In this paper the transverse mode coupling instability is studied in the two particle model in the presence of reactive feedback. If the feedback is used to increase the frequency of mode '0', by a fixed frequency shift of $\omega_s$, then the threshold is increased by a factor of two. If the feedback is used to decrease the frequency of mode '0' by a fixed frequency shift of $\omega_s$ so that mode '0' and '-1' just fall on opposite sides of each other, then the modes separate at small currents and only become degenerate and unstable at much larger currents. In this case the threshold increases by about a factor of 4. The limitations of the two particle model are also discussed.

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Introduction

It has been suggested at CERN that the "strong" transverse instability which occurs in bunched beams might be cured with a reactive feedback system; i.e. one which changes only the frequency of the coherent oscillations of the bunch (no damping). This instability is due to mode crossing; thus, the basic idea is to use the reactive feedback to prevent this crossing.

However, the problem is not quite so simple. The two modes which cross (in the simplest case of a very short bunch) are modes 0 and -1, that is modes with initial frequency $\omega_0$ and $\omega_B - \omega_0$. If we now alter the frequency of mode zero near the threshold of instability, this must change the frequency of mode -1 also. This is true because the modes are strongly coupled due to the transverse impedance. In fact it is just this strong coupling which causes the instability. Therefore one must see how all the modes are affected by feedback.

In practice the threshold for this instability and the basic physics is qualitatively described by a two particle model. Therefore, it is reasonable to first understand how feedback operates in this simple model. In the first section the two particle model is introduced. The general solution is calculated and the transfer matrix for one synchrotron oscillation is introduced. The eigenvalues and the stability threshold are then calculated. In the next section reactive feedback is introduced into the model. Again the transfer matrix and eigenvalues are calculated. In the last two sections we compare the two alternative types of reactive feedback and discuss the limitations of the model.

Two particle model

Many people have written about the two particle model; however, we will follow the basic philosophy of Ref(4) with some modifications, that is to calculate a transfer matrix, the iterates of which describe the motion.

The wakefield in this model is taken to be a constant, $W_0$, independent of the separation of the two particles. The particles perform synchrotron oscillations so that for one half of the synchrotron period particle 1 leaves a wakefield while particle 2 feels the force of this wakefield and for the last half of a synchrotron period the reverse is true. Thus the model is
The solution for the next half period is simply

\[ T_2(t) = \begin{pmatrix} A(t) & B(t) \\ 0 & A(t) \end{pmatrix} \]
Thus the matrix for an entire synchrotron oscillation is given by

\[
T = \begin{pmatrix} A(t_s/2) & 0 \\ B(t_s/2) & A(t_s/2) \\ A(t_s/2) & B(t_s/2) \end{pmatrix} \begin{pmatrix} A(t_s/2) & 0 \\ 0 & A(t_s/2) \end{pmatrix}
\]

To determine stability we need to find the eigenvalues of \( T \). If they all lie on the unit circle, then we have stability. Before continuing it is useful (for a simple analytical analysis) to make one approximation. The terms in \( B(t_s/2) \) are of two types

1) \( \sim \frac{a}{\omega^2} \)

2) \( \sim \frac{at_s}{2\omega} = \frac{a2\pi}{2\omega\omega_s} \).

The ratio of term (1) to term (2) is thus

\[
\frac{1)}{2)} = \frac{\omega_s}{\pi\omega}
\]

For most accelerators or storage rings of interest this ratio is very small (LEP = .0005); thus the terms of the first type can be neglected. If we perform the matrix multiplications, we find \( T \) is of the form

\[
T = \begin{pmatrix} C(t_s) & \eta D(t_s) \\ \eta D(t_s) & (1-\eta^2)C(t_s) \end{pmatrix}
\]

where

\[
\eta = \frac{at_s}{4\omega}
\]

\[
C(t_s) = \begin{pmatrix} \cos(\omega t_s) & \frac{\sin(\omega t_s)}{\omega} \\ -\omega \sin(\omega t_s) & \cos(\omega t_s) \end{pmatrix}
\]

\[
D(t_s) = \begin{pmatrix} \sin(\omega t_s) & \frac{-\cos(\omega t_s)}{\omega} \\ \omega \cos(\omega t_s) & \sin(\omega t_s) \end{pmatrix}
\]
Note that \( \text{Det}(T) \) is unity but that \( T \) itself is not symplectic (it is part of a larger symplectic matrix). The next step is to calculate the eigenvalues of \( T \). The characteristic equation for eigenvalue \( x \) can be shown to be

\[
x^4 + c_1 x^3 + c_2 x^2 + c_1 x + 1 = 0 \quad (12)
\]

where

\[
c_1 = (2n^2 - 4) \cos(\omega t_s) \quad (13)
\]

\[
c_2 = (n^2 - 2)^2 + 4\cos^2(\omega t_s) - 2 .
\]

Searching for eigenvalues on the unit circle of the form

\[
x = e^{\pm i\mu}
\]

we find

\[
\cos \mu = \cos \omega t_s \left(1 - \frac{n^2}{2}\right) \pm \sin \omega t_s \left[ \frac{n^2}{2} - \frac{n^4}{4}\right]^{1/2}
\]

or if we let

\[
\cos(\Delta \omega t_s) = 1 - \frac{n^2}{2} \quad (16)
\]

we find

\[
\cos \mu = \cos[(\omega \Delta \omega) t_s] .
\]

From Eq. (16) it is apparent that there is instability if and only if

\[
|1 - \frac{n^2}{2}| > 1 \quad (18)
\]

or

\[
|n| > 2 . \quad (19)
\]

This instability corresponds to a degeneracy in the two modes of oscillation of the two particle system.
It is also interesting to examine the frequency spectrum of the two modes of the two particle system. This is straightforward\(^4\) and yields an infinite number of modes which divide into two classes, even and odd, as shown in Fig. (1). The modes "0" and "-1" (mod \(2\omega_S\)) change in frequency and finally become degenerate and unstable at \(\eta=2\).

It must be kept in mind that the two particle model can yield misleading results for higher modes. This is true because each macro-particle in the model actually has internal freedom to oscillate, and it is just this internal oscillation which contributes to higher modes. Since the bunch is frozen into two particles by the model, these internal oscillations are eliminated. Therefore, in subsequent figures these higher modes are suppressed. To reconstruct them it is useful to think of the frequency spectrum (vs. \(\eta\)) as a cylinder with edges at 0 and \(-2\omega_S\) attached. This reminds one that there are really only two independent modes of oscillation.

Reactive Feedback

To include reactive feedback in the two particle model it is necessary to make a few assumptions. First, assume that each bunch can be treated independently of the others. Thus the kicker/pickup must have a response time less than the time between successive bunch passages at the location of the pickup. This sets a lower bound on the bandwidth of the kicker/pickup. Secondly, assume that the kicker/pickup can only measure and act on the average position of the two particles. This yields an upper bound on the bandwidth. Thirdly, assume that the feedback is adjusted to be purely reactive; that is, that there is no damping caused by the feedback system. For feedback with one kicker and pickup, this is accomplished by a phase advance of \(\pi\) between pickup and kicker (for more details see the Appendix). Lastly, assume that the betatron oscillation with the feedback system present can be smoothed and represented by a differential equation (rather than a map). This is a good approximation if the synchrotron oscillation tune is small compared to an integer. Of course the feedback must be included in the betatron transfer matrix before smoothing (see Appendix). With these assumptions the two particle model with reactive feedback becomes
The parameter $\sigma$ of the feedback system is easily identified if we consider the case for small current ($\alpha = 0$). In this case

\[
\sigma = -\frac{\Delta \omega_{\text{FB}}}{\omega_\beta} \quad (\alpha = 0)
\]

\[
= -\Delta Q_{\text{FB}} \omega_\beta
\]

thus, $\sigma$ is just proportional to the shift in oscillation frequency of the coherent mode $(y_1 + y_2)$ due to the feedback. Note that for small current the other mode $(y_1 - y_2)$ oscillates at the frequency $\omega_\beta$, unaffected by the feedback system. The parameter $\Delta Q_{\text{FB}}$, the shift in coherent tune due to feedback, is simply related to the kick as shown in the Appendix.

To solve Equation (20) and find the transfer matrix first search for a solution,

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A e^{i \lambda t}
\]

With this substitution we find the characteristic equation

\[
\lambda^4 - 2 \omega^2 \lambda^2 + \omega^4 - \sigma(\sigma + \alpha) = 0
\]

where

\[
\omega^2 \equiv \omega_\beta^2 - \sigma
\]

This has solutions

\[
\lambda^2 = \omega^2 \pm \sqrt{\sigma(\sigma + \alpha)}
\]
and thus we find the four frequencies,

$$\lambda = \pm\left[\omega \pm \sqrt{\sigma + a}\right] \pm i\lambda \pm \sqrt{2\omega}$$

(26)

where the square root has been expanded to first order. It is important to note here that the quantity in the square root can be negative. This possibility cannot be discarded as unstable since it is the total transfer matrix which determines stability.

To find the transfer matrix for the first half we also need the eigenvectors for $\lambda_\pm$. These are given by

$$\lambda_\pm, A_\pm \sim \begin{pmatrix} \frac{a}{\sqrt{\sigma + a}} \cr 1 \end{pmatrix}$$

(27)

where

$$a \equiv \frac{\sigma}{\sqrt{\sigma + a}}$$

(28)

therefore, the general solution to Equation (20) is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -a \\ 1 \end{pmatrix} (b_1 \cos \lambda_+ t + b_2 \sin \lambda_+ t)$$

$$+ \begin{pmatrix} a \\ 1 \end{pmatrix} (b_3 \cos \lambda_- t + b_4 \sin \lambda_- t),$$

(29)

where $b_1, b_2$ are linear functions of the initial conditions. After a bit of algebra, one can show that the transfer matrix is of the form

$$M = \begin{pmatrix} A & aB \\ B/a & A \end{pmatrix}$$

(30)

where

$$A(t) = \frac{1}{2} \begin{pmatrix} \cos(\lambda_+ t) + \cos(\lambda_- t) & \sin(\lambda_+ t) + \sin(\lambda_- t) \\ \lambda_+ \sin(\lambda_+ t) + \lambda_- \sin(\lambda_- t) & \cos(\lambda_+ t) + \cos(\lambda_- t) \end{pmatrix}$$

(31)

$$B(t) = \frac{1}{2} \begin{pmatrix} \cos(\lambda_- t) - \cos(\lambda_+ t) & \sin(\lambda_- t) - \sin(\lambda_+ t) \\ \lambda_+ \sin(\lambda_+ t) - \lambda_- \sin(\lambda_- t) & \cos(\lambda_- t) - \cos(\lambda_+ t) \end{pmatrix}.$$
If we again neglect terms of order $\omega_s/\omega$ (only outside the sin's & cos's) then $A$ and $B$, take on the simpler form,

\begin{align*}
A &= \cos(\delta \omega t_s/2) C(t_s/2) \\
B &= \sin(\delta \omega t_s/2) D(t_s/2)
\end{align*}

where

\begin{equation}
\delta \omega = \sqrt{\gamma(\sigma + \alpha)} / 2\omega
\end{equation}

$C(t_s/2)$ and $D(t_s/2)$ are the matrices in Eq.(11) with $t_s$ replaced by $t_s/2$.

For the next time period, $t_s/2 < t < t_s$, the only change comes in the calculation of the eigenvectors. It is easy to see that if we simply replace $a$ by $1/a$ in Eq.(30), we obtain the matrix for the second half. Thus the total matrix becomes

\begin{equation}
M = M_1 M_2 = \begin{pmatrix}
A & ab \\
B/a & A
\end{pmatrix} \begin{pmatrix}
A & B/a \\
ab & A
\end{pmatrix}
\end{equation}

\begin{equation}
M = \begin{pmatrix}
A^2 + a^2B^2 & \frac{AB + aBA}{a} \\
\frac{BA + aAB}{a} & A^2 + B^2/a^2
\end{pmatrix}
\end{equation}

which is of the form

\begin{equation}
\begin{bmatrix}
\cos^2(\delta \omega t_s/2) - a^2\sin^2(\delta \omega t_s/2) C(t_s) & (a+1/a)\sin(\delta \omega t_s/2) \cos(\delta \omega t_s) D(t_s) \\
(a+1/a)\sin(\delta \omega t_s) \cos(\delta \omega t_s) D(t_s) & \cos^2(\delta \omega t_s/2) - \frac{\sin^2(\delta \omega t_s)}{a^2} C(t_s)
\end{bmatrix}
\end{equation}
Note that the frequency which occurs in C & D is understood to be
\[ \omega = \sqrt{\frac{\omega^2}{\beta} - \frac{\sigma^2}{2\omega^2}}. \] (36)

If the eigenvalues of M lie on the unit circle, then we have stability. The characteristic equation of M is given by
\[ x^4 + b_1x^3 + b_2x^2 + b_1x + 1 = 0 \] (37)
where
\[
\begin{align*}
b_1 &= 2\cos(\omega t_s)\sin^2(\delta t_s/2)\left(a+1/a\right)^2 - 4\cos(\omega t_s) \quad (38) \\
b_2 &= \sin^4(\delta t_s/2)\left(a+1/a\right)^4 - 4\sin^4(\delta t_s/2)\left[(a+1/a)^2+1\right] + 6.
\end{align*}
\]

If we search for a solution on the unit circle \(x=\pm e^{i\mu}\), then we find
\[ \cos \mu = \cos(\omega t_s)\left[1-\sin^2(\delta t_s/2)\left(a+1/a\right)^2/2\right] \]
\[ \pm \sin(\omega t_s)\left[\sin^2(\delta t_s/2)\left(a+1/a\right)^2 - \sin^4(\delta t_s/2)\left(a+1/a\right)^4/4\right]^{1/2}. \] (39)

Therefore if we define \(\Delta \omega\) by
\[ \cos(\Delta \omega t_s) = 1-\sin^2(\delta t_s/2)\left(a+1/a\right)^2/2, \]
then we find
\[ \cos \mu = \cos((\omega t_s + \Delta \omega) t_s). \] (40)

Thus, the frequency of the two modes is given by
\[ \omega = \omega - \frac{\sigma}{2\omega^2} \pm \Delta \omega. \]

Notice the marked similarity to the case without feedback. In fact the formulas from the first section can all be recovered by taking the limit as \(\sigma \to 0\).
In the case with feedback the stability criterion has been changed to

\[ |1 - \sin^2(\delta \omega t_s/2)(a + 1/a)^2/2| < 1 \] (42)

or more simply

\[ |\sin(\delta \omega t_s/2)(a + 1/a)| < 2. \] (43)

Recall that

\[
\delta \omega = \frac{\sigma(\sigma + \alpha)}{2\omega}, \quad a = \frac{\sigma}{\sqrt{\alpha(\sigma + \alpha)}}.
\]

All the equations derived so far are also valid for \(\delta \omega\) and "\(\alpha\)" pure imaginary. In that case the stability criterion becomes

\[ |\sinh(|\delta \omega| t_s/2)|\left|\frac{|a| - 1/|a|}\right| < 2. \] (45)

(\(\delta \omega\) and \(\alpha\) pure imaginary)

In the next section we discuss the consequences of Eq (43) and examine various modes of operation for a reactive feedback system.

\textbf{Regions of Stability/Beat them or join them?}

To understand the regions of stability it is convenient to introduce a dimensionless parameter defined by

\[
\xi = \frac{\sigma t_s}{4\omega_\beta},
\]

\[
\eta = \frac{\alpha t_s}{4\omega_\beta},
\]

where we have recalled the definition of \(\eta\) from Eq.(10) to note that \(\xi\) is simply \(\sigma\) in the same (dimensionless) units as \(\eta\). Recall again the \(\eta\) is just proportional to the current while \(\xi\) is at our disposal subject to the limits of the feedback system.
The stability with feedback is only a function of these two parameters, thus two dimensional plots are useful. In Figure 2 we plot the regions of stability in \((\xi, \eta)\) space. The open areas correspond to stability while those hatched with lines correspond to unstable areas. The dark line at \(\xi=0\) from \(\eta=0\) to 2 corresponds to normal running in the absence of feedback. There is, as was demonstrated in the Section 1, a threshold for instability at \(\eta=2, \xi=0\). For \(\xi\) and \(\eta\) of the same sign, the frequency shifts of the two tend to add while the reverse is true when they are of opposite sign. The sign of the slope of the stability line which crosses \(\xi=0\) is easy to understand. For \(\xi>0\) the modes are pushed close together and so collide sooner as the current (and \(n\)) increases, while for \(\xi<0\) the two modes are separated and so collide later. This is illustrated in Figure 3. In fact by continuity the entire stability line which crosses at \(\eta=2, \xi=0\) is due to the collision of mode 0 with mode -1 from above.

On the other hand there is another method of operation (abnormal) which is evident from the stable area for \(\xi\) positive in Fig. (2). In this case the frequency shift due to the current is enhanced but such that modes "0" and "-1" fall on opposite sides of each other. Thus at small currents the modes tend to separate while only at much larger currents do they come together (Fig. (4)). Notice that there is different behaviour for the modes with an initial shift of mode 0 which is \(1.1\omega_0\). In this case the modes which couple are the "-1" with the shifted "+2" and the shifted "0" with the "-3". However, one must keep in mind that in this model these are not really separate modes but a manifestation that the frequency shift versus \(\eta\) graph is really a cylinder. For the case of \(\xi = 1.1\pi/2\) in Fig.(4), modes "0" and "-1" again become degenerate, but they do so by going around the cylinder rather than by staying on the same side. Also notice that for \(\xi = 0.9\pi/2\) it is necessary to turn on the feedback at a finite \(\eta\) because for small \(\eta\) there is a degeneracy of modes that causes instability. To compare the two methods consider \(\xi = \pm \pi/2\) which yields a frequency shift of \(\pm \omega_0\) independent of current. Then in Fig (3 & 4) you see that "normal" feedback yields about a factor of 2 in the threshold while "abnormal" feedback yields about a factor of 4. This strongly suggests that for large currents the abnormal method may be superior.
Discussion

The model described here has limitations in that when $|\xi| > \pi/2$, higher modes interact, and it is just these higher modes that are misrepresented by a simple two-particle model. A manifestation of these limitations is that there are two lines in $(\xi, \eta)$ space on which there is no instability. These are plotted in Fig. 2 and are given by

$$\sqrt{\xi(\xi + \eta)} = \pi$$  \hspace{1cm} (48)

$$\xi = -\eta/2.$$  \hspace{1cm} (49)

The unstable areas approach these lines but do not cross. In order to interpret the model with these limitations we have compared $\xi = \pm \pi/2$ in the previous section. This is a compromise because for normal feedback there is instability near $\eta = 0$ for $\xi > \pi/2$, while for abnormal feedback there is instability near $\eta = 0$ for $\xi < \pi/2$.

In an actual feedback system with fixed electronic gain, the parameter $\xi$ is just proportional to the current (or $N$, the number of particles in a bunch). Thus for normal operation of the feedback system, it is not necessary to provide a $\xi = \pi/2$ at all currents. In fact a gain sufficient to yield $\xi = \pi/4$ at threshold current is enough because this linear increase provides enough mode separation until $\eta = 4$.

The situation is different for abnormal feedback because it is necessary to turn on the system suddenly near threshold in order to avoid instability near $\eta = 0$. After the feedback is on the natural increase of $\xi$ can be cancelled by adjusting the gain.

One can also imagine a combined mode of operation in which the normal feedback is used to double the threshold, after which a rapid switch ($< 1/(2\pi Q_s)$ turns) to abnormal feedback yields another factor of two. In this last case the gain is limited by the normal mode of operation. The gain necessary for such a mode of operation is that which yields $\Delta Q_{FB} > Q_s/2$ at the threshold current ($\eta = 2$).  \hspace{1cm} (50)

where $\Delta Q_{FB}$ is calculated assuming no mode coupling as in the Appendix.
To summarize there are several messages from the two particle model:

1. Reactive feedback can yield large increases in the threshold current provided the gain is sufficient to satisfy Eq. (50) above.

2. There are two distinct modes of operation, and it seems desirable to be able to switch quickly from one to the other.

3. At large currents and frequency shifts the higher modes become important, and thus reactive feedback should be studied using the Vlasov Eq. in order to systematically include these higher modes.

4. Reactive feedback must be compared with resistive feedback to judge the relative effectiveness for a given gain.

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References

Appendix

A reactive feedback system is conceptually the same as a resistive one except that the phase advance between pickup and kicker must be an even multiple of \( \pi/2 \) rather than an odd multiple. However, in practice the pickup and kicker are usually placed physically close so that the bunch must traverse almost the entire ring after being detected before it reaches the kicker to be kicked (this is especially true for a large machine like LEP). This means that the coherent phase advance will be shifted (depending upon the current) due to the transverse impedance of the ring. Therefore, the resulting feedback will be shifted from reactive to reactive plus resistive (or anti-resistive).

This effect can be compensated by the use of two pickups placed \( \pi/2 \) apart and then combined linearly at the kicker. To see this let the kick at the kicker be given by

\[
\Delta y_k' = g_1 y_1 + g_2 y_2 \quad (A.1)
\]

where \( \Delta y_k' \) is the change in slope at the kicker, and \( y_1 \) and \( y_2 \) are the positions of the bunch measured at the first and second pickup respectively. Let the second pickup be \( \pi/2 \) in phase after the first. Then the two important quantities are the determinant and trace of the "coherent" transfer matrix, \( M \). With the kick in (A.1) it is straightforward to show that

\[
\text{Det}(M) = 1 - g_1 \sqrt{\beta_1 \beta_k} \sin(\mu_{1k}) + g_2 \sqrt{\beta_2 \beta_k} \cos(\mu_{1k}) ,
\]

\[
\text{Trace}(M) = \cos \mu_0 + \frac{1}{2} (g_1 \sqrt{\beta_1 \beta_k} \cos(\mu_{1k}) + g_2 \sqrt{\beta_2 \beta_k} \sin(\mu_{1k})) \sin \mu_0
\]

\[
\quad + \frac{1}{2} (-g_1 \sqrt{\beta_1 \beta_k} \sin(\mu_{1k}) + g_2 \sqrt{\beta_2 \beta_k} \cos(\mu_{1k})) \cos \mu_0 \quad (A.2)
\]

where the subscripts 1, 2 and \( k \) denote pickup 1, 2 and the kicker, \( \mu_{1k} \) is the coherent phase advance between pickup 1 and kicker and \( \mu_0 \) is the coherent phase advance of the entire machine without feedback,

\[
\mu_0 = 2 \pi Q_0 \quad (A.4)
\]
The condition for reactive feedback is

\[ \text{Det } M = 1 \quad \text{(A.5)} \]

or

\[ g_2 \cos(\mu_{1k}) = \sqrt{\beta_1/\beta_2} g_1 \sin(\mu_{1k}) \quad \text{(A.6)} \]

when Eq. (A.6) is satisfied, the phase advance with feedback is calculated by the inverse cosine of Eq. (A.3). If the shift is small, we have

\[ 2\pi \Delta Q_{FB} = \Delta \mu_{FB} = -\frac{1}{2} \left( g_1 \sqrt{\beta_1/\beta_2} \cos(\mu_{1k}) + g_2 \sqrt{\beta_2/\beta_1} \sin(\mu_{1k}) \right) \]

or using Eq. (A.4)

\[ g_1 = \frac{-4\pi \Delta Q_{FB} \cos(\mu_{1k})}{\sqrt{\beta_1/\beta_2}} \quad \text{(A.7)} \]

\[ g_2 = \frac{-4\pi \Delta Q_{FB} \sin(\mu_{1k})}{\sqrt{\beta_2/\beta_1}} \quad \text{(A.8)} \]

For \( \Delta Q_{FB} \) large simply make the replacement

\[ \Delta Q_{FB} \rightarrow \frac{\left[ \cos(2\pi Q_0) - \cos(2\pi (Q_0 + \Delta Q_{FB})) \right]}{2\pi \sin(2\pi Q_0)} \quad \text{(A.9)} \]

Equations (A.6-8) indicate the change in the gains which are necessary to maintain reactive feedback of a given \( \Delta Q \). Once \( \mu_{1k} \) as a function of current is either measured or calculated, one can shift the gains accordingly. Also note that it is, of course, necessary to avoid integer and half integer resonances with the restriction.

\[ | \cos(2\pi (Q_0 + \Delta Q_{FB})) | < 1 \quad \text{(A.10)} \]
In addition there are analogous formulas for the case of "pure resistive" feedback. In this case the real frequency, although not exactly zero, is second order in the damping rate if the coefficient of $\sin(\mu_0)$ in Eq. (A.3) is set to zero. That is

$$g_1 \cos(\mu_1k) = -\sqrt{\beta_1/\beta_2} \ g_2 \sin(\mu_1k).$$  \hspace{1cm} \text{(A.11)}$$

This yields a "damping rate", $1/\tau_{FB}$, defined by

$$[1-\text{Det}(M)] = \frac{2t_{\text{rev}}}{\tau_{FB}} \ g_1 \sqrt{\beta_1\beta_k} \sin(\mu_1k) - g_2 \sqrt{\beta_2\beta_k} \cos(\mu_1k)$$ \hspace{1cm} \text{(A.12)}$$
or using Eq. (A.12)

$$g_2 = \frac{-2t_{\text{rev}} \cos(\mu_1k)}{\tau_{FB} \sqrt{\beta_2\beta_k}}$$ \hspace{1cm} \text{(A.13)}$$

$$g_1 = \frac{+2t_{\text{rev}} \sin(\mu_1k)}{\tau_{FB} \sqrt{\beta_1\beta_k}}$$ \hspace{1cm} \text{(A.14)}$$

A combination of damping and reactive feedback can be obtained by simply assigning appropriate values to Eq. (A.2) and (A.3) and then solving for $g_1$ and $g_2$. 
Two particle modes in frequency space. This pattern repeats indefinitely. It is best to view the frequency space as a cylinder by identifying $0$ and $-2\omega_s$; in this way the higher modes are recovered by going around the cylinder one way for increasing frequency and the other way for decreasing frequency. The instability threshold occurs at $\eta=2$ when the two modes become degenerate.
Fig. 2  Regions of stability in (ξ,η) space. η is proportional to current, ξ is proportional to the frequency shift caused by the feedback for small current.
Fig. 3  Mode Coupling for different values of the feedback parameter, $\xi$, in the "normal" method. The threshold is increased by separating the modes. The higher modes have been suppressed.
Mode Coupling in the "abnormal" method. Mode "O" is labelled with the $\xi$ parameter. In this case the shifted mode "0" is below mode -1. The modes separate at small $\eta$ and only come together at much larger $\eta$. Higher modes are suppressed.