix} \tag{3}$$

where

$$\cos\delta_n = \frac{k_n}{\varepsilon_n} \quad \text{and} \quad \sin\delta_n = \frac{m}{\varepsilon_n} \tag{3a}$$

We label all these positive energy solutions by $\eta_p(x)$ with $p = 1,2,...$

(2) Negative energy solutions. These are obtained from the positive energy solutions by changing the sign of one component, and have energy $(-\varepsilon_p)$:

$$\tilde{\eta}_p(x) = \sigma_3 \; \eta_p(x) \qquad ; \; p = 1, 2, \cdots \tag{4}$$

(3) Zero energy solutions. There are two of these:

$$\eta_0(x) \equiv \left(\frac{m}{1 - e^{-mL}}\right)^{1/2} \begin{pmatrix} e^{-m|x|} \\ 0 \end{pmatrix}$$

and

$$\tilde{\eta}_0(x) \equiv \left(\frac{m}{1 - e^{-mL}}\right)^{1/2} \begin{pmatrix} 0 \\ e^{m(|x| - L/2)} \end{pmatrix} \tag{5}$$

The second zero mode is centered on the antikink which is automatically introduced by periodic boundary conditions. It is there, however large L is made.

The set of solutions η_p and $\tilde{\eta}_p$ with p = 0,1,2,.... together form a complete orthonormal set

$$\sum_{p=0}^{\infty} \eta_{pi}(x)\eta_{pj}(y) + \tilde{\eta}_{pi}(x)\,\tilde{\eta}_{pj}(y) = \delta_{ij}\,\delta(x-y) \tag{6}$$

where i,j are spinor indices, and p=0 includes both zero modes.

The second quantized Dirac field may be expanded in the usual way

$$\Psi_i(x,t) = \sum_0^{\infty} b_p\,\eta_{pi}(x)\,e^{-i\varepsilon_p t} + d_p^{+}\,\tilde{\eta}_{pi}(x)\,e^{i\varepsilon_p t} \tag{7}$$

The usual anticommution rules hold:

$$\left\{ \Psi_i(x,t)\,,\ \Psi_j^{+}(y,t) \right\} = \delta_{ij}\,\delta(x-y)$$

$$\{b_p\,,b_q\}_+ = \{d_p\,,d_q\}_+ = \{b_p\,,d_q\}_+ = \{b_p\,,d_q^{+}\}_+ = 0$$

$$\{b_p\,,b_q^{+}\}_+ = \{d_p\,,d_q^{+}\}_+ = \delta_{pq}$$

Now consider the antisymmetrized charge density operator at t=0:

$$\rho(x) = \frac{1}{2}\left[\Psi^{+}(x)\,,\ \Psi(x)\right]_{t=0} \tag{8}$$

$$= \frac{1}{2}\sum_{p,q=0}^{\infty}\Big([b_p^+, b_q]\eta_p(x)\eta_q(x) + [d_p, d_q^+]\tilde{\eta}_p(x)\tilde{\eta}_q(x) + [b_p^+, d_q^+]\eta_p(x)\tilde{\eta}_q(x) + [d_p, b_q]\tilde{\eta}_p(x)\eta_q(x)\Big)$$

$$= \sum_{p,q=0}^{\infty}\Big(b_p^+ b_q \eta_p\eta_q - d_q^+ d_p \tilde{\eta}_p\tilde{\eta}_q + b_p^+ d_q^+ \eta_p\tilde{\eta}_q + d_p b_q \tilde{\eta}_p\eta_q\Big) + \frac{1}{2}\Big(\tilde{\eta}_0(x)\tilde{\eta}_0(x) - \eta_0(x)\eta_0(x)\Big) \qquad (9)$$

where summation over suppressed spinor indices is understood. In the last step, where the anticommutators (8) have been used, the c-number term $\frac{1}{2}(\tilde{\eta}_0^2-\eta_0^2)$ survives because these two zero modes are not charge conjugates of one another. However the non zero modes, which satisy

$$\tilde{\eta}_p(x)\,\eta_q(x) = \eta_p(x)\,\tilde{\eta}_q(x)$$

and

$$\eta_p(x)\,\eta_q(x) = \tilde{\eta}_p(x)\,\tilde{\eta}_q(x) \quad , \text{ for all } p,q \neq 0$$

cancel out, and do not appear in the last term of (9). The total charge operator is

$$Q \equiv \int_{-L/2}^{L/2} \rho(x)\,dx \qquad (10)$$

Inserting the density (9), and using orthogonality of wavefunctions,

$$Q = \sum_{p=0}^{\infty}\left(b_p^+ b_p - d_p^+ d_p\right) + \frac{1}{2}\int_{-\frac{L}{2}}^{L/2} dx\,\left(\tilde{\eta}_0(x)\tilde{\eta}_0(x) - \eta_0(x)\eta_0(x)\right)$$

$$= \sum_{0}^{\infty}\left(b_p^+ b_p - d_p^+ d_p\right) \qquad (11)$$

The contributions of $1/2$ from the two zero modes cancel in the total charge Q:

The total charge has integral eigenvalues.

There are four zero energy states in the second quantized theory. Define the state $|0\rangle$ by

$$b_p|0\rangle = d_p|0\rangle = 0 \qquad \text{for all } p = 0, 1, 2, \cdots \tag{12}$$

Then the states

$$|0\rangle \quad , \quad b_0^\dagger|0\rangle \quad , \quad d_0^\dagger|0\rangle \quad \text{and} \quad b_0^\dagger d_0^\dagger|0\rangle$$

have all zero energy, and are eigenstates of charge with eigenvalues

$$0 \quad , \quad 1 \quad , \quad -1 \quad , \quad \text{and} \quad 0$$

respectively.

We repeat: the total charge Q has integral eigenvalues. But how about a partial charge P that does not involve the remote boundary and the second soliton? Define

$$P \equiv \int_{-l/2}^{l/2} \rho(x)\,dx, \qquad \text{with } L \gg l \gg m^{-1} \tag{13}$$

Using (9),

$$\langle 0|\rho(x)|0\rangle = \frac{1}{2}\left(\tilde{\eta}_0(x)\tilde{\eta}_0(x) - \eta_0(x)\eta_0(x)\right) \tag{14}$$

Thus

$$\langle 0 | P | 0 \rangle = \frac{1}{2}\int_{-\ell/2}^{\ell/2} \left(\tilde{\eta}_0 \tilde{\eta}_0 - \eta_0 \eta_0 \right) dx$$

$$\simeq -\frac{1}{2} \int_{-\ell/2}^{\ell/2} \eta_0 \eta_0 \, dx \qquad = -1/2 \tag{15}$$

for L>>ℓ>>1/m. The contribution of the second zero mode is excluded by the region of support of P. Similarly

$$\langle d_0^{+} 0 | P | d_0^{+} 0 \rangle = -1/2 \quad , \text{ and}$$

$$\langle b_0^{+} 0 | P | b_0^{+} 0 \rangle = \langle b_0^{+} d_0^{+} 0 | P | b_0^{+} d_0^{+} 0 \rangle = 1/2 \tag{16}$$

But these half integral values are only expectation values, not eigenvalues.

Neither the charge density ρ nor the partial charge P commutes with the Hamiltonian. They contain terms of the form

$$b_p^{+} d_q^{+} \eta_p(x) \tilde{\eta}_q(x)$$

which do not vanish when integrated only over part of the space. They cause excitation when acting on energy eigenstates. For example let us calculate the mean square fluctuation of P in the state |0>. We have, using (9) and (13),

$$(\Delta P)^2 \equiv \langle 0 | (P - \langle 0|P|0\rangle)^2 | 0 \rangle$$

$$= \sum_{p,q=0}^{\infty} \left| \int_{-\ell/2}^{\ell/2} dx \, \tilde{\eta}_q(x) \eta_p(x) \right|^2 \tag{17}$$

This is a sum over nonnegative terms. An equivalent form is obtained from the continuity equation:

$$(\Delta P)^2 = \sum_{p,q} \left| \langle pq | \frac{dP}{dt} | 0 \rangle \right|^2 (\varepsilon_p + \varepsilon_q)^{-2}$$

$$= \sum_{p,q} \left| \left[\eta_p(x) \, i \overleftrightarrow{\partial} \, \tilde{\eta}_q(x) \right]_{-\ell/2}^{\ell/2} \right|^2 (\varepsilon_p + \varepsilon_q)^{-2} \tag{18}$$

Inserting the wavefunctions (3-5), and with $L \gg \ell \gg m^{-1}$ one obtains in the limit of infinite L, ℓ,

$$(\Delta P)^2 = \int_0^{\infty} \frac{dk}{2\pi} \int_0^{\infty} \frac{dk'}{2\pi} \, f(k,k') \, (\varepsilon_k + \varepsilon_{k'})^{-2} \tag{19}$$

where

$$f(k,k') \equiv 2\left[\left(\frac{k}{\varepsilon_k}\right)^2 + \left(\frac{k'}{\varepsilon_{k'}}\right)^2 + m^2 \left(\frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_{k'}} \right)^2 \right]$$

$$= 4\left(1 + \frac{m^2}{\varepsilon_k \varepsilon_{k'}}\right) \tag{20}$$

Notice that these fluctuations persist at infinite L, and are then independent of ℓ when the latter is large compared with 1/m. This is not surprising in view of the relation of the effect, in (18), to currents at the boundary of the region over which P is defined. Notice, by the way that (19) is logarithmically divergent.

We see that although there is a partial charge P with half integral expectation values , the states in question are not eigenstates of P.

There are large fluctuations of P. However such fluctuations are normal in quantum field theory, and would appear even in the normal (soliton-less) vacuum. As is well known the fluctuations can be diminished either by time averaging, or by making diffuse the boundaries of the region over which the charge is defined. Moreover physical observations do not refer either to instants of time or to regions with perfectly sharply defined boundaries.

Consider for example softening the boundaries by inserting, into the definition (13) of P, a weight function which falls to zero, not sharply at $|x| = \ell/2$, but linearly over a distance d. Because the matrix elements

$$\eta_p(x) \; i\alpha \; \tilde{\eta}_q(x)$$

are linear combinations (with coefficients of order unity) of trigonometric functions of

$$\left(k_p \pm k_q\right) x$$

the result is simply the introduction of extra factors

$$\frac{\sin (k+k')d}{(k+k')d}$$

into the matrix elements. These not only remove the logarithmic divergence, but in the large d limit,

$$d \to \infty \quad , \text{ with } L \gg \ell \gg d,$$

suppress the fluctuations completely.

Turning to time averaging, the fluctuation time scale is set by

$$G(t) \equiv \langle 0 \left| \Big(P(t) - \langle P(t) \rangle\Big)\Big(P(0) - \langle P(0) \rangle\Big) \right| 0 \rangle$$

$$= \int_0^\infty \frac{dk}{2\pi} \int_0^\infty \frac{dk'}{2\pi} \; e^{i(\varepsilon_k + \varepsilon_{k'})t} \; f(k,k') \; (\varepsilon_k + \varepsilon_{k'})^{-2} \tag{21}$$

This G(t) has a width of order m^{-1}. If an experiment has a resolution time longer than this, fluctuations will not be seen. This must be borne in mind when interpreting the experimental evidence [5] cited in this connection.

In conclusion, the states in question are eigenstates, with fractional eigenvalue, only of sufficiently 'soft' partial charge operators. Such softness is presumably implicit in the definitions of the authors cited.

Some of the details of the argument are different with different boundary conditions , and in the atomic 'polyacetylene' model [2,3](a Kogut- Susskind [6] lattice-regulated version of the continuum model). We return to this in another paper.

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