HARDT

CONDITION FOR

SUPERPOSITION OF

SEPARATRICES

presented by

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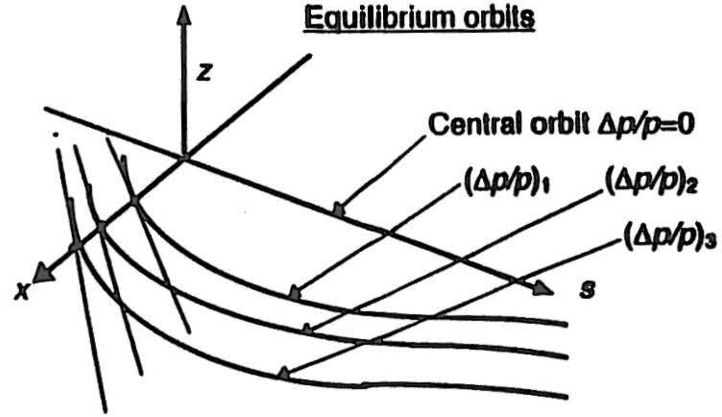
PHASE-SPACE REPRESENTATION OF A BEAM (1)

The motion in an accelerator is parameterised by defining an equilibrium orbit for each momentum and describing particle motions (betatron oscillations) about this equilibrium orbit. Equilibrium orbits for momenta different from the central orbit are parameterised by the dispersion function, *D*.



Positions and angles of orbits are given by

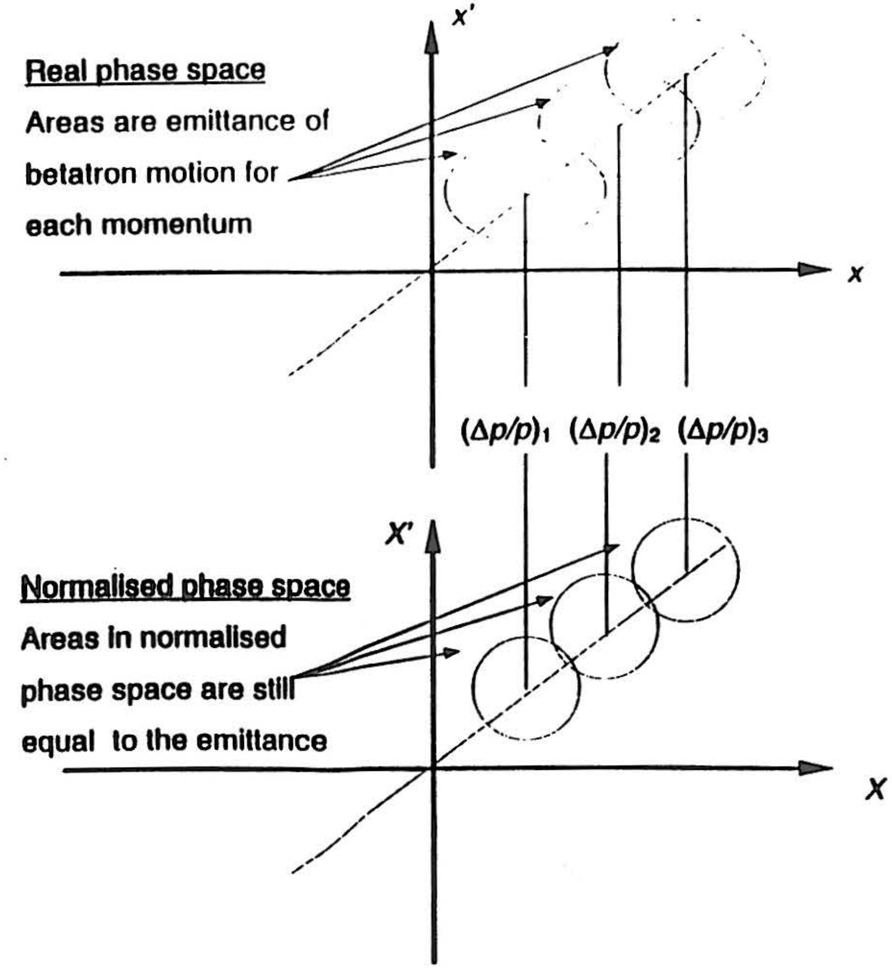
$$x(\Delta p)=D_x\frac{\Delta p}{p_0} \text{ and } \frac{dx}{ds}=D'_x\frac{\Delta p}{p_0} \quad \text{Real phase space}$$

$$X(\Delta p)=D_{n,x}\frac{\Delta p}{p_0} \text{ and } \frac{dX}{d\mu}=D'_{n,x}\frac{\Delta p}{p_0} \quad \text{Normalised phase space}$$

(The conversion between the normalised and real phase spaces was given in the basic lecture)

PHASE-SPACE REPRESENTATION OF A BEAM (2)

At a given position in the machine the beam can be represented in phase space by a series of ellipses (circles in normalized phase space) centred around the dispersion vector $(D,D')\cdot\Delta p/p$



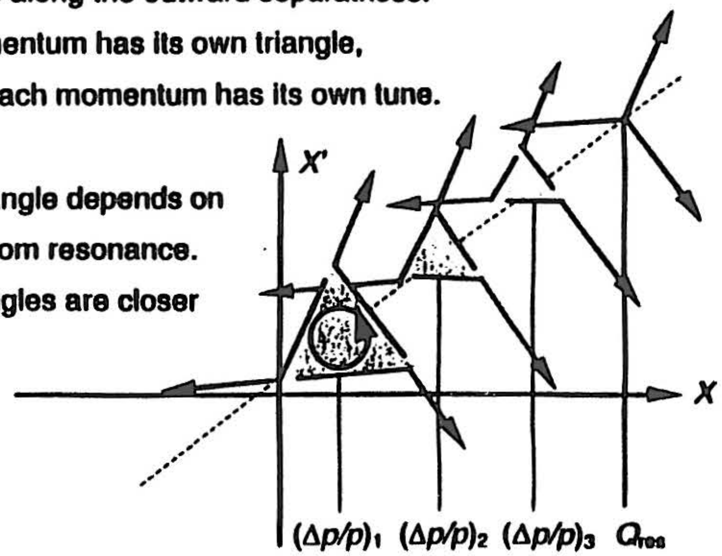
PHASE SPACE REPRESENTATION OF THE RESONANCE (1)

The circles that represent the beam emittance in normalised phase space become triangles (of the same area) under the influence of the resonance.

The resonance for particles of each momentum is represented by a triangle corresponding to the last stable orbit and the extensions along the outward separatrices.

Each momentum has its own triangle, because each momentum has its own tune.

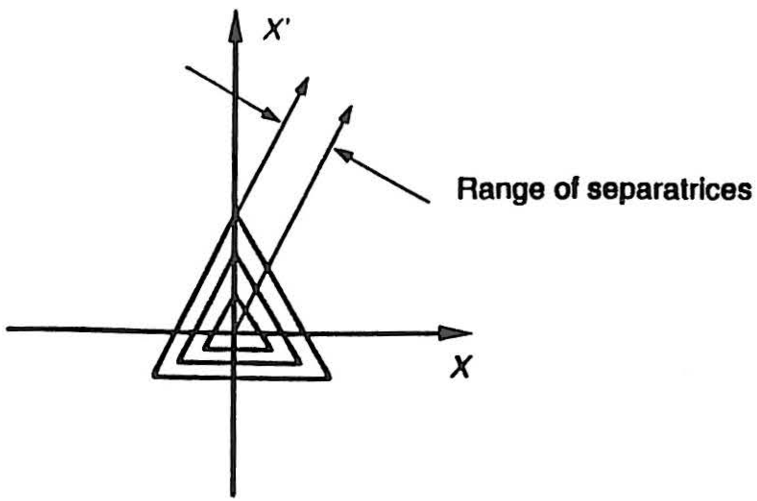
Size of triangle depends on distance from resonance. Small triangles are closer



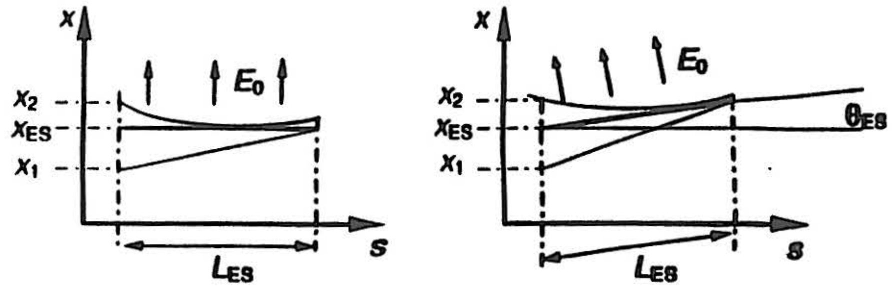
The rotation of the triangles depends on the betatron phase advance from the resonance sextupole to the observation point.

PHASE SPACE REPRESENTATION OF THE RESONANCE (2)

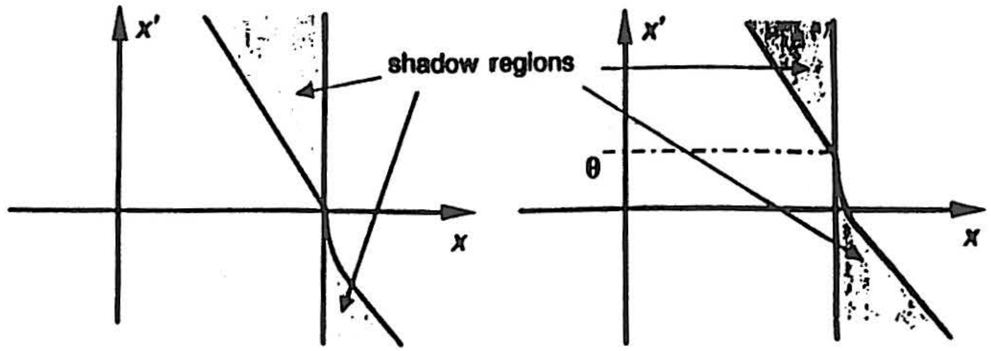
For a monoenergetic beam with increasing sextupole strength, the stable triangles shrink. The size of the stable triangles is characterised by *H* (see simply theory), which is proportional to $\delta Q/S$. By increasing the sextupole strength, particles with smaller and smaller amplitude get extracted.



LOSSES AT THE ELECTROSTATIC SEPTUM IN REAL PHASE SPACE

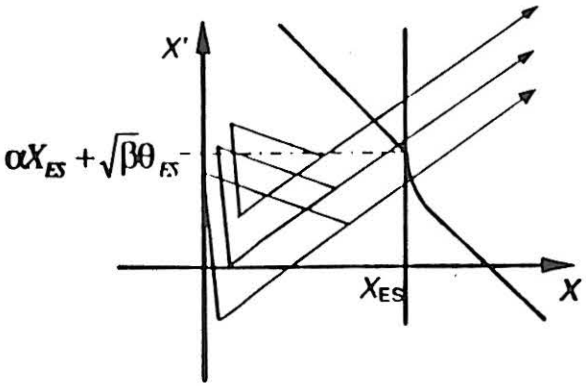


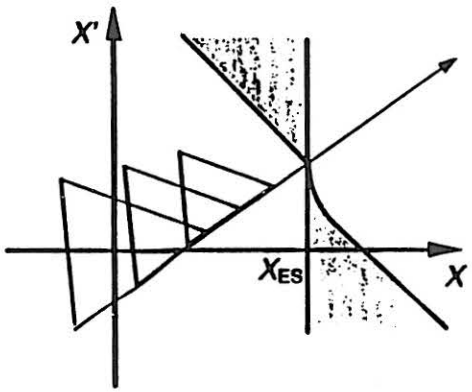
For particles starting at positions x_1 and x_2, not all angles are allowed. In the first drawing, a particle with a divergence greater than $(x_{ES} - x_1)/L_{ES}$ will hit the septum. A similar situation exists for particles inside the septum, except that the electric field curves their orbits and modifies the forbidden angles. The forbidden <u>shadow regions</u> are:



LOSSES AT THE ELECTROSTATIC SEPTUM

IN NORMALISED PHASE SPACE





REASONS FOR THE HARDT CONDITION

The Hardt Condition aligns the separatrices belonging to different momenta and amplitudes at the electrostatic septum to avoid particle losses due to the "shadow regions" shown above, while chromaticity and dispersion are finite. This then makes it possible to keep the bulk of the waiting beam stable and to control the spill.

If the chromaticity and dispersion were zero, then all triangles would have a common centre, but particles with different initial amplitudes would leave the machine along different separatrices and would arrive at the ES with different angles, leading to losses in the shadow regions. <u>Due to the zero chromaticity the waiting beam could become unstable and the spill would be difficult to control.</u>

In any other case, if the Hardt Condition is not fulfilled, particles with different momenta and different amplitudes arrive at the ES with different angles, causing losses in the shadow regions.

<u>Note:</u> With the Hardt Condition fulfilled, all separatrices are coincident and collinear and aperture requirements in the electrostatic septum are an absolute minimum.

REVISION - EQUATION OF A SEPARATRIX

Re-express the separatrix equation with *H* in terms of *Q'* and $\Delta p/p$

$$\left(X - D_n \frac{\Delta p}{p}\right)\cos(\alpha - \Delta\mu) + \left(X' - D'_n \frac{\Delta p}{p}\right)\sin(\alpha - \Delta\mu) = H$$

$$\left(X - D_n \frac{\Delta p}{p}\right)\cos(\alpha - \Delta\mu) + \left(X' - D'_n \frac{\Delta p}{p}\right)\sin(\alpha - \Delta\mu) = \frac{4\pi}{S}\delta Q$$

$$\left(X - D_n \frac{\Delta p}{p}\right)\cos(\alpha - \Delta\mu) + \left(X' - D'_n \frac{\Delta p}{p}\right)\sin(\alpha - \Delta\mu) = \frac{4\pi}{S}Q'\frac{\Delta p}{p}$$

The momentum dependence of the separatrix is now apparent in each of the three main terms. Adjusting the momentum dependent terms in a way that they sum to zero, establishes the Hardt Condition.*

* The above assumes that the beam is moved into the resonance without changing transverse beam parameters. The alternative of sweeping the resonance through the stack would require an additional time variation to be taken into account.

HARDT CONDITION

To remove the momentum dependence from the separatrix, the equation below must be satisfied.

$$D_n \cos(\alpha - \Delta\mu) + D'_n \sin(\alpha - \Delta\mu) = -\frac{4\pi}{S}Q'$$

The above equation may appear very flexible, but there are some boundary conditions on the parameters.

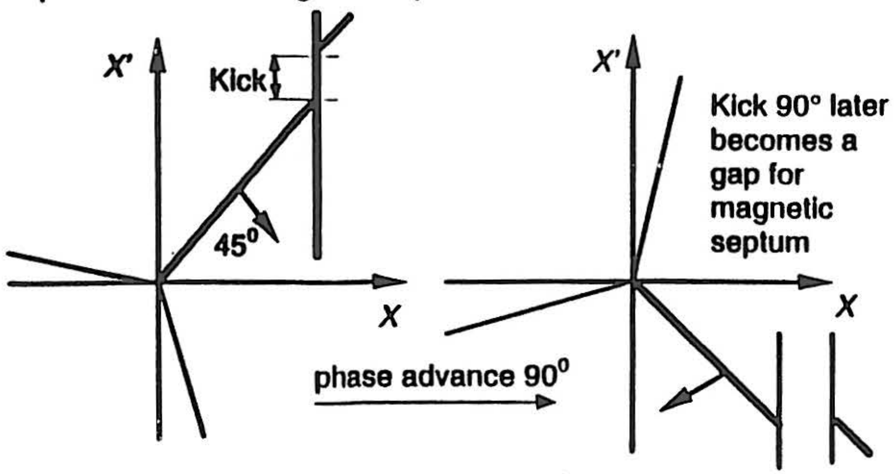
- The choice of the angle (α-$\Delta\mu$) is restrained by the geometry of the extraction. This will be discussed in Section A.
- D_n and D'_n depend on the lattice design. This will be discussed in Section B.
- *Q'* is the main variable, but for machines working below transition energy the stability of a coasting beam is improved by making *Q'* negative. This will be discussed in Section C.
- *S* has been set, somewhat arbitrarily, to give an increase of 10 mm in the amplitude of the separatrix over 3 turns at the electrostatic septum. This is discussed in Section D.
- Finally, numerical values will be discussed in Section E.

SECTION A

SEPARATRIX GEOMETRY FOR EXTRACTION (1)

Figures below show a logical layout in phase space for the septa and separatrices with the extraction separatrix at 45° and -45°. This foresees a 90° phase advance between the septa, which maximises the effect of the electrostatic septum's kick.

The other two separatrices are 120° apart in phase space. Thus, the extracting separatrix could be moved by a maximum of 15° anticlockwise before the preceding separatrix hits the electrostatic septum, or 15° clockwise before the following separatrix hits the magnetic septum.

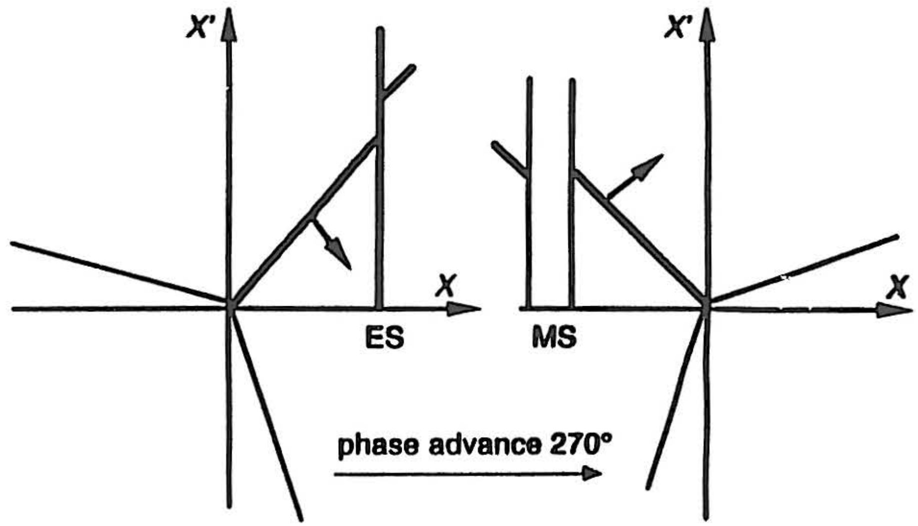


WHAT VARIATIONS EXIST ON THIS THEME ?

SEPARATRIX GEOMETRY FOR EXTRACTION (2)

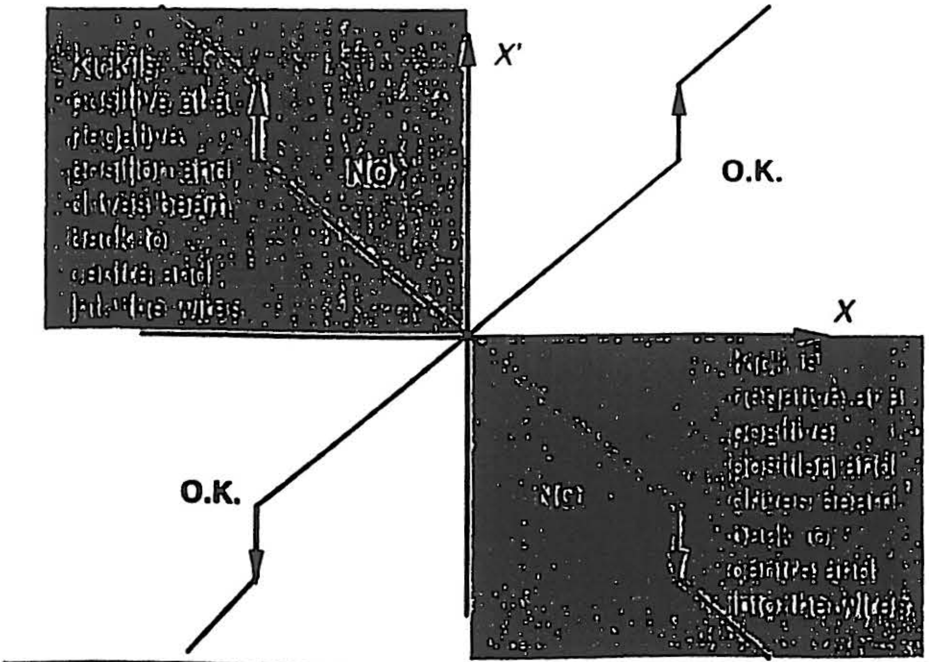
The alternatives to the layout on the previous page are limited. The first is to accept a much smaller phase advance of say 40° or less. This situation may be imposed by lack of space and requires much stronger electrostatic and magnetic septa. In the case of the electrostatic septum, this may have implications for the reliability.

The second alternative is to allow a 270° phase advance.



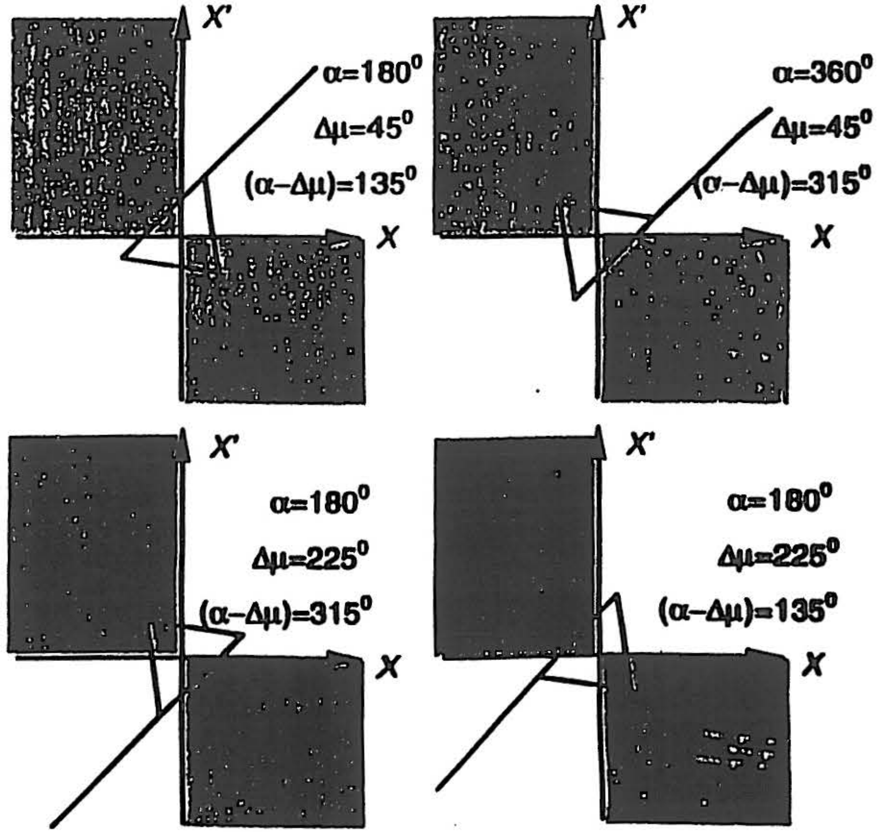
SEPARATRIX GEOMETRY FOR EXTRACTION (3)

So far the kick of the electrostatic septum has been shown in the first quadrant only. In fact only the first and third quadrants are usable. In the second and fourth quadrants, the kick takes the beam back into the septum wires. The use of the third quadrant gives one extra possibility.



For ideal working conditions, the 1st or 3rd quadrants must be used, with 90^0 or 270^0 phase advance between the septa, and with a tolerance of $\pm 15^0$ in the separatrix positions (for much less phase advance one must rely on very strong kicks).

STABLE-TRIANGLE GEOMETRY FOR 1ST & 3RD QUADRANT OPERATION

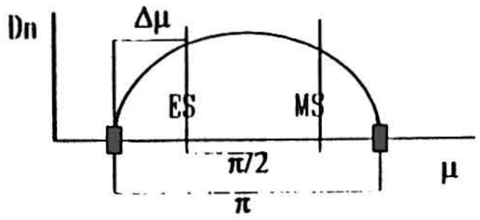


Thus, for the Hardt condition there are only two values for the phase term $(\alpha - \Delta\mu)$ i.e. 315 ± 15^0 and 135 ± 15^0.

SECTION B

The normalised dispersion terms take a simple form in dispersion bumps, 360^0 achromatic arcs and regular FODO or doublet structures that are common in small machines.

Dispersion bumps



where, $D_n = D_{n,0} \sin \Delta\mu$, $D'_n = D_{n,0} \cos \Delta\mu$.

Achromatic arcs

The variations of D_n and D'_n are very much the same as for the dispersion bump except that the curve is less smooth due to the distributed dipoles.

Regular FODO and Doublet structures

In these structures the normalised dispersion function takes a form close to a sine wave with a dc offset.

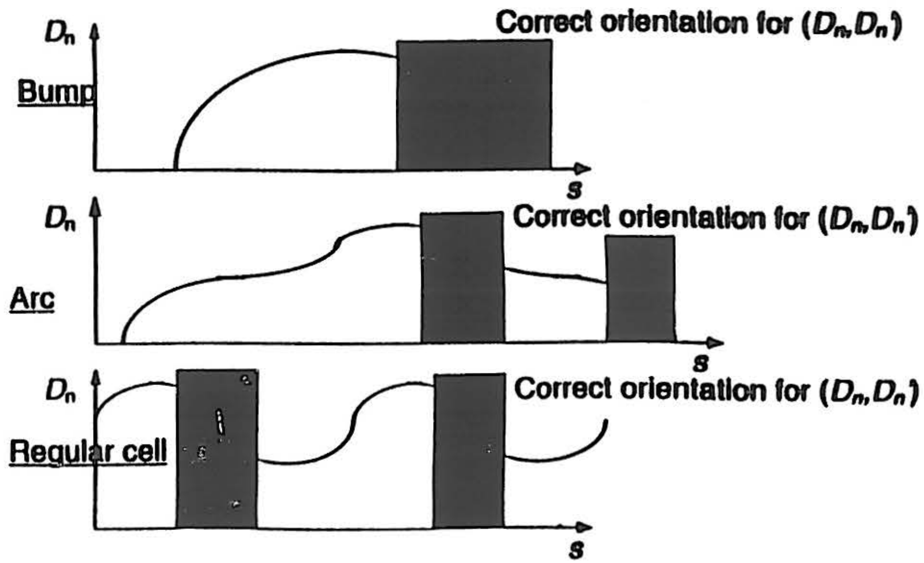
The relative usefulness of these structures can be demonstrated by numerical calculations.

WHAT IS NEEDED OF THE (D_n, D_n') VECTOR

The (D_n, D_n') vector moves the stable triangles for different momenta to superposition the separatrices.

The separatrices that can be used are at 45^0±15^0 in the first quadrant and at 225^0±15^0 in the third quadrant. Thus, the most efficient (D_n, D_n') vectors for corrections will be at 135^0±15^0 or 315^0±15^0, i.e. approximately at rightangles to the separatrices.

Thus, for bumps, arcs, regular lattices etc. the favoured position for an electrostatic septum is on the downward slope of (D,D'), but this is the least efficient for providing space for the magnetic septum without crossing dipoles between the septa.



SECTION C

STABILITY OF THE WAITING BEAM

Since any fluctuation in the waiting stack will appear, greatly magnified, in the spill, the stability of the waiting beam is of prime importance.

Below transition, beams are intrinsically stable longitudinally. The transverse stability is of more concern. The criterion for stability is,

$$|Z_T| \le F_2 \frac{E_0}{e} \frac{4Q_0\gamma\beta}{IR} \frac{\Delta p_{\mathrm{fwhm}}}{p_0} |(n-Q)\eta - Q'|$$

where F_2 is a form factor (close to unity), E_0 is the proton rest energy, R is the average machine radius (R/Q_0 can probably be replaced by the average betatron amplitude function), I is the current, n is the azimuthal mode number and η is the revolution frequency spread $[(df/f)/(dp/p)=1/\gamma^2-1/\gamma_t^2]$. The mode to be considered will probably be n close to Q with $n>Q$. This means that for maximum stability of the waiting stack Q' should be negative.

LEAR works very successfully with positive chromaticity, but LEAR has an active feed-back system and works with very small emittances achieved by cooling. Medical rings have larger emittances, especially with multi-turn injection. Octupoles may also provide some stability via an amplitude-frequency spread, but this distorts the resonance line.

SECTION D

SPIRAL STEP OR PITCH

REVISION

The motion in phase space over time intervals of 3 turns was derived as,

$$\Delta X_3 = \varepsilon X_0' + \frac{3}{2} S X_0 X_0'$$
$$\Delta X_3' = -\varepsilon X_0 + \frac{3}{4} S\left(X_0^2 - X_0'^2\right)$$

If the motion is restricted to the vertical separatrix $X=-(2/3)(\varepsilon/S)$, then

$$\Delta X_3 = \varepsilon X_0' - \varepsilon X_0' = 0$$
$$\Delta X' = \frac{2}{3}\frac{\varepsilon^2}{S} + \left(\frac{1}{3}\frac{\varepsilon^2}{S} - \frac{3}{4} S X_0'^2\right)$$

If this vector is allowed to rotate 90^0, it changes from an angular step to a positional step given by,

Spiral step

$$\Delta X_3' = 0$$
$$\Delta X = \frac{\varepsilon^2}{S} - \frac{3S}{4} X^2$$

SECTION E

NUMERICAL CALCULATIONS - STRATEGY

- To determine S on the basis of a 10 mm projection of the spiral step onto the x axis at the electrostatic septum.
- To use this value to determine the chromaticity via the Hardt Condition with whatever lattice functions are being proposed.
- To determine the δQ corresponding to the last stable orbit for the maximum emittance in the waiting stack.

First, assume that the spiral step along the separatrix is dominated by the second term, so as to eliminate the circular reference to ε.

$$\Delta R \cong \frac{3S}{4} X^2$$

Set $\Delta R = 0.01/[\cos(\phi)\sqrt{\beta}]$ [$m^{1/2}$] where ϕ is the angle between the separatrix and the X axis and, initially, $X = 0.03/\sqrt{\beta}$ [$m^{1/2}$].Then apply the Hardt Condition,

$$D_n \cos(\alpha - \Delta\mu) + D'_n \sin(\alpha - \Delta\mu) = \frac{4\pi}{S} Q'$$

with $(\alpha - \Delta\mu) = 135^0$ or 315^0.

Finally, find the tune shift for last stable triangle,

$$E_{\mathrm{Beam}} = E_{\mathrm{Last\ stable\ triangle}} = \frac{48\sqrt{3}\pi}{S^2} (\delta Q)^2 \pi$$