Appendix A

Computation of Lyapunov Exponents: The Benettin Algorithm

A convenient and widely used algorithm for evaluation of Lyapunov exponents (Benettin et al., 1980) is shortly explained here. To be concrete, we discuss a continuous-time autonomous system governed by equation

$$\frac{dx}{dt} = F(x).$$ (A.1)

Generalizations to the non-autonomous case and to diffeomorphisms are simple and straightforward.

First, we undertake the numerical solution of Eq. (A.1) up to arrival of the orbit at the attractor.

Then, to compute the largest Lyapunov exponent we proceed with simultaneous integration of Eq. (A.1) together with the variation equation

$$\frac{d\tilde{x}}{dt} = A(x(t)) \tilde{x}. \quad (A.2)$$

Here $A(x(t))$ is a matrix derivative composed of the partial derivatives of components of the vector function $F(x)$ over the components of the vector $x$. We perform integration on some appropriately chosen time interval $T$ starting at $t = t_0$ from $x(t_0)$ and with some vector $\tilde{x}(t_0)$ of certain norm $\|\tilde{x}(t_0)\| = \sqrt{\sum_{i=1}^{N} \tilde{x}_i^2(t_0)}$. As $\tilde{x}$ obeys the linear equation, this norm can be chosen arbitrarily, say, $\|\tilde{x}(t_0)\| = 1$. At $t_1 = t_0 + T$ we redefine the perturbation to have the norm equal to the initial value but keeping direction of the vector unchanged, namely, $\tilde{x}^0(t_1) = \tilde{x}(t_1) / \|\tilde{x}(t_1)\|$ (Fig. A.1). Now, continue the procedure with numerical solution from the initial point $x(t_1)$ with the perturbation $\tilde{x}^0(t_1)$. Next, obtain the state $x(t_2)$ at $t_2 = t_0 + 2T$ and perturbation $\tilde{x}(2T)$, and redefine the perturbation as $\tilde{x}^0(t_2) = \tilde{x}(t_2) / \|\tilde{x}(t_2)\|$, and so on. For $M$ steps of the algorithm the total factor determining the change of the perturbation norm will be $P = \prod_{k=1}^{M} \|\tilde{x}_k(t_0 + kT)\|$. Obviously, evolution of the magnitude of the perturbation is determined by the largest Lyapunov exponent, so, it is estimated as
\[ \lambda_1 \approx \frac{1}{MT} \ln P = \frac{1}{MT} \sum_{k=1}^{M} \ln \| \tilde{x}_k(t_0 + kT) \|, \quad (A.3) \]

where \( M \) is supposed to be large enough. This successive renormalization of the perturbation vectors is essential feature of the computational algorithm.

**Fig. A.1** An illustration for Benettin’s algorithm of evaluation of the largest Lyapunov exponent (see text).

To compute more than one Lyapunov exponent we monitor a collection of the respective number of perturbation vectors evolving in time along the reference phase trajectory and at each step of the algorithm redefine the perturbation vectors applying the Gram-Schmidt orthogonalization process.

It works as follows. Suppose, at some step of algorithm the perturbation vectors evolve from \( x_0^1(t_{k-1}), x_0^2(t_{k-1}), x_0^3(t_{k-1}), \ldots \) at \( t_{k-1} = t_0 + (k-1)T \) to \( x_1(t_k), x_2(t_k), \ldots \) at \( t_k = t_0 + kT \). Then, we set

\[
\tilde{x}_0^1(t_k) = \frac{x_1(t_k)}{\| x_1(t_k) \|}, \quad \tilde{x}_0^2(t_k) = \frac{a_2}{\| a_2 \|}, \quad a_2 = \tilde{x}_2(t_k) - \langle \tilde{x}_2(t_k), \tilde{x}_1(t_k) \rangle \tilde{x}_0^1(t_k), \\
\tilde{x}_0^3(t_k) = \frac{a_3}{\| a_3 \|}, \quad a_3 = \tilde{x}_3(t_k) - \langle \tilde{x}_3(t_k), \tilde{x}_1(t_k) \rangle \tilde{x}_0^1(t_k) - \langle \tilde{x}_3(t_k), \tilde{x}_2(t_k) \rangle \tilde{x}_0^2(t_k), \\
\ldots \ldots \]

(A.4)

Here the brackets designate the dot product: \( \langle u, v \rangle = u \cdot v = \sum_{i=1}^{N} u_i v_i \). The obtained vectors \( x_1^0, x_2^0, \ldots \) are used to start a next step of the algorithm at \( t_k = kT \). In the course of the computations, the accumulated sums \( S_m(M) = \sum_{i=1}^{M} \ln \| \tilde{x}_m(kT) \| \) are calculated, and the Lyapunov exponents are evaluated from the relations

\[ \lambda_m \approx S_m(M) / MT. \quad (A.5) \]
References

Appendix B
Hénon and Ikeda Maps

In some physical systems description by iterated maps appears in a very natural way. Say, considering a particle moving under action of periodic pulses (kicks), one can express the coordinate and velocity just before the next kick via the state before the previous kick and consider dynamics in discrete time with this mapping. Particularly, well-known maps of Hénon (Hénon, 1976) and Ikeda (Ikeda et al., 1980) may be obtained in appropriate setup; it reveals physical significance of these models.

Accounting framework of the current book, it is worth stressing that chaotic attractors in the Hénon and Ikeda maps do not relate to the class of uniformly hyperbolic ones. Transformation of the phase space volumes in the course of dynamical evolution is similar to that mentioned in Chap. 1 in the context of Fig. 1.4 accompanied by formation of local singularities of the density of the “phase fluid”. It is associated with an inevitable presence of tangencies between stable and unstable manifolds of orbits on the attractors as illustrated in Chap. 7.

Consider the following simple physical system (Fig. B.1). Let a particle of mass \( m \) be able to move along the \( x \) axis with friction force proportional to the velocity, \( f = -kv \). Suppose further that pulsed kicks effect the particle with period \( T \) of intensity depending on the instantaneous position of the particle, i.e. the momentum transferred is expressed by some function \( P(x) \).

![Fig. B.1 Mechanical system whose dynamics are described by the Hénon map.](image)

Assume that just before the \( n \)-th kick the particle coordinate is \( x_n \), and the velocity is \( v_n \). Immediately after the kick the velocity is \( v_n + P(x_n) / m \) and then it
decreases exponentially according to the expression \( v(t) = (v_n + P(x_n)/m)e^{-kt/m} \).

At the instant before the next kick we get

\[
v_{n+1} = (v_n + P(x_n)/m)e^{-kT/m};
\]
\[
x_{n+1} = x_n + \int_0^T v(t)dt = x_n + (mv_n + P(x_n)) \left( 1 - e^{-kT/m} \right) k^{-1}.
\] (B.1)

In terms of variable \( y = x - mk^{-1}(e^{kT/m} - 1)v \) introduced instead of \( v \), Equations (B.1) may be rewritten as

\[
x_{n+1} = f(x_n) + by_n, \quad y_{n+1} = x_n,
\] (B.2)

where

\[
b = -e^{-kT/m} \text{ and } f(x) = x(1-b) + P(x)(1+b)k^{-1}.
\] (B.3)

Now, let us concretize the spatial distribution of the force field in such a way that \( f(x) = 1 - ax^2 \). Then, (B.2) reads

\[
x_{n+1} = 1 - ax_n^2 + by_n, \quad y_{n+1} = x_n.
\] (B.4)

In 1976 Hénon introduced this map from purely abstract motivation to get a simple model with chaotic attractor. Now, it is evident that this map may serve for description of simple physical systems, one of which just has been considered. Other examples are dissipative oscillator and rotator under pulsed periodic driving (Heagy, 1992). In physical setup negative values are natural for the parameter \( b \), although in the original work of Hénon occasionally the sign of that parameter was set opposite.

Observe that the map is invertible: one can easily express \( x_n \) and \( y_n \) via \( x_{n+1} \) and \( y_{n+1} \) uniquely.

Evaluation of the Jacobi determinant for the Hénon map yields

\[
J = \left| \begin{array}{cc}
\frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial x_n} \\
\frac{\partial x_{n+1}}{\partial y_n} & \frac{\partial x_{n+1}}{\partial y_n}
\end{array} \right| = \left| \begin{array}{cc}
-2a & b \\
1 & 0
\end{array} \right| = -b.
\] (B.5)

If we consider a cloud of representative points occupying an area \( S \) on the phase plane \((x,y)\), then, at each successive step of iterations the area will be multiplied by factor \(|b|\) so, at \(|b| < 1\) the system is dissipative.

Figure B.2 shows attractor of the map at parameters \( a = 1.4, b = 0.3 \) corresponding to the original work of Hénon.

Figure B.3 shows a chart of dynamical regimes on the parameter plane \((a,b)\). Distinct gray scales correspond to observed periodic dynamics of different periods. One special gray tone is used for the chaos. The bottom part of the diagram corresponds to presence of attractive stable fixed point. As the parameter \( a \) grows, attractors corresponding to periodic oscillations arise out of period 2, then periods 4, 8, etc.; there is a sequence of period doubling bifurcations. The bifurcation lines in the parameter plane represent borders of respective areas of these periodic regimes. They accumulate according to the Feigenbaum law to the limit critical line, which represents the border of chaos (Feigenbaum, 1979; Vul et al., 1984; Schuster and
Just, 2005). Observe small areas of periodic motions scattered over the region occupied by chaos; because of their characteristic shape, they are called the “shrimps” (Gallas, 1993, 1994). Presence of these formations is a visually illustrative feature of non-hyperbolic nature of chaos.

Figure B.4 shows the larger one of two Lyapunov exponents of the Hénon map (B.4) obtained numerically and plotted versus parameter $a$ at constant $b = 0.3$. Observe a characteristic feature of the dependence intrinsic to the non-hyperbolic chaos. On a background of a set of positive Lyapunov exponent indicating chaos there occur densely deep narrow drops to negative values of the Lyapunov exponent in the windows of periodicity (or, windows of regularity). In the chart of dynamical regimes of Fig. B.3 the diagram B.4 corresponds to a path on the parameter plane along a vertical line $b = −0.3$; there occur the drops to negative Lyapunov exponent values as the path crosses the “shrimps”.

Now, let us turn to another system, an oscillator with cubic nonlinearity driven by periodic short pulses. The model equation is
$\ddot{x} + \gamma \dot{x} + \omega_0^2 x + \beta x^3 = \Sigma C \delta(t - nT), \quad (B.6)$

where $x$ is the dynamical variable, $\omega_0$ is natural frequency, $\gamma$ is a coefficient of dissipation, and $\beta$ is parameter of nonlinearity. External driving is represented by a sequence of pulses accounted by Dirac delta function, which follow with period $T$ and have magnitude characterized by constant $C$. We suppose for simplicity that the period of pulses contains an integer number of periods of natural frequency, $\omega_0 T / 2\pi = N$.

In assumption that $\omega_0$ and $T$ are relatively large, while $\gamma$, $\beta$, and $C$ are relatively small, one can use the slow amplitude method in intervals between the kicks. For this, we introduce the complex amplitude $a(t)$ and set $x = ae^{i\omega_0 t} + a^* e^{-i\omega_0 t}$, $\dot{x} = i\omega_0 (ae^{i\omega_0 t} - a^* e^{-i\omega_0 t})$. \quad (B.7)

These relations suggest that the complex amplitude satisfies a condition

$\dot{a}e^{i\omega_0 t} + \dot{a}^* e^{-i\omega_0 t} = 0. \quad (B.8)$

Substituting the relations (B.7) into (B.6) with zero right-hand part, and accounting (B.8), after averaging over a period of fast oscillations we obtain

$\dot{a} = -\frac{\gamma}{2} a + \frac{3}{2} i \frac{\beta}{\omega_0} |a|^2 a. \quad (B.9)$

Now, suppose that just after the $n$-th kick at $t=nT$ the complex amplitude is $a_n = \frac{1}{2} (x_n - i\omega_0^{-1} \dot{x}_n)$. To obtain the amplitude just before the next pulse, we integrate Eq. (B.9) in period $T$ with the initial condition $a(nT + 0) = a_n$. It can be done analytically. Indeed, substitution $a = re^{i\phi}$ yields $\dot{r} + ir\dot{\phi} = \frac{1}{2} \gamma r + \frac{3}{2} i \beta \omega_0^{-1} r^3$, and after separation of the real and imaginary parts,
\[ \dot{r} = -\frac{1}{2} \gamma r, \quad \dot{\phi} = \frac{3}{2} \beta \omega_0^{-1} r^2. \]  
(B.10)

The following hold

\[
\begin{align*}
    r(\Delta t) &= r_n e^{-\frac{1}{2} \gamma \Delta t}, \\
    \varphi(\Delta t) &= \varphi_n + \frac{3}{2} \beta \omega_0^{-1} r_n^2 \int_0^{\Delta t} e^{-\frac{1}{2} \gamma \tau} d\tau = \varphi_n + 3 \beta \omega_0^{-1} \gamma^{-1} (1 - e^{-\frac{1}{2} \gamma \Delta t}) r_n^2.
\end{align*}
\]  
(B.11)

Substituting \( \Delta t = T \), we obtain

\[
\begin{align*}
    a((n+1)T - 0) &= r_n e^{-\frac{1}{2} \gamma T} e^{i \varphi_0 + 3 \beta \omega_0^{-1} \gamma^{-1} (1 - e^{-\frac{1}{2} \gamma T}) r_n^2} \\
    &= a_n e^{-\frac{1}{2} \gamma T} e^{3 \beta \omega_0^{-1} \gamma^{-1} (1 - e^{-\frac{1}{2} \gamma T}) |a_n|^2}.
\end{align*}
\]  
(B.12)

The delta-function kick is accompanied by instant transfer of the momentum; as seen from (B.6) the velocity \( \dot{x} \) is changed by the value \( C \), while the coordinate \( x \) just after the kick retains its value. Thus, we have to set

\[
a_{n+1} = \frac{1}{2} (x(nT - 0) - i \omega_0^{-1} (\dot{x}(nT - 0) + C)) = a(nT - 0) - \frac{1}{2} i \omega_0^{-1} C. \quad (B.13)
\]

Combining with (B.12) implies that

\[
a_{n+1} = -\frac{1}{2} i \omega_0^{-1} C + a_n e^{-\frac{1}{2} \gamma T} e^{3 \beta \omega_0^{-1} \gamma^{-1} (1 - e^{-\frac{1}{2} \gamma T}) |a_n|^2}. \quad (B.14)
\]

Finally, by the change of variable and parameters

\[
Z = ia \left( \frac{3 \beta (1 - e^{-\gamma T/2})}{\omega_0 \gamma} \right)^{1/2}, \quad A = \frac{C}{2 \omega_0} \left( \frac{3 \beta (1 - e^{-\gamma T/2})}{\omega_0 \gamma} \right)^{1/2}, \quad B = e^{-\gamma T/2}.
\]  
(B.15)

we arrive at the convenient form of the map

\[
Z_{n+1} = A + BZ_n e^{i|Z_n|^2}. \quad (B.16)
\]

This map was introduced firstly for rather different physical setup, namely, for optical circular resonator filled by medium with the refractive index varying proportional to squared amplitude of electric field of the propagating electromagnetic wave (Ikeda et al., 1980). After that, this map called the Ikeda map became a popular model; it was studied extensively and used often in literature for illustrations of phenomena of chaotic dynamics.

It is worth noticing that applicability of the map (B.16) to quantitative description of the nonlinear optical system is doubtful. Correctly, that system is described by the time-delayed equations written out and examined by the
same authors (Ikeda et al., 1980). As to the map (B.16), it was introduced heuristically, and never was derived for the nonlinear optical system on a firm theoretical basis. By contrast, for the nonlinear oscillator this map really delivers a reasonable description valid in appropriate asymptotic (Kuznetsov et al., 2008).

The map (B.16) is represented as a single relation for the complex variable $Z$, but in terms of real variables it is in fact two-dimensional. Evaluation of Jacobi matrix for this map yields $J = B^2$; so, at $B < 1$ the map is dissipative (area compressing). Figure B.5 shows portraits of attractors for the Ikeda map at different values of parameter $A$ at $B=0.2$.

![Fig. B.5 Strange chaotic attractor of the Ikeda map (B.16) at $B = 0.2$ for three different values of the intensity of external driving.](image)

In Fig. B.6 a chart of dynamical regimes on the parameter plane ($A$, $B$) is presented. If one increases the parameter of intensity of driving $A$ at fixed dissipation $B$, the transition to chaos occurs, as a rule, through the period-doubling bifurca-

![Fig. B.6 Chart of dynamic regimes on the parameter plane for the Ikeda map (B.16).](image)
tion cascade. However, globally the border of chaos is arranged in complex manner. Particularly, there are narrow strips of periodic dynamics entering deeply inside area occupied by chaos and containing scattered small domains of periodicity, the “shrimps”.

Figure B.7 shows the larger one of two Lyapunov exponents of the Ikeda map (B.4) obtained numerically and plotted versus parameter $A$ at constant $B = 0.2$. Again, one can see, in a wide range mostly occupied by chaos, a set of deep narrow drops to negative values of the Lyapunov exponent associated with windows of periodicity. In the chart of dynamical regimes it corresponds to a path on the parameter plane along a horizontal line with the drops to negative Lyapunov exponent values as the pass crosses the “strips” and the “shrimps”.

![Figure B.7](image)

**Fig. B.7** The largest Lyapunov exponent of the Ikeda map (B.16) versus parameter $A$ at constant value of $B = 0.2$.

In Hénon map and Ikeda map under appropriate selection of parameters the configuration of mapping corresponding to the Smale horseshoe (Appendix C) can be verified. It proves complex and chaotic nature of trajectories in the phase space, but chaotic attractors, if occur, do not relate to the uniformly hyperbolic class. A visible indicator of non-hyperbolicity is presence of tiny domains of periodicity surrounded by chaos on the charts of dynamical regimes of Figs. B.3 and B.5. It indicates the absence of the structural stability. In Chap. 7 the Hénon map and Ikeda map are used as examples of non-hyperbolic behavior to oppose them against the systems with uniformly hyperbolic attractors.

References

Appendix C
Smale’s Horseshoe and Homoclinic Tangle

In Cha. 1 we mentioned complex dynamics associated with stretching, folding, and compression of phase space volume in two-dimensional setting (Fig. 1.4). Formalization of that construction is the so-called Smale horseshoe (Smale, 1967; Guckenheimer and Holmes, 1983; Anosov et al., 1995; Devaney, 2003; Schuster and Just, 2005). It is a simple model map invented by Smale as a counterexample to assumption that in high dimensions structurally stable systems are typical. There is no uniformly hyperbolic attractor in this map. Instead, there is a non-trivial chaotic invariant set of saddle type. Presence of the horseshoe is an indicator of complexity of dynamics and chaos on some set of orbits, although physically it may be unobservable as the invariant set is not attractive.

Consider a region in the form of a stadium, which consists of three parts, the square $S$, and half-disks, $D_0$ and $D_1$ docked on two sides. Let the action of the map correspond to stretching this area horizontally, compression in the vertical direction, so it becomes long and narrow strip. Next, we deform it so that it takes a form of a horseshoe, and impose it into the original area, as shown in Fig. C.1. This is Smale’s horseshoe map.

All the points belonging to the domains $D_0$ and $D_1$, are mapped inside $D_0$ and eventually attracted to a stable fixed point located in this area. Non-trivial dynamics corresponds to a set of orbits starting from domain $S$. It is clear that after the first iteration of the map some of them leave the domain of $S$ and fall into the $D_0$ or $D_1$; they are not our interest.

Where are the starting points placed, which remain inside the domain $S$ after the first iteration? Obviously, they occupy two narrow vertical bands $V_0$ and $V_1$, which, when stretched horizontally, just become equal in width to the area $S$. And how do placed images of points belong to $S$ at the previous time step? They occupy two narrow horizontal bands $H_0$ and $H_1$, which correspond to the two halves of the horseshoe. At the intersection of these bands the points live that remain in $S$ for iterating twice.

Now consider the horseshoe map in two steps of the iterations (see bottom picture in Fig. C.2). Now, there are four vertical stripes, two of which are located in a strip $V_0$ (denote them by $V_{00}$ and $V_{01}$) and two are situated in the band $V_1$ (denote them by
Points belonging to these strips survive in the region $S$ with twice iterating. Next, there are four horizontal stripes, two of which are located in the strip $H_0$ (they are $H_{00}$ and $H_{10}$) and two in the strip $H_1$ (they are $H_{01}$ and $H_{11}$). They represent a set where the orbits arrive started in $S$ after twice iterating. The union of overlaps of these strips determines a set of starting points of orbits, which do not leave the region $S$ with four times iterating. With increase of number of considered iterations, each time within each vertical and horizontal strip two more narrow sub-strips are selected and left for following steps of the construction. In the limit of infinite number of iterating, the set of vertical strips and of the horizontal ones form a Cantor set. The intersection of these two sets is a “Cantor grid”, which is an invariant set $\Omega$, whose elements survive in the area of $S$ forever, for an arbitrarily large number of iterating of the horseshoe map. Complete description of the invariant set $\Omega$ in terms of symbolic dynamics is achieved by introducing appropriate binary numbering of horizontal and vertical lines, like that introduced in Fig. C.2 for the first two steps of the construction.

Elements of the set $\Omega$ are encoded by two-side infinite binary sequences, and dynamics in this representation is determined as a shift by one step at each once iterating (Bernoulli shift). It implies the presence of chaos. Indeed, given an element of this set by a random, say, with a coin toss, we can obtain two sequences of random digits and use them as head and tail of the two-side binary sequence determining position of some point belonging to the set $\Omega$ in the domain $S$. Starting at that point, and considering dynamics in forward and in inverse time we observe visiting the horizontal and vertical bands of Fig. C.2(a) in accordance with prescribed random sequences. Periodic symbolic sequences correspond to periodic orbits of
Appendix C Smale’s Horseshoe and Homoclinic Tangle

**Fig. C.2** Initial two steps of constructing invariant set $\Omega$ of points surviving in the region $S$. The right-hand pictures on panels (a) and (b) show in dark gray the sets of survivors with twice and four times iterating, respectively.

-the horseshoe map; they form an infinite countable set. The set of all nonperiodic sequences is of cardinality of continuum.

It should be noted that all conclusions about the complex dynamics of the horseshoe map are based essentially only on topological arguments. Configuration of regions may look not exactly like that in Fig. C.1; only topological consistency is needed. Deduction for complex dynamics associated with existence of chaotic orbits and of an infinite number of periodic orbits is in any case a significant result, although, in the case of chaos established by the horseshoe map argumentation chaotic motion as sustained physically relevant regime may not occur because of non-attractive nature of the invariant set $\Omega$.

Existence of the invariant set $\Omega$ is closely linked with the remarkable object called the homoclinic tangle (Guckenheimer and Holmes, 1983; Neimark and Landa, 1992; Anosov et al., 1995; Hilborn, 2000; Ott, 2002). Figure C.3 shows disposition of stable and unstable manifolds for a saddle-fixed point existing in the domain $S$ for the horseshoe map. Existence of intersection of these manifolds at the homoclinic point $\Gamma$ implies existence of an infinite collection of other homoclinic points! Indeed, the point $\Gamma$ is mapped to some other point, which must lay at intersection of the stable and unstable manifolds; that point, in turn, is mapped to a point of the same nature and so on. These multiple intersections are seen in panel (b), where prolonged segments of the manifolds are shown.
Occurrence of the homoclinic tangle is an equivalent indicator of complex dynamics in a sense of presence of non-attractive chaotic invariant set in the phase space of the system. Simple examples of physically relevant models, in which these phenomena take place, are the Hénon map and Ikeda map discussed in Appendix B.

Fig. C.3 Disposition of stable and unstable manifolds for a saddle-fixed point of the horseshoe map intersecting at the homoclinic point $\Gamma$ (a) and illustration of formation of the homoclinic tangle (b).

References


Essential attribute of chaotic attractors is intrinsic Cantor-like transversal structure provided by the repetitive transformations of stretching, folding, and transversal compression for the phase space volume (see Chap. 1). A tool for quantitatively characterizing this structure is delivered by the fractal geometry (Mandelbrot, 1982; Beck and Schlogl, 1993; Ott, 2002; Schuster and Just, 2005). Fractal dimension generalizes the familiar notion of dimension (1 for curves, 2 for surfaces, 3 for volumes), but for non-trivial sets, like strange attractors, it may be non-integer containing a fractional part.

The simplest definition relates the so-called box-counting dimension, or capacity. It is based on covering the set by boxes of equal size ε without overlap (Fig. D.1), and considering dependence on the needed number of the elements of this size. The dimension is defined then as

\[
D_0 = - \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon}.
\]  

(Fig. D.1) Covering of an attractor by boxes of fixed size ε (a) and constructing classic Cantor set by successive steps of removing the middle thirds of segments obtained in the previous steps (b).
For example, a classic Cantor set (which appears just in the above dissipative baker map in cross-section of the picture in Fig. 1.3 by a vertical line) obviously can be covered by $N = 2^n$ intervals of length $\varepsilon = 3^{-n}$ (it just corresponds to the $n$-th step of its construction). Taking the limit $n \to \infty$, which corresponds to $\varepsilon \to 0$, we obtain from (1.13) $D = \log 2 / \log 3 = 0.6309 \ldots$ Dimension of the attractor of the baker map may be easily found to be larger by one, namely, $D_b = 1.6309 \ldots$

As it is, to characterize strange attractors with sufficient completeness, it is necessary to introduce not one but a set of dimensions. Why is it so that a single dimension $D_0$ is not enough?

Imagine that the attractor is essentially not uniform: some of elements of covering are visited more frequently than others. This is in no way reflected in the definition of $D_0$, although, evidently, it should be significant for such a quantitative characterization of the attractor, which pretends to be a dimension.

To account the different probabilities of visiting the cells one needs to account the invariant measure. As noted in Chap. 1, the natural invariant measure of any region of phase space $S$ may be defined as the probability of residence in the area for a typical orbit: $p(S) = \lim_{T \to \infty} (\tau(S, T) / T)$, where $\tau(S, T)$ is the residence time in $S$ at observation interval $T$. Assuming that the invariant measure is defined for the attractor, we attribute certain measure $p_i$ to each $i$-th cell, which is probability of residence there of a typical orbit. Now, consider a sum $I(\varepsilon) = - \sum_{i=1}^{N(\varepsilon)} p_i \log p_i$. It may be interpreted as amount of information needed to specify a cell of residence at given instant. It is clear that the sum will increase with decreasing size of the cells. We can assume that this growth should obey a power law, $I(\varepsilon) \approx \varepsilon^{-D_1}$, or, equivalently that one can define a limit

$$D_1 = - \lim_{\varepsilon \to 0} (I(\varepsilon) / \log \varepsilon) = \lim_{\varepsilon \to 0} \left( \frac{N(\varepsilon)}{\sum_{i=1}^{N(\varepsilon)} p_i \log p_i} \right). \quad (D.2)$$

It is called the information dimension.

Moreover, in 1983 an infinite family of generalized dimensions was introduced in use in the nonlinear dynamics called the Rényi dimensions on behalf of the Hungarian mathematician who proposed them earlier in another context (Grassberger, 1983; Hentschel and Procaccia, 1983).

Assume that the size of boxes covering the attractor is $\varepsilon$, and the probability of residence in the $i$-th cell is $p_i$. Then for any real number $q$, the dimension $D_q$ is defined as the limit

$$D_q = \frac{1}{q-1} \lim_{\varepsilon \to 0} \left( \log \varepsilon \sum_{i=1}^{N(\varepsilon)} p_i^q \right). \quad (D.3)$$

Note that at $q = 0$ it turns to the capacity dimension, and at $q = 1$ it is the information dimension. The last follows from evaluating the limit $q \to 1$ by means of the l’Hôpital’s rule.
One more special case is the so-called *correlation dimension* $D_2$, which is distinguished because of existence of relatively simple procedure of estimate in computations suggested by (Grassberger and Procaccia, 1983).

In more accurate mathematical analysis, improved definitions were developed accounting possibility of covering sets with elements of different forms and sizes (Hausdorff dimension) (Guckenheimer and Holmes, 1983; Beck and Schlogl, 1993; Ott, 2002).

Remind now that formation of fractal Cantor-like structure in strange attractors occurs due to stretching and compressions of phase volume, which are associated with presence of positive and negative Lyapunov exponents, respectively. Hence, one can ask about a relation between the dimensions and the Lyapunov spectrum. An answer for this question is somewhat given by the *Kaplan-Yorke formula* (Kaplan and Yorke, 1979).

Let the dimension of the phase space of a dissipative system be $N$, then we have $N$ Lyapunov exponents, which are assumed to be numbered in descending order: $\lambda_1, \lambda_2, \ldots, \lambda_N$. If the attractor is chaotic, the largest exponent is positive. On the other hand, the sum of all Lyapunov exponents is negative. Therefore, calculating the sums $S_m = \sum_{k=1}^{m} \lambda_k$ for $m = 1, 2, \ldots$ we will get a result at first positive and then negative values (Fig. D.2). We select such $m$ for which $S_m > 0$, but $S_{m+1} < 0$. Consider the subspace spanning over the perturbation vectors associated with the first $m$ exponents. In this space, the phase volume will increase in the course of the dynamics. On the other hand, in the subspace spanning over $m + 1$ vectors the phase volume will be compressed. Along other dimensions the compression is even stronger, so, we will ignore those dimensions.

![Fig. D.2](image)

**Fig. D.2** Illustration of dependence of a sum of Lyapunov exponents on the number of terms accounted. Particularly, $S_1$ is the largest Lyapunov exponent $\lambda_1$, the maximum value of $S_m$ is the Kolmogorov-Sinai entropy $h$, $S_N$ is negative constant $d$, characterizing the contraction rate for approach to the attractor. A point of intersection of the graph with the horizontal axis yields the Kaplan-Yorke dimension according to formula (D.7).

In the subspace of dimension $m + 1$ consider covering of the attractor by boxes of size $\varepsilon$ assuming the orientations of the edges correspond to the directions associated with the Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$. Let the number of cells
needed be \( N(\varepsilon) \). Due to dynamics of representative points along respective orbits, the domain covered by the boxes after some time \( T \) transforms to another domain, strongly elongated in the direction corresponding to the largest exponent \( \lambda_1 \) and flattened in the direction associated with the last one taken into consideration, \( \lambda_{m+1} \) (Fig. D.3). Each box transforms to a parallelepiped of size \( \varepsilon_1 \times \varepsilon_2 \times \varepsilon_3 \times \cdots \times \varepsilon_{m+1} \), where \( \varepsilon_i = \varepsilon e^{\lambda_i T} \). Covering of one parallelepiped by boxes of size \( \varepsilon_{m+1} \) requires

\[
n = \prod_{i=1}^{m} \left( \frac{\varepsilon_i}{\varepsilon_{m+1}} \right) = \prod_{i=1}^{m} \left( \frac{\varepsilon e^{\lambda_i T}}{e^{-|\lambda_{m+1}| T}} \right) \text{ cells.}
\]

Doing so with all \( N(\varepsilon) \) parallelepipeds we obtain a new covering of the attractor with the number of cells

\[
N(\varepsilon_{m+1}) = nN(\varepsilon) = N(\varepsilon) \prod_{i=1}^{m} \left( \varepsilon e^{\lambda_i T} / e^{-|\lambda_{m+1}| T} \right)
\]

\[
= N(\varepsilon) \prod_{i=1}^{m} \left( e^{\lambda_i T} / e^{-|\lambda_{m+1}| T} \right).
\]

Fig. D.3 To the derivation of the Kaplan-Yorke formula.

Estimate of the dimension according to (D.1) yields

\[
D \cong -\frac{\log N(\varepsilon_{m+1})}{\log \varepsilon_{m+1}} = \frac{\log N(\varepsilon) + \log \prod_{i=1}^{m} e^{\lambda_i T} - m \log e^{-|\lambda_{m+1}| T}}{-\log \varepsilon + |\lambda_{m+1}| T}.
\]

(D.5)

Multiplying the left-and the right-hand parts by \( -\log \varepsilon + |\lambda_{m+1}| T \) we obtain

\[
-D \log \varepsilon + D|\lambda_{m+1}| T = \log N(\varepsilon) + \sum_{i=1}^{m} \lambda_i T + m|\lambda_{m+1}| T.
\]

(D.6)

Accounting that \( D \cong -\log N(\varepsilon) / \log \varepsilon \), we cancel terms in both parts of the relation and finally obtain the Kaplan-Yorke formula

\[
D = m + \frac{\sum_{i=1}^{m} \lambda_i}{|\lambda_{m+1}|}.
\]

(D.7)
Here, remind that the number \( m \) is determined in such a way that \( S_m = \sum_{i=1}^{m} \lambda_i > 0 \), but already \( S_{m+1} = \sum_{i=1}^{m+1} \lambda_i < 0 \).

The initial conjecture was that this formula evaluates the information dimension of strange attractors. Indeed, it was proven for attractors of two-dimensional maps. But in general case the conjecture was not confirmed and, generally speaking, it is not valid. Nevertheless, empirically, this formula provides usually sufficiently good results, remarkably close to correct dimensions of attractors in many cases.

It the framework of the present book, one should specially outline that the above derivation exploits essentially assumption of uniform expansion and contraction of the phase volume elements in the course of the dynamical evolution. Hence, we expect that among various chaotic attractors, this formula is especially appropriate just for the uniformly hyperbolic attractors.

To conclude, observe how meaningful the diagram of Fig. D.2 is, which depicts the sums \( S_m \) versus the number of accounted Lyapunov exponents \( m \). From this plot one can see immediately a set of relevant quantifiers for chaotic attractors: \( S_1 \) equals the positive Lyapunov exponent responsible for chaos, \( S_{\text{max}} \) is the Kolmogorov-Sinai entropy (in accordance with the Pesin formula), the intersection of the plot with the abscissa axis determines the Kaplan-Yorke dimension, and the value \( S_N \) determines the rate of compression of the phase volume in the course of approach to the attractor.

References

Appendix E
Hunt’s Model: Formal Definition

Hunt’s model is an artificially constructed example of a flow non-autonomous system with attractor of Plykin type in stroboscopic Poincaré section developed by T. Hunt in his PhD thesis under supervision of Prof. Robert MacKay (Hunt, 2000). The set of differential equations is formulated for two variables \(x\) and \(y\), with the right-hand parts depending on \(x\), \(y\), and the time variable \(t\):

\[
\frac{dx}{dt} = f_*(x,y,t), \quad \frac{dy}{dt} = g_*(x,y,t). \tag{E.1}
\]

Here the functions \(f_*\) and \(g_*\) are continuously differentiable, and have period \(2\pi\) in respect to the argument \(t\).

Dynamics of variables \(x\), \(y\) in the time interval \(\Delta t = 2\pi\) is represented in three successive stages, the duration of each is \(2\pi/3\). In formulation of the relations, fictitious time \(s\) is used, which varies in each stage from 0 to 1. To ensure smoothness of the flow, relationship between \(s\) and \(t\) is chosen in such a way that the motion of the representative points on the plane of \(x\) and \(y\) is of zero velocity at the edges of the stages.

In description of the flow, three special curvilinear coordinate systems \((r, \theta)^i\), \(i = 1, 2, 3\) are used in domains \(D_{1,2,3}\) (Fig. 2.6), being linked with the Cartesian coordinates \(x\), \(y\) by relations

\[
x = X + S_i(\pi/2 - |\theta|), \quad y = Y + r\text{sgn}\theta, \quad |\theta| \geq \pi/2, \\
x = X + S_ir\cos f_r(\theta), \quad y = Y + r\sin f_r(\theta), \quad |\theta| < \pi/2, \tag{E.2}
\]

where \(f_r(\theta) = 2\pi^2r\theta/(\pi^2(r + 1) + 4(r - 1)\theta^2)\), \(S_1 = -1\) and \(S_{2,3} = 1\). Variables \(X\) and \(Y\) determine location of the origin for the \(i\)-th coordinate system in the Cartesian coordinates. For brevity, we designate the transformations to new coordinates and back, respectively, as \((r, \theta)^i = (R(x,y,X,Y,S_i),\Theta(x,y,X,Y,S_i))\) and \((x,y) = (F^i(r, \theta, X, Y, S_i), G^i(r, \theta, X, Y, S_i))\). The explicit expressions are derived easily from (E.2).

Let us set \(\lambda = (3 + \sqrt{5})/2\), \(\mu = (3 - \sqrt{5})/2\), and introduce some other constants determining geometrical disposition of the construction on the plane \((x,y)\):
\[
X_1 = \frac{1}{4} \left( 6\sqrt{5} - 6 - (3 - \sqrt{5})\pi \right), \quad X_2 = 3, \quad X_3 = \frac{1}{4} \left( 18 - 6\sqrt{5} - (\sqrt{5} - 1)\pi \right),
\]
\[
Y_1 = \frac{1}{2} (1 + \sqrt{5}), \quad Y_2 = 1, \quad Y_3 = \frac{1}{2} (3 + \sqrt{5}),
\]
\[
R_1 = \frac{1}{2} (1 + \sqrt{5}), \quad R_2 = 1, \quad R_3 = \frac{1}{2} (\sqrt{5} - 1).
\]

1. At the first stage \( t \in [0, 2\pi/3] \) we set \( s = \sin^2 \left( \frac{3}{4} \right) \). The origins for the special coordinates move in dependence on \( s \) in accordance with

\[
X_i(s) = \lambda s(X_i + \pi/2) - \pi/2, \quad Y_i = \mu s Y_i, \quad i = 1, 2, 3.
\]

Besides, we set \( R_{11}(s) = \mu^s R_i \). Given certain \( x, y, z \), we determine the vector \( f(x,y,z) = (f(x,y,z), g(x,y,z)) \) via the following procedure.

(a) If \( x \leq 0 \), perform transformation to the coordinate system number 1:

\[
(r, \theta)^1 = (R(x,y,-X_{11}(s),Y_{11}(s),-1), \Theta(x,y,-X_{11}(s),Y_{11}(s),-1)),
\]

and, designating by dot the derivative in respect to \( s \), set

\[
\dot{\theta} = (\ln \lambda) \theta, \quad \dot{r} = (\gamma(\theta,X_{11}(s))) h_{12}(r,R_{11}(s)) + (1 - \gamma(\theta,X_{11}(s))) (\ln \mu) r.
\]

Here the functions are introduced:

\[
\gamma(\theta,X) = \begin{cases}
1, & |\theta| \leq \frac{\pi}{2}, \\
\cos^{2} \frac{1}{4} \pi(2|\theta| - \pi)^{-1} X^{-1}, & \frac{\pi}{2} \leq |\theta| \leq \frac{\pi}{2} + X, \\
0, & |\theta| \geq \frac{\pi}{2} + X, \\
\cos \pi(1 - r/\epsilon), & r < \epsilon, \\
1, & \epsilon \leq r \leq R - \epsilon, \\
\end{cases}
\]

\[
h_{12}(r,R) = \begin{cases}
\frac{5}{2} + \frac{1}{4} \cos \frac{1}{2\epsilon} \pi(r - R), & |r - R| < \epsilon, \\
\frac{3}{2}, & r \geq R + \epsilon.
\end{cases}
\]

Then, compute components of the vector field in Cartesian coordinates \((f_1, g_1)\) as

\[
f_i = \begin{cases}
\dot{X} - S \text{sgn}(\theta) \dot{\theta}, & |\theta| \geq \pi/2, \\
\dot{X} + S \dot{r} \cos f_r(\theta) - S r \sin f_r(\theta) (\dot{r} \partial f_r(\theta)/\partial r + \dot{\theta} \partial f_r(\theta)/\partial \theta), & |\theta| \leq \pi/2,
\end{cases}
\]

\[
g_i = \begin{cases}
\dot{Y} + \text{sgn}(\theta) \dot{r}, & |\theta| \geq \pi/2, \\
\dot{Y} + \dot{r} \dot{\sin f_r(\theta)} + r \cos f_r(\theta) (\dot{r} \partial f_r(\theta)/\partial r + \dot{\theta} \partial f_r(\theta)/\partial \theta), & |\theta| \leq \pi/2,
\end{cases}
\]

where \( i = 1, S = S_1 = -1, \dot{X} = (\ln \lambda)(X_{11} + \pi/2), \dot{Y} = (\ln \mu) Y_{11} \).
(b) If \( x > 0 \), perform analogous computations in the coordinate systems 2 and 3. First, find out
\[
(r, \theta)^2 = (R(x, y, X_{21}(s), Y_{21}(s), 1), \Theta(x, y, X_{21}(s), Y_{21}(s), 1)).
\] (E.9)
Then, set
\[
\dot{\theta} = (\ln \lambda) \theta,
\]
\[
\dot{r} = (\gamma(\theta, X_{21}(s) - X_{31}(s))h_{12}(r, R_{21}(s))
\]
\[
+ (1 - \gamma(\theta, X_{21}(s) - X_{31}(s))))(\ln \mu)r,
\]
and with the formula (E.8), where \( i = 2, S = S_2 = 1 \), \( \bar{X} = (\ln \lambda)(X_{21} + \pi/2) \), \( \bar{Y} = (\ln \mu)Y_{21} \) compute the components of the vector field \((f_2, g_2)\).

Next, find out
\[
(r, \theta)^3 = (R(x, y, X_{31}(s), Y_{31}(s), 1), \Theta(x, y, X_{31}(s), Y_{31}(s), 1))
\] (E.11)
and set
\[
\dot{\theta} = (\ln \lambda) \theta, \quad \dot{r} = (\gamma(\theta, X_{31}(s))h_{3}(r, R_{31}(s)) + (1 - \gamma(\theta, X_{31}(s))))(\ln \mu)r,
\] (E.12)
where
\[
h_{3}(r) = \begin{cases} 
\cos \pi(1 - r/\varepsilon), & r < \varepsilon, \\
1, & r \geq \varepsilon. 
\end{cases}
\] (E.13)

With the formula (E.8), where \( i = 3, S = S_3 = 1 \), \( \bar{X} = (\ln \lambda)(X_{31} + \pi/2) \), \( \bar{Y} = (\ln \mu)Y_{31} \), obtain components of the vector field \((f_3, g_3)\).

(c) Now, determine the vector field
\[
\tilde{f}(x, y, s) = \begin{cases} 
(f_1, g_1), & x \leq 0, \\
w(d_3, d_2) \cdot (f_2, g_2) + w(d_2, d_3) \cdot (f_3, g_3), & x > 0,
\end{cases}
\] (E.14)
where the function is introduced
\[
w = w(u, v) = \sin^2 \left( \frac{1}{2} \pi u(v + 1)^{-1} \right),
\] (E.15)
with arguments expressed as \( d_\alpha = \max \{ R^\alpha(x, y, X_{\alpha1}(s), Y_{\alpha1}(s)) - \mu^2 R_\alpha, 0 \} \), \( \alpha = 2, 3 \).

(d) As a final step of computations at the stage 1 set
\[
f(x, y, s) = \tilde{f}(x, y, s) + \sum_{i=1}^{3} \beta(||x - X_{i1}||)(\dot{X}_{i1}(s) - \tilde{f}(x, y, s) + (\ln \lambda)(x - X_{i1})).
\] (E.16)
Here \( x = (x, y), X_{i1}(s) = (S_i X_{i1}, Y_{i1}), \dot{X}_{i1}(s) = (S_i \dot{X}_{i1}, \dot{Y}_{i1}) \), and the function is introduced.
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\[
\beta(\rho) = \begin{cases} 
1, & \rho \leq \varepsilon/4, \\
\cos^2 \pi(2\rho/\varepsilon - 1/2), & \varepsilon/4 < \rho < \varepsilon/2, \\
0, & \rho \geq \varepsilon/2.
\end{cases}
\]  
(E.17)

Here an inaccuracy occurs in the Hunt’s work (Aidarova and Kuznetsov, 2009). While the definitions in his text correspond to (E.16) and (E.17), in the code of the program in Mathematical the argument of the function \(\beta\) has been incorrectly written as squared norm \(\|x - X\|_2^2\). Computations indicate its significance: with substitution of the squared norm the hyperbolicity disappears! On the other hand, with the definitions (E.16) and (E.17), the attractor really is detected as a hyperbolic one. However, at \(\varepsilon = 0.05\) selected by Hunt, it appears difficult to observe the fractal transversal structure of the attractor in illustrations. Thus, in computations discussed in Chap. 2, an increased parameter value \(\varepsilon = 0.17\) was taken, although within the allowable range specified in the work of Hunt.

2. At the second stage \(t \in [2\pi/3, 4\pi/3]\) set \(s = \sin^2 \left(\frac{3}{4}t - \frac{1}{2}\pi\right)\). Taking as the origin the point

\[
\begin{align*}
X_{22}(s) &= X_2 + (1-s)D, \\
Y_{22}(s) &= Y_2 + (1-s)D, \\
D &= (\lambda - 1)(X_2 + \pi/2) + \mu R_2,
\end{align*}
\]  
(E.18)

we represent an instantaneous state \((x, y)\) in new coordinates as

\[
(r, \theta) = (R(x, y, X_{22}(s), Y_{22}(s), 1), \Theta(x, y, X_{22}(s), Y_{22}(s), 1))
\]  
(E.19)

and set

\[
\dot{r} = -D, \quad \dot{\theta} = D.
\]  
(E.20)

Backward transformation to Cartesian coordinates is performed with formula (E.8), where \(S = 1, \dot{X} = -D, \dot{Y} = -D\), and components of the vector field \((f_0, g_0)\) are obtained. Finally, set

\[
f(x, y, s) = w(a_2, b_2) \cdot (f_0, g_0),
\]  
(E.21)

where the function \(w\) is given by (E.14), and its arguments are expressed as

\[
\begin{align*}
a_2 &= \max \left\{ R(x, y, X_{22}(s), \frac{1}{2}(Y_2 + Y_{22}(s)), 1) - \frac{1}{2}(-Y_2 + Y_{22}(s)) - \mu R_3, 0 \right\}, \\
b_2 &= \max \left\{ Y_{22}(s) - 2R_2\mu - R(x, y, X_{22}(s), Y_{22}(s), 1), 0 \right\}.
\end{align*}
\]  
(E.22)

3. At the third stage \(t \in [4\pi/3, 2\pi]\) set \(s = \sin^2 \left(\frac{3}{4}t - \pi\right)\). Take the point
\[ X_{13}(s) = X_1 + (1 - s)D, \quad Y_{13}(s) = Y_1 + (1 - s)D, \quad D = (\lambda - 1)(X_1 + \pi/2) + \mu R_1 \]  
(E.23)
as the origin. Transform initial values \((x, y)\) to coordinates \((r, \theta)\):

\[
(r, \theta) = (R(x, y, -X_{13}(s), Y_{13}(s), -1), \Theta(x, y, -X_{13}(s), Y_{13}(s), -1)).
\]  
(E.24)

Next, define the flow by formula (E.20) and pass to the Cartesian coordinates by means of (E.8), setting \(S = -1, \dot{X} = -D, \dot{Y} = -D\). Now,

\[
f(x, y, s) = w(a_3, b_3) \cdot (f_0, g_0),
\]  
(E.25)

where the function \(w\) is defined by (E.15), and

\[
a_3 = \max \left\{ R(x, y, -X_{13}(s), \frac{1}{2}(Y_1 + Y_{13}(s)), -1) - \frac{1}{2}(-Y_1 + Y_{13}(s)) - \mu R_2, 0 \right\},
\]

\[
b_3 = \max \{ Y_{13}(s) - 2R_1 \mu - R(x, y, -X_{13}(s), Y_{13}(s), -1), 0 \}.
\]

4. Finally, setting \(f(x, y, s) = (f(x, y, s), g(x, y, s))\), we obtain the equations for the continuous-time model valid at all three stages as follows:

\[
\frac{dx}{dt} = \frac{3}{4} |\sin \frac{3}{2}t| f(x, y, s(t)), \quad \frac{dy}{dt} = \frac{3}{4} |\sin \frac{3}{2}t| g(x, y, s(t)).
\]  
(E.26)

In the right-hand parts a factor \(s'(t) = \frac{3}{4} |\sin \frac{3}{2}t|\) is taken into account, arising because of the passage from fictitious time \(s\) to natural time \(t\).

References


Appendix F
Geodesics on a Compact Surface of Negative Curvature

Free mechanical motion of a particle in curved space takes place along the geodesic lines of metric identified by a positive-definite quadratic form with smooth coordinate-dependent coefficients. In two dimensions, the square of the infinitesimal displacement is expressed by the relation

$$ds^2 = E(x,y)dx^2 + F(x,y)dxdy + G(x,y)dy^2.$$  \hspace{1cm} (F.1)

Through the coefficients of the quadratic form we can obtain the Gaussian curvature using the Brioschi formula (Struik, 1988)

$$K = \frac{1}{(EG - F^2)^2} \cdot \begin{vmatrix}
-\frac{1}{2}E_{yy} + F_{xy} - \frac{1}{2}G_{xx} & \frac{1}{2}E_x & F - \frac{1}{2}E_y & 0 \\
F_y - \frac{1}{2}G_x & E & F & -\frac{1}{2}E_y \\
\frac{1}{2}G_y & F & G & \frac{1}{2}G_x \\
\end{vmatrix}.$$  \hspace{1cm} (F.2)

If the curvature is negative, the geodesics are characterized by instability with respect to transversal perturbations; then, if the motion takes place in bounded domain, it is chaotic. Analysis of the dynamics in this situation goes back to Hadamard and others (Anosov, 1967). This problem was one of the main sources of inspiration for the creation of the hyperbolic theory. The purpose of this Appendix is to illustrate the main features of the motion on a compact manifold of negative curvature for a specific example, which follows (Balazs and Voros, 1986).

Surface of constant negative curvature (the Lobachevsky plane or pseudosphere) can be represented by a model of Poincaré, in which points of the surface are mapped as points of the unit disk, and straight lines (i.e. geodesics) are represented by circular arcs orthogonal to the edge of the disc (Fig. F.1(a)). Metric given by the Poincaré model
Fig. F.1 Unit disc representing the Lobachevsky plane in the model of Poincaré with shown representatives of the geodesics. Domain D for consideration of motion of a particle and a set of equivalent domains obtained from D via the variable change (F.7) with \( m = 0, 1, \ldots, 7 \).

\[
\frac{ds^2}{ds^2} = 4\left( \frac{dx^2 + dy^2}{1 - x^2 - y^2} \right) \quad (F.3)
\]
corresponds to a constant negative curvature \( K = -1 \) that may be verified by formula (F.2).

The Lagrange function for the free particle motion contains only the term of the kinetic energy; accounting (F.3) it is expressed as

\[
L = W = \frac{2\mu (\dot{x}^2 + \dot{y}^2)}{(1 - x^2 - y^2)^2}, \quad (F.4)
\]
where \( \mu \) is the mass of the particle. The Euler-Lagrange equations governing the motion of the particle then are obtained in the form

\[
\ddot{x} = 2\frac{xy^2 - x\dot{x}^2 - 2y\dot{x}\dot{y}}{(1 - x^2 - y^2)^2}, \quad \ddot{y} = 2\frac{yx^2 - y\dot{y}^2 - 2x\dot{x}\dot{y}}{(1 - x^2 - y^2)^2}. \quad (F.5)
\]
Motion according these equations conserves the kinetic energy \( W = 4\mu (\dot{x}^2 + \dot{y}^2)(1 - x^2 - y^2)^{-2} \) and, consequently, occurs along the geodesics of the metric determined by relation (F.3). The velocity, as the rate of increase of distance traveling along the geodesic line, is expressed as

\[
\dot{s}^2 = \frac{4(\dot{x}^2 + \dot{y}^2)}{(1 - x^2 - y^2)^2} = 2W\mu^{-1}. \quad (F.6)
\]

Consider motion of the particle in a domain of an octagon D bounded by circular arcs (Fig. F.1(b)). Because of symmetry inherent to the pseudosphere, the octagon can be considered equivalent to each of the eight regions attached to the sides, which in the Poincaré model look like distorted octagons of smaller size. The variable changes corresponding to the transformation of each of these regions in the original octagon are given by the relations
\[ x' + iy' = \frac{\sqrt{1 + \sqrt{2}(x + iy)} + e^{\pi m/4} \sqrt{2 + 2\sqrt{2}}}{\sqrt{1 + \sqrt{2} - \sqrt{2} + \sqrt{2}e^{-\pi m/4}(x + iy)}}, \quad (F.7) \]

where \( m \) is an index of the respective domain. In the course of the motion, each time when the particle crosses a border of the octagon \( D \), one can think of it as appearing on the opposite side with coordinates reassigned according to (F.7) and velocity components obtained as a derivative of this relation:

\[ x' + iy' = \frac{(3 + 3\sqrt{2})(\dot{x} + i\dot{y})}{\left(\sqrt{1 + \sqrt{2} - \sqrt{2} + \sqrt{2}e^{-\pi m/4}(x + iy)}\right)^2}. \quad (F.8) \]

Points of every two opposite segments of the border of the domain \( D \) may be identified; one can think of them as being glued to each other. Topologically, the resulting object may be thought of as a doughnut with two holes, or a sphere with two attached handles. According to the classification adopted in the topology, this is a manifold of genus 2.

Figure F.2 shows a trajectory of free motion of a particle in the domain \( D \) obtained from numerical solution of the differential equations (F.5) supplemented with the rules of the border cross-sections (F.7) and (F.8). For longer integration time the orbit is observed to cover the domain densely.

To be sure in its chaotic nature it is worth examining the Lyapunov exponents. Numerically, they may be computed with standard Benettin method solving the equations of motion (F.5) together with a collection of four sets of linearized equations

\[
\begin{align*}
\ddot{x} &= 2\dddot{x}^2 + 2xy\dot{y} - \ddot{x}^2 - 2x\dot{x}\dddot{x} - 2y\dot{x}\dot{y} - 2y\dddot{y} - 2xy\dot{y} \\
&\quad + 4\left(\frac{x\dot{y}^2 - x\dddot{y} - 2y\dot{y}\dddot{y}}{(1 - x^2 - y^2)^2} - \frac{2y\dot{y}}{(1 - x^2 - y^2)^3}\right), \\
\ddot{y} &= 2\dddot{y}^2 + 2yy\dot{x} - \ddot{y}^2 - 2y\dot{y}\dddot{y} - 2\dot{x}\dddot{y} - 2\dddot{x}y - 2x\dot{x}\dddot{y} \\
&\quad + 4\left(\frac{y\dot{x}^2 - y\dddot{x} - 2x\dot{x}\dddot{x}}{(1 - x^2 - y^2)^2} - \frac{2x\dot{x}}{(1 - x^2 - y^2)^3}\right). \\
\end{align*}
\quad (F.9)\]

Fortunately, due to special nature of the problem under consideration, there is an alternative approach. One can represent the vector \( \tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \) in locally determined special basis, \((\mathbf{e}_\parallel, \mathbf{e}_\perp)\), where \( \mathbf{e}_\parallel \) is a unit vector directed along the geodesic line, and \( \mathbf{e}_\perp \) is a unit vector orthogonal to the previous one. Setting \( \tilde{\mathbf{x}} = J_\parallel \mathbf{e}_\parallel + J_\perp \mathbf{e}_\perp \) one can derive the following equations for the coefficients:

\[ J_\parallel = 0, \quad J_\perp = -Ks^2J_\perp, \quad (F.10) \]

where \( K \) is the Gaussian curvature. These equations hold even for a general case of free motion of a particle in metrics (F.1), with arbitrary dependence of the coeffi-
Cients on coordinates; then $K$ should be understood as the local curvature depending on the instant coordinates $x, y$ in accordance with the formula (F.2).

Fig. F.2 A representative trajectory of free motion of a particle in compact domain on a surface of constant curvature in accordance with Eqs. (F.5) supplemented with the rules of border cross-sections (F.7) and (F.8).

General solution of the first equation (F.10) is a linear combination of functions 1 and $t$. Hence, it contributes to the spectrum of Lyapunov exponents with two zero values. As to the second equation, for the system under consideration the value $s^2$ is equal to the doubled ratio of a constant kinetic energy of the particle to its mass, $2W\mu^{-1}$ (see (F.3)). With the constant negative curvature $K$, the fundamental solutions are $e^{\sqrt{2KW\mu^{-1}}t}$ and $e^{-\sqrt{2KW\mu^{-1}}t}$. So, the Lyapunov spectrum contains one positive exponent, two zero ones, and one negative one, namely,

$$\lambda_1 = \sqrt{2KW\mu^{-1}}, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = -\sqrt{2KW\mu^{-1}}.$$  (F.11)

Presence of a positive exponent indicates chaotic nature of the dynamics. Observe that the sum of all the exponents is zero that reflects the conservative nature of the system we consider here.

Results of direct numerical calculation of the Lyapunov exponents by the Benettin method are in good agreement with the theoretical result (F.11). For example, for a trajectory with initial conditions $x = 0, y = 0, \dot{x} = 0.6, \dot{y} = 0.8$ we have $W = 2\mu$, and accounting $K = -1$, obtain $\sqrt{2KW\mu^{-1}} = 2$. The result of computation of the Lyapunov exponents in time interval $\Delta t = 1000$ is

$$\lambda_1 = 2.0004, \quad \lambda_2 = 0.0038, \quad \lambda_3 = -0.0038, \quad \lambda_4 = -2.0004.$$  (F.12)

Obviously, deflections from (F.11) should be interpreted as inaccuracy of the numerical computations.
References


Appendix G
Effect of Noise in a System with a Hyperbolic Attractor

In this appendix some results are presented concerning effect of noise in a physically realizable flow system with a hyperbolic chaotic attractor of the Smale-Williams type (Jalnine and Kuznetsov, 2008).

Let us examine a system considered in Sect. 4.2 and add random external driving to get a set of stochastic equations

\[
\begin{align*}
\ddot{x} - (A \cos(2\pi t/T) - x^2)\dot{x} + \omega_0^2 x &= \varepsilon y \cos \omega_0 t + D\xi(t), \\
\ddot{y} - (-A \cos(2\pi t/T) - y^2)\dot{y} + 4\omega_0^2 y &= \varepsilon x^2 + D\xi(t).
\end{align*}
\]  

(G.1)

Here \( \xi(t) \) represents the Gaussian noise; it is assumed that \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)\xi(t-\tau) \rangle = \delta(\tau) \). Parameter \( D \) characterizes the noise intensity and can be varied in a wide range.

For numerical solution of the stochastic equations (G.1) we exploited a second-order method described in (Mannella and Palleschi, 1989). A plot in Fig. G.1(a) shows the results of computation for the noisy system at the parameters \( A = 3, \varepsilon = 0.5, \omega_0 = 2\pi, T = 10 \) and at the noise intensity \( D = 0.01 \). In gray color we show 100 superimposed samples of the process under effect of noise starting from identical initial conditions. Due to the noise, in the final part of the interval of observation the states appear to be essentially different because of instability intrinsic to the orbits on the chaotic attractor. As a result, the picture becomes fuzzy. For comparison, the curve shown in black color is related to the system without noise, starting from the same initial conditions.

It is easy to demonstrate qualitatively that a very similar picture is observed in the noiseless system if one considers an ensemble of samples with a slight deviation of the initial conditions. In Fig. G.1(b) we show a set of 100 samples for the system without noise launched from initial conditions with a small random variation near the same initial state as in the previous diagram. The range of the random variations is specially selected to obtain close degree of mutual divergence of the orbits in the considered time interval in comparison with that produced by the effect of noise.

As shown, for the noiseless system the phases determined at successive stages of activity for one of the subsystems obey approximately the Bernoulli map. It is
Fig. G.1 Time dependence for the variable $x$ of the model (G.1): (a) superimposed 100 samples obtained at the noise level $D = 0.02$ (gray) and the trajectory without noise (black), all with the same initial conditions; (b) superimposed 100 samples for the system without noise ($D = 0$) for the ensemble of slightly different initial conditions. Here and hereafter the parameters are chosen to be $A = 3.0$, $\varepsilon = 0.5$, $\omega_0 = 2\pi$, and $T = 10$.

Interesting to investigate the influence of noise on the iteration diagrams for phases. Taking into account that during the active stage the oscillations of $x(t)$ are close to sinusoidal ones with modulated amplitude and floating phase, we determine the phases at the time moments $t_n = t_0 + nT$ as $\phi_n = \arg(\dot{x}(t_n) + i\omega_0x(t_n))$. Figure G.2 shows the iteration diagrams for the phases in the presence of noise (gray) and without noise (black). Note that the presence of noise of sufficiently low intensity does not change the topological nature of the phase map, which remains in the same class as the Bernoulli map $\phi_{n+1} = 2\phi_n + \text{const}$.

Fig. G.2 Iteration diagrams for the phase of the first subsystem of the model (G.1) with noise $D = 0.1$. 
Another approach that allows us to compare the noisy and the noiseless dynamics is based on the analysis of the Lyapunov exponents. For the model with noise they can be computed using the same linearized variation equations as in Chap. 4, but integrated along the reference trajectory in the noisy system (G.1).

In Fig. G.3 the Lyapunov exponents are plotted versus the noise intensity parameter $D$. At low intensity of noise, the largest Lyapunov exponent is close to $T^{-1} \ln 2$, which corresponds to the approximate description of the dynamics by the Bernoulli map. A notable deflection appears only at a sufficiently high level of noise, namely, at $D \geq 0.2$. At $D \sim 0.5$ the effect of noise is already very relevant, the largest Lyapunov exponent crosses zero and becomes negative indicating suppression of the intrinsic dynamical chaos by the external noise. Dependence of other Lyapunov exponents on the noise intensity is not noticeable at all.

![Fig. G.3 Lyapunov exponents versus intensity of noise for the model (G.1).](image)

For a system with hyperbolic chaotic attractor, presence of noise of low intensity is in some sense inessential: under the slightly varying initial conditions, we can ensure that the phase trajectory of the system without noise remains close to the trajectory of the system with noise over arbitrarily large time interval. This assertion, based on the so-called shadowing lemma (Guckenheimer and Holmes, 1983), is justified in the context of the mathematical analysis of noise in (Kiefer, 1974). We have demonstrated this property in computations for the system (G.1).

Suppose two trajectories are launched from identical initial conditions, one in the “pure” system without noise and the other in the system with noise. Obviously, the “noisy” orbit will diverge from the “pure” one. Now, let us try to vary the initial conditions for the pure trajectory to get an approximation for the noisy orbit in the best way at a long time interval. The whole construction is performed for a definite sample of the noisy orbit obtained with the same sample of noise. To explain the method of selection of the initial conditions, it is appropriate to consider the Poincaré map produced by a period-$T$ stroboscopic section of the flow system (G.1) without noise $x_{n+1} = T(x_n)$. In Fig. G.4(a) one can see a portrait of the attractor of the map projected onto the plane $(x, \dot{x}/\omega_0)$. This is solenoid manifesting filaments of an infinite number of wraps possessing the Cantor-like structure in the cross-section. Solid
black dots in the picture denote four successive points of the stroboscopic section for one specially chosen trajectory on the attractor. The black lines passing through these dots designate the respective unstable direction $D^u$, which are tangent to the unstable manifolds $W^u$ at the given points. These directions can be approximated numerically from long-time evolution of an arbitrarily chosen unit perturbation vector $V_n = (\tilde{x}, \dot{\tilde{x}}/\omega_0, \tilde{y}, \dot{\tilde{y}}/2\omega_0)$ at $t_n = t_0 + nT$, since such a vector governed by the linearized variation equations tends to the unstable direction associated with a single positive Lyapunov exponent.

Now one should take a note of how a cloud of representative points evolves from the same initial conditions in the presence of noise. An illustration is shown in Fig. G.4(b). The dots pictured in gray are obtained in the Poincaré cross-section, i.e., stroboscopically at each next period $T$ from 104 sample orbits of the noisy system at $D = 0.02$. The black dots correspond to the trajectory of the system without noise ($D = 0$) launched from the same initial conditions. Note that the cloud stretches along the unstable direction tangent to the filaments forming the attractor, but it does not grow in the radial direction.

The algorithm for localization of the “pure” trajectory, which approximates a noisy orbit, consists in the following. Let us denote the original noisy orbit as $V_{\text{noisy}}(t)$, while the pure trajectory in zero approximation is denoted as $V^{(0)}(t)$. We start first from the initial conditions at $t = t_0$ identical to the noisy and the pure orbit: $V_{\text{noisy}}(t_0) = V^{(0)}(t_0)$. The starting point is supposed to belong to the attractor of the system without noise. Then, computing the two trajectories $V_{\text{noisy}}(t)$ and $V^{(0)}(t)$, we consider the norm of the difference vector $\|\Delta V^{(0)}(t_1)\| = \|V_{\text{noisy}}(t_1) - V^{(0)}(t_1)\|$ at $t_1 = t_0 + T$. Next, we slightly modify the initial condition for the pure orbit
from $V^{(0)}(t_0)$ to $V^{(G.1)}(t_0)$ with the purpose to minimize the norm of the difference $\|V_{\text{noisy}}(t_1) - V^{(1)}(t_1)\|$. It is provided by variation of the phase of the partial oscillator active at $t = t_0$. For our system it corresponds to variation of the initial state along the unstable direction associated with the point on the attractor, or, that is the same, along a filament of the attractor containing the initial point. The procedure of the search for new initial conditions is explained in detail in (Jalnine and Kuznetsov, 2008).

Suppose we take a new initial condition for the pure orbit $V^{(1)}(t_0)$. Now with the same sample of the noise we again trace two trajectories, the noisy one starting from $V_{\text{noisy}}(t_0)$ and the pure one starting from $V^{(1)}(t_0)$ for a longer time interval up to $t_2 = t_0 + 2T$, and obtain the norm of the difference vector $\|\Delta V^{(1)}(t_2)\| = \|V_{\text{noisy}}(t_2) - V^{(1)}(t_2)\|$.

Then, modifying again the initial condition for the pure orbit by variation of the initial phase of the active oscillator, at this time in closer neighborhood of $V^{(1)}(t_0)$, we try to minimize $\|V_{\text{noisy}}(t_2) - V^{(2)}(t_2)\|$ and obtain the new initial condition $V^{(2)}(t_0)$. The procedure of variation of the initial conditions is the same as the above-described, with only difference being that the maximum variation $\Delta V^{(2)}$ is chosen from the relation of $\Delta V^{(2)} = \|\Delta V^{(1)}(t_2)\| \exp(-2\lambda_1 T)$. Step by step we successively increase the time interval $nT$ and select the initial conditions for the pure orbit $V^{(n)}(t_0)$, which deliver minimal values for the norms of the differences $\|V_{\text{noisy}}(t_n) - V^{(n)}(t_n)\|$. Note that the maximum variation of the initial condition at the $n$-th step of the algorithm decays as $\|\Delta V^{(n)}\| \sim \exp(-n\lambda_1 T)$. Due to the recurrent nature of the algorithm, it appears that the approximation of the noisy orbit by the pure one holds uniformly during the whole time interval $[t_0, t_0 + nT]$.

Figure G.5 illustrates results of application of several steps of the above algorithm. Parameters of the system are taken the same as those in the previous examples of computations, and the noise intensity is $D = 0.1$.

The first plot in Fig. G.5 corresponds to the initial step of the algorithm: the noisy (gray) and pure (black) trajectories start from identical initial conditions. One can see their sufficiently fast divergence: the phase synchronism disappears already after one period of the parameter modulation $T$. After the first modification of the initial conditions for the pure orbit the divergence is delayed and the phase synchronism persists over $1 - 2$ periods of $T$, but then the orbits diverge. At the next steps of the algorithm the time intervals of existence of the phase synchronism become longer and longer and occupy finally the whole range in the diagram. Figure G.6 shows in gray color 20 dots of the projection of the noisy 5 orbit at $D = 0.04$ on the plane $(x, \dot{x}/\omega_0)$. In black color 20 dots are shown in the same stroboscopic cross-section of the approximating pure trajectory recovered by the above method.

To characterize the degree of closeness of a noisy orbit to a shadowing pure orbit, we use the following value

$$\rho_k = \frac{1}{kT} \int_{t_0}^{t_0+kT} \|V_{\text{noisy}}(t) - V^{(k)}(t)\| dt, \quad (G.2)$$

where $k$ designates the duration of the considered time interval in units of the modulation period $T$. Averaging this quantity over the ensemble of initial conditions and
the noise samples we obtain a mean deviation $\langle \rho_k \rangle$, characterizing the degree of closeness of the noisy and shadowing trajectories on the attractor. In Fig. G.7(a) we present plots of the mean deviation $\langle \rho_k \rangle$ versus $k$ for different noise levels. It can be seen in the figure that the mean deviation depends on $k$ very weakly in the range $k = 8, \ldots, 50$. At $k \sim 50 \div 60$ the errors become noticeable due to the finite-digital arithmetic and for the considered level of accuracy further observation of the shadowing becomes impossible.

Figure G.7(b) shows a plot of the mean deviation $\langle \rho_k \rangle$ computed for $k = 9$ in dependence on the parameter of the noise intensity $D$. For each $D$ we consider a set of 100 orbits launched from different initial conditions and calculated for different noise samples. Each noisy orbit was approximated by the shadowing trajectory with the method described above, and the mean value $\langle \rho_9 \rangle$ was evaluated by averaging over the set of orbits. As seen from the figure, the dependence $\langle \rho_9 \rangle$ versus $D$ is nearly linear in the range from zero to 0.1.

**Fig. G.5** Successive steps of constructing the shadowing trajectory (black) to the given noisy orbits (gray). At the initial step both orbits start from the identical initial conditions. At the last presented diagram the shadowing takes place over the time range of six modulation periods.
**Fig. G.6** Stroboscopic cross-section of the noisy (light gray) and the shadowing noiseless (black) trajectories on a background of the attractor (dark gray) in projection onto the phase plane of the first subsystem; the noise intensity \( D = 0.04 \).

**Fig. G.7** Mean average mutual deviation of noise and shadowing trajectories versus the number of modulation periods taken for the computations at the three noise levels \( D = 0.002, 0.01 \) and 0.02 \((a)\). The dependence of \( < \rho_k > \) on the noise intensity \( D \) for \( k = 9 \) \((b)\).

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