ACHROMATIC TRANSFER
BETWEEN
ELECTROSTATIC SEPTUM
AND MAGNETIC SEPTUM

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PARTICLE MOTION IN A LINEAR LATTICE

Particle motion in a linear lattice (containing only dipoles and quadrupoles) can be described with transfer matrices.

Transfer matrix for an on-momentum particle

The motion of an on-momentum ($\delta p = 0$) particle between two lattice elements 1 and 2 with a betatron phase advance $\mu$ is described by a 2x2 transfer matrix $M$ (Twiss-matrix), where

$$M = \left( \begin{array}{cc}
\sqrt{p_2}(\cos\phi_1 + \sin\phi_1) & \sqrt{p_1}\sin\phi_2 \\
-1 & \sqrt{p_1}\sqrt{p_2} - \sin\phi_1 \cos\phi_2
\end{array} \right)$$

The horizontal position and angle of the particle at element 2 is then given by

$$x_2 = m_{11} \cdot x_1 + m_{12} \cdot x'_1$$
$$x'_2 = m_{21} \cdot x_1 + m_{22} \cdot x'_1$$
Transfer matrix for a particle with momentum deviation

The transfer matrix for a particle with a momentum error $\delta p$ is a 3x3 matrix:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

with $m_{11}$, $m_{12}$, $m_{21}$, $m_{13}$ elements from the $2 \times 2$ matrix and $m_{13}$, $m_{23}$ depending on the dispersion function.

The horizontal position and angle of the particle at element 2 now depend on the momentum deviation $\delta p$.

$$x_2 = m_{11}x_1 + m_{12}x_1' + m_{13}\frac{\delta p}{p}$$

$$x'_2 = m_{21}x_1 + m_{22}x_1' + m_{23}\frac{\delta p}{p}$$

Expression for $m_{13}$ and $m_{23}$

Consider a particle with a momentum error $\delta p$ moving on the closed orbit belonging to $\delta p$. The horizontal position and angle of the particle at any position $s$ in the machine is given by

$$x(s, \delta p) = D(s) \frac{\delta p}{p}$$

and at elements 1 and 2

$$x_1(\delta p) = D_1 \frac{\delta p}{p}$$

$$x_2(\delta p) = D_2 \frac{\delta p}{p}$$

By comparison with the transfer matrix $M$, expressions for $m_{13}$ and $m_{23}$ are derived

$$D_2 = m_{11}D_1 + m_{12}D_1 + m_{13}$$

$$D_2 = m_{21}D_1 + m_{22}D_1 + m_{23}$$

and

$$m_{13} = D_2 - D_1 \begin{pmatrix} \frac{\beta_2}{\beta_1} \\ \frac{\beta_2}{\beta_1} \end{pmatrix} (\cos \alpha_1 + \sin \alpha_1 \sin \mu) - D_1 \sqrt{\frac{\beta_1}{\beta_2}} \sin \mu$$

$$m_{23} = D_2 + \frac{D_1}{\sqrt{\beta_1 \beta_2}} \left[ (\sin \alpha_2 + \cos \alpha_2 \cos \mu) - D_1 \sqrt{\frac{\beta_1}{\beta_2}} (\cos \alpha_2 \sin \mu) \right]$$

or when using $D_n$ and $D'_n$

$$m_{13} = \sqrt{\beta_2} (D_n - D_n \cos \mu - D'_n \sin \mu)$$

$$m_{23} = \frac{1}{\sqrt{\beta_2}} (D'_n - D_n \alpha_2 + D_n (\sin \mu \alpha_2 \cos \mu) - D'_n (\cos \mu \alpha_2 \sin \mu))$$
**Effect of the Electrostatic Septum**

The electrostatic septum cuts off particles from the separatrices. Particles inside the septum are kicked by the electrostatic field by a certain angle $\varphi$. This difference in angle between particles inside the septum and those remaining on the separatrix transforms into a gap further downstream, where the magnetic septum is positioned.

\[
\begin{align*}
\text{Electrostatic Septum} & \quad \text{Magnetic Septum} \\
\text{particle A} & \quad x'_{\text{MS}} = m_{11}x_{\text{ES}} + m_{12}x'_{\text{ES}} + m_{12}\varphi \\
\text{particle B} & \quad x'_{\text{MS}} = m_{21}x_{\text{ES}} + m_{22}x'_{\text{ES}} + m_{22}\varphi 
\end{align*}
\]

Thus, the effect of the kick of the electrostatic septum seen at the magnetic septum gives a difference in position and angle of the particles

\[
\Delta x_{\text{MS}} = m_{12}\varphi, \quad \Delta x'_{\text{MS}} = m_{22}\varphi
\]

where $\Delta x_{\text{MS}}$ is the gap for the thicker magnetic septum.

\[
\Delta x_{\text{MS}} = \varphi \sqrt{\beta_1 \beta_2} \sin \mu
\]
To make full use of the kick provided by the ES:

- look for a phase advance of $90^\circ + n \cdot 360^\circ$ (septa on same side of vacuum chamber) or
- $270^\circ + n \cdot 360^\circ$ (septa on opposite sides)
- look for reasonable values of $\beta_{ES}$ and $\beta_{MS}$

Generally, during the extraction process particles with different momenta are extracted at the same time. (This is not the case when using a transport mechanism that cuts slices of particles with equal momenta from the waiting stack and brings them into the resonance.)

If the Hardt Condition is fulfilled, separatrices for particles with different momenta and amplitudes are superimposed. Therefore, all extracted particles reach the electrostatic septum on the same separatrix. As the momentum spread is small (approximately 0.1%) all particles inside the electrostatic septum get almost the same kick.

Transfer from ES to MS for particles with $\delta p_{max}$

Particle C starts just inside, particle D just outside the ES

Separatrix is aligned on origin by the Hardt Condition

Hardt Condition no longer applies at MS so separatrix moves away from origin
Using the 3x3 transfer matrix between the two septa, one finds:

\[
\begin{align*}
\text{particle C:} & \quad x_{MS} &= m_{11}x_{ES} + m_{12}x'_{ES} + m_{13}\phi + m_{13}\frac{\delta p}{p} \\
& \quad x'_{MS} &= m_{21}x_{ES} + m_{22}x'_{ES} + m_{23}\phi + m_{23}\frac{\delta p}{p}
\end{align*}
\]

\[
\begin{align*}
\text{particle D:} & \quad x_{MS} &= m_{11}x_{ES} + m_{12}x'_{ES} + m_{13}\phi + m_{13}\frac{\delta p}{p} \\
& \quad x'_{MS} &= m_{21}x_{ES} + m_{22}x'_{ES} + m_{23}\phi + m_{23}\frac{\delta p}{p}
\end{align*}
\]

and \[
\Delta x_{MS} = m_{12}\phi \quad \Delta x'_{MS} = m_{22}\phi
\]

The gap created by the ES is the same as for an on-momentum particle, but it appears at a different position and angle. The shift in position reduces the effective gap width for the magnetic septum.
EFFECTS OF NON-ZERO $m_{13}$ AND $m_{23}$

A non-zero $m_{13}$ causes a loss of space for the MS and has to be corrected with a stronger kick of the ES

$$\text{gap}_e = m_{12} \cdot \varphi - \left| m_{13} \cdot \frac{\delta \rho}{\rho} \right|$$

The extracted part of the beam becomes longer and requires a larger horizontal aperture in the MS

$$\Delta l_e = m_{13} \cdot \left| \frac{\delta \rho}{\rho} \right|$$

A non-zero $m_{23}$ is leading to a bigger divergence of the extracted beam at the MS and also requires a larger horizontal aperture in the MS

$$\Delta \text{div}_{ex} = m_{23} \cdot \left| \frac{\delta \rho}{\rho} \right|$$

Note: At the ES any angle error will lead to losses, but at the MS there will be a small clearance of say 1 mm and angular spreads up to 1 mrad (approx.) will not lead to losses. For this reason, only the $m_{13}$ will be considered further.

MINIMISATION OF EFFECTS OF $m_{13}$

- To fulfill the Hardt Condition, the ES must be in a region with dispersion (to date, we have only considered positive dispersion at the ES). Fulfilling the Hardt Condition fixes the $\delta \rho / \rho_{\text{max}}$ of the extracted beam and therefore it cannot be used to compensate the effects of a non-zero $m_{13}$.

- The loss of space for the magnetic septum due to $m_{13}$ being non zero is proportional to $\sqrt{\beta_{MS}}$. Decreasing $\beta_{MS}$ reduces the influence of $m_{13}$, but the gap created by the electrostatic septum is also proportional to $\sqrt{\beta_{MS}}$ and becomes smaller. Overall the effective gap at the magnetic septum is reduced by decreasing $\beta_{MS}$.

- The only effective approach is to reduce $m_{13}$ directly
MINIMISATION OF $m_{13}$

1 Both septa in a bending-free dispersion region

In a bending free-region, the dispersion behaves like a betatron oscillation and can therefore be described with a 2x2 transfer matrix

\[
\begin{pmatrix}
D
\end{pmatrix}
=\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
D
\end{pmatrix}_{ES}.
\]

Using this transformation for $D$ and $D'$ it follows directly that $m_{13}$ and $m_{23}$ are zero and therefore:

The transfer via a dispersion region without crossing bending magnets is always achromatic with respect to position and angle.

2 Both septa in regions with dispersion and bending

The transfer element $m_{13}$ is given by:

\[
m_{13} = \sqrt{\beta_{MS}} (D_{n,MS} - D_{n,EB} \cos \mu - D'_{n,EB} \sin \mu)
\]

To make full use of the kick provided by the ES make the phase advance either $\mu = 90^\circ + n \cdot 360^\circ$ or $\mu = 270^\circ + n \cdot 360^\circ$.

For $\mu = 90^\circ + n \cdot 360^\circ$ it follows

\[
m_{13} = \sqrt{\beta_{MS}} (D_{n,MS} - D'_{n,EB})
\]

and therefore to make $m_{13} = 0$

\[
D_{n,MS} = D'_{n,EB} \text{ Septa same side } 90^\circ
\]

is required. But as shown in the presentation of the Hardt Condition, one needs to work with a negative $D'_{n,ES}$ so $m_{13}$ can only be made zero by having negative dispersion at the magnetic septum.
For $\mu = 270^\circ + n \cdot 360^\circ$ it follows:

$$m_{13} = \sqrt{\beta_{MS}} (D_{n,MS} + D_{n,ES})$$

and therefore to make $m_{13} = 0$

$$D_{n,MS} = -D_{n,ES} \quad \text{Septa opposite sides } 270^\circ$$

is required. In this case, $m_{13}$ can be made zero by having a positive $D_{n,MS}$ and a negative $D_{n,ES}$ just as required by the Hardt Condition. A disadvantage of this solution might be that the particles which are extracted have to be transported for a longer distance in the machine (e.g. crossing of sextupoles between the two septa would be more difficult to avoid).\(^1\)

\(^1\) If a sextupole is crossed (either resonance or chromaticity) between the ES and the MS, then there is a variable optical element in the extraction channel. Any change in the Q' or resonance strength alters the extraction geometry.

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### 3 Electrostatic septum in a dispersion region and magnetic septum in a zero-dispersion region

For $D_{MS} = 0$, the transfer element $m_{13}$ is given by

$$m_{13} = -\sqrt{\beta_{MS}} (D_{n,ES} \cos \mu + D_{n,EB} \sin \mu)$$

Position the ES in a $180^\circ$ dispersion bump. If the bump was created by single kicks, $D_n$ and $D'_n$ can be described as follows:

$$D_n(\theta) = D_{n,0} \cdot \sin \theta \quad D'_n(\theta) = D_{n,0} \cdot \cos \theta$$

and a simple expression for $m_{13}$ is derived

$$m_{13} = -\sqrt{\beta_{MS}} D_{n,EB} \sin(\theta + \mu) \quad \text{coming out of a}$$

$$m_{13} = 0 \quad \text{for } (\theta + \mu) = n \cdot 180^\circ \quad \text{dispersion bump}$$

It is difficult to use $n=1$, since this gives exactly the position of the dipole which is closing the bump. To keep $m_{13}$ small, the MS has to be positioned as close to the dipole as possible. For larger $n$, there is again the problem of transporting the extracted part of the beam through a larger distance in the machine.
4 Transfer for un-fulfilled Hardt Condition

Fulfilling the Hardt Condition fixes the chromaticity and the \( \delta p_{\text{max}} \) of the extracted particles.

If one does not fulfill the Hardt Condition, the chromaticity can be used to adjust the \( \delta p_{\text{max}} \) in a way that particles with different momenta arrive at the ES with different angles in order to compensate the effect of a non-zero \( m_{13} \).
This method is used in the present PS slow extraction scheme.

5 Transfers from zero dispersion regions to zero dispersion regions are always achromatic